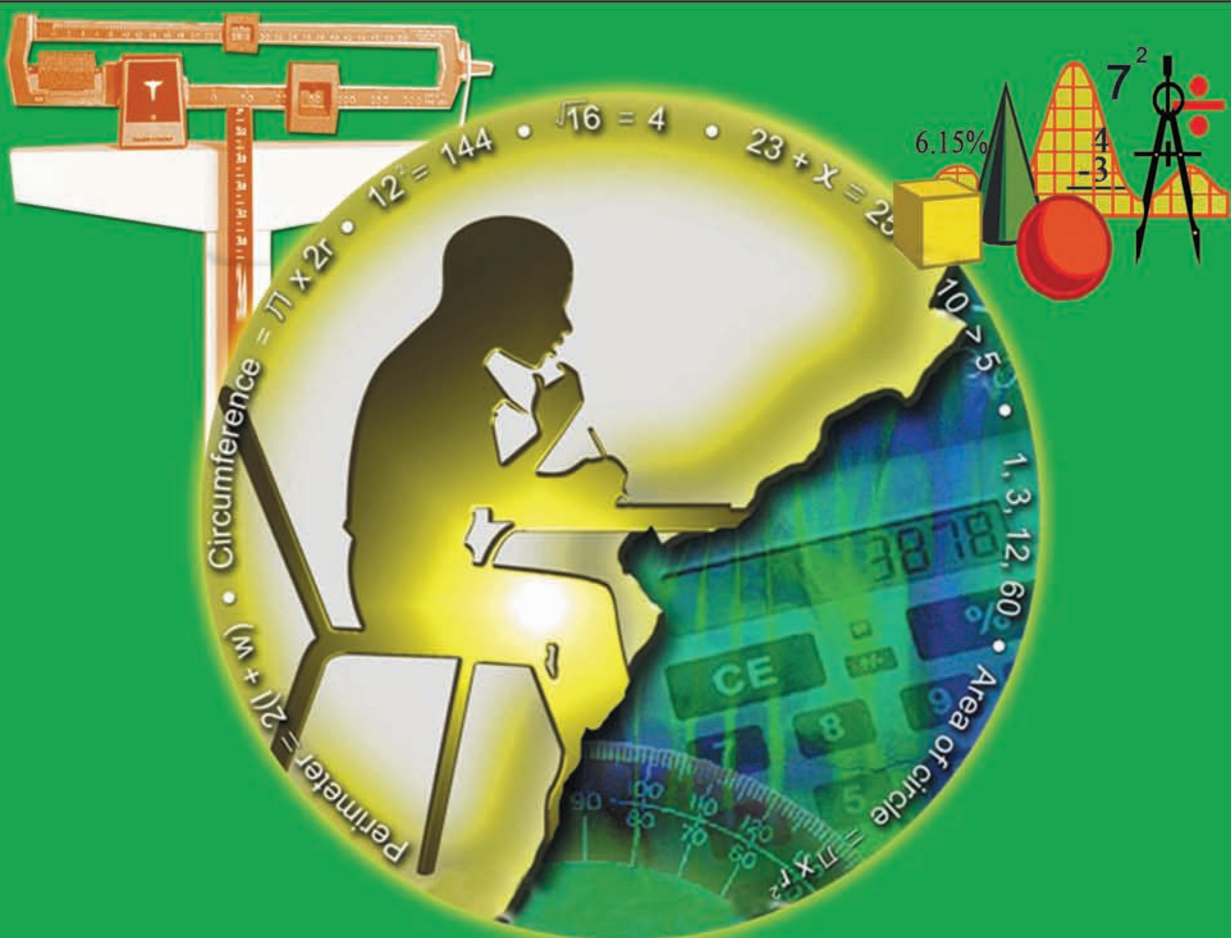
MATHEMATICS

STUDENT TEXTBOOK

GRADE 9



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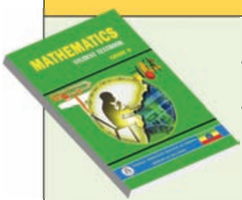

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GRADE 9

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Contents

Unit 1

	Numerical Value							
Arabic Numeral	1	2	3	5	10	20	21	100
Babylonian	▾	▾▾	▾▾▾	▾▾▾▾▾	<	<<	<<▾	▾▸
Egyptian Hieroglyphic	I	II	III	IIIII	Λ	ΛΛ	IΛΛ	ϑ
Greek Herodianic	I	II	III	Γ	Δ	ΔΔ	ΔΔI	H
Roman	I	II	III	V	X	XX	XXI	C
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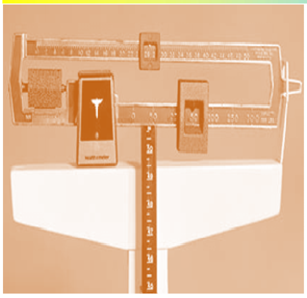

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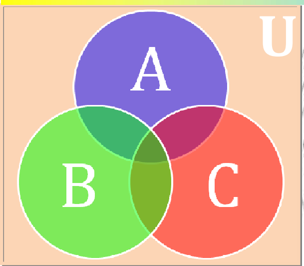

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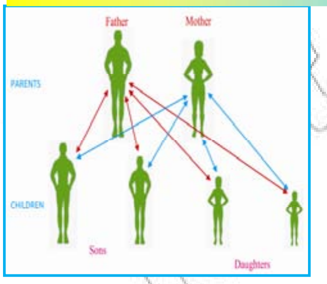

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Unit 1

	Numerical Value							
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Egyptian Hieroglyphic	I	II	III	IIIII	Λ	ΛΛ	IΛΛ	ϑ
Greek Herodianic	I	II	III	Γ	Δ	ΔΔ	ΔΔI	H
Roman	I	II	III	V	X	XX	XXI	C
Ethiopian Geez	፩	፪	፫	፭	፲	፳	፳፩	፻

THE NUMBER SYSTEM

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts and important facts about real numbers.*
- *justify methods and procedures in computation with real numbers.*
- *solve mathematical problems involving real numbers.*

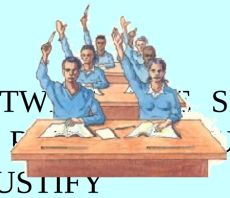
Main Contents

INTRODUCTION

IN EARLIER GRADES, YOU HAVE LEARNT ABOUT RATIONAL NUMBERS, THEIR PROPERTIES AND BASIC MATHEMATICAL OPERATIONS UPON THEM. AFTER A REVIEW OF YOUR KNOWLEDGE A NUMBERS, YOU WILL CONTINUE STUDYING THE NUMBER SYSTEMS IN THE PRESENT UN WILL LEARN ABOUT IRRATIONAL NUMBERS AND REAL NUMBERS, THEIR PROPERT OPERATIONS UPON THEM. ALSO, YOU WILL DISCUSS SOME RELATED CONCEPTS APPROXIMATION, ACCURACY, AND SCIENTIFIC NOTATION.

1.1 REVISION ON THE SET OF RATIONAL NUMBERS

ACTIVITY 1.1



THE DIAGRAM BELOW SHOWS THE RELATIONSHIPS BETWEEN THE SETS OF NATURAL NUMBERS, WHOLE NUMBERS, INTEGERS AND RATIONAL NUMBERS. USE THIS DIAGRAM TO ANSWER QUESTIONS 1 AND 2 GIVEN BELOW. JUSTIFY YOUR ANSWERS.

1 TO WHICH SET(S) OF NUMBERS DOES EACH OF THE FOLLOWING NUMBERS BELONG?

A 27 **B** -17 **C** $-7\frac{2}{3}$

D 0.625 **E** $0.\overline{615}$

2 **I** DEFINE THE SET OF:

A NATURAL NUMBERS

B WHOLE NUMBERS

C INTEGERS

D RATIONAL NUMBERS

II WHAT RELATIONS DO THESE SETS HAVE?

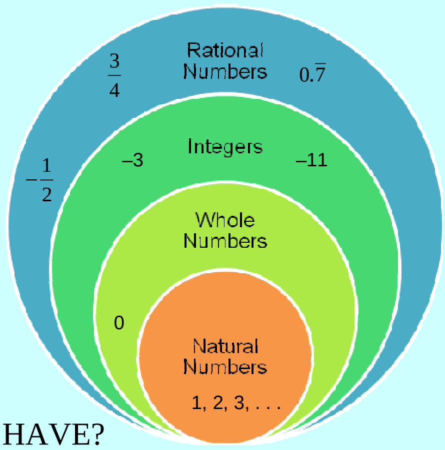


Figure 1.1

1.1.1 Natural Numbers, Integers, Prime Numbers and Composite Numbers

IN THIS SUBSECTION, YOU WILL REVISE IMPORTANT FACTS ABOUT NATURAL NUMBERS, PRIME NUMBERS, COMPOSITE NUMBERS AND INTEGERS. YOU HAVE LEARNT SEVERAL FACTS A IN PREVIOUS GRADES, IN GRADE 7 IN PARTICULAR. WORKING THROUGH ACTIVITY 1.2 BELOW WILL REFRESH YOUR MEMORY!

ACTIVITY 1.2

1 FOR EACH OF THE FOLLOWING STATEMENTS, WRITE "true" IF THE STATEMENT IS CORRECT OR "false" OTHERWISE. IF YOUR ANSWER IS "false", JUSTIFY IT BY GIVING A COUNTER EXAMPLE OR REASON.

A THE SET {1, 2, 3, . . .} DESCRIBES THE SET OF NATURAL NUMBERS.

B THE SET {1, 2, 3, . . .} ∪ {. . . −3, −2, −1} DESCRIBES THE SET OF INTEGERS.

C 57 IS A COMPOSITE NUMBER.

D {1} ∩ {PRIME NUMBERS} = ∅

E {PRIME NUMBERS} ∪ {COMPOSITE NUMBER} = {1, 2, 3, . . .}.

F {ODD NUMBERS} ∩ {COMPOSITE NUMBERS} ≠ ∅

G 48 IS A MULTIPLE OF 12.

H 5 IS A FACTOR OF 72.

I 621 IS DIVISIBLE BY 3.

J {FACTORS OF 24} ∩ {FACTORS OF 87} = {1, 2, 3}.

K {MULTIPLES OF 6} ∩ {MULTIPLES OF 4} = {12, 24}.

L $2^2 \times 3^2 \times 5$ IS THE PRIME FACTORIZATION OF 180.

2 GIVEN TWO NATURAL NUMBERS *a* AND *b*, WHAT IS MEANT BY:

A *a* IS A FACTOR OF *b* **B** *a* IS DIVISIBLE BY *b* **C** *a* IS A MULTIPLE OF *b*

FROM YOUR LOWER GRADE MATHEMATICS, RECALL THAT;

✓ THE SET OF NATURAL NUMBERS, DENOTED BY ℕ, IS DESCRIBED BY ℕ = {1, 2, 3,…}

✓ THE SET OF WHOLE NUMBERS, DENOTED BY W, IS DESCRIBED BY W = {0, 1, 2, 3,…}

✓ THE SET OF INTEGERS, DENOTED BY ℤ, IS DESCRIBED BY ℤ = {...,–3, –2, –1, 0, 1, 2, 3,...}

✓ GIVEN TWO NATURAL NUMBERS m AND p, m IS CALLED A multiple of p IF THERE IS A NATURAL NUMBER q SUCH THAT

$$m = p \times q.$$

IN THIS CASE, p IS CALLED A factor OR divisor OF m. WE ALSO SAY m IS DIVISIBLE BY p. SIMILARLY, q IS ALSO A FACTOR OR DIVISOR OF m AND m IS DIVISIBLE BY q.

FOR EXAMPLE, 621 IS A MULTIPLE OF 3 BECAUSE 621 = 3 × 207.

Definition 1.1 **Prime numbers and composite numbers**

- A natural number that has exactly two distinct factors, namely 1 and itself, is called a prime number.
- A natural number that has more than two factors is called a composite number.

Note: 1 IS NEITHER PRIME NOR COMPOSITE.

Group Work 1.1

1 LIST ALL FACTORS OF 24. HOW MANY FACTORS DI

2 THE AREA OF A RECTANGLE IS 432 SQ UNITS. THE ME S OFTHE LENGTH AND WIDTH OF THE RECTANGLE ARE EXPRESSED BY NATURAL NUMBERS.

FIND ALL THE POSSIBLE DIMENSIONS (LENGTH AND WIDTH) OF THE RECTANGLE.

3 FIND THE PRIME FACTORIZATION OF 360.

THE FOLLOWING RULES CAN HELP YOU TO DETERMINE WHETHER A NUMBER IS DIVISI 5, 6, 8, 9 OR 10.

Divisibility test

A NUMBER IS DIVISIBLE BY:

- ✓ 2, IF ITS UNIT'S DIGIT IS DIVISIBLE BY 2.
- ✓ 3, IF THE SUM OF ITS DIGITS IS DIVISIBLE BY 3.
- ✓ 4, IF THE NUMBER FORMED BY ITS LAST TWO DIGITS IS DIVISIBLE BY 4.
- ✓ 5, IF ITS UNIT'S DIGIT IS EITHER 0 OR 5.
- ✓ 6, IF IT IS DIVISIBLE BY 2 AND 3.
- ✓ 8, IF THE NUMBER FORMED BY ITS LAST THREE DIGITS IS DIVISIBLE BY 8.
- ✓ 9, IF THE SUM OF ITS DIGITS IS DIVISIBLE BY 9.
- ✓ 10, IF ITS UNIT'S DIGIT IS 0.

OBSERVE THAT DIVISIBILITY TEST FOR 7 IS NOT STATED HERE AS IT IS BEYOND THE SCOPE OF YOUR PRESENT LEVEL.

EXAMPLE 1 USE THE DIVISIBILITY TEST TO DETERMINE WHETHER 2,416 IS DIVISIBLE BY 2, 3, 4, 5, 6, 8, 9 AND 10.

SOLUTION:

- 2, 416 IS DIVISIBLE BY 2 BECAUSE THE UNIT'S DIGIT 6 IS DIVISIBLE BY 2.
- 2,416 IS DIVISIBLE BY 4 BECAUSE 16 (THE NUMBER FORMED BY THE LAST TWO IS DIVISIBLE BY 4.
- 2,416 IS DIVISIBLE BY 8 BECAUSE THE NUMBER FORMED BY THE LAST THREE DIGITS (416) IS DIVISIBLE BY 8.
- 2,416 IS NOT DIVISIBLE BY 5 BECAUSE THE UNIT'S DIGIT IS NOT 0 OR 5.
- SIMILARLY YOU CAN CHECKTHAT 2,416 IS NOT DIVISIBLE BY 3, 6, 9, AND 10.

THEREFORE, 2,416 IS DIVISIBLE BY 2, 4 AND 8 BUT NOT BY 3, 5, 6, 9 AND 10.

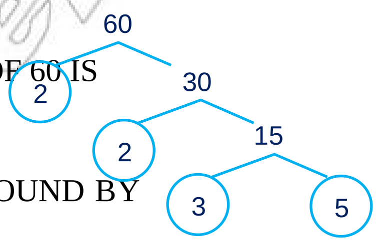
A FACTOR OF A COMPOSITE NUMBER IS CALLED A **prime factor**, IF IT IS A PRIME NUMBER. FOR INSTANCE, 2 AND 5 ARE BOTH PRIME FACTORS OF 20.

EVERY COMPOSITE NUMBER CAN BE WRITTEN AS A PRODUCT OF PRIME NUMBERS. TO FIN FACTORS OF ANY COMPOSITE NUMBER, BEGIN BY EXPRESSING THE NUMBER AS A PRO FACTORS WHERE AT LEAST ONE OF THE FACTORS IS PRIME. THEN, CONTINUE TO FACTORI COMPOSITE FACTOR UNTIL ALL THE FACTORS ARE PRIME NUMBERS.

WHEN A NUMBER IS EXPRESSED AS A PRODUCT OF ITS PRIME FACTORS, THE EXPRESSION **prime factorization** OF THE NUMBER.

FOR EXAMPLE, THE PRIME FACTORIZATION OF 60 IS

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5.$$



THE PRIME FACTORIZATION OF 60 IS ALSO FOUND BY USING A FACTORING TREE.

NOTE THAT THE SET {2, 3, 5} IS A SET OF PRIME FACTORS OF 60. IS THIS SET UNIQUE? THIS PROPERTY LEADS US TO STATE THE **Fundamental Theorem of Arithmetic**.

> **Theorem 1.1 Fundamental theorem of arithmetic**
>
> Every composite number can be expressed (factorized) as a product of primes. This factorization is unique, apart from the order in which the prime factors occur.

YOU CAN USE THE DIVISIBILITY TESTS TO CHECK WHETHER OR NOT A PRIME NUMBE GIVEN NUMBER.

EXAMPLE 2 FIND THE PRIME FACTORIZATION OF 1,530.

SOLUTION: START DIVIDING 1,530 BY ITS SMALLEST PRIME FACTOR. IF THE QUOTIENT IS A COMPOSITE NUMBER, FIND A PRIME FACTOR OF THE QUOTIENT IN THE SAME

REPEAT THE PROCEDURE UNTIL THE QUOTIENT IS A PRIME NUMBER AS SHOWN BEL

PRIME FACTORS

↓

$1,530 \div 2 = 765$

$765 \div 3 = 255$

$255 \div 3 = 85$

$85 \div 5 = 17$; AND 17 IS A PRIME NUMBE

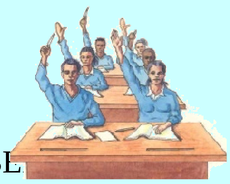
THEREFORE, $1,530 = 2 \times 3^2 \times 5 \times 17$.

1.1.2 Common Factors and Common Multiples

IN THIS SUBSECTION, YOU WILL REVISE THE CONCEPTS OF COMMON FACTORS AN MULTIPLES OF TWO OR MORE NATURAL NUMBERS. RELATED TO THIS, YOU WILL AI GREATEST COMMON FACTOR AND THE LEAST COMMON MULTIPLE OF TWO OR MORE NA

A Common factors and the greatest common factor

ACTIVITY 1.3



1 GIVEN THE NUMBERS 30 AND 45,

A FIND THE COMMON FACTORS OF THE TWO NUMBE

B FIND THE GREATEST COMMON FACTOR OF THE TWO NUMBERS.

2 GIVEN THE NUMBERS 36, 42 AND 48,

A FIND THE COMMON FACTORS OF THE THREE NUMBERS.

B FIND THE GREATEST COMMON FACTOR OF THE THREE NUMBERS.

GIVEN TWO OR MORE NATURAL NUMBERS, A NUMBER WHICH IS A FACTOR OF ALL OF T common factor. NUMBERS MAY HAVE MORE THAN ONE COMMON FACTOR. THE GREATEST OF THE COMMON FACTORS IS CALLED THE greatest common factor (GCF) OR THE highest common factor (HCF) OF THE NUMBERS.

➢ THE GREATEST COMMON FACTOR OF TWO NUMBERS a AND b IS DENOTED BY GCF (a, b).

EXAMPLE 1 FIND THE GREATEST COMMON FACTOR OF:

A 36 AND 60. **B** 32 AND 27.

SOLUTION:

A FIRST, MAKE LISTS OF THE FACTORS OF 36 AND 60, USING SETS.

LET F_{36} AND F_{60} BE THE SETS OF FACTORS OF 36 AND 60, RESPECTIVELY. THEN,

$F_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$F_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

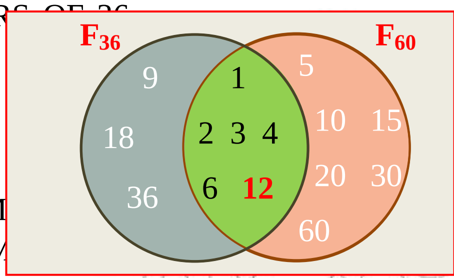


Figure 1.2

YOU CAN USE THE DIAGRAM TO SUM INFORMATION. NOTICE THAT THE COM ARE SHADED IN *green*. THEY ARE 1, 2, 3, 4, 6 AND 12 AND THE GREATEST IS 12.

I.E., $F_{36} \cap F_{60} = \{1, 2, 3, 4, 6, 12\}$

THEREFORE, GCF (36, 60) = 12.

B SIMILARLY,

$F_{32} = \{1, 2, 4, 8, 16, 32\}$ AND

$F_{27} = \{1, 3, 9, 27\}$

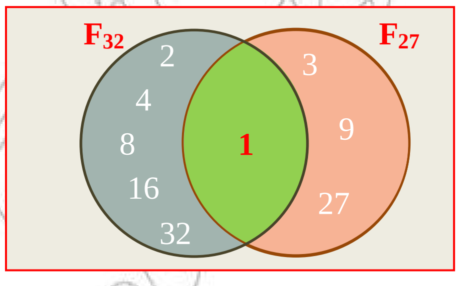


Figure 1.3

THEREFORE, $F_{32} \cap F_{27} = \{1\}$

THUS, GCF (32, 27) = 1

➢ TWO OR MORE NATURAL NUMBERS THAT HAVE A GCF OF 1 ARE CALLED **relatively prime**.

Definition 1.2

The greatest common factor (GCF) of two or more natural numbers is the greatest natural number that is a factor of all of the given numbers.

Group Work 1.2

LET a = 1800 AND b = 756

1 WRITE:

A THE PRIME FACTORIZATION OF a AND b

B THE PRIME FACTORS THAT ARE COMMON TO BOTH a AND b.

NOW LOOK AT THESE COMMON PRIME FACTORS; THE LOWEST POWER OF EACH (IN EITHER OF THE TWO PRIME FACTORIZATIONS) SHOULD BE 2^2 AND 3^2.

C WHAT IS THE PRODUCT OF THESE LOWEST POWERS?

D WRITE DOWN THE HIGHEST POWERS OF THE COMMON PRIME FACTORS.

E WHAT IS THE PRODUCT OF THESE HIGHEST POWERS?

2 **A** COMPARE THE RESULT OF 1C WITH THE GCF OF THE GIVEN NUMBERS. ARE THEY THE SAME?

B COMPARE THE RESULT OF 1E WITH THE GCF OF THE GIVEN NUMBERS. ARE THEY THE SAME?

THE ABOVE GROUP WORK LEADS YOU TO ANOTHER ALTERNATIVE METHOD TO FIND THE GCF OF NUMBERS. THIS METHOD (WHICH IS A QUICKER WAY TO FIND THE GCF) IS CALLED THE **prime factorization method**. IN THIS METHOD, THE GCF OF A GIVEN SET OF NUMBERS IS THE PRODUCT OF THEIR COMMON PRIME FACTORS, EACH POWER TO THE SMALLEST NUMBER APPEARS IN THE PRIME FACTORIZATION OF ANY OF THE NUMBERS.

EXAMPLE 2 USE THE PRIME FACTORIZATION METHOD TO FIND GCF (180, 216, 540).

SOLUTION:

Step 1 EXPRESS THE NUMBERS 180, 216 AND 540 IN THEIR PRIME FACTORIZATION.

$180 = 2^2 \times 3^2 \times 5$; $216 = 2^3 \times 3^3$; $540 = 2^2 \times 3^3 \times 5$

Step 2 AS YOU SEE FROM THE PRIME FACTORIZATIONS OF 180, 216 AND 540, THE NUMBERS 2 AND 3 ARE COMMON PRIME FACTORS.

SO, GCF (180, 216, 540) IS THE PRODUCT OF THESE COMMON PRIME FACTORS WITH THE SMALLEST RESPECTIVE EXPONENTS IN ANY OF THE NUMBERS.

$\therefore$ GCF (180, 216, 540) $= 2^2 \times 3^2 = 36$.

B Common multiples and the least common multiple

Group Work 1.3

FOR THIS GROUP WORK, YOU NEED 2 COLOURED PENCILS.

Work with a partner

Try this:

* LIST THE NATURAL NUMBERS FROM 1 TO 100 ON A SHEET OF PAPER.
* CROSS OUT ALL THE MULTIPLES OF 10.
* USING A DIFFERENT COLOUR, CROSS OUT ALL THE MULTIPLES OF 8.

Discuss:

1 WHICH NUMBERS WERE CROSSED OUT BY BOTH COLOURS?

2 HOW WOULD YOU DESCRIBE THESE NUMBERS?

3 WHAT IS THE LEAST NUMBER CROSSED OUT BY BOTH COLOURS? WHAT DO YOU CALL THIS NUMBER?

Definition 1.3

For any two natural numbers a and b, the least common multiple of a and b denoted by LCM (a, b), is the smallest multiple of both a and b.

EXAMPLE 3 FIND LCM (8, 9).

SOLUTION: LET M_8 AND M_9 BE THE SETS OF MULTIPLES OF 8 AND 9 RESPECTIVELY.

$$M_8 = \{8, 16, 24, 32, 40, 48, 56, 64, \boxed{72}, 80, 88, \ldots\}$$

$$M_9 = \{9, 18, 27, 36, 45, 54, 63, \boxed{72}, 81, 90, \ldots\}$$

THEREFORE LCM (8, 9) = 72

PRIME FACTORIZATION CAN ALSO BE USED TO FIND THE LCM OF A SET OF TWO OR MC NUMBERS. A COMMON MULTIPLE CONTAINS ALL THE PRIME FACTORS OF EACH NUMB THE LCM IS THE PRODUCT OF EACH OF THESE PRIME FACTORS TO THE GREATEST NUME APPEARS IN THE PRIME FACTORIZATION OF THE NUMBERS.

EXAMPLE 4 USE THE PRIME FACTORIZATION METHOD TO FIND LCM (9, 21, 24).

SOLUTION:

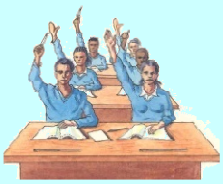
$$\left.\begin{aligned} 9 &= 3 \times 3 = 3^2 \\ 21 &= 3 \times 7 \\ 24 &= 2 \times 2 \times 2 \times 3 = 2^3 \times 3 \end{aligned}\right\}$$

THE PRIME FACTORS THAT APPEAR IN THESE FACTORIZATIONS ARE 2, 3 AND 7.

CONSIDERING THE GREATEST NUMBER OF TIMES EACH PRIME FACTOR APPEARS, WE CAN GET 2^3, 3^2 AND 7, RESPECTIVELY.

THEREFORE, LCM (9, 21, 24) = $2^3 \times 3^2 \times 7 = 504$.

ACTIVITY 1.4



1 FIND:

- **A** THE GCF AND LCM OF 36 AND 48
- **B** GCF (36, 48) × LCM (36, 48)
- **C** 36 × 48

2 DISCUSS AND GENERALIZE YOUR RESULTS.

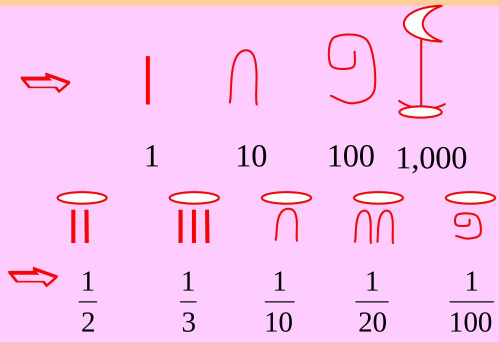
➢ FOR ANY NATURAL NUMBERS a AND b, GCF (a, b) × LCM (a, b) = $a \times b$.

1.1.3 Rational Numbers

HISTORICAL NOTE:

About 5,000 years ago, Egyptians used hieroglyphics to represent numbers.

The Egyptian concept of fractions was mostly limited to fractions with numerator 1. The hieroglyphic was placed under the symbol to indicate the number as a denominator. Study the examples of Egyptian fractions.



RECALL THAT THE SET OF INTEGERS IS GIVEN BY

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

USING THE SET OF INTEGERS, WE DEFINE THE SET OF RATIONAL NUMBERS AS FOLLOWS:

Definition 1.4 **Rational number**

Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, is called a rational number. The set of rational numbers, denoted by $\mathbb{Q}$, is the set described by

$$\mathbb{Q} = \left\{\frac{a}{b} : a \text{ AND } b \text{ ARE INTEGERS AND } b \neq 0\right\}.$$

THROUGH THE FOLLOWING DIAGRAM, YOU CA
SHOW HOW SETS WITHIN RATIONAL NUMBE
RELATED TO EACH OTHER. NOTE THAT
NUMBERS, WHOLE NUMBERS AND INTE
INCLUDED IN THE SET OF RATIONAL NU
IS BECAUSE INTEGERS SUCH AS 4 AND –7 CAN BE
WRITTEN AS $\frac{4}{1}$ AND $\frac{-7}{1}$.

THE SET OF RATIONAL NUMBERS ALSO
TERMINATING AND REPEATING DECIMAL
BECAUSE TERMINATING AND REPEATING D
CAN BE WRITTEN AS FRACTIONS.

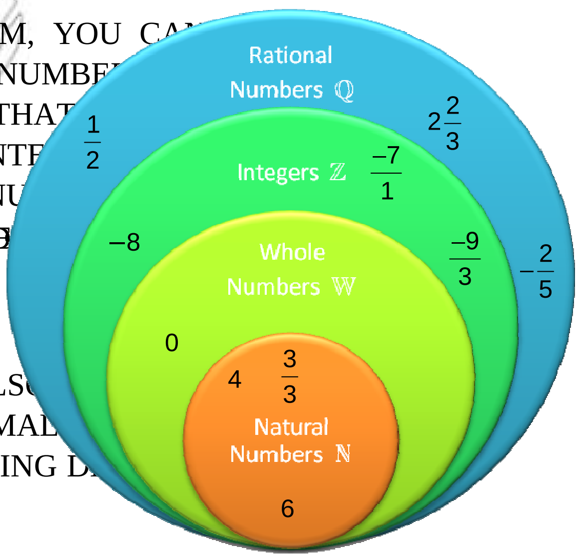


Figure 1.4

FOR EXAMPLE, −1.3 CAN BE WRITTEN AS $\frac{-13}{10}$ AND −0.29 AS $\frac{-29}{100}$.

MIXED NUMBERS ARE ALSO INCLUDED IN THE SET OF RATIONAL NUMBERS BECAUSE ANY

$\frac{30}{45} = \frac{2\times3\times5}{3\times3\times5} = \frac{2}{3}$. CAN BE WRITTEN AS AN improper fraction.

FOR EXAMPLE, $2\frac{2}{3}$ CAN BE WRITTEN AS $\frac{8}{3}$.

WHEN A RATIONAL NUMBER IS EXPRESSED AS A FRACTION, IT IS OFTEN EXPRESSED IN SIMPLEST FORM (LOWEST TERMS). A FRACTION $\frac{a}{b}$ IS IN SIMPLEST FORM WHEN GCF $(a, b) = 1$.

EXAMPLE 1 WRITE $\frac{30}{45}$ IN SIMPLEST FORM.

SOLUTION: $\frac{30}{45} = \frac{2\times3\times5}{3\times3\times5} = \frac{2}{3}$. (BY FACTORIZATION AND CANCELLATION)

HENCE, $\frac{30}{45}$ WHEN EXPRESSED IN LOWEST TERMS (SIMPLEST FORM) IS $\frac{2}{3}$.

Exercise 1.1

1. DETERMINE WHETHER EACH OF THE FOLLOWING NUMBERS IS PRIME OR COMPOSITE:
   - **A** 45 **B** 23 **C** 91 **D** 153
2. PRIME NUMBERS THAT DIFFER BY TWO ARE CALLED TWIN PRIMES.
   - **I** WHICH OF THE FOLLOWING PAIRS ARE TWIN PRIMES?
     - **A** 3 AND 5 **B** 13 AND 17 **C** 5 AND 7
   - **II** LIST ALL PAIRS OF TWIN PRIMES THAT ARE LESS THAN 30
3. DETERMINE WHETHER EACH OF THE FOLLOWING NUMBERS IS DIVISIBLE BY 2, 3, 4, 5, 6, 8, 9 OR 10:
   - **A** 48 **B** 153 **C** 2,470
   - **D** 144 **E** 12,357
4. **A** IS 3 A FACTOR OF 777? **B** IS 989 DIVISIBLE BY 9?
   - **C** IS 2,348 DIVISIBLE BY 4?
5. FIND THREE DIFFERENT WAYS TO WRITE 84 AS A PRODUCT OF TWO NATURAL NUMBERS.
6. FIND THE PRIME FACTORIZATION OF:
   - **A** 25 **B** 36 **C** 117 **D** 3,825

7 IS THE VALUE OF $a + 3b$ PRIME OR COMPOSITE WHEN $a = 11$ AND $b = 7$?

8 WRITE ALL THE COMMON FACTORS OF 30 AND 42.

9 FIND:

A GCF (24, 36) **B** GCF (35, 49, 84)

10 FIND THE GCF OF $2^2 \times 3^3 \times 5^2$ AND $2^3 \times 3 \times 5^2$.

11 WRITE THREE NUMBERS THAT HAVE A GCF OF 7.

12 LIST THE FIRST SIX MULTIPLES OF EACH OF THE FOLLOWING NUMBERS:

A 7 **B** 5 **C** 14 **D** 25 **E** 150

13 FIND:

A LCM (12, 16) **B** LCM (10, 12, 14)

C LCM (15, 18) **D** LCM (7, 10)

14 WHEN WILL THE LCM OF TWO NUMBERS BE THE PRODUCT OF THE NUMBERS?

15 WRITE EACH OF THE FOLLOWING FRACTIONS IN SIMPLEST FORM:

A $-\frac{3}{9}$ **B** $\frac{24}{120}$ **C** $\frac{48}{72}$ **D** $\frac{72}{98}$

16 HOW MANY FACTORS DOES EACH OF THE FOLLOWING NUMBERS HAVE?

A 12 **B** 18 **C** 24 **D** 72

17 FIND THE VALUE OF AN ODD NATURAL NUMBER x IF LCM $(x, 40) = 1400$.

18 THERE ARE BETWEEN 50 AND 60 EGGS IN A BASKET. WHEN ASKALE MAMMEN COUNTS BY 3'S, THERE ARE 2 EGGS LEFT OVER. WHEN HE COUNTS BY 5'S THERE ARE 4 LEFT OVER. HOW MANY EGGS ARE THERE IN THE BASKET?

19 THE GCF OF TWO NUMBERS IS 3 AND THE LCM IS 180. IF ONE OF THE NUMBERS IS 45, WHAT IS THE OTHER NUMBER?

20 **I** LET a, b, c, d BE NON-ZERO INTEGERS. SHOW THAT EACH OF THE FOLLOWING IS A RATIONAL NUMBER:

A $\frac{a}{b}+\frac{c}{d}$ **B** $\frac{a}{b}-\frac{c}{d}$ **C** $\frac{a}{b}\times\frac{c}{d}$ **D** $\frac{a}{b}\div\frac{c}{d}$

WHAT DO YOU CONCLUDE FROM THESE RESULTS?

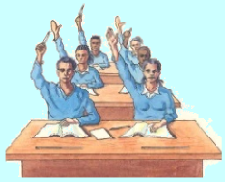
II FIND TWO RATIONAL NUMBERS BETWEEN $\frac{1}{3}$ AND $\frac{3}{4}$.

1.2 THE REAL NUMBER SYSTEM

1.2.1 Representation of Rational Numbers by Decimals

IN THIS SUBSECTION, YOU WILL LEARN HOW TO EXPRESS RATIONAL NUMBERS IN THE FORM OF FRACTIONS AND DECIMALS.

ACTIVITY 1.5



1. A WHAT DO WE MEAN BY A 'DECIMAL NUMBER'?
   B GIVE SOME EXAMPLES OF DECIMAL NUMBERS.
2. HOW DO YOU REPRESENT $\frac{3}{4}$ AND $\frac{1}{3}$ AS DECIMALS?
3. CAN YOU WRITE 0.4 AND 1.34 AS THE RATIO (QUOTIENT) OF TWO INTEGERS?

REMEMBER THAT A FRACTION IS ANOTHER WAY OF WRITING DIVISION OF ONE QUANTI ANY FRACTION OF NATURAL NUMBERS CAN BE EXPRESSED AS A DECIMAL BY D NUMERATOR BY THE DENOMINATOR.

EXAMPLE 1 SHOW THAT $\frac{3}{8}$ AND $\frac{7}{12}$ CAN EACH BE EXPRESSED AS A DECIMAL.

SOLUTION: $\frac{3}{8}$ MEANS $3 \div 8$ $\qquad$ $\frac{7}{12}$ MEANS $7 \div 12$

$$\begin{array}{r|l} & 0.375 \\ \hline 8 & 3.000 \\ & \underline{24} \\ & \;\;60 \\ & \;\;\underline{56} \\ & \;\;\;\;40 \\ & \;\;\;\;\underline{40} \\ & \;\;\;\;\;\;0 \end{array} \qquad \begin{array}{r|l} & 0.5833\ldots \\ \hline 12 & 7.0000 \\ & \underline{60} \\ & 100 \\ & \;\;\underline{96} \\ & \;\;\;\;40 \\ & \;\;\;\;\underline{36} \\ & \;\;\;\;\;\;40 \\ & \;\;\;\;\;\;\underline{36} \\ & \;\;\;\;\;\;\;\;4 \end{array}$$

$\therefore \frac{3}{8} = 0.375$ $\qquad$ $\therefore \frac{7}{12} = 0.5833\ldots$

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THE FRACTION (RATIONAL NUMBER) $\frac{3}{8}$ CAN BE EXPRESSED AS THE DECIMAL 0.375. A DECIMAL LIKE 0.375 IS CALLED A terminating decimal BECAUSE THE DIVISION ENDS OR TERMINATES, WHEN THE REMAINDER IS ZERO.

THE FRACTION $\frac{7}{12}$ CAN BE EXPRESSED AS THE DECIMAL 0.58333… (HERE, THE DIGIT 3 REPEATS AND THE DIVISION DOES NOT TERMINATE.) A DECIMAL LIKE 0.58333… IS CALLED A repeating decimal. TO SHOW A REPEATING DIGIT OR A BLOCK OF REPEATING DIGITS IN A REPEATING DECIMAL NUMBER, WE PUT A BAR ABOVE THE REPEATING DIGIT (OR BLOCK OF DIGITS). FOR EXAMPLE, 0.58333… CAN BE WRITTEN AS $0.58\overline{3}$, AND 0.0818181… CAN BE WRITTEN AS $0.0\overline{81}$. THIS METHOD OF WRITING A REPEATING DECIMAL IS KNOWN AS bar notation.

THE PORTION OF A DECIMAL THAT REPEATS IS CALLED THE repetend. FOR EXAMPLE,

IN 0.583333… $= 0.58\overline{3}$, THE REPETEND IS 3.

IN 1.777… $= 1.\overline{7}$, THE REPETEND IS 7.

IN 0.00454545… $= 0.00\overline{45}$, THE REPETEND IS 45.

TO GENERALIZE:

ANY RATIONAL NUMBER $\frac{a}{b}$ CAN BE EXPRESSED AS A DECIMAL BY DIVIDING THE NUMERATOR a BY THE DENOMINATOR b.

WHEN YOU DIVIDE a BY b, ONE OF THE FOLLOWING TWO CASES WILL OCCUR.

Case 1 THE DIVISION PROCESS ENDS OR TERMINATES WHEN A REMAINDER OF ZERO IS OBTAINED. IN THIS CASE, THE DECIMAL IS CALLED A terminating decimal.

Case 2 THE DIVISION PROCESS DOES NOT TERMINATE AND THE REMAINDER NEVER BECOMES ZERO. SUCH A DECIMAL IS CALLED A repeating decimal.

Expressing terminating and repeating decimals as fractions

- EVERY TERMINATING DECIMAL CAN BE EXPRESSED AS A FRACTION (A RATIO OF TWO INTEGERS) WITH A DENOMINATOR OF 10, 100, 1000 AND SO ON.

EXAMPLE 2 EXPRESS EACH OF THE FOLLOWING DECIMALS AS A FRACTION IN ITS SIMPLEST FORM (LOWEST TERMS):

A 0.85 **B** 1.3456

SOLUTION:

A $0.85 = 0.85 \times \frac{100}{100} = \frac{85}{100} = \frac{17}{20}$ (WHY?)

B $1.3456 = 1.3456 \times \frac{10000}{10000} = 1.3456 \times \frac{10^4}{10^4} = \frac{13456}{10000} = \frac{841}{625}$

➢ IF d IS A TERMINATING DECIMAL NUMBER THAT HAS n DIGITS AFTER A DECIMAL POINT, THEN WE REWRITE d AS

$$d = \frac{10^n \times d}{10^n}$$

THE RIGHT SIDE OF THE EQUATION GIVES THE FRACTIONAL FORM OF d.

FOR EXAMPLE, IF $d = 2.128$, THEN $n = 3$.

$$\therefore \quad 2.128 = \frac{10^3 \times 2.128}{10^3} = \frac{2128}{1000} = \frac{266}{125}$$

✓ REPEATING DECIMALS CAN ALSO BE EXPRESSED AS FRACTIONS (RATIO OF TWO INTEGERS).

EXAMPLE 3 EXPRESS EACH OF THE FOLLOWING DECIMALS AS A FRACTION (RATIO OF TWO INTEGERS):

A $0.\overline{7}$ **B** $0.\overline{25}$

SOLUTION: A LET $d = 0.\overline{7} = 0.777\ldots$ THEN,

$10d = 7.777\ldots$ (*multiplying d by* 10 *because* 1 *digit repeats*)

SUBTRACT $d = 0.777\ldots$ (*to eliminate the repeating part* 0.777…)

$10d = 7.777\ldots$ 1

$1d = 0.777\ldots$ 2 ($d = 1d$)

$9d = 7$ (*subtracting expression* 2 *from expression* 1)

$\therefore d = \frac{7}{9}$ (*dividing both sides by* 9)

HENCE $0.\overline{7} = \frac{7}{9}$

B LET $d = 0.\overline{25} = 0.252525\ldots$

THEN, $100d = 25.2525\ldots$ (*multiplying d by* 100 *because* 2 *digits repeat*)

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$$\begin{aligned} 100d &= 25.252525\ldots \\ 1d &= \ \ 0.252525\ldots \\ \hline 99d &= 25 \\ \therefore d &= \frac{25}{99} \end{aligned}$$

(*subtracting* 1*d* *from* 100*d* *eliminates the repeating part* 0.2525…)

$$\text{SO, } 0.\overline{25} = \frac{25}{99}$$

IN EXAMPLE 3A, ONE DIGIT REPEATS. SO, YOU MULTIPLIED d BY 10. IN EXAMPLE 3B, TWO DIGITS REPEAT. SO YOU MULTIPLIED d BY 100.

THE ALGEBRA USED IN THE ABOVE EXAMPLE CAN BE GENERALIZED AS FOLLOWS:

➢ IN GENERAL, IF d IS A REPEATING DECIMAL WITH k NON-REPEATING AND p REPEATING DIGITS AFTER THE DECIMAL POINT, THEN THE FORMULA

$$d = \frac{d\left(10^{k+p} - 10^{k}\right)}{10^{k+p} - 10^{k}}$$

IS USED TO CHANGE THE DECIMAL TO THE FRACTIONAL FORM OF d.

EXAMPLE 4 EXPRESS THE DECIMAL $0.3\overline{75}$ AS A FRACTION.

SOLUTION: LET $d = 0.3\overline{75}$, THEN,

$k = 1$ (*number of non-repeating digits*)

$p = 2$ (*number of repeating digits*) AND

$k + p = 1 + 2 = 3.$

$$\Rightarrow d = \frac{d\left(10^{k+p} - 10^{k}\right)}{\left(10^{k+p} - 10^{k}\right)} = \frac{d\left(10^{3} - 10^{1}\right)}{\left(10^{3} - 10^{1}\right)} = \frac{10^{3}d - 10d}{10^{3} - 10}$$

$$= \frac{10^{3} \times 0.3\overline{75} - 10 \times 0.3\overline{75}}{990}$$

$$= \frac{375.\overline{75} - 3.\overline{75}}{990} = \frac{372}{990}$$

FROM EXAMPLES 1, 2, 3 AND 4, YOU CONCLUDE THE FOLLOWING:

I EVERY RATIONAL NUMBER CAN BE EXPRESSED AS EITHER A TERMINATING DECIMAL OR A REPEATING DECIMAL.

II EVERY TERMINATING OR REPEATING DECIMAL REPRESENTS A RATIONAL NUMBER.

Exercise 1.2

1 EXPRESS EACH OF THE FOLLOWING RATIONAL NUMBERS AS A DECIMAL:

A $\frac{4}{9}$ **B** $\frac{3}{25}$ **C** $\frac{11}{7}$ **D** $-5\frac{2}{3}$ **E** $\frac{3706}{100}$ **F** $\frac{22}{7}$

2 WRITE EACH OF THE FOLLOWING AS A DECIMAL AND THEN AS A FRACTION IN ITS LOWEST TERM:

A THREE TENTHS **B** FOUR THOUSANDTHS

C TWELVE HUNDREDTHS **D** THREE HUNDRED AND SIXTY NINE THOUSANDTHS.

3 WRITE EACH OF THE FOLLOWING IN METRES FIRST AS A FRACTION AND THEN AS A DECIMAL:

A 4 MM **B** 6 CM AND 4 MM **C** 56 CM AND 4 MM

Hint: RECALL THAT 1 METRE(M) = 100 CENTIMETRES(CM) = 1000 MILLIMETRES(MM).

4 FROM EACH OF THE FOLLOWING FRACTIONS, IDENTIFY THOSE THAT CAN BE EXPRESSED AS TERMINATING DECIMALS:

A $\frac{5}{13}$ **B** $\frac{7}{10}$ **C** $\frac{69}{64}$ **D** $\frac{11}{60}$

E $\frac{11}{80}$ **F** $\frac{17}{125}$ **G** $\frac{5}{12}$ **H** $\frac{4}{11}$

GENERALIZE YOUR OBSERVATION.

5 EXPRESS EACH OF THE FOLLOWING DECIMALS AS A FRACTION OR MIXED NUMBER IN SIMPLEST FORM:

A 0.88 **B** $0.7\overline{7}$ **C** $0.8\overline{3}$ **D** 7.08 **E** $0.52\overline{52}$ **F** $-1.00\overline{3}$

6 EXPRESS EACH OF THE FOLLOWING DECIMALS USING BAR NOTATION:

A 0.454545… **B** 0.1345345…

7 EXPRESS EACH OF THE FOLLOWING DECIMALS WITHOUT USING BAR NOTATION. (IN EACH CASE USE AT LEAST TEN DIGITS AFTER THE DECIMAL POINT)

A $0.\overline{13}$ **B** $-0.\overline{305}$ **C** $0.3\overline{81}$

8 VERIFY EACH OF THE FOLLOWING COMPUTATIONS BY CONVERTING THE DECIMALS TO FRACTIONS:

A $0.\overline{275} + 0.\overline{714} = 0.\overline{989}$ **B** $0.\overline{6} - 1.\overline{142857} = -0.\overline{476190}$

1.2.2 Irrational Numbers

REMEMBER THAT TERMINATING OR REPEATING DECIMALS ARE RATIONAL NUMBERS, SI BE EXPRESSED AS FRACTIONS. THE SQUARE ROOTS OF PERFECT SQUARES ARE ALSO RAT FOR EXAMPLE, $\sqrt{4}$ IS A RATIONAL NUMBER SINCE $\sqrt{4} = 2 = \frac{2}{1}$. SIMILARLY, $\sqrt{0.09}$ IS A RATIONAL NUMBER BECAUSE $\sqrt{0.09} = 0.3$ IS A RATIONAL NUMBER.

IF $x^2 = 4$, THEN WHAT DO YOU THINK IS THE VALUE OF x? $x = \pm\sqrt{4} = \pm 2$. THEREFORE x IS A RATIONAL NUMBER. WHAT IF $x^2 = 3$?

IN FIGURE 1.4 OF SECTION 1.1.3, WHERE DO NUMBERS LIKE $\sqrt{2}$ AND $\sqrt{5}$ FIT? NOTICE WHAT HAPPENS WHEN YOU FIND $\sqrt{2}$ AND $\sqrt{5}$ WITH YOUR CALCULATOR:

Study Hint

MOST CALCULATORS ROUND ANSWERS BUT SOME TRUNCATE ANSWERS. I.E., THEY CUT OFF AT A CERTAIN POINT, IGNORING SUBSEQUENT DIGITS.

If you first press the button 2 and then the square-root button, you will find $\sqrt{2}$ *on the display.*

I.E., $\sqrt{2}$: 2 √ = 1.414213562…

$\sqrt{5}$: 5 √ = 2.236067977…

NOTE THAT MANY SCIENTIFIC CALCULATORS, SUCH AS CASIO ONES, WORK THE SAME AS THE WRITTEN ORDER, I.E., INSTEAD OF PRES 2 AND THEN THE √ BUTTON, YOU PRESS THE √ BUTTON AND THEN 2. BEFORE USING ANY CALCULATOR, IT IS ALWAYS ADVISABLE T THE USER'S MANUAL.

NOTE THAT THE DECIMAL NUMBERS FOR $\sqrt{2}$ AND $\sqrt{5}$ DO NOT TERMINATE, NOR DO THEY HAVE A PATTERN OF REPEATING DIGITS. THEREFORE, THESE NUMBERS ARE NOT RATIONAL N NUMBERS ARE CALLED irrational numbers. IN GENERAL, IF a IS A NATURAL NUMBER THAT IS NOT A PERFECT SQUARE, THEN $\sqrt{a}$ IS AN IRRATIONAL NUMBER.

EXAMPLE 1 DETERMINE WHETHER EACH OF THE FOLLOWING NUMBERS IS RATIONAL OR IRRATIONAL.

A 0.16666 . . . **B** 0.1611611161111611111 6 . . . **C**

SOLUTION: **A** IN 0.16666 . . . THE DECIMAL HAS A REPEATING PATTERN. IT IS A RATIONAL NUMBER AND CAN BE EXPRESSED AS $\frac{1}{6}$.

B THIS DECIMAL HAS A PATTERN THAT NEITHER TERMINATES NOR REPEATS. IT IS AN IRRATIONAL NUMBER.

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C = 3.1415926… THIS DECIMAL DOES NOT REPEAT OR TERMINATE. IT IS A IRRATIONAL NUMBER. (*The fraction* $\frac{22}{7}$ *is an approximation to the value of* . *It is not the exact value!*).

IN EXAMPLE 1, B AND C LEAD US TO THE FOLLOWING FACT:

➢ A DECIMAL NUMBER THAT IS NEITHER TERMINATING NOR REPEATING IS AN irrational number.

1 Locating irrational numbers on the number line

Group Work 1.4

You will need a compass and straight edge to perform the following:

1 To locate $\sqrt{2}$ on the number line:

- DRAW A NUMBER LINE. AT THE POINT CORRESPONDING TO 1 ON THE NUMBER LINE, CONSTRUCT A PERPENDICULAR LINE SEGMENT 1 UNIT LONG.
- DRAW A LINE SEGMENT FROM THE POINT CORRESPONDING TO 0 TO THE TOP OF THE 1 UNIT SEGMENT AND LABEL IT AS c.

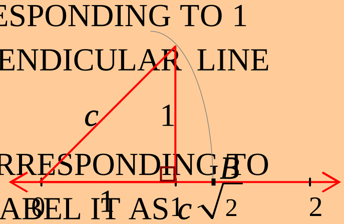


Figure 1.5

- USE THE PYTHAGOREAN THEOREM TO SHOW THAT c IS $\sqrt{2}$ UNITS LONG.
- OPEN THE COMPASS TO THE LENGTH OF c. WITH THE TIP OF THE COMPASS AT THE POINT CORRESPONDING TO 0, DRAW AN ARC THAT INTERSECTS THE NUMBER LINE AT B. THE DISTANCE FROM THE POINT CORRESPONDING TO 0 TO B IS $\sqrt{2}$ UNITS.

2 To locate $\sqrt{5}$ on the number line:

- FIND TWO NUMBERS WHOSE SQUARES HAVE A SUM OF 5. ONE PAIR THAT WORKS IS 1 AND 2, SINCE $1^2 + 2^2 = 5$.
- DRAW A NUMBER LINE. AT THE POINT CORRESPONDING TO 2, ON THE NUMBER LINE, CONSTRUCT A PERPENDICULAR LINE SEGMENT 1 UNIT LONG.
- DRAW THE LINE SEGMENT SHOWN FROM THE POINT CORRESPONDING TO 0 TO THE TOP OF THE 1 UNIT SEGMENT. LABEL IT AS c.

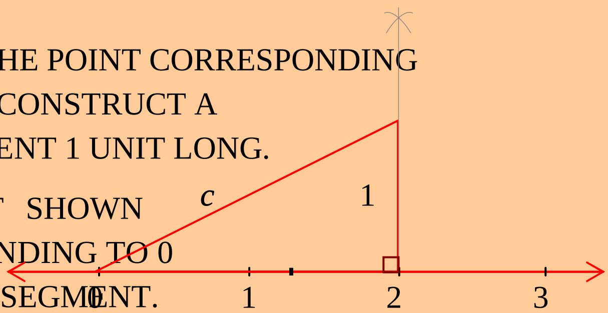


Figure 1.6

- THE PYTHAGOREAN THEOREM CAN BE USED TO SHOW THAT c IS $\sqrt{5}$ UNITS LONG.

$$c^2 = 1^2 + 2^2 = 5$$

$$c = \sqrt{5}$$

- OPEN THE COMPASS TO THE LENGTH OF C. WITH THE TIP OF THE COMPASS AT THE POINT CORRESPONDING TO 0, DRAW AN ARC INTERSECTING THE NUMBER LINE AT B. THE DISTANCE FROM THE POINT CORRESPONDING TO 0 TO B IS $\sqrt{5}$ UNITS.

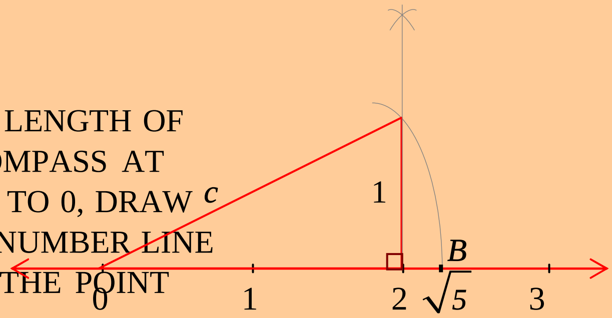
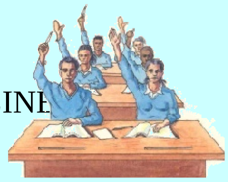


Figure 1.7

Definition 1.5 Irrational number

An irrational number is a number that cannot be expressed as $\frac{a}{b}$, such that a and b are integers and $b \neq 0$.

ACTIVITY 1.6



1 LOCATE EACH OF THE FOLLOWING ON THE NUMBER LINE, USING GEOMETRICAL CONSTRUCTION:

A $\sqrt{3}$ **B** $-\sqrt{2}$ **C** $\sqrt{6}$

2 EXPLAIN HOW $\sqrt{2}$ CAN BE USED TO LOCATE:

A $\sqrt{3}$ **B** $\sqrt{6}$

3 LOCATE EACH OF THE FOLLOWING ON THE NUMBER LINE:

A $1 + \sqrt{2}$ **B** $-2 + \sqrt{2}$ **C** $3 - \sqrt{2}$

EXAMPLE 2 SHOW THAT $3 + \sqrt{2}$ IS AN IRRATIONAL NUMBER.

SOLUTION: TO SHOW THAT $3 + \sqrt{2}$ IS NOT A RATIONAL NUMBER, LET US BEGIN BY ASSUMING THAT $3 + \sqrt{2}$ IS RATIONAL, I.E., $3 + \sqrt{2} = \frac{a}{b}$ WHERE a AND b ARE INTEGERS, $b \neq 0$.

THEN $\sqrt{2} = \frac{a}{b} - 3 = \frac{a-3b}{b}$.

SINCE $a - 3b$ AND b ARE INTEGERS (WHY?), $\frac{a-3b}{b}$ IS A RATIONAL NUMBER, MEANING THAT $\sqrt{2} = \left(\frac{a-3b}{b}\right)$ IS RATIONAL, WHICH IS FALSE. AS THE ASSUMPTION THAT $3+\sqrt{2}$ IS RATIONAL HAS LED TO A FALSE CONCLUSION, THE ASSUMPTION MUST BE FALSE. THEREFORE, $3+\sqrt{2}$ IS AN IRRATIONAL NUMBER.

ACTIVITY 1.7

EVALUATE THE FOLLOWING:

1 $0.3030030003\ldots + 0.1414414441\ldots$

2 $0.5757757775\ldots - 0.242442444\ldots$

3 $\left(3+\sqrt{2}\right) \times \left(3-\sqrt{2}\right)$

4 $\sqrt{12} \div \sqrt{3}$

FROM EXAMPLE 2 AND ACTIVITY 1.7, YOU CAN GENERALIZE THE FOLLOWING FACTS:

I THE SUM OF ANY RATIONAL NUMBER AND ANY IRRATIONAL NUMBER IS AN IRRATIONAL NUMBER.

II THE SET OF IRRATIONAL NUMBERS IS NOT CLOSED UNDER ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION.

III IF p IS A POSITIVE INTEGER THAT IS NOT A PERFECT SQUARE, THEN $a + b\sqrt{p}$ IS IRRATIONAL, WHERE a AND b ARE INTEGERS AND $b \neq 0$. FOR EXAMPLE, $3+\sqrt{2}$ AND $2-2\sqrt{3}$ ARE IRRATIONAL NUMBERS.

Exercise 1.3

1 IDENTIFY EACH OF THE FOLLOWING NUMBERS AS RATIONAL OR IRRATIONAL:

A $\frac{5}{6}$ B $2.\overline{34}$ C $-0.1213141516\ldots$

D $\sqrt{0.81}$ E $0.121121112\ldots$ F $\sqrt{5}-\sqrt{2}$

G $\sqrt[3]{72}$ H $1+\sqrt{3}$

2 GIVE TWO EXAMPLES OF IRRATIONAL NUMBERS, ONE IN THE FORM OF A RADICAL AND THE OTHER IN THE FORM OF A NON-TERMINATING DECIMAL.

3 FOR EACH OF THE FOLLOWING, DECIDE WHETHER THE STATEMENT IS 'true' OR 'false'. IF YOUR ANSWER IS 'false', GIVE A COUNTER EXAMPLE TO JUSTIFY.

A THE SUM OF ANY TWO IRRATIONAL NUMBERS IS AN IRRATIONAL NUMBER.

B THE SUM OF ANY TWO RATIONAL NUMBERS IS A RATIONAL NUMBER.

C THE SUM OF ANY TWO TERMINATING DECIMALS IS A TERMINATING DECIMAL.

D THE PRODUCT OF A RATIONAL NUMBER AND AN IRRATIONAL NUMBER IS IRRATIONAL.

1.2.3 Real Numbers

IN SECTION 1.2.1 YOU OBSERVED THAT EVERY RATIONAL NUMBER IS EITHER A TERMINATING DECIMAL OR A REPEATING DECIMAL. CONVERSELY, ANY TERMINATING OR REPEATIN RATIONAL NUMBER. MOREOVER, IN SECTION 1.2.2 YOU LEARNED THAT DECIMALS WHICH ARE NEITHER TERMINATING NOR REPEATING EXIST. FOR EXAMPLE, 0.1313313331… SUCH DI ARE DEFINED TO BE irrational numbers. SO A DECIMAL NUMBER CAN BE A RATIONAL OR AN IRRATIONAL NUMBER.

IT CAN BE SHOWN THAT EVERY DECIMAL NUMBER, BE IT RATIONAL OR IRRATIONAL, CA WITH A UNIQUE POINT ON THE NUMBER LINE AND CONVERSELY THAT EVERY POINT O LINE CAN BE ASSOCIATED WITH A UNIQUE DECIMAL NUMBER, EITHER RATIONAL OR IRR USUALLY EXPRESSED BY SAYING THAT THERE EXISTS A ONE-TO-ONE CORRESPONDEN SETS C AND D WHERE THESE SETS ARE DEFINED AS FOLLOWS.

C = {P : P IS A POINT ON THE NUMBER LINE}

D = {D : D IS A DECIMAL NUMBER }

THE ABOVE DISCUSSION LEADS US TO THE FOLLOWING DEFINITION.

Definition 1.6 Real numbers

A number is called a real number, if and only if it is either a rational number or an irrational number.

The set of real numbers, denoted by $\mathbb{R}$, can be described as the union of the sets of rational and irrational numbers.

$\mathbb{R}$ = { x : x is a rational number or an irrational number.}

THE SET OF REAL NUMBERS AND ITS SUBSETS ARE SHOWN IN THE ADJACENT DIAGRA

FROM THE PRECEDING DISCUSSION, Y SEE THAT THERE EXISTS A ONE CORRESPONDENCE BETWEEN $\mathbb{R}$ AND THE SE

THE SET C = {P:P IS A POINT ON THE NU LINE}.

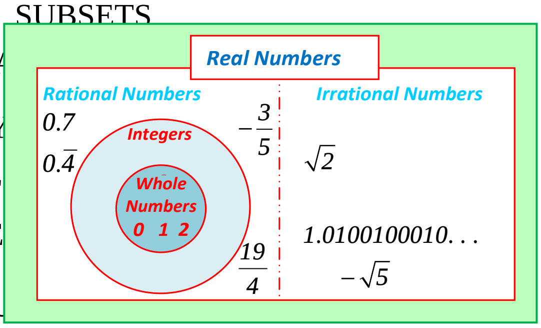


Figure 1.8

IT IS GOOD TO UNDERSTAND AND APPRECIATE THE EXISTENCE OF A ONE-TO-ONE CORR BETWEEN ANY TWO OF THE FOLLOWING SETS.

1 D = $\{x : x$ IS A DECIMAL NUMBER$\}$

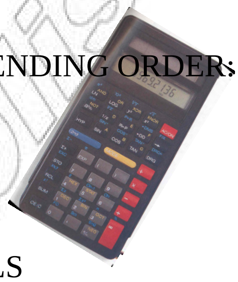
2 P = $\{x : x$ IS A POINT ON THE NUMBER LINE$\}$

3 $\mathbb{R} = \{x : x$ IS A REAL NUMBER$\}$

SINCE ALL REAL NUMBERS CAN BE LOCATED ON THE NUMBER LINE, THE NUMBER LINE COMPARE AND ORDER ALL REAL NUMBERS. FOR EXAMPLE, USING THE NUMBER LINE THAT

$$-3 < 0, \quad \sqrt{2} < 2.$$

EXAMPLE 1 ARRANGE THE FOLLOWING NUMBERS IN ASCENDING ORDER:



$$\frac{5}{6},\ 0.8,\ \frac{\sqrt{3}}{2}.$$

SOLUTION: USE A CALCULATOR TO CONVERT $\frac{5}{6}$ AND $\frac{\sqrt{3}}{2}$ TO DECIMALS

$5 \boxed{\div} 6 \boxed{=} 0.83333\ldots$ AND

$3 \boxed{\sqrt{\ }} \boxed{\div} 2 \boxed{=} 0.866025\ldots$

SINCE $0.8 < 0.8\overline{3} < 0.866025\ldots$, THE NUMBERS WHEN ARRANGED IN ASCENDING ORDER

$$0.8,\ \frac{5}{6},\ \frac{\sqrt{3}}{2}.$$

HOWEVER, THERE ARE ALGEBRAIC METHODS OF COMPARING AND ORDERING REAL

HERE ARE TWO IMPORTANT PROPERTIES OF ORDER.

1 Trichotomy property

FOR ANY TWO REAL NUMBERS a AND b, ONE AND ONLY ONE OF THE FOLLOWING IS TRUE

$a < b$ OR $a = b$ OR $a > b$.

2 Transitive property of order

FOR ANY THREE REAL NUMBERS a, b AND c, IF $a < b$ AND $b < c$, THEN $a < c$.

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A THIRD PROPERTY, STATED BELOW, CAN BE DERIVED FROM THE TRICHOTOMY PROPERTY AND THE TRANSITIVE PROPERTY OF ORDER.

➢ FOR ANY TWO NON-NEGATIVE REAL NUMBERS a AND b, IF $a^2 < b^2$, THEN $a < b$.

YOU CAN USE THIS PROPERTY TO COMPARE TWO NUMBERS WITHOUT USING A CALCULA

FOR EXAMPLE, LET US COMPARE $\frac{5}{6}$ AND $\frac{\sqrt{3}}{2}$.

$$\left(\frac{5}{6}\right)^2 = \frac{25}{36}, \quad \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} = \frac{27}{36}$$

SINCE $\left(\frac{5}{6}\right)^2 < \left(\frac{\sqrt{3}}{2}\right)^2$, IT FOLLOWS THAT $\frac{5}{6} < \frac{\sqrt{3}}{2}$.

Exercise 1.4

1 COMPARE THE NUMBERS a AND b USING THE SYMBOL < OR >.

A $a = \frac{\sqrt{6}}{4}$, $b = 0.\overline{6}$

B $a = 0.432$, $b = 0.437$

C $a = -0.128$, $b = -0.123$

2 STATE WHETHER EACH SET (A – E GIVEN BELOW) IS CLOSED UNDER EACH OF THE FOLLOWIN OPERATIONS:

I ADDITION **II** SUBTRACTION **III** MULTIPLICATION **IV** DIVISION

A ℕ THE SET OF NATURAL NUMBERS. **B** ℤ THE SET OF INTEGERS.

C ℚ THE SET OF RATIONAL NUMBERS. **D** 𝕀 THE SET OF IRRATIONAL NUMBERS.

E ℝ THE SET OF REAL NUMBERS.

1.2.4 Exponents and Radicals

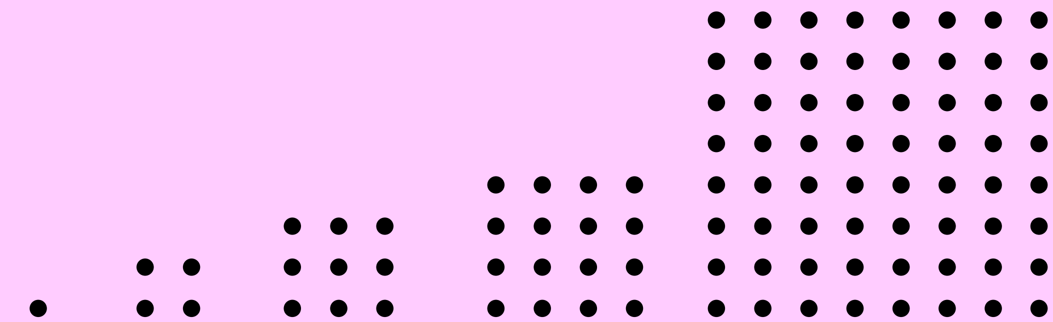
A Roots and radicals

IN THIS SUBSECTION, YOU WILL DEFINE THE ROOTS OF NUMBERS AND DISCUSS THEIR PROPERTIES. COMPUTATIONS OF EXPRESSIONS INVOLVING RADICALS AND FRACTIONAL ALSO CONSIDERED.

Roots

HISTORICAL NOTE:

The Pythagorean School of ancient Greece focused on the study of philosophy, mathematics and natural science. The students, called Pythagoreans, made many advances in these fields. One of their studies was to symbolize numbers. By drawing pictures of various numbers, patterns can be discovered. For example, some whole numbers can be represented by drawing dots arranged in squares.



NUMBERS THAT CAN BE PICTURED IN SQUARES OF DOTS ARE CALLED perfect squares OR square numbers. THE NUMBER OF DOTS IN EACH ROW OR COLUMN IN THE SQUARE IS A square root OF THE PERFECT SQUARE. THE PERFECT SQUARE 9 HAS A SQUARE ROOT OF 3, BECAUSE THER
AND 3 COLUMNS. YOU SAY 8 IS A SQUARE ROOT OF 64, BECAUSE 64 = 8 × 8 OR 8^2

Definition 1.7 Square root

For any two real numbers a and b, if $a^2 = b$, then a is a square root of b.

PERFECT SQUARES ALSO INCLUDE DECIMALS AND FRACTIONS LIKE 0.09 AND $\frac{4}{9}$ SINCE $(0.3)^2 = 0.09$ AND $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, IT IS ALSO TRUE THAT $(-8)^2 = 64$ AND $(-12)^2 = 144$.

SO, YOU MAY SAY THAT –8 IS ALSO A SQUARE ROOT OF 64 AND –12 IS A SQUARE ROOT OF

THE POSITIVE SQUARE ROOT OF A NUMBER IS CALLED THE principal square root.

THE SYMBOL $\sqrt{\ }$, CALLED A radical sign, IS USED TO INDICATE THE PRINCIPAL SQUARE ROOT.

THE SYMBOL $\sqrt{25}$ IS READ AS "the principal square root of 25" OR JUST "the square root of 25" AND $-\sqrt{25}$ IS READ AS "the negative square root of 25". IF b IS A POSITIVE REAL NUMBER, $\sqrt{b}$ IS A POSITIVE REAL NUMBER. NEGATIVE REAL NUMBERS DO NOT HAVE SQUA THE SET OF REAL NUMBERS SINCE $a^2 \geq 0$ FOR ANY NUMBER a. THE SQUARE ROOT OF ZERO IS ZERO.

SIMILARLY, SINCE $4^3 = 64$, YOU SAY THAT 64 IS THE CUBE OF 4 AND 4 IS THE CUBE ROOT OF THAT IS WRITTEN AS $4 = \sqrt[3]{64}$.

THE SYMBOL $4 = \sqrt[3]{64}$ IS READ AS "the principal cube root of 64" OR JUST "the cube root of 64".

- EACH REAL NUMBER HAS EXACTLY ONE CUBE ROOT.

$(-3)^3 = -27$ SO, $\sqrt[3]{-27} = -3$ $\qquad$ $0^3 = 0$ SO, $\sqrt[3]{0} = 0$.

YOU MAY NOW GENERALIZE AS FOLLOWS:

Definition 1.8 The n^{th} root

For any two real numbers a and b, and positive integer n, if $a^n = b$, then a is called an n^{th} root of b.

EXAMPLE 1

A -3 IS A CUBE ROOT OF -27 BECAUSE $(-3)^3 = -27$

B 4 IS A CUBE ROOT OF 64 BECAUSE $4^3 = 64$

Definition 1.9 Principal n^{th} root

If b is any real number and n is a positive integer greater than 1, then, the principal n^{th} root of b, denoted by $\sqrt[n]{b}$ is defined as

$$\sqrt[n]{b} = \begin{cases} \text{THE POSITIVE } n^{th} \text{ ROOT OF } b, \text{ IF } b > 0. \\ \text{THE NEGATIVE } n^{th} \text{ ROOT OF } b, \text{ IF } b < 0 \text{ AN} \\ 0, \text{ IF } b = 0. \end{cases}$$

I IF $b < 0$ AND n IS EVEN, THERE IS NO REAL n^{th} ROOT OF b, BECAUSE AN EVEN POWER OF ANY REAL NUMBER IS A NON-NEGATIVE NUMBER.

II THE SYMBOL $\sqrt[n]{\ }$ IS CALLED A RADICAL SIGN. THE EXPRESSION $\sqrt[n]{b}$ IS CALLED A radical, n IS CALLED THE index AND b IS CALLED THE radicand. WHEN NO INDEX IS WRITTEN, THE RADICAL SIGN INDICATES SQUARE ROOT.

EXAMPLE 2

A $\sqrt[4]{16} = 2$ BECAUSE $2^4 = 16$

B $\sqrt{0.04} = 0.2$ BECAUSE $(0.2)^2 = 0.04$

C $\sqrt[3]{-1000} = -10$ BECAUSE $(-10)^3 = -1000$

NUMBERS SUCH AS $\sqrt{23}$, $\sqrt[3]{35}$ AND $\sqrt[3]{1}$ ARE IRRATIONAL NUMBERS AND CANNOT BE WRITTEN TERMINATING OR REPEATING DECIMALS. HOWEVER, IT IS POSSIBLE TO APPROXIMATE NUMBERS AS CLOSELY AS DESIRED USING DECIMALS. THESE rational approximations CAN BE FOUND THROUGH SUCCESSIVE TRIALS, USING A SCIENTIFIC CALCULATOR. THE METHOD *successive trials* USES THE FOLLOWING PROPERTY:

➢ FOR ANY THREE POSITIVE REAL NUMBERS a, b AND c AND A POSITIVE INTEGER n

IF $a^n < b < c^n$, THEN $a < \sqrt[n]{b} < c$.

EXAMPLE 3 FIND A RATIONAL APPROXIMATION TO $\sqrt{43}$ TO THE NEAREST HUNDREDTH.

SOLUTION: USE THE ABOVE PROPERTY AND DIVIDE-AND-AVERAGE ON A CALCULATOR.

SINCE $6^2 = 36 < 43 < 49 = 7^2$

$6 < \sqrt{43} < 7$

ESTIMATE $\sqrt{43}$ TO TENTHS, $\sqrt{43} \approx 6.5$

DIVIDE 43 BY 6.5

$$6.5 \overline{)\,43.000} = 6.615$$

AVERAGE THE DIVISOR AND THE QUOTIENT $\dfrac{6.5 + 6.615}{2} = 6.558$

DIVIDE 43 BY 6.558

$$6.558 \overline{)\,43.000} = 6.557$$

NOW YOU CAN CHECK THAT $(6.557)^2 < 43 < (6.558)^2$. THEREFORE $\sqrt{43}$ IS BETWEEN 6.557 AND 6.558. IT IS 6.56 TO THE NEAREST HUNDREDTH.

EXAMPLE 4 THROUGH SUCCESSIVE TRIALS ON A CALCULATOR, COMPUTE $\sqrt[3]{53}$ TO THE NEAREST TENTH.

SOLUTION:

$3^3 = 27 < 53 < 64 = 4^3$. THAT IS, $3^3 < 53 < 4^3$. SO $3 < \sqrt[3]{53} < 4$

TRY 3.5:	$3.5^3 = 42.875$	SO $3.5 < \sqrt[3]{53} < 4$
TRY 3.7:	$3.7^3 = 50.653$	SO $3.7 < \sqrt[3]{53} < 4$
TRY 3.8:	$3.8^3 = 54.872$	SO $3.7 < \sqrt[3]{53} < 3.8$
TRY 3.75:	$3.75^3 = 52.734375$	SO $3.75 < \sqrt[3]{53} < 3.8$

THEREFORE, $\sqrt[3]{53}$ IS 3.8 TO THE NEAREST TENTH.

B Meaning of fractional exponents

ACTIVITY 1.8

1. STATE ANOTHER NAME FOR $2^{\frac{1}{2}}$.
2. WHAT MEANING CAN YOU GIVE TO $2^{\frac{1}{2}}$ OR $2^{0.5}$?
3. SHOW THAT THERE IS AT MOST ONE POSITIVE NUMBER WHOSE FIFTH POWER IS 2.

BY CONSIDERING A TABLE OF POWERS OF 3 AND USING A CALCULATOR, YOU CAN DEFINE $3^{\frac{1}{5}}$ AS $\sqrt[5]{3}$. THIS CHOICE WOULD RETAIN THE PROPERTY OF EXPONENTS BY WHICH $\left(3^{\frac{1}{5}}\right)^5 = 3^{\left(\frac{1}{5}\right)\times 5} = 3$.

SIMILARLY, YOU CAN DEFINE $5^{\frac{1}{n}}$, WHERE n IS A POSITIVE INTEGER GREATER THAN 1, AS $\sqrt[n]{5}$. IN GENERAL, YOU CAN DEFINE $b^{\frac{1}{n}}$ FOR ANY $b \in \mathbb{R}$ AND n A POSITIVE INTEGER TO BE $\sqrt[n]{b}$ WHENEVER $\sqrt[n]{b}$ IS A REAL NUMBER.

Definition 1.10 The n^{th} power

If $b \in \mathbb{R}$ and n is a positive integer greater than 1, then

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

EXAMPLE 5 WRITE THE FOLLOWING IN EXPONENTIAL FORM:

A $\sqrt{7}$ **B** $\dfrac{1}{\sqrt[3]{10}}$

SOLUTION:

A $\sqrt{7} = 7^{\frac{1}{2}}$ **B** $\dfrac{1}{\sqrt[3]{10}} = \dfrac{1}{10^{\frac{1}{3}}} = 10^{-\frac{1}{3}}$

EXAMPLE 6 SIMPLIFY:

A $25^{\frac{1}{2}}$ **B** $(-8)^{\frac{1}{3}}$ **C** $64^{\frac{1}{6}}$

SOLUTION:

A $25^{\frac{1}{2}} = \sqrt{25} = 5$ (SINCE $5^2 = 25$)

B $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$ (SINCE $(-2)^3 = -8$)

C $64^{\frac{1}{6}} = \sqrt[6]{64} = 2$ (SINCE $2^6 = 64$)

Group Work 1.5

SIMPLIFY:

I	**A**	$(8\times27)^{\frac{1}{3}}$	**B**	$8^{\frac{1}{3}}\times27^{\frac{1}{3}}$
II	**A**	$\sqrt[3]{8\times27}$	**B**	$\sqrt[3]{8}\times\sqrt[3]{27}$
III	**A**	$(36\times49)^{\frac{1}{2}}$	**B**	$36^{\frac{1}{2}}\times49^{\frac{1}{2}}$
IV	**A**	$\sqrt{36\times49}$	**B**	$\sqrt{36}\times\sqrt{49}$

WHAT RELATIONSHIP DO YOU OBSERVE BETWEEN A AND B IN I, II, III AND IV?

THE OBSERVATIONS FROM THE ABOVE GROUP WORK LEAD YOU TO THINK THAT $8^{\frac{1}{3}}\times27^{\frac{1}{3}} = (8\times27)^{\frac{1}{3}}$.

THIS PARTICULAR CASE SUGGESTS THE FOLLOWING GENERAL PROPERTY (THEOREM).

Theorem 1.2

For any two real numbers a and b and for all integers $n \geq 2$, $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$

EXAMPLE 7 SIMPLIFY EACH OF THE FOLLOWING.

A $9^{\frac{1}{3}}\times3^{\frac{1}{3}}$ **B** $\sqrt[5]{16}\times\sqrt[5]{2}$

SOLUTION:

A $9^{\frac{1}{3}}\times3^{\frac{1}{3}} = (9\times3)^{\frac{1}{3}}$ (BY THEOREM 1.2)

$= (27)^{\frac{1}{3}}$ (MULTIPLICATION)

$= 3$ ($3^3 = 27$)

B $\sqrt[5]{16}\times\sqrt[5]{2} = \sqrt[5]{16\times2}$

$= \sqrt[5]{32}$

$= 2$

ACTIVITY 1.9

SIMPLIFY:

I **A** $\dfrac{64^{\frac{1}{5}}}{2^{\frac{1}{5}}}$ **B** $\left(\dfrac{64}{2}\right)^{\frac{1}{5}}$

II **A** $\dfrac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}}$ **B** $\left(\dfrac{8}{2}\right)^{\frac{1}{2}}$

III **A** $\dfrac{27^{\frac{1}{3}}}{729^{\frac{1}{3}}}$ **B** $\left(\dfrac{27}{729}\right)^{\frac{1}{3}}$

WHAT RELATIONSHIP DO YOU OBSERVE BETWEEN A AND B IN I, II AND III?

THE OBSERVATIONS FROM THE ACTIVITY ABOVE LEAD US TO THE FOLLOWING THEOREM:

Theorem 1.3

For any two real numbers a and b where $b \neq 0$ and for all integers $n \geq 2$,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$$

EXAMPLE 8 SIMPLIFY **A** $\dfrac{16^{\frac{1}{3}}}{2^{\frac{1}{3}}}$ **B** $\dfrac{\sqrt[6]{128}}{\sqrt[6]{2}}$

SOLUTION:

A $\dfrac{16^{\frac{1}{3}}}{2^{\frac{1}{3}}} = \left(\dfrac{16}{2}\right)^{\frac{1}{3}}$ (BY THEOREM 1.3)

$= 8^{\frac{1}{3}} = 2$ (SINCE $2^3 = 8$)

B $\dfrac{\sqrt[6]{128}}{\sqrt[6]{2}} = \sqrt[6]{\dfrac{128}{2}}$

$= \sqrt[6]{64}$

$= 2$ (BECAUSE $2^6 = 64$)

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ACTIVITY 1.10

1 SUGGEST, WITH REASONS, A MEANING FOR

A $2^{\frac{7}{2}}$ B $2^{\frac{9}{2}}$ IN TERMS OF $2^{\frac{1}{2}}$

2 SUGGEST A RELATION BETWEEN $5^{\frac{3}{2}}$ AND $5^{\frac{1}{2}}$

APPLYING THE PROPERTY $(a^m)^n = a^{mn}$, YOU CAN WRITE $\left(7^{\frac{1}{10}}\right)^9$ AS $7^{\frac{9}{10}}$. IN GENERAL, YOU CAN SAY $\left(a^{\frac{1}{q}}\right)^p = a^{\frac{p}{q}}$, WHERE p AND q ARE POSITIVE INTEGERS AND $a \geq 0$. THUS, YOU HAVE THE FOLLOWING DEFINITION:

Definition 1.11

For $a \geq 0$ and p and q any two positive integers, $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p$

Exercise 1.5

1 SHOW THAT: A $64^{\frac{1}{3}} = 4$ B $256^{\frac{1}{8}} = 2$ C $125^{\frac{1}{3}} = 5$

2 EXPRESS EACH OF THE FOLLOWING IN TERMS OF RATIONAL EXPONENT WITHOUT RADICAL SIGNS:

A $81^{\frac{1}{4}}$ B $9^{\frac{1}{2}}$ C $\sqrt[6]{64}$ D $\left(\frac{27}{8}\right)^{\frac{1}{3}}$

E $(0.00032)^{\frac{1}{5}}$ F $\sqrt[4]{0.0016}$ G $\sqrt[6]{729}$

3 EXPLAIN EACH STEP OF THE FOLLOWING:

$$(27\times125)^{\frac{1}{3}} = \left[(3\times3\times3)\times(5\times5\times5)\right]^{\frac{1}{3}} = \left[(3\times5)\times(3\times5)\times(3\times5)\right]^{\frac{1}{3}}$$
$$= 3\times5 = 15$$

4 IN THE SAME MANNER AS IN QUESTION 3, SIMPLIFY EACH OF THE FOLLOWING:

A $(25\times121)^{\frac{1}{2}}$ B $(625\times16)^{\frac{1}{4}}$ C $(1024\times243)^{\frac{1}{5}}$

5 EXPRESS THEOREM 1.2 USING RADICAL NOTATION.

6 SHOW THAT:

A $7^{\frac{1}{4}}\times5^{\frac{1}{4}} = (7\times5)^{\frac{1}{4}}$ B $\sqrt{5}\times\sqrt{3} = \sqrt{5\times3}$

C $\sqrt[3]{7}\times\sqrt[3]{9} = \sqrt[3]{7\times9}$ D $11^{\frac{1}{7}}\times6^{\frac{1}{7}} = (11\times6)^{\frac{1}{7}}$

7 EXPRESS IN THE SIMPLEST FORM:

A $32^{\frac{1}{6}} \times 2^{\frac{1}{6}}$ **B** $9^{\frac{1}{3}} \times 3^{\frac{1}{3}}$ **C** $128^{\frac{1}{6}} \times \left(\frac{1}{2}\right)^{\frac{1}{6}}$

D $\sqrt[5]{16} \times \sqrt[5]{2}$ **E** $\sqrt[3]{16} \times \sqrt[3]{4}$ **F** $32^{\frac{1}{7}} \times 4^{\frac{1}{7}}$

G $5^{\frac{1}{8}} \times 27^{\frac{1}{5}} \times \left(\frac{1}{5}\right)^{\frac{1}{8}} \times 9^{\frac{1}{5}}$ **H** $\sqrt[3]{5} \times \sqrt[5]{8} \times \sqrt[3]{\frac{1}{5}} \times \sqrt[5]{4}$

8 EXPRESS THEOREM 1.3 USING RADICAL NOTATION.

9 SIMPLIFY:

A $\dfrac{128^{\frac{1}{5}}}{4^{\frac{1}{5}}}$ **B** $\dfrac{9^{\frac{1}{3}}}{243^{\frac{1}{3}}}$ **C** $\dfrac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}}$ **D** $\dfrac{32^{\frac{1}{4}}}{162^{\frac{1}{4}}}$

E $\dfrac{\sqrt[3]{16}}{\sqrt[3]{2}}$ **F** $\dfrac{\sqrt[5]{64}}{\sqrt[5]{2}}$ **G** $\dfrac{\sqrt[6]{512}}{\sqrt[6]{8}}$ **H** $\dfrac{\sqrt[3]{625}}{\sqrt[3]{5}}$

10 REWRITE EACH OF THE FOLLOWING IN THE FORM $a^{\frac{p}{q}}$.

A $\left(13^{\frac{1}{5}}\right)^{9}$ **B** $\left(12^{\frac{1}{5}}\right)^{11}$ **C** $\left(11^{\frac{1}{6}}\right)^{5}$

11 REWRITE THE FOLLOWING IN THE FORM $\left(a^{\frac{1}{q}}\right)^{p}$

A $3^{\frac{7}{5}}$ **B** $5^{\frac{6}{3}}$ **C** $64^{\frac{5}{6}}$ **D** $729^{\frac{2}{3}}$

12 REWRITE THE EXPRESSIONS IN QUESTION 10 USING RADICALS.

13 REWRITE THE EXPRESSIONS IN QUESTION 11 USING RADICALS.

14 EXPRESS THE FOLLOWING WITHOUT FRACTIONAL EXPONENTS OR RADICAL SIGNS

A $\left(27^{\frac{1}{3}}\right)^{5}$ **B** $27^{\frac{5}{3}}$ **C** $8^{\frac{1}{3}}$

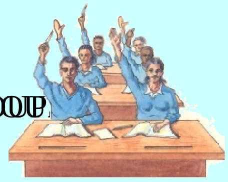
15 SIMPLIFY EACH OF THE FOLLOWING:

A $64^{\frac{1}{6}}$ **B** $81^{\frac{3}{2}}$ **C** $64^{\frac{3}{18}}$

D $81^{\frac{6}{4}}$ **E** $512^{\frac{2}{3}}$ **F** $512^{\frac{6}{9}}$

C Simplification of radicals

ACTIVITY 1.11



1 EVALUATE EACH OF THE FOLLOWING AND DISCUSS IN YOUR GROUP.

A $\sqrt[3]{(-2)^3}$ **B** $\sqrt{(-3)^2}$ **C** $\sqrt[4]{(-5)^4}$

D $\sqrt[5]{4^5}$ **E** $\sqrt{2^2}$ **F** $\sqrt[7]{(-1)^7}$

2 DOES THE SIGN OF YOUR RESULT DEPEND ON WHETHER THE INDEX IS ODD OR EVEN?

CAN YOU GIVE A GENERAL RULE FOR $\sqrt[n]{a^n}$ WHERE a IS A REAL NUMBER AND

I n IS AN ODD INTEGER? **II** n IS AN EVEN INTEGER?

TO COMPUTE AND SIMPLIFY EXPRESSIONS INVOLVING RADICALS, IT IS OFTEN NE DISTINGUISH BETWEEN ROOTS WITH ODD INDICES AND THOSE WITH EVEN INDICES.

FOR ANY REAL NUMBER a AND A POSITIVE INTEGER n,

$\sqrt[n]{a^n} = a$, IF n IS ODD.

$\sqrt[n]{a^n} = |a|$, IF n IS EVEN.

$$\sqrt[5]{(-2)^5} = -2, \quad \sqrt[3]{x^3} = x, \quad \sqrt{(-2)^2} = |-2| = 2$$

$$\sqrt{x^2} = |x|, \quad \sqrt[4]{(-2)^4} = |-2| = 2, \quad \sqrt[4]{x^4} = |x|$$

EXAMPLE 9 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt{y^2}$ **B** $\sqrt[3]{-27x^3}$ **C** $\sqrt{25x^4}$ **D** $\sqrt[6]{x^6}$ **E** $\sqrt[4]{x^3}$

SOLUTION:

A $\sqrt{y^2} = |y|$ **B** $\sqrt[3]{-27x^3} = \sqrt[3]{(-3x)^3} = -3x$

C $\sqrt{25x^4} = |5x^2| = 5x^2$ **D** $\sqrt[6]{x^6} = |x|$ **E** $\sqrt[4]{x^3} = \left(x^3\right)^{\frac{1}{4}} = x^{\frac{3}{4}}$

A RADICAL $\sqrt[n]{a}$ IS IN SIMPLEST FORM, IF THE RADICAND a CONTAINS NO FACTOR THAT CAN BE EXPRESSED AS AN n^{th} POWER. FOR EXAMPLE $\sqrt[3]{54}$ IS NOT IN SIMPLEST FORM BECAUSE 3^3 IS A FACTOR OF 54.

USING THIS FACT AND THE RADICAL NOTATIONS OF THEOREM 1.2 AND THEOREM 1.3, YOU CAN SIMPLIFY RADICALS.

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EXAMPLE 10 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt{48}$ **B** $\sqrt[3]{9}\times\sqrt[3]{3}$ **C** $\sqrt[4]{\dfrac{32}{81}}$

SOLUTION:

A $\sqrt{48}=\sqrt{16\times 3}=\sqrt{16}\times\sqrt{3}=4\sqrt{3}$

B $\sqrt[3]{9}\times\sqrt[3]{3}=\sqrt[3]{9\times 3}=\sqrt[3]{27}=3$

C $\sqrt[4]{\dfrac{32}{81}}=\sqrt[4]{\dfrac{16\times 2}{81}}=\sqrt[4]{\dfrac{16}{81}}\times\sqrt[4]{2}=\dfrac{\sqrt[4]{16}}{\sqrt[4]{81}}\times\sqrt[4]{2}=\dfrac{2}{3}\sqrt[4]{2}$

Exercise 1.6

1 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt{8}$ **B** $5\sqrt{32}$ **C** $3\sqrt{8x^2}$ **D** $\sqrt{363}$

E $\sqrt[3]{512}$ **F** $\dfrac{1}{3}\sqrt{27x^3y^2}$ **G** $\sqrt[4]{405}$

2 SIMPLIFY EACH OF THE FOLLOWING IF POSSIBLE. STATE RESTRICTIONS WHERE NECESSARY.

A $\sqrt{50}$ **B** $2\sqrt{36}$ **C** $\dfrac{1}{3}\sqrt{72}$ **D** $3\sqrt{8x^2}$ **E** $\sqrt{a^3}$

F $\sqrt{0.27}$ **G** $-\sqrt{63}$ **H** $\dfrac{\sqrt{180}}{9}$ **I** $\sqrt[3]{16}$ **J** $\sqrt[3]{-54}$

3 IDENTIFY THE ERROR AND WRITE THE CORRECT SOLUTION IN EACH OF THE FOLLOWING CASES:

A A STUDENT SIMPLIFIED $\sqrt{28}$ TO $\sqrt{25+3}$ AND THEN TO $5\sqrt{3}$

B A STUDENT SIMPLIFIED $\sqrt{72}$ TO $\sqrt{4}\sqrt{18}$ AND THEN TO $4\sqrt{3}$

C A STUDENT SIMPLIFIED $\sqrt{7x^9}$ AND GOT $x^3\sqrt{7}$

4 SIMPLIFY EACH OF THE FOLLOWING:

A $8\sqrt{250}$ **B** $\sqrt[3]{16}\times\sqrt[3]{5}$ **C** $\sqrt[4]{5}\times\sqrt[4]{125}$

D $\dfrac{\sqrt{2}}{7}\times\sqrt{7}\times\sqrt{14}$ **E** $\dfrac{\sqrt[3]{81}}{\sqrt[3]{3}}$ **F** $\dfrac{12\sqrt{96}}{3\sqrt{6}}$

G $\dfrac{2\sqrt{98x^3y^2}}{14\sqrt{xy}}$ $x>0, y>0.$ **H** $4\sqrt{3}\times 2\sqrt{18}$

5 THE NUMBER OF UNITS N PRODUCED BY A COMPANY FROM THE USE OF K UNITS OF CAPITAL AND L UNITS OF LABOUR IS GIVEN BY $N = 12\sqrt{LK}$.

- **A** WHAT IS THE NUMBER OF UNITS PRODUCED, IF THERE ARE 625 UNITS OF LABOUR AND 1024 UNITS OF CAPITAL?
- **B** DISCUSS THE EFFECT ON THE PRODUCTION, IF BOTH LABOUR AND CAPITAL ARE DOUBLED.

Addition and subtraction of radicals

WHICH OF THE FOLLOWING DO YOU THINK IS CORRECT?

1 $\sqrt{2}+\sqrt{8}=\sqrt{10}$ **2** $\sqrt{19}-\sqrt{3}=4$ **3** $5\sqrt{2}+7\sqrt{2}=12\sqrt{2}$

THE ABOVE PROBLEMS INVOLVE ADDITION AND SUBTRACTION OF RADICALS. YOU DEF CONCEPT OF LIKE RADICALS WHICH IS COMMONLY USED FOR THIS PURPOSE.

Definition 1.12

Radicals that have the same index and the same radicand are said to be like radicals.

FOR EXAMPLE,

- **I** $3\sqrt{5},\ -\frac{1}{2}\sqrt{5}$ AND $\sqrt{\ }$ ARE LIKE RADICALS.
- **II** $\sqrt{5}$ AND $\sqrt[3]{\ }$ ARE NOT LIKE RADICALS.
- **III** $\sqrt{11}$ AND $\sqrt{\ }$ ARE NOT LIKE RADICALS.

BY TREATING LIKE RADICALS AS LIKE TERMS, YOU CAN ADD OR SUBTRACT LIKE RADICA THEM AS A SINGLE RADICAL. ON THE OTHER HAND, THE SUM OF UNLIKE RADICALS CAN EXPRESSED AS A SINGLE RADICAL UNLESS THEY CAN BE TRANSFORMED INTO LIKE RAD

EXAMPLE 11 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt{2}+\sqrt{8}$ **B** $3\sqrt{12}-\sqrt{3}+2\sqrt{\frac{1}{3}}+\frac{1}{9}\sqrt{27}$

SOLUTION:

A $$\sqrt{2}+\sqrt{8}=\sqrt{2}+\sqrt{2\times 4}=\sqrt{2}+\sqrt{4}\sqrt{2}=\sqrt{2}+2\sqrt{2}$$
$$=(1+2)\sqrt{2}=3\sqrt{2}$$

B $$3\sqrt{12}-\sqrt{3}+2\sqrt{\frac{1}{3}}+\frac{1}{9}\sqrt{27}=3\sqrt{4\times3}-\sqrt{3}+2\sqrt{\frac{1}{3}\times\frac{3}{3}}+\frac{1}{9}\sqrt{9\times3}$$

$$=3\sqrt{4}\times\sqrt{3}-\sqrt{3}+2\frac{\sqrt{3}}{\sqrt{9}}+\frac{1}{9}\sqrt{9}\times\sqrt{3}$$

$$=6\sqrt{3}-\sqrt{3}+\frac{2}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}$$

$$=\left(6-1+\frac{2}{3}+\frac{1}{3}\right)\sqrt{3}=6\sqrt{3}$$

Exercise 1.7

SIMPLIFY EACH OF THE FOLLOWING IF POSSIBLE. STATE RESTRICTIONS WHERE NECESSA

1 **A** $\sqrt{2}\times\sqrt{5}$ **B** $\sqrt{3}\times\sqrt{6}$ **C** $\sqrt{21}\times\sqrt{5}$ **D** $\sqrt{2x}\times\sqrt{8x}$

E $\frac{\sqrt{2}}{\sqrt{2}}$ **F** $\frac{\sqrt{10}}{4\sqrt{3}}$ **G** $\sqrt{50y^3}\div\sqrt{2y}$ **H** $\frac{9\sqrt{40}}{3\sqrt{10}}$

I $4\sqrt[3]{16}\div2\sqrt[3]{2}$ **J** $\frac{9\sqrt{24}\div15\sqrt{75}}{3\sqrt{3}}$

2 **A** $2\sqrt{3}+5\sqrt{3}$ **B** $9\sqrt{2}-5\sqrt{2}$ **C** $\sqrt{3}+\sqrt{12}$

D $\sqrt{63}-\sqrt{28}$ **E** $\sqrt{75}-\sqrt{48}$ **F** $\sqrt{6}\left(\sqrt{12}-\sqrt{3}\right)$

G $\sqrt{2x^2}-\sqrt{50x^2}$ **H** $5\sqrt[3]{54}-2\sqrt[3]{2}$ **I** $8\sqrt{24}+\frac{2}{3}\sqrt{54}-2\sqrt{96}$

J $\frac{\sqrt{a+2\sqrt{ab}+b}}{\sqrt{a}+\sqrt{b}}$ **K** $\left(\sqrt{a}-\sqrt{b}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)$

3 **A** FIND THE SQUARE ROOT OF $7-2\sqrt{10}$.

B SIMPLIFY EACH OF THE FOLLOWING:

I $\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$ **II** $\frac{\sqrt{7+\sqrt{24}}}{2}+\frac{\sqrt{7-\sqrt{24}}}{2}$

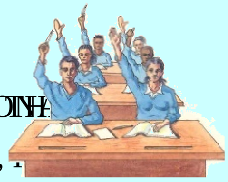
III $\left(\sqrt{p^2+1}-\sqrt{p^2-1}\right)\left(\sqrt{p^2-1}+\sqrt{p^2+1}\right)$

4 SUPPOSE THE BRAKING DISTANCE d FOR A GIVEN AUTOMOBILE WHEN IT IS TRAVELLIN v KM/HR IS APPROXIMATED BY $d = 0.00021\sqrt[3]{v^5}$ M. APPROXIMATE THE BRAKING DISTANCE WHEN THE CAR IS TRAVELLING 64 KM/HR.

1.2.5 The Four Operations on Real Numbers

THE FOLLOWING ACTIVITY IS DESIGNED TO HELP YOU THE FOUR OPERATIONS ON THE SET OF RATIONAL NUMBERS WHICH YOU HAVE DONE IN YOUR PREVIOUS GRADES.

ACTIVITY 1.12



1 APPLY THE PROPERTIES OF THE FOUR OPERATIONS IN THE SET OF RATIONAL NUMBERS TO COMPUTE THE FOLLOWING (MENTALLY, ...).

A $\frac{2}{9}+\left(\frac{3}{5}+\frac{7}{9}\right)$

B $\frac{3}{7}\times\left(\frac{-11}{21}\right)+\left(\frac{-3}{7}\right)\left(\frac{-11}{21}\right)$

C $\frac{3}{7}+\left(\frac{5}{6}+\frac{-3}{7}\right)$

D $\left(\frac{-9}{7}\times\frac{23}{-27}\right)\times\left(\frac{-7}{9}\right)$

2 STATE A PROPERTY THAT JUSTIFIES EACH OF THE FOLLOWING STATEMENTS.

A $\frac{-2}{3}\left(\frac{3}{2}\times\frac{3}{5}\right)=\left(\frac{-2}{3}\times\frac{3}{2}\right)\times\frac{3}{5}$

B $\frac{-7}{9}\left(\frac{3}{2}+\frac{-4}{5}\right)=\frac{-7}{9}\left(\frac{-4}{5}+\frac{3}{2}\right)$

C $\left(\frac{-3}{5}\right)+\left(\frac{-5}{6}\right)<\left(\frac{-1}{5}\right)+\left(\frac{-5}{6}\right)$, SINCE $\frac{-3}{5}<\frac{-1}{5}$

IN THIS SECTION, YOU WILL DISCUSS OPERATIONS ON THE SET OF REAL NUMBERS. TH YOU HAVE STUDIED SO FAR WILL HELP YOU TO INVESTIGATE MANY OTHER PROPERTI REAL NUMBERS.

Group Work 1.6

Work with a partner

Required:- SCIENTIFIC CALCULATOR

1 **Try this**

COPY AND COMPLETE THE FOLLOWING TABLE. USE A CALCULATOR TO FIND EACH PRODUCT AND COMPLETE THE TABLE.

Factors	product	product written as a power
$2^3 \times 2^2$		
$10^1 \times 10^1$		
$\left(\frac{-1}{5}\right)\times\left(\frac{-1}{5}\right)^3$		

2 Try this

COPY THE FOLLOWING TABLE. USE A CALCULATOR TO FIND THE QUOTIENT AND COMPLETE THE TABLE.

Division	Quotient	Quotient written as a power
$10^5 \div 10^1$		
$3^5 \div 3^2$		
$\left(\frac{1}{2}\right)^4 \div \left(\frac{1}{2}\right)^2$		

Discuss the two tables:

I A COMPARE THE EXPONENTS OF THE FACTORS TO THE EXPONENT IN THE PRODUCT. WHAT DO YOU OBSERVE?

B WRITE A RULE FOR DETERMINING THE EXPONENT OF THE PRODUCT WHEN YOU MULTIPLY POWERS. CHECK YOUR RULE BY MULTIPLYING $3^2 \times 3^3$ USING A CALCULATOR.

II A COMPARE THE EXPONENTS OF THE DIVISION EXPRESSIONS TO THE EXPONENTS IN THE QUOTIENTS. WHAT PATTERN DO YOU OBSERVE?

B WRITE A RULE FOR DETERMINING THE EXPONENT IN THE QUOTIENT WHEN YOU DIVIDE POWERS. CHECK YOUR RULE BY DIVIDING $7^5 \div 7^3$ USING A CALCULATOR.

3 INDICATE WHETHER EACH STATEMENT IS TRUE OR FALSE. IF FALSE, EXPLAIN:

A BETWEEN ANY TWO RATIONAL NUMBERS, THERE IS ALWAYS A RATIONAL NUMBER.

B THE SET OF REAL NUMBERS IS THE UNION OF THE SET OF RATIONAL NUMBERS AND THE SET OF IRRATIONAL NUMBERS.

C THE SET OF RATIONAL NUMBERS IS CLOSED UNDER ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION EXCLUDING DIVISION BY ZERO.

D THE SET OF IRRATIONAL NUMBERS IS CLOSED UNDER ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION.

4 GIVE EXAMPLES TO SHOW EACH OF THE FOLLOWING:

A THE PRODUCT OF TWO IRRATIONAL NUMBERS MAY BE RATIONAL OR IRRATIONAL.

B THE SUM OF TWO IRRATIONAL NUMBERS MAY BE RATIONAL OR IRRATIONAL.

C THE DIFFERENCE OF TWO IRRATIONAL NUMBERS MAY BE RATIONAL OR IRRATIONAL.

D THE QUOTIENT OF TWO IRRATIONAL NUMBERS MAY BE RATIONAL OR IRRATIONAL.

5 DEMONSTRATE WITH AN EXAMPLE THAT THE SUM OF AN IRRATIONAL NUMBER AND A RATIONAL NUMBER IS IRRATIONAL.

6 DEMONSTRATE WITH AN EXAMPLE THAT THE PRODUCT OF AN IRRATIONAL NUMBER AND A NON-ZERO RATIONAL NUMBER IS IRRATIONAL.

7 COMPLETE THE FOLLOWING CHART USING THE WORDS ‘YES’ OR ‘NO’.

Number	Rational number	Irrational number	Real number
2			
$\sqrt{3}$			
$-\frac{2}{3}$			
$\frac{\sqrt{3}}{2}$			
$1.2\overline{3}$			
$1.20220222\ldots$			
$-\frac{2}{3}\times 1.2\overline{3}$			
$\sqrt{75}+1.2\overline{3}$			
$\sqrt{75}-\sqrt{3}$			
$1.20220222\ldots + 0.13113111\ldots$			

QUESTIONS 3, 4, 5 AND IN PARTICULAR QUESTION 7 OF THE ABOVE GROUP WORK LEAD YOU TO CONCLUDE THAT THE SET OF REAL NUMBERS IS CLOSED UNDER ADDITION, S MULTIPLICATION AND DIVISION, EXCLUDING DIVISION BY ZERO.

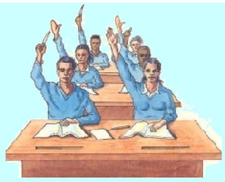
YOU RECALL THAT THE SET OF RATIONAL NUMBERS SATISFY THE COMMUTATIVE, A DISTRIBUTIVE LAWS FOR ADDITION AND MULTIPLICATION.

IF YOU ADD, SUBTRACT, MULTIPLY OR DIVIDE (EXCEPT BY 0) TWO RATIONAL NUMBE RATIONAL NUMBER, THAT IS, THE SET OF RATIONAL NUMBERS IS CLOSED WITH RESP SUBTRACTION, MULTIPLICATION AND DIVISION.

FROM GROUP WORK 1.6 YOU MAY HAVE REALIZED THAT THE SET OF IRRATIONAL NUMBERS IS NOT CLOSED UNDER ALL THE FOUR OPERATIONS, NAMELY ADDITION, SUBTRACTION, MULTIPLICATION

DO THE FOLLOWING ACTIVITY AND DISCUSS YOUR RESULTS.

ACTIVITY 1.13



1 FIND $a + b$, IF

A $a = 3+\sqrt{2}$ AND $b = 3-\sqrt{2}$

B $a = 3+\sqrt{3}$ AND $b = 2-\sqrt{3}$

2 FIND $a - b$, IF

A $a = \sqrt{3}$ AND $b = \sqrt{\ }$ **B** $a = \sqrt{5}$ AND $b = \sqrt{\ }$

3 FIND ab, IF

A $a = \sqrt{3} - 1$ AND $b = \sqrt{\ }\ 3$ **B** $a = 2\sqrt{3}$ AND $b = \sqrt{3}$

4 FIND $a \div b$, IF

A $a = 5\sqrt{2}$ AND $b = \sqrt{3}$ **B** $a = 6\sqrt{6}$ AND $b = \sqrt{2}$

LET US SEE SOME EXAMPLES OF THE FOUR OPERATIONS ON REAL NUMBERS.

EXAMPLE 1 ADD $a = 2\sqrt{3} + 3\sqrt{2}$ AND $\sqrt{2} - 3\sqrt{3}$

SOLUTION

$$\left(2\sqrt{3}+3\sqrt{2}\right)+\left(\sqrt{2}-3\sqrt{3}\right) = 2\sqrt{3}+3\sqrt{2}+\sqrt{2}-3\sqrt{3}$$
$$= \sqrt{3}(2-3)+\sqrt{2}(3+1)$$
$$= -\sqrt{3}+4\sqrt{2}$$

EXAMPLE 2 SUBTRACT $3\sqrt{2}+\sqrt{5}$ FROM $3\sqrt{5}-2\sqrt{2}$

SOLUTION:

$$\left(3\sqrt{5}-2\sqrt{2}\right)-\left(3\sqrt{2}+\sqrt{5}\right) = 3\sqrt{5}-2\sqrt{2}-3\sqrt{2}-\sqrt{5}$$
$$= \sqrt{5}(3-1)+\sqrt{2}(-2-3)$$
$$= 2\sqrt{5}-5\sqrt{2}$$

EXAMPLE 3 MULTIPLY

A $2\sqrt{3}$ BY $3\sqrt{2}$ **B** $2\sqrt{5}$ BY $3\sqrt{5}$

SOLUTION:

A $2\sqrt{3}\times 3\sqrt{2} = 6\sqrt{6}$ **B** $2\sqrt{5}\times 3\sqrt{5} = 2\times 3\times\left(\sqrt{5}\right)^2 = 30$

EXAMPLE 4 DIVIDE

A $8\sqrt{6}$ BY $2\sqrt{3}$ **B** $12\sqrt{6}$ BY $\left(\sqrt{2}\times\sqrt{3}\right)$

SOLUTION:

A $$8\sqrt{6}\div 2\sqrt{3} = \frac{8\sqrt{6}}{2\sqrt{3}} = \frac{8}{2}\times\sqrt{\frac{6}{3}} = 4\sqrt{2}$$

B $$12\sqrt{6}\div\left(\sqrt{2}\times\sqrt{3}\right) = \frac{12\sqrt{6}}{\sqrt{2}\times\sqrt{3}} = \frac{12\sqrt{6}}{\sqrt{6}} = 12$$

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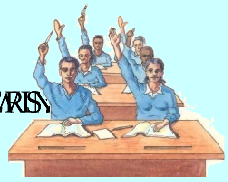
RULES OF EXPONENTS HOLD FOR REAL NUMBERS. IF a AND b ARE NON-ZERO NUMBERS AND m AND n ARE REAL NUMBERS, THEN WHENEVER THE POWERS ARE DEFINED, YOU HAVE THE LAWS OF EXPONENTS.

1 $a^m \times a^n = a^{m+n}$ **2** $\left(a^m\right)^n = a^{mn}$ **3** $\dfrac{a^m}{a^n} = a^{m-n}$

4 $a^n \times b^n = (ab)^n$ **5** $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$, $b \neq 0$.

ACTIVITY 1.14



1 FIND THE ADDITIVE INVERSE OF EACH OF THE FOLLOWING NUMBERS:

A 5 **B** $-\dfrac{1}{2}$ **C** $\sqrt{2}+1$

D $2.4\overline{5}$ **E** $2.1010010001\ldots$

2 FIND THE MULTIPLICATIVE INVERSE OF EACH OF THE FOLLOWING REAL NUMBERS:

A 3 **B** $\sqrt{5}$ **C** $1-\sqrt{3}$ **D** $2^{\frac{1}{6}}$

E 1.71 **F** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **G** $1.\overline{3}$

3 EXPLAIN EACH OF THE FOLLOWING STEPS:

$$\left(\sqrt{6}-2\sqrt{15}\right)\times\frac{\sqrt{3}}{3}+\sqrt{20}=\frac{\sqrt{3}}{3}\times\left(\sqrt{6}-2\sqrt{15}\right)+\sqrt{20}$$

$$=\left(\frac{\sqrt{3}}{3}\times\sqrt{6}-\frac{\sqrt{3}}{3}\times2\sqrt{15}\right)+\sqrt{20}$$

$$=\left(\frac{\sqrt{18}}{3}-\frac{2\sqrt{45}}{3}\right)+\sqrt{20}$$

$$=\left(\frac{\sqrt{9}\times\sqrt{2}}{3}-\frac{2\sqrt{9}\times\sqrt{5}}{3}\right)+\sqrt{20}$$

$$=\left(\frac{3\times\sqrt{2}}{3}-\frac{2\times3\times\sqrt{5}}{3}\right)+\sqrt{20}$$

$$=\left(\sqrt{2}-2\sqrt{5}\right)+\sqrt{20}$$

$$=\sqrt{2}+\left[\left(-2\sqrt{5}\right)+2\sqrt{5}\right]$$

$$=\sqrt{2}$$

LET US NOW EXAMINE THE BASIC PROPERTIES THAT GOVERN ADDITION AND MULTIPLI
NUMBERS. YOU CAN LIST THESE BASIC PROPERTIES AS FOLLOWS:

✓ **Closure property:**

THE SET $\mathbb{R}$ OF REAL NUMBERS IS CLOSED UNDER ADDITION AND MULTIPLICATION. THIS

THE SUM AND PRODUCT OF TWO REAL NUMBERS IS A REAL NUMBER; THAT IS, FOR AL $a, b \in \mathbb{R}$,

$a + b \in \mathbb{R}$ AND $ab \in \mathbb{R}$

✓ **Addition and multiplication are commutative in $\mathbb{R}$:**

THAT IS, FOR ALL $a, b \in \mathbb{R}$,

I $a + b = b + a$

II $ab = ba$

✓ **Addition and multiplication are associative in $\mathbb{R}$:**

THAT IS, FOR ALL $a, b, c \in \mathbb{R}$,

I $(a + b) + c = a + (b + c)$

II $(ab)c = a(bc)$

✓ **Existence of additive and multiplicative identities:**

THERE ARE REAL NUMBERS 0 AND 1 SUCH THAT:

I $a + 0 = 0 + a = a$, FOR ALL $a \in \mathbb{R}$.

II $a \cdot 1 = 1 \cdot a = a$, FOR ALL $a \in \mathbb{R}$.

✓ **Existence of additive and multiplicative inverses:**

I FOR EACH $a \in \mathbb{R}$ THERE EXISTS $-a \in \mathbb{R}$ SUCH THAT $a + (-a) = 0 = (-a) + a$, AND $-a$ IS CALLED THE ADDITIVE INVERSE OF a.

II FOR EACH NON-ZERO $a \in \mathbb{R}$, THERE EXISTS $\frac{1}{a} \in \mathbb{R}$ SUCH THAT $a \times \left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right) \times a$,

AND $\frac{1}{a}$ IS CALLED THE MULTIPLICATIVE INVERSE OR RECIPROCAL OF a

✓ **Distributive property:**

MULTIPLICATION IS DISTRIBUTIVE OVER ADDITION; THAT IS, IF $a, b, c \in \mathbb{R}$, THEN:

i $a(b + c) = ab + ac$

ii $(b + c)a = ba + ca$

Exercise 1.8

1 FIND THE NUMERICAL VALUE OF EACH OF THE FOLLOWING:

A $\left(4^{-1}\right)^4 \times 2^5 \times \left(\frac{1}{16}\right)^3 \times \left(8^{-2}\right)^5 \times \left(64^2\right)^3$ B $\sqrt{176} - 2\sqrt{275} + \sqrt{1584} - \sqrt{891}$

C $15\sqrt{1.04} - \frac{3}{5}\sqrt{5\frac{5}{9}} + 6\sqrt{\frac{1}{18}} - \left(5\sqrt{0.02} - \sqrt{300}\right)$

D $\sqrt[4]{0.0001} - \sqrt[5]{0.00032}$ E $2\sqrt[3]{0.125} + \sqrt[4]{0.0016}$

2 SIMPLIFY EACH OF THE FOLLOWING

A $(216)^{\frac{1}{3}}$ B $2^{\frac{2}{3}} \times 2^{\frac{3}{5}}$ C $\left(3^{\frac{1}{2}}\right)^5$ D $\dfrac{7^{\frac{3}{4}}}{49^{\frac{1}{4}}}$

E $3^{\frac{1}{4}} \times 25^{\frac{1}{8}}$ F $16^{\frac{1}{4}} \div 2$ G $\sqrt[4]{\sqrt[3]{7}}$ H $\dfrac{\sqrt[5]{32}}{\sqrt[5]{243}}$

3 WHAT SHOULD BE ADDED TO EACH OF THE FOLLOWING TO MAKE IT A RATIONAL NUMBER? (THERE ARE MANY POSSIBLE ANSWERS. IN EACH CASE, GIVE TWO ANSWI

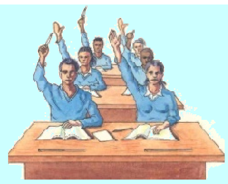
A $5 - \sqrt{3}$ B $-2 - \sqrt{5}$ C $4.383383338\ldots$

D $6.123456\ldots$ E $10.3030003\ldots$

1.2.6 Limits of Accuracy

IN THIS SUBSECTION, YOU SHALL DISCUSS CERTAIN CONCEPTS SUCH AS APPROXIMATIC MEASUREMENTS, SIGNIFICANT FIGURES (S.F), DECIMAL PLACES (D.P) AND ROUNDING O IN ADDITION TO THIS, YOU SHALL DISCUSS HOW TO GIVE APPROPRIATE UPPER AND L FOR DATA TO A SPECIFIED ACCURACY (FOR EXAMPLE MEASURED LENGTHS).

ACTIVITY 1.15



1 ROUND OFF THE NUMBER 28617 TO THE NEAREST

A 10,000 B 1000 C 100

2 WRITE THE NUMBER I 7.864 II $6.\overline{437}$ III $4.56556555\ldots$

A TO ONE DECIMAL PLACE B TO TWO DECIMAL PLACES

3 WRITE THE NUMBER 43.25 TO

A TWO SIGNIFICANT FIGURES B THREE SIGNIFICANT FIGURES

4 THE WEIGHT OF AN OBJECT IS 5.4 KG.

GIVE THE LOWER AND UPPER BOUNDS WITHIN WHICH THE WEIGHT OF THE OBJECT

1 Counting and measuring

COUNTING AND MEASURING ARE AN INTEGRAL PART OF OUR LIFE. MOST OF US DO SO FOR VARIOUS REASONS AND AT VARIOUS OCCASIONS. FOR EXAMPLE YOU CAN COUNT TH RECEIVE FROM SOMEONE, A TAILOR MEASURES THE LENGTH OF THE SHIRT HE/SHE MA A CARPENTER COUNTS THE NUMBER OF SCREWS REQUIRED TO MAKE A DESK

Counting: THE PROCESS OF COUNTING INVOLVES FINDING THE EXACT NUMBER OF THINGS. FOR EXAMPLE, YOU DO COUNTING TO FIND OUT THE NUMBER OF STUDENTS IN A CLASS IS AN EXACT NUMBER AND IS EITHER CORRECT OR, IF YOU HAVE MADE A MISTAKE, IN MANY OCCASIONS, JUST AN ESTIMATE IS SUFFICIENT AND THE EXACT NUMBER IS NO IMPORTANT.

Measuring: IF YOU ARE FINDING THE LENGTH OF A FOOTBALL FIELD, THE WEIGHT OF A PERSON OR THE TIME IT TAKES TO WALK DOWN TO SCHOOL, YOU ARE MEASURING. THE ANSW EXACT NUMBERS BECAUSE THERE COULD BE ERRORS IN MEASUREMENTS.

2 Estimation

IN MANY INSTANCES, EXACT NUMBERS ARE NOT NECESSARY OR EVEN DESIRABLE. IN T CONDITIONS, APPROXIMATIONS ARE GIVEN. THE APPROXIMATIONS CAN TAKE SEVERAL HERE YOU SHALL DEAL WITH THE COMMON TYPES OF APPROXIMATIONS.

A Rounding

IF 38,518 PEOPLE ATTEND A FOOTBALL GAME THIS FIGURE CAN BE REPORTED TO VARIOUS LEVELS OF ACCURACY.

TO THE NEAREST 10,000 THIS FIGURE WOULD BE ROUNDED UP TO 40,000.

TO THE NEAREST 1000 THIS FIGURE WOULD BE ROUNDED UP TO 39,000.

TO THE NEAREST 100 THIS FIGURE WOULD BE ROUNDED DOWN TO 38,500

IN THIS TYPE OF SITUATION, IT IS UNLIKELY THAT THE EXACT NUMBER WOULD BE REPO

B Decimal places

A NUMBER CAN ALSO BE APPROXIMATED TO A GIVEN NUMBER OF DECIMAL PLACES (D.P). THIS REFERS TO THE NUMBER OF FIGURES WRITTEN AFTER A DECIMAL POINT.

EXAMPLE 1

A WRITE 7.864 TO 1 D.P. **B** WRITE 5.574 TO 2 D.P.

SOLUTION:

A THE ANSWER NEEDS TO BE WRITTEN WITH ONE NUMBER AFTER THE DECIMAL POINT. HOWEVER, TO DO THIS, THE SECOND NUMBER AFTER THE DECIMAL POINT ALS BE CONSIDERED. IF IT IS 5 OR MORE, THEN THE FIRST NUMBER IS ROUNDED UP

THAT IS 7.864 IS WRITTEN AS 7.9 TO 1 D.P

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B THE ANSWER HERE IS TO BE GIVEN WITH TWO NUMBERS AFTER THE DECIMAL POINT. IN THIS CASE, THE THIRD NUMBER AFTER THE DECIMAL POINT NEEDS TO BE CO AS THE THIRD NUMBER AFTER THE DECIMAL POINT IS LESS THAN 5, THE NUMBER IS NOT ROUNDED UP.

THAT IS 5.574 IS WRITTEN AS 5.57 TO 2 D.P.

NOTE THAT TO APPROXIMATE A NUMBER TO 1 D.P MEANS TO APPROXIMATE THE NUMBE NEAREST TENTH. SIMILARLY APPROXIMATING A NUMBER TO 2 DECIMAL PLACES MEAN APPROXIMATE TO THE NEAREST HUNDREDTH.

C Significant figures

NUMBERS CAN ALSO BE APPROXIMATED TO A GIVEN NUMBER OF SIGNIFICANT FIGURES NUMBER 43.25 THE 4 IS THE MOST SIGNIFICANT FIGURE AS IT HAS A VALUE OF 40. IN CO 5 IS THE LEAST SIGNIFICANT AS IT ONLY HAS A VALUE OF 5 HUNDREDTHS. WHEN WE SIGNIFICANT FIGURES TO INDICATE THE ACCURACY OF APPROXIMATION, WE COUNT DIGITS IN THE NUMBER FROM LEFT TO RIGHT, BEGINNING AT THE FIRST NON-ZERO KNOWN AS THE NUMBER OF SIGNIFICANT FIGURES.

EXAMPLE 2

A WRITE 43.25 TO 3 S.F. **B** WRITE 0.0043 TO 1 S.F.

SOLUTION:

A WE WANT TO WRITE ONLY THE THREE MOST SIGNIFICANT DIGITS. HOWEVER, THE FOURTH DIGIT NEEDS TO BE CONSIDERED TO SEE WHETHER THE THIRD DIGIT IS TO B UP OR NOT.

THAT IS, 43.25 IS WRITTEN AS 43.3 TO 3 S.F.

B NOTICE THAT IN THIS CASE 4 AND 3 ARE THE ONLY SIGNIFICANT DIGITS. THE NUMBER 4 IS THE MOST SIGNIFICANT DIGIT AND IS THEREFORE THE ONLY ONE OF THE WRITTEN IN THE ANSWER.

THAT IS 0.0043 IS WRITTEN AS 0.004 TO 1 S.F.

3 Accuracy

IN THE PREVIOUS LESSON, YOU HAVE STUDIED THAT NUMBERS CAN BE APPROXIMATED

A BY ROUNDING UP

B BY WRITING TO A GIVEN NUMBER OF DECIMAL PLACE AND

C BY EXPRESSING TO A GIVEN NUMBER OF SIGNIFICANT FIGURE.

IN THIS LESSON, YOU WILL LEARN HOW TO GIVE APPROPRIATE upper and lower bounds FOR DATA TO A SPECIFIED ACCURACY (FOR EXAMPLE, NUMBERS ROUNDED OFF OR NUMBERS EXPR GIVEN NUMBER OF SIGNIFICANT FIGURES).

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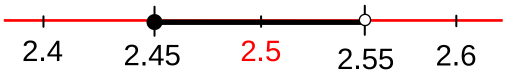
NUMBERS CAN BE WRITTEN TO DIFFERENT DEGREES OF ACCURACY.

FOR EXAMPLE, ALTHOUGH 2.5, 2.50 AND 2.500 MAY APPEAR TO REPRESENT THE SAME NU
THEY ACTUALLY DO NOT. THIS IS BECAUSE THEY ARE WRITTEN TO DIFFERENT DEGREE

2.5 IS ROUNDED TO ONE DECIMAL PLACE (OR TO THE NEAREST TENTHS) AND THEREFOR
FROM 2.45 UP TO BUT NOT INCLUDING 2.55 WOULD BE ROUNDED TO 2.5. ON THE NUMBER
WOULD BE REPRESENTED AS



AS AN INEQUALITY, IT WOULD BE EXPRESSED AS

$2.45 \leq 2.5 < 2.55$

2.45 IS KNOWN AS THE lower bound OF 2.5, WHILE

2.55 IS KNOWN AS THE upper bound.

2.50 ON THE OTHER HAND IS WRITTEN TO TWO DECIMAL PLACES AND THEREFORE ONLY
2.495 UP TO BUT NOT INCLUDING 2.505 WOULD BE ROUNDED TO 2.50. THIS, THEREFORE,
REPRESENTS A MUCH SMALLER RANGE OF NUMBERS THAN THAT BEING ROUNDED TO 2.
THE RANGE OF NUMBERS BEING ROUNDED TO 2.500 WOULD BE EVEN SMALLER.

EXAMPLE 3 A GIRL'S HEIGHT IS GIVEN AS 162 CM TO THE NEAREST CENTIMETRE.

I WORKOUT THE LOWER AND UPPER BOUNDS WITHIN WHICH THE HEIGHT LIE.

II REPRESENT THIS RANGE OF NUMBERS ON A NUMBER LINE.

III IF THE GIRL'S HEIGHT IS h CM, EXPRESS THIS RANGE AS AN INEQUALITY.

SOLUTION:

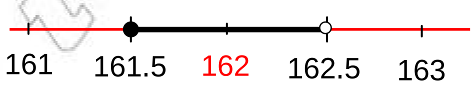
I 162 CM IS ROUNDED TO THE NEAREST CENTIMETRE AND THEREFORE ANY MEASUREMENT OF CM FROM 161.5 CM UP TO AND NOT INCLUDING 162.5 CM WOULD BE ROUND 162 CM.

THUS,

LOWER BOUND = 161.5 CM

UPPER BOUND = 162.5 CM

II RANGE OF NUMBERS ON THE NUMBER LINE IS REPRESENTED AS



III WHEN THE GIRL'S HEIGHT h CM IS EXPRESSED AS AN INEQUALITY, IT IS GIVEN BY

$161.5 \leq h < 162.5$.

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Effect of approximated numbers on calculations

WHEN APPROXIMATED NUMBERS ARE ADDED, SUBTRACTED OR MULTIPLIED, THEIR SUMS, DIFFERENCES AND PRODUCTS GIVE A RANGE OF POSSIBLE ANSWERS.

EXAMPLE 4 THE LENGTH AND WIDTH OF A RECTANGLE ARE 6.7 CM AND 4.4 CM RESPECTIVELY. FIND THEIR SUM.

SOLUTION: IF THE LENGTH l = 6.7 CM AND THE WIDTH w = 4.4 CM

THEN $6.65 \le l < 6.75$ AND $4.35 \le w < 4.45$

THE LOWER BOUND OF THE SUM IS OBTAINED BY ADDING THE TWO LOWER BOUND THEREFORE, THE MINIMUM SUM IS 6.65 + 4.35 THAT IS 11.00.

THE UPPER BOUND OF THE SUM IS OBTAINED BY ADDING THE TWO UPPER BOUNDS. THEREFORE, THE MAXIMUM SUM IS 6.75 + 4.45 THAT IS 11.20.

SO, THE SUM LIES BETWEEN 11.00 CM AND 11.20 CM.

EXAMPLE 5 FIND THE LOWER AND UPPER BOUNDS FOR THE FOLLOWING CALCULATION, GIVEN THAT EACH NUMBER IS GIVEN TO 1 DECIMAL PLACE.

$$3.4 \times 7.6$$

SOLUTION:

IF $x = 3.4$ AND $y = 7.6$ THEN $3.35 \le x < 3.45$ AND $7.55 \le y < 7.65$

THE LOWER BOUND OF THE PRODUCT IS OBTAINED BY MULTIPLYING THE TWO LO THEREFORE, THE MINIMUM PRODUCT IS 3.35×7.55 THAT IS 25.2925

THE UPPER BOUND OF THE PRODUCT IS OBTAINED BY MULTIPLYING THE TWO UPP THEREFORE, THE MAXIMUM PRODUCT IS 3.45×7.65 THAT IS 26.3925.

SO THE PRODUCT LIES BETWEEN 25.2925 AND 26.3925.

EXAMPLE 6 CALCULATE THE UPPER AND LOWER BOUNDS TO $\frac{54.5}{36.0}$, GIVEN THAT EACH OF THE NUMBERS IS ACCURATE TO 1 DECIMAL PLACE.

SOLUTION: 54.5 LIES IN THE RANGE $54.45 \le x < 54.55$

36.0 LIES IN THE RANGE $35.95 \le x < 36.05$

THE LOWER BOUND OF THE CALCULATION IS OBTAINED BY DIVIDING THE LOWER NUMERATOR BY THE UPPER BOUND OF THE DENOMINATOR.

SO, THE MINIMUM VALUE IS $54.45 \div 36.05$ I.E., 1.51 (2 DECIMAL PLACES).

THE UPPER BOUND OF THE CALCULATION IS OBTAINED BY DIVIDING THE UPPER BO NUMERATOR BY THE LOWER BOUND OF THE DENOMINATOR.

SO, THE MAXIMUM VALUE IS $54.55 \div 35.95$ I.E., 1.52 (2 DECIMAL PLACES).

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Exercise 1.9

1 ROUND THE FOLLOWING NUMBERS TO THE NEAREST 1000.

A 6856 **B** 74245 **C** 89000 **D** 99500

2 ROUND THE FOLLOWING NUMBERS TO THE NEAREST 100.

A 78540 **B** 950 **C** 14099 **D** 2984

3 ROUND THE FOLLOWING NUMBERS TO THE NEAREST 10.

A 485 **B** 692 **C** 8847 **D** 4 **E** 83

4 **I** GIVE THE FOLLOWING TO 1 D.P.

A 5.58 **B** 4.04 **C** 157.39 **D** 15.045

II ROUND THE FOLLOWING TO THE NEAREST TENTH.

A 157.39 **B** 12.049 **C** 0.98 **D** 2.95

III GIVE THE FOLLOWING TO 2 D.P.

A 6.473 **B** 9.587 **C** 0.014 **D** 99.996

IV ROUND THE FOLLOWING TO THE NEAREST HUNDREDTH.

A 16.476 **B** 3.0037 **C** 9.3048 **D** 12.049

5 WRITE EACH OF THE FOLLOWING TO THE NUMBER OF SIGNIFICANT FIGURES INDICATED IN BRACKETS.

A 48599 (1 S.F) **B** 48599 (3 S.F) **C** 2.5728 (3 S.F)

D 2045 (2 S.F) **E** 0.08562 (1 S.F) **F** 0.954 (2 S.F)

G 0.00305 (2 S.F) **H** 0.954 (1 S.F)

6 EACH OF THE FOLLOWING NUMBERS IS EXPRESSED TO THE NEAREST WHOLE NUMBER.

I GIVE THE UPPER AND LOWER BOUNDS OF EACH.

II USING x AS THE NUMBER, EXPRESS THE RANGE IN WHICH THE NUMBER LIES INEQUALITY.

A 6 **B** 83 **C** 151 **D** 1000

7 EACH OF THE FOLLOWING NUMBERS IS CORRECT TO ONE DECIMAL PLACE.

I GIVE THE UPPER AND LOWER BOUNDS OF EACH.

II USING x AS THE NUMBER, EXPRESS THE RANGE IN WHICH THE NUMBER LIES INEQUALITY.

A 3.8 **B** 15.6 **C** 1.0 **D** 0.3 **E** –0.2

8 EACH OF THE FOLLOWING NUMBERS IS CORRECT TO TWO SIGNIFICANT FIGURES.

I GIVE THE UPPER AND LOWER BOUNDS OF EACH.

II USING x AS THE NUMBER, EXPRESS THE RANGE IN WHICH THE NUMBER LIES INEQUALITY.

A 4.2 **B** 0.84 **C** 420 **D** 5000 **E** 0.045

9 CALCULATE THE UPPER AND LOWER BOUNDS FOR THE FOLLOWING CALCULATIONS, IF EACH OF THE NUMBERS IS GIVEN TO 1 DECIMAL PLACE.

A 9.5×7.6 **B** 11.0×15.6 **C** $\frac{46.5}{32.0}$

D $\frac{25.4}{8.2}$ **E** $\frac{4.9+6.4}{2.6}$

10 THE MASS OF A SACK OF VEGETABLES IS GIVEN AS 5.4 KG.

A ILLUSTRATE THE LOWER AND UPPER BOUNDS ON A NUMBER LINE.

B USING M KG FOR THE MASS, EXPRESS THE RANGE OF VALUES IN WHICH M MUST LIE, AS AN INEQUALITY.

11 THE MASSES TO THE NEAREST 0.5 KG OF TWO PARCELS ARE 1.5 KG AND 2.5 KG. CALCULATE THE LOWER AND UPPER BOUNDS OF THEIR COMBINED MASS.

12 CALCULATE UPPER AND LOWER BOUNDS FOR THE PERIMETER OF THE FOOTBALL FIELD SHOWN, IF ITS DIMENSIONS ARE CORRECT TO 1 DECIMAL PLACE.

Figure 1.9

13 CALCULATE UPPER AND LOWER BOUNDS FOR THE LENGTH MARKED x OF THE RECTANGLE SHOWN. THE AREA AND LENGTH ARE BOTH GIVEN TO 1 DECIMAL PLACE.

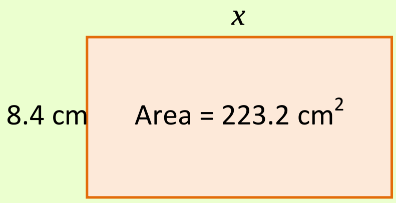


Figure 1.10

1.2.7 Scientific Notation (Standard form)

IN SCIENCE AND TECHNOLOGY, IT IS USUAL TO SEE VERY LARGE AND VERY SMALL INSTANCE:

THE AREA OF THE AFRICAN CONTINENT IS ABOUT 30,000,000 KM^2.

THE DIAMETER OF A HUMAN CELL IS ABOUT 0.0000002 M.

VERY LARGE NUMBERS AND VERY SMALL NUMBERS MAY SOMETIMES BE DIFFICULT TO WRITE. HENCE YOU OFTEN WRITE VERY LARGE OR VERY SMALL NUMBERS IN **scientific notation**, ALSO CALLED **standard form**.

EXAMPLE 1 1.86×10^{-6} IS WRITTEN IN SCIENTIFIC NOTATION.

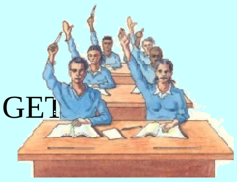
Number from 1 up to but not including 10.

Times 10 to a power.

8.735×10^4 AND 7.08×10^{-3} ARE WRITTEN IN SCIENTIFIC NOTATION.

14.73×10^{-1}, 0.0863×10^4 AND 3.86^4 ARE NOT WRITTEN IN STANDARD FORM (SCIENTIF NOTATION).

ACTIVITY 1.16



1 BY WHAT POWERS OF 10 MUST YOU MULTIPLY 1.3 TO GET:

A 13? **B** 130? **C** 1300?

COPY AND COMPLETE THIS TABLE.

$13 = 1.3 \times 10^1$
$130 = 1.3 \times 10^2$
$1,300 = 1.3 \times$
$13,000 =$
$1,300,000 =$

2 CAN YOU WRITE NUMBERS BETWEEN 0 AND 1 IN SCIENTIFIC NOTATION, FOR EXAMPLE 0.00013?

COPY AND COMPLETE THE FOLLOWING TABLE.

$13.0 = 1.3 \times 10 = 1.3 \times 10^1$
$1.3 = 1.3 \times 1 = 1.3 \times 10^0$
$0.13 = 1.3 \times \frac{1}{10} = 1.3 \times 10^{-1}$
$0.013 = 1.3 \times \frac{1}{100} =$
$0.0013 =$
$0.00013 =$
$0.000013 =$
$0.0000013 =$

NOTE THAT IF n IS A POSITIVE INTEGER, MULTIPLYING A NUMBER BY 10^n MOVES ITS DECIMAL POINT n PLACES TO THE RIGHT, AND MULTIPLYING IT BY 10^{-n} MOVES THE DECIMAL POINT n PLACES TO THE LEFT.

Definition 1.13

A number is said to be in scientific notation (or standard form), if it is written as a product of the form

$$a \times 10^k$$

where $1 \leq a < 10$ and k is an integer.

EXAMPLE 2 EXPRESS EACH OF THE FOLLOWING NUMBERS IN SCIENTIFIC NOTATION.

A 243, 900,000 **B** 0.000000595

SOLUTION:

A $243{,}900{,}000 = 2.439 \times 10^8$.

THE DECIMAL POINT MOVES 8 PLACES TO THE LEFT.

B $0.000000595 = 5.95 \times 10^{-7}$.

THE DECIMAL POINT MOVES 7 PLACES TO THE RIGHT.

EXAMPLE 3 EXPRESS 2.483×10^5 IN ORDINARY DECIMAL NOTATION.

SOLUTION: $2.483 \times 10^5 = 2.483 \times 100{,}000 = 248{,}300$.

EXAMPLE 4 THE DIAMETER OF A RED BLOOD CELL IS ABOUT 7.4×10^{-4} CM. WRITE THIS DIAMETER IN ORDINARY DECIMAL NOTATION.

SOLUTION: $7.4 \times 10^{-4} = 7.4 \times \frac{1}{10^4} = 7.4 \times \frac{1}{10{,}000} = 7.4 \times 0.0001 = 0.00074$.

SO, THE DIAMETER OF A RED BLOOD CELL IS ABOUT 0.00074 CM.

CALCULATORS AND COMPUTERS ALSO USE SCIENTIFIC NOTATION TO DISPLAY LARG SMALL NUMBERS BUT SOMETIMES ONLY THE EXPONENT OF 10 IS SHOWN. CALCULATOR BEFORE THE EXPONENT, WHILE COMPUTERS USE THE LETTER E.

➤ THE CALCULATOR DISPLAY 5.23 06 MEANS 5.23×10^6 (5,230,000).

THE FOLLOWING EXAMPLE SHOWS HOW TO ENTER A NUMBER WITH TOO MANY DIGITS TO FIT ON THE DISPLAY SCREEN INTO A CALCULATOR.

EXAMPLE 5 ENTER 0.00000000627 INTO A CALCULATOR.

SOLUTION: FIRST, WRITE THE NUMBER IN SCIENTIFIC NOTATION.

$0.00000000627 = 6.27 \times 10^{-9}$

THEN, ENTER THE NUMBER.

6.27 $\boxed{\text{EXP}}$ 9 $\boxed{+/-}$ GIVING 6.27 – 09

Decimal notation	Scientific notation	Calculator display	Computer display
250,000	2.5×10^5	2.5 05	2.5 E + 5
0.00047	4.7×10^{-4}	4.7 – 04	4.7 E – 4

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Exercise 1.10

1 EXPRESS EACH OF THE FOLLOWING NUMBERS IN SCIENTIFIC NOTATION:

A 0.00767 **B** 5,750,000,000 **C** 0.00083

D 400,400 **E** 0.054

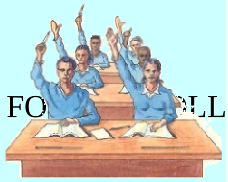
2 EXPRESS EACH OF THE FOLLOWING NUMBERS IN ORDINARY NOTATION:

A 4.882×10^5 **B** 1.19×10^{-5} **C** 2.021×10^2

3 EXPRESS THE DIAMETER OF AN ELECTRON WHICH IS ABOUT 0.0000000000004 CM IN SCIENTIFIC NOTATION.

1.2.8 Rationalization

ACTIVITY 1.17



FIND AN APPROXIMATE VALUE, TO TWO DECIMAL PLACES, OF THE FOLLOWING:

I $\frac{1}{\sqrt{2}}$ **II** $\frac{\sqrt{2}}{2}$

IN CALCULATING THIS, THE FIRST STEP IS TO FIND AN APPROXIMATION OF $\sqrt{2}$ IN A REFERENCE BOOK OR OTHER REFERENCE MATERIAL (1.414214…). THEN THE CALCULATION OF $\frac{1}{\sqrt{2}}$ IS 1 DIVIDED BY 1.414214… WHICH IS A DIFFICULT TASK. HOWEVER, EVALUATING $\frac{\sqrt{2}}{2}$ AS $\frac{1.414214}{2} \approx 0.707107$ IS EASY.

SINCE $\frac{1}{\sqrt{2}}$ IS EQUIVALENT TO $\frac{\sqrt{2}}{2}$ (HOW?), YOU SEE THAT IN ORDER TO EVALUATE AN EXPRESS WITH A RADICAL IN THE DENOMINATOR, FIRST YOU SHOULD TRANSFORM THE EXPR EQUIVALENT EXPRESSION WITH A RATIONAL NUMBER IN THE DENOMINATOR.

THE TECHNIQUE OF TRANSFERRING THE RADICAL EXPRESSION FROM THE DENOMI NUMERATOR IS CALLED **rationalizing the denominator** (CHANGING THE DENOMINATOR INTO A RATIONAL NUMBER).

THE NUMBER THAT CAN BE USED AS A MULTIPLIER TO RATIONALIZE THE DENOMINATO **rationalizing factor.** THIS IS EQUIVALENT TO 1.

FOR INSTANCE, IF $\sqrt{n}$ IS AN IRRATIONAL NUMBER THEN $\frac{1}{\sqrt{n}}$ CAN BE RATIONALIZED BY MULTIPLYING IT BY $\frac{\sqrt{n}}{\sqrt{n}} = 1$. SO, $\frac{\sqrt{n}}{\sqrt{n}}$ IS THE **rationalizing factor.**

EXAMPLE 1 RATIONALIZE THE DENOMINATOR IN EACH OF THE FOLLOWING.

A $\frac{5\sqrt{3}}{8\sqrt{5}}$ **B** $\frac{6}{\sqrt{3}}$ **C** $\frac{3}{\sqrt[3]{2}}$

SOLUTION:

A THE RATIONALIZING FACTOR IS $\frac{\sqrt{5}}{\sqrt{5}}$

SO, $\frac{5\sqrt{3}}{8\sqrt{5}} = \frac{5\sqrt{3}}{8\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{15}}{8\sqrt{25}} = \frac{5\sqrt{15}}{8\sqrt{5^2}} = \frac{5\sqrt{15}}{8\times5} = \frac{\sqrt{15}}{8}$

B THE RATIONALIZING FACTOR IS $\frac{\sqrt{3}}{\sqrt{3}}$

SO, $\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3^2}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

C THE RATIONALIZING FACTOR IS $\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}$ BECAUSE $\sqrt[3]{2} \times \sqrt[3]{4} = \sqrt[3]{8} = 2$

SO, $\frac{3}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{3\sqrt[3]{4}}{\sqrt[3]{2^3}} = \frac{3\sqrt[3]{4}}{2}$

IF A RADICAND ITSELF IS A FRACTION $\left(\text{FOR EXAMPLE } \sqrt{\frac{2}{3}}\right)$, THEN, IT CAN BE WRITTEN IN THE EQUIVALENT FORM $\frac{\sqrt{2}}{\sqrt{3}}$ SO THAT THE PROCEDURE DESCRIBED ABOVE CAN BE APPLIED TO RATIONALIZE THE DENOMINATOR. THEREFORE,

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{\sqrt{3^2}} = \frac{\sqrt{6}}{3}$$

IN GENERAL,

FOR ANY NON-NEGATIVE INTEGERS a, b ($b \neq 0$)

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\sqrt{b}}{\sqrt{b}\sqrt{b}} = \frac{\sqrt{ab}}{b}.$$

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Exercise 1.11

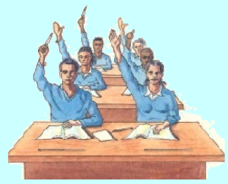
SIMPLIFY EACH OF THE FOLLOWING. STATE RESTRICTIONS WHERE NECESSARY. IN EA THE RATIONALIZING FACTOR YOU USE AND EXPRESS THE FINAL RESULT WITH A RATIO IN ITS LOWEST TERM.

A $\frac{2}{\sqrt{2}}$ **B** $\frac{\sqrt{2}}{\sqrt{6}}$ **C** $\frac{5\sqrt{2}}{4\sqrt{10}}$ **D** $\frac{12}{\sqrt{27}}$ **E** $\sqrt{\frac{5}{18}}$

F $\frac{3}{2\sqrt[3]{3}}$ **G** $\sqrt[3]{\frac{1}{4}}$ **H** $\sqrt{\frac{9}{a^2}}$ **I** $\frac{\sqrt[3]{20}}{\sqrt[3]{4}}$ **J** $\sqrt{\frac{4}{5}}$

More on rationalizations of denominators

ACTIVITY 1.18



FIND THE PRODUCT OF EACH OF THE FOLLOWING:

1 $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$ **2** $\left(5+3\sqrt{2}\right)\left(5-3\sqrt{2}\right)$

3 $\left(\sqrt{5}-\frac{1}{2}\sqrt{3}\right)\left(\sqrt{5}+\frac{1}{2}\sqrt{3}\right)$

YOU MIGHT HAVE OBSERVED THAT THE RESULTS OF ALL OF THE ABOVE PRODUCT NUMBERS.

THIS LEADS YOU TO THE FOLLOWING CONCLUSION:

USING THE FACT THAT

$$(a - b)(a + b) = a^2 - b^2,$$

YOU CAN RATIONALIZE THE DENOMINATORS OF EXPRESSIONS SUCH AS

$\frac{1}{a+\sqrt{b}}, \frac{1}{\sqrt{a}-b}, \frac{1}{\sqrt{a}-\sqrt{b}}$ WHERE $\sqrt{a}, \sqrt{b}$ ARE IRRATIONAL NUMBERS AS FOLLOWS.

I $$\frac{1}{a+\sqrt{b}} = \frac{1}{\left(a+\sqrt{b}\right)}\left(\frac{a-\sqrt{b}}{a-\sqrt{b}}\right) = \frac{a-\sqrt{b}}{a^2-\left(\sqrt{b}\right)^2} = \frac{a-\sqrt{b}}{a^2-b}$$

II $$\frac{1}{\sqrt{a}-b} = \frac{1}{\sqrt{a}-b}\left(\frac{\sqrt{a}+b}{\sqrt{a}+b}\right) = \frac{\sqrt{a}+b}{\left(\sqrt{a}\right)^2-b^2} = \frac{\sqrt{a}+b}{a-b^2}$$

III $$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{1}{\left(\sqrt{a}-\sqrt{b}\right)}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right) = \frac{\sqrt{a}+\sqrt{b}}{\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

EXAMPLE 2 RATIONALIZE THE DENOMINATOR OF EACH OF THE FOLLOWING

A $\frac{5}{1-\sqrt{2}}$ **B** $\frac{3}{\sqrt{6}+3\sqrt{2}}$

SOLUTION:

A THE RATIONALIZING FACTOR IS $\frac{1+\sqrt{2}}{1+\sqrt{2}}$

SO $\frac{5}{1-\sqrt{2}} = \frac{5(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{5+5\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2}$

$= \frac{5+5\sqrt{2}}{1-2} = -5-5\sqrt{2}$

B THE RATIONALIZING FACTOR IS $\frac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}}$

SO $\frac{3}{\sqrt{6}+3\sqrt{2}} = \frac{3}{\left(\sqrt{6}+3\sqrt{2}\right)}\frac{\sqrt{6}-3\sqrt{2}}{\sqrt{6}-3\sqrt{2}} = \frac{3\left(\sqrt{6}-3\sqrt{2}\right)}{\left(\sqrt{6}\right)^2-\left(3\sqrt{2}\right)^2}$

$= \frac{3\left(\sqrt{6}-3\sqrt{2}\right)}{6-18} = -\frac{1}{4}\left(\sqrt{6}-3\sqrt{2}\right)$

$= \frac{3\sqrt{2}-\sqrt{6}}{4}$

Exercise 1.12

RATIONALIZE THE DENOMINATOR OF EACH OF THE FOLLOWING:

A $\frac{1}{3-\sqrt{5}}$ **B** $\frac{\sqrt{18}}{\sqrt{5}-3}$ **C** $\frac{2}{\sqrt{5}-\sqrt{3}}$

D $\frac{\sqrt{3}+4}{\sqrt{3}-2}$ **E** $\frac{10}{\sqrt{7}-\sqrt{2}}$ **F** $\frac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

G $\frac{1}{\sqrt{2}+\sqrt{3}-1}$

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1.2.9 Euclid's Division Algorithm

A The division algorithm

ACTIVITY 1.19

1. IS THE SET OF NON-NEGATIVE INTEGERS (WHOLE NUMBERS) CLOSED UNDER DIVISION?
2. CONSIDER ANY TWO NON-NEGATIVE INTEGERS a AND b.
   - A WHAT DOES THE STATEMENT "a IS A MULTIPLE OF b" MEAN?
   - B IS IT ALWAYS POSSIBLE TO FIND A NON-NEGATIVE INTEGER c SUCH THAT $a = bc$?

IF a AND b ARE ANY TWO NON-NEGATIVE INTEGERS, THEN $a \div b$ $(b \neq 0)$ IS SOME NON-NEGATIVE INTEGER c (IF IT EXISTS) SUCH THAT $a = bc$. HOWEVER, SINCE THE SET OF NON-NEGATIVE INTEGERS IS NOT CLOSED UNDER DIVISION, IT IS CLEAR THAT EXACT DIVISION IS NOT POSSIBLE FOR EVERY PAIR (INTEGERS.

FOR EXAMPLE, IT IS NOT POSSIBLE TO COMPUTE $17 \div 5$ IN THE SET OF NON-NEGATIVE INTEGERS, AS $17 \div 5$ IS NOT A NON-NEGATIVE INTEGER.

$15 = 3 \times 5$ AND $20 = 4 \times 5$. SINCE THERE IS NO NON-NEGATIVE INTEGER BETWEEN 3 AND 4, SINCE 17 LIES BETWEEN 15 AND 20, YOU CONCLUDE THAT THERE IS NO NON-NEGATIV c SUCH THAT $17 = c \times 5$.

YOU OBSERVE, HOWEVER, THAT BY ADDING 2 TO EACH SIDE OF THE EQUATION $15 = 3 \times 5$ EXPRESS IT AS $17 = (3 \times 5) + 2$. FURTHERMORE, SUCH AN EQUATION IS USEFUL. FOR INSTAN WILL PROVIDE A CORRECT ANSWER TO A PROBLEM SUCH AS: IF 5 GIRLS HAVE BIRR 17 T MANY BIRR WILL EACH GIRL GET? EXAMPLES OF THIS SORT LEAD TO THE FOLLOWING THE Division Algorithm.

Theorem 1.4 Division algorithm

Let a and b be two non-negative integers and $b \neq 0$, then there exist unique non-negative integers q and r, such that,

$$a = (q \times b) + r \quad \text{with } 0 \leq r < b.$$

IN THE THEOREM, a IS CALLED THE dividend, q IS CALLED THE quotient, b IS CALLED THE divisor, AND r IS CALLED THE remainder.

EXAMPLE 1 WRITE a IN THE FORM $b \times q + r$ WHERE $0 \leq r < b$,

A IF $a = 47$ AND $b = 7$ B IF $a = 111$ AND $b = 3$ C IF $a = 5$ AND $b = 8$

SOLUTION:

A

$$\begin{array}{r|l} & 6 \\ \hline 7 & 47 \\ & 42 \\ \hline & \;\;5 \end{array}$$

$q = 6$ AND $r = 5$

$\therefore$ $47 = 7\,(6) + 5$

B

$$\begin{array}{r|l} & 37 \\ \hline 3 & 111 \\ & 9 \\ \hline & 21 \\ & 21 \\ \hline & 0 \end{array}$$

$q = 37$ AND $r = 0$

$\therefore$ $111 = 3\,(37) + 0$

C

$$\begin{array}{r|l} & 0 \\ \hline 8 & 5 \\ & 0 \\ \hline & 5 \end{array}$$

$q = 0$ AND $r = 5$

$\therefore$ $5 = 8\,(0) + 5$.

Exercise 1.13

FOR EACH OF THE FOLLOWING PAIRS OF NUMBERS, LET a BE THE FIRST NUMBER OF THE PAIR AND b THE SECOND NUMBER. FIND q AND r FOR EACH PAIR SUCH THAT $a = b \times q + r$, WHERE $0 \le r < b$:

A	72, 11	**B**	16, 9	**C**	11, 18
D	106, 13	**E**	176, 21	**F**	25, 39

B The Euclidean algorithm

ACTIVITY 1.20

GIVEN TWO NUMBERS 60 AND 36

1. FIND GCF (60, 36).
2. DIVIDE 60 BY 36 AND FIND THE GCF OF 36 AND THE REMAINDER.
3. DIVIDE 36 BY THE REMAINDER YOU GOT IN STEP 2. THEN, FIND THE GCF OF THE TWO REMAINDERS, THAT IS, THE REMAINDER YOU GOT IN STEP 2 AND THE ONE YOU GOT IN STEP 3.
4. COMPARE THE THREE GCFS YOU GOT.
5. GENERALIZE YOUR RESULTS.

THE ABOVE ACTIVITY LEADS YOU TO ANOTHER METHOD FOR FINDING THE GCF OF TWO NUMBERS, WHICH IS CALLED THE Euclidean algorithm. WE STATE THIS ALGORITHM AS A THEOREM.

Theorem 1.5 Euclidean algorithm

If a, b, q and r are positive integers such that

$a = q \times b + r$, then, GCF (a, b) = GCF (b, r).

EXAMPLE 2 FIND GCF (224, 84).

SOLUTION: TO FIND GCF (224, 84), YOU FIRST DIVIDE 224 BY 84. THE divisor AND remainder OF THIS DIVISION ARE THEN USED AS dividend AND divisor, RESPECTIVELY, IN A SUCCEEDING DIVISION. THE PROCESS IS REPEATED UNTIL A REMAINDER 0 IS OBTAINED.

THE COMPLETE PROCESS TO FIND GCF (224, 84) IS SHOWN BELOW.

Euclidean algorithm

Computation	Division algorithm form	Application of Euclidean Algorithm
$\begin{array}{r|l} & 2 \\ \hline 84 & 224 \\ & 168 \\ \hline & 56 \end{array}$	$224 = (2 \times 84) + 56$	GCF (224, 84) = GCF (84, 56)
$\begin{array}{r|l} & 1 \\ \hline 56 & 84 \\ & 56 \\ \hline & 28 \end{array}$	$84 = (1 \times 56) + 28$	GCF (84, 56) = GCF (56, 28)
$\begin{array}{r|l} & 2 \\ \hline 28 & 56 \\ & 56 \\ \hline & 0 \end{array}$	$56 = (2 \times 28) + 0$	GCF (56, 28) = 28 *(by inspection)*

CONCLUSION GCF (224, 84) = 28.

Exercise 1.14

1 FOR THE ABOVE EXAMPLE, VERIFY DIRECTLY THAT

GCF (224, 84) = GCF (84, 56) = GCF (56, 28).

2 FIND THE GCF OF EACH OF THE FOLLOWING PAIRS OF NUMBERS BY USING THE EUCLIDEAN ALGORITHM:

A 18; 12 **B** 269; 88 **C** 143; 39

D 1295; 407 **E** 85; 68 **F** 7286; 1684

Key Terms

bar notation
composite number
divisible
division algorithm
factor
fundamental theorem of arithmetic
greatest common factor (GCF)
irrational number
least common multiple (LCM)
multiple
perfect square
prime factorization
prime number
principal n^{th} root
principal square root
radical sign
radicand
rational number
rationalization
real number
repeating decimal
repetend
scientific notation
significant digits
significant figures
terminating decimal

Summary

1 THE SETS OF NATURAL NUMBERS, WHOLE NUMBERS, INTEGERS AND RATIONAL NUMBERS DENOTED BY $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, AND $\mathbb{Q}$, RESPECTIVELY ARE DESCRIBED BY

$$\mathbb{N} = \{1, 2, 3,\ldots\} \qquad \mathbb{W} = \{0, 1, 2,\ldots\} \qquad \mathbb{Z} = \{\ldots,-3, -2, -1, 0, 1, 2, 3,\ldots\}$$

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{Z},\ b \neq 0\right\}$$

2 **A** A composite number IS A NATURAL NUMBER THAT HAS MORE THAN TWO FACT

B A prime number IS A NATURAL NUMBER THAT HAS EXACTLY TWO DISTINCT FACTORS, 1 AND ITSELF.

C PRIME NUMBERS THAT DIFFER BY TWO ARE CALLED twin primes.

D WHEN A NATURAL NUMBER IS EXPRESSED AS A PRODUCT OF FACTORS THAT ARE ALL PRIME, THEN THE EXPRESSION IS CALLED the prime factorization OF THE NUMBER.

E Fundamental theorem of arithmetic.

EVERY COMPOSITE NUMBER CAN BE EXPRESSED (FACTORIZED) AS A PROD PRIMES, AND THIS FACTORIZATION IS UNIQUE, APART FROM THE ORDER IN W PRIME FACTORS OCCUR.

3 **A** THE greatest common factor (GCF) OF TWO OR MORE NUMBERS IS THE GREATEST FACTOR THAT IS COMMON TO ALL NUMBERS.

B THE least common multiple (LCM) OF TWO OR MORE NUMBERS IS THE SMALLEST OR LEAST OF THE COMMON MULTIPLES OF THE NUMBERS.

4 **A** ANY RATIONAL NUMBER CAN BE EXPRESSED AS A repeating decimal OR A terminating decimal.

B ANY TERMINATING DECIMAL OR REPEATING DECIMAL IS A RATIONAL NUMBER.

5 IRRATIONAL NUMBERS ARE DECIMAL NUMBERS THAT NEITHER REPEAT NOR TERMINATE.

6 THE SET OF REAL NUMBERS DENOTED BY $\mathbb{R}$ IS DEFINED BY

$$\mathbb{R} = \{x : x \text{ IS RATIONAL OR } x \text{ IS IRRATIONAL}\}$$

7 THE SET OF IRRATIONAL NUMBERS IS NOT CLOSED UNDER ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION.

8 THE SUM OF AN IRRATIONAL AND A RATIONAL NUMBER IS ALWAYS AN IRRATIONAL NUMBER.

9 FOR ANY REAL NUMBER b AND POSITIVE INTEGER $n > 1$

$$b^{\frac{1}{n}} = \sqrt[n]{b} \text{ (WHENEVER } \sqrt[n]{b} \text{ IS A REAL NUMBER)}$$

10 FOR ALL REAL NUMBERS a AND $b \neq 0$ FOR WHICH THE RADICALS ARE DEFINED AND FOR ALL INTEGERS $n \geq 2$:

I $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ **II** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

11 A NUMBER IS SAID TO BE WRITTEN IN SCIENTIFIC NOTATION (STANDARD NOTATION), IF IT IS WRITTEN IN THE FORM $a \times 10^k$ WHERE $1 \leq a < 10$ AND k IS AN INTEGER.

12 LET a AND b BE TWO NON-NEGATIVE INTEGERS AND $b \neq 0$, THEN THERE EXIST UNIQUE NON-NEGATIVE INTEGERS q AND r SUCH THAT $a = (q \times b) + r$ WITH $0 \leq r < b$.

13 IF a, b, q AND r ARE POSITIVE INTEGERS SUCH THAT $a = q \times b + r$, THEN

GCF (a, b) = GCF (b, r).

Review Exercises on Unit 1

1 DETERMINE WHETHER EACH OF THE FOLLOWING NUMBERS IS DIVISIBLE BY 2, 3, 4, 5, 6, 8, 9 OR 10:

A 533 **B** 4, 299 **C** 111

2 FIND THE PRIME FACTORIZATION OF:

A 150 **B** 202 **C** 63

3 FIND THE GCF FOR EACH SET OF NUMBERS GIVEN BELOW:

A 16; 64 **B** 160; 320; 480

4 EXPRESS EACH OF THE FOLLOWING FRACTIONS OR MIXED NUMBERS AS A DECIMAL:

A $\frac{5}{8}$ **B** $\frac{16}{33}$ **C** $5\frac{4}{9}$ **D** $3\frac{1}{7}$

5 EXPRESS EACH OF THE FOLLOWING DECIMALS AS A FRACTION OR MIXED NUMBER IN ITS SIMPLEST FORM:

A 0.65 **B** -0.075 **C** $0.\overline{16}$ **D** $-24.\overline{54}$ **E** $-0.\overline{02}$

6 ARRANGE EACH OF THE FOLLOWING SETS OF RATIONAL NUMBERS IN INCREASING ORDER:

A $\frac{71}{100}, -\frac{3}{2}, \frac{23}{30}$ **B** $3.2, 3.\overline{22}, 3.\overline{23}, 3.2\overline{3}$

C $\frac{2}{3}, \frac{11}{18}, \frac{16}{27}, \frac{67}{100}$

7 WRITE EACH OF THE FOLLOWING EXPRESSIONS IN ITS SIMPLEST FORM:

A $\sqrt{180}$ **B** $\sqrt{\frac{169}{196}}$ **C** $\sqrt[3]{250}$ **D** $2\sqrt{3}+3\sqrt{2}+\sqrt{180}$

8 GIVE EQUIVALENT EXPRESSION, CONTAINING FRACTIONAL EXPONENTS, FOR EACH OF THE FOLLOWING:

A $\sqrt{15}$ **B** $\sqrt{a+b}$ **C** $\sqrt[3]{x-y}$ **D** $\sqrt[4]{\frac{13}{16}}$

9 EXPRESS THE FOLLOWING NUMBERS AS FRACTIONS WITH RATIONAL DENOMINATORS:

A $\frac{1}{\sqrt{2}+1}$ **B** $\sqrt{\frac{5}{3}}$ **C** $\frac{-5}{\sqrt{3}+\sqrt{7}}$ **D** $\sqrt[4]{\frac{13}{16}}$

10 SIMPLIFY

A $\left(3+\sqrt{7}\right)+\left(2\sqrt{7}-12\right)$ **B** $\left(2+\sqrt{5}\right)+\left(2-\sqrt{5}\right)$

C $2\sqrt{6}\div 3\sqrt{54}$ **D** $2\left(3+\sqrt{7}\right)-2\sqrt{7}$

11 IF $\sqrt{5}\approx 2.236$ AND $\sqrt{10}\approx 3.16$, FIND THE VALUE OF $\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}$

12 IF $\dfrac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=x+\sqrt{6}y$, FIND THE VALUES OF x AND y.

13 EXPRESS EACH OF THE FOLLOWING NUMBERS IN SCIENTIFIC NOTATION:

A 7,410,00 **B** 0.0000648 **C** 0.002056 **D** 12.4×10^{-6}

14 SIMPLIFY EACH OF THE FOLLOWING AND GIVE THE ANSWER IN SCIENTIFIC NOTATION:

A $10^{9}\times 10^{-6}\times 27$ **B** $\dfrac{796\times 10^{4}\times 10^{-2}}{10^{-7}}$ **C** 0.00032×0.002

15 THE FORMULA $d = 3.56\sqrt{h}$ KM ESTIMATES THE DISTANCE A PERSON CAN SEE TO THE HOR WHERE h IS THE HEIGHT OF THE EYES OF THE PERSON FROM THE GROUND IN METERS. SUPPOSE YOU ARE IN A BUILDING SUCH THAT YOUR EYE LEVEL IS 20 M ABOVE THE GROUN HOW FAR YOU CAN SEE TO THE HORIZON.

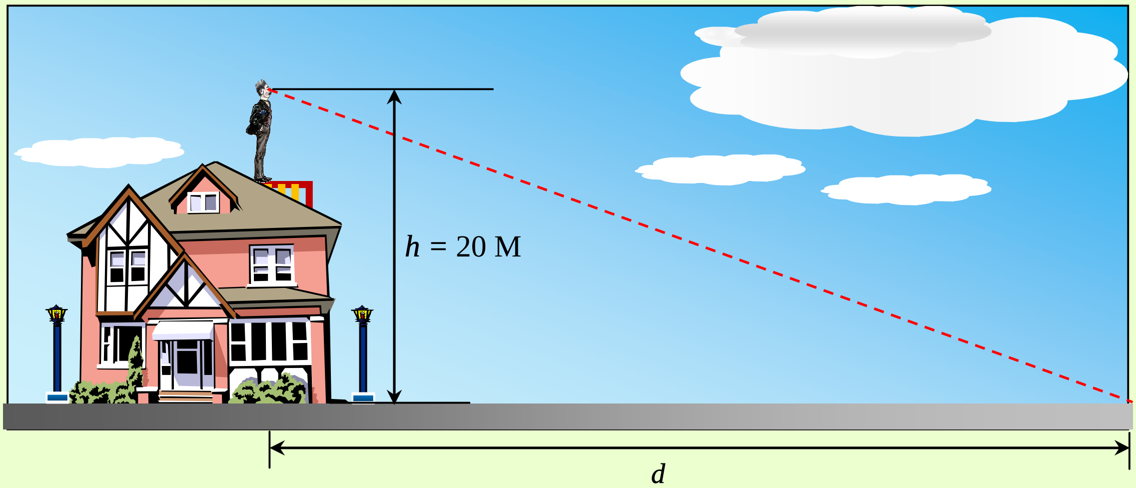
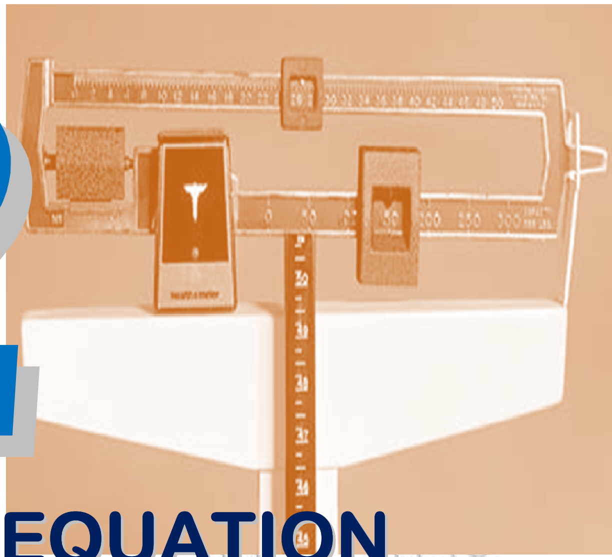


Figure 1.11

Unit 2



SOLUTION OF EQUATION

Unit Outcomes:

After completing this unit, you should be able to:

- *identify equations involving exponents and radicals, systems of two linear equations, equations involving absolute values and quadratic equations.*
- *solve each of these equations.*

Main Contents

2.1 Equations involving exponents and radicals

2.2 Systems of linear equations in two variables

2.3 Equations involving absolute value

2.4 Quadratic equations

Key Terms

Summary

Review Exercises

INTRODUCTION

IN EARLIER GRADES, YOU HAVE L ABOUT ALGEBRAIC EQUATIONS AND THE YOU ALSO LEARNED ABOUT LINEAR EQUATIONS IN ONE VARIABLE AND THE METHO PRESENT UNIT, WE DISCUSS FURTHER ABOUT EQUATIONS INVOLVING ABSOLUTE VALUES. YOU SHALL ALSO LEARN ABOUT SYSTEMS OF LINEAR EQUAT QUADRATIC EQUATIONS IN SINGLE VARIABLE, AND THE ME

2.1 EQUATIONS INVOLVING EXPONENTS AND RADICALS

EQUATIONS ARE EQUALITY OF EXPRESSIONS. THERE ARE DIFFERENT TYPI ON THE VARIABLE(S) CONSIDERED. WHEN THE HAS AN EXPONI OTHER THAN 1, IT IS SAID TO BE AN EQUATION INVOLVI

ACTIVITY 2.1

1 DETERMINE WHETHER OR NOT EACH OF THE FOLLC

A $2^4 \times 2^5 = 2^{20}$ **B** $(3^2)^3 = 3^6$ **C** $\left(5^2\right)^{\frac{1}{2}} = 5$

D $2^n \times 2^2 = 2^{2n}$ **E** $2^x = 8$ IS EQUIVALENT TO $x = 3$.

2 EXPRESS EACH OF FOLLOW NUMBERS IN POWER FORM.

A 8 **B** 27 **C** 625 **D** 343

THE ABOVE ACTIVITY LEADS YOU TO REVISE THE RULES THAT YOU DISCUSS UNIT 1.

EXAMPLE 1 SOLVE EACH OF THE FOLLOWIN

A $\sqrt{x} = 3$ **B** $x^3 = 8$ **C** $2^x = 16$

SOLUTION:

A

$$\sqrt{x} = 3$$

$$\sqrt{x}^2 = 3^2 \qquad \textit{Squaring both sides}$$

$$x = 9$$

THEREFORE, $x = 9$.

B TO SOLVE $x^3 = 8$, RECALL THAT FOR ANY REAL NUMBER a AND b, IF $a^n = b$, THEN a IS THE n^{TH} ROOT OF b.

$$x^3 = 8$$

$$x = \sqrt[3]{8} = 2$$

C TO SOLVE $2^x = 16$, FIRST EXPRESS 16 AS 2^4; $2^x = 2^4$.

IN $2^x = 2^4$, THE BASES ARE EQUAL. HENCE THE EXPONENTS MUST B

THEREFORE $x = 4$ IS THE SOLUT

ACTIVITY 2.2

SOLVE EACH OF THE FOLLOWING EQU

A $8^x = 2^{2x+2}$ **B** $4^{x+1} = 2^x$ **C** $\sqrt{5} = 25^{2x}$

THE FOLLOWING RULE IS VERY USEFUL IN SOLVI

Rule: FOR $a > 0$, $a^x = a^y$, IF AND ONLY IF $x = y$.

EXAMPLE 2 SOLVE $3^{2x+1} = 3^{x-2}$

SOLUTION: BY USING THE RULE, SINCE $3^{2x+1} = 3^{x-2}$, IF AND ONLY IF THE EX

$2x + 1 = x - 2$. FROM THIS WE CAN SEE THAT THE $x = -3$.

EXAMPLE 3 SOLVE EACH OF THE FOLLOWIN

A $8^x = 2^{2x+1}$ **B** $9^{x-3} = 27^{3x}$ **C** $\sqrt[3]{3^x} = 3^{2x+5}$

SOLUTION:

A $8^x = 2^{2x+1}$

$(2^3)^x = 2^{2x+1}$ *Expressing 8 as a power of 2*

$2^{3x} = 2^{2x+1}$ *Applying laws of exponents*

$3x = 2x + 1$

THEREFORE $x = 1$.

B $9^{x-3} = 27^{3x}$

$(3^2)^{x-3} = (3^3)^{3x}$ *Expressing 9 and 27 as powers of 3*

$3^{2(x-3)} = 3^{3(3x)}$ *Applying laws of exponents*

$3^{2x-6} = 3^{9x}$

$2x - 6 = 9x$

$7x = -6$

THEREFORE $x = -\frac{6}{7}$.

C $\sqrt[3]{3^x} = 3^{2x+5}$

$(3^x)^{\frac{1}{3}} = 3^{2x+5}$ *Applying laws of exponents*

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$$3^{\frac{x}{3}} = 3^{2x+5}$$

$$\frac{x}{3} = 2x+5$$

$$x = 3(2x+5)$$

$$x = 6x + 15$$

$$-5x = 15$$

THEREFORE $x = -3$.

Exercise 2.1

1 SOLVE EACH OF THE FOLLOWING.

A $3^x = 27$ **B** $\left(\frac{1}{4}\right)^x = 16$ **C** $\left(\frac{1}{16}\right)^{3x-1} = 32$

D $81^{5x+2} = \frac{1}{243}$ **E** $9^{2x} = 27^{2x+1}$ **F** $16^{x+4} = 2^{3x}$

G $(3x+1)^3 = 64$ **H** $\sqrt[3]{81^{2x-1}} = 3^x$

2 SOLVE $(2x+3)^2 = (3x-1)^2$.

3 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $9^{2x}27^{1-x} = 81^{2x+1}$ **B** $9^{2x+2}\left(\frac{1}{81}\right)^{x+2} = 243^{-3x-2}$

C $16^{3x+4} = 2^{3x}64^{-4x+1}$

2.2 SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

RECALL THAT FOR REAL NUMBERS a AND b, ANY EQUATION OF THE FORM $ax + b = 0$, WHERE $a \neq 0$ IS CALLED a **linear equation**. THE NUMBERS a AND b ARE CALLED **coefficients** OF THE EQUATION.

ACTIVITY 2.3

1 SOLVE EACH OF THE FOLLOWING LINEAR EQUATIONS.

A $x - 2 = 7$ **B** $x + 7 = 3$

C $2x = 4$ **D** $2x - 5 = 7$ **E** $3x + 5 = 14$

2 HOW MANY SOLUTIONS GET FOR EACH EQUATION?

OBSERVE THAT EACH EQUATION HAS EXACTLY ONE SOLUTION. IN GENN ONE VARIABLEHAS ONE SOLU

Definition 2.1

Any equation that can be reduced to the form $ax + b = 0$, where $a, b \in \mathbb{R}$ and $a \neq 0$, is called a linear equation in one variable.

Group Work 2.1

FORM A GROUP AND DO THE .

1 SOLVE EACH OF THE FOLLOWIN.

A $7x-3=2(3x+2)$ **B** $-3(2x+4)=2(-3x-6)$

C $2x+4=2(x+5)$

2 HOW MANY SOLUTIONS DOFOR EACH EQUATION?

3 WHAT CAN YOU CONCLUDE ABOUT NUMBI

FROM THE GROUP WORK, OBSERVE TSUCH EQUATIONS CAN HAVE ONE SOLUTION SOLUTIONS OR NO SOLUTION.

Linear equations in two variables

WE DISCUSSED HOW WE EQUATIONS WITH ONE VARIABLE THAT CAN IFORM $ax + b = 0$. WHAT DO YOU THINKTHE S, IF THE EQUATION IS ($y = ax + b$?

ACTIVITY 2.4

1 WHICH OF THE FOLLOWING ARE LINEAR EQUATION

A $2x - y = 5$ **B** $-x + 7 = y$ **C** $2x+3 = 4$

D $2x - y^2 = 7$ **E** $\frac{1}{x}+\frac{1}{y}=6$

2 HOW MANY SOLUARE THERE FOR EACH OF THE LINEAR EQUATIONS IN T?

3 A HOUSE WAS REFO BIRR 2,000 PER MONTH PLUS BIRR 2 FOR WATER CONS PER M^3

A WRITE AN EQUATION FOR THE x-YEARS RENT AND m^3 OF WATER USED.

B IF THE TOTAIFOR x-YEARS RENT AND y m^3 OF WATER USEIRR 10,000 WRITE AN EQUATIC

NOTE THAT $ax + b = 0$, IS A PARTICULAR CASE OF $y = ax + b$ WHEN $y = 0$. THIS MEANS, FOR DIFFERENT VALUES OF y THERE WILL BE DIFFERENT EQUATIONS WITH THEIR OWN SOLUTIONS.

AN EQUATION OF THE TYPE $cx + dy = e$, WHERE c, d AND e ARE ARBITRARY CONSTANTS AND $c \neq 0$, $d \neq 0$, IS CALLED A linear equation in two variables. AN EQUATION IN TWO VARIABLES OF THE FORM $cx + dy = e$ CAN BE REDUCED TO THE FORM $y = ax + b$.

EXAMPLE 1

A GIVE SOLUTIONS TO $y = 2x + 1$ WHERE y ASSUMES VALUES 0, 1, 2 AND 3.

B PLOT SOME OF THE ORDERED PAIRS THAT MAKE $y = 2x + 1$ TRUE ON THE xy-COORDINATE SYSTEM.

SOLUTION:

A LET US CONSIDER $y = 2x + 1$.

WHEN $y = 0$, THE EQUATION BECOMES $0 = 2x + 1$ AND ITS SOLUTION IS $x = -\frac{1}{2}$.

WHEN $y = 1$, THE EQUATION BECOMES $1 = 2x + 1$ AND ITS SOLUTION IS $x = 0$.

WHEN $y = 2$, THE EQUATION BECOMES $2 = 2x + 1$ AND ITS SOLUTION IS $x = \frac{1}{2}$.

WHEN $y = 3$, THE EQUATION BECOMES $3 = 2x + 1$ AND ITS SOLUTION IS $x = 1$.

OBSERVE THAT FOR EACH VALUE OF y, THERE IS ONE CORRESPONDING VALUE OF x. THIS RELATION IS REPRESENTED BY AN ORDERED PAIR (x, y). THE SET OF ALL THOSE ORDERED PAIRS THAT SATISFY EQUATION $y = 2x + 1$ IS THE SOLUTION TO THE EQUATION $y = 2x + 1$.

B FROM THE FOUR PARTICULAR CASES CONSIDERED ABOVE FOR $y = 2x + 1$, WHERE y ASSUMES VALUES 0, 1, 2 AND 3, WE CAN SEE THAT THE SOLUTION IS

$$\left\{\left(-\frac{1}{2}, 0\right), (0, 1), \left(\frac{1}{2}, 2\right), (1, 3)\right\}.$$

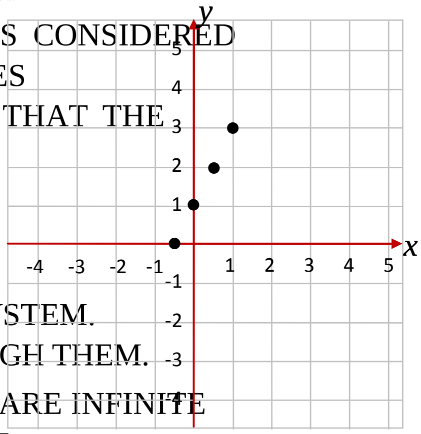

Figure 2.1

NOW LET US PLOT THESE POINTS ON THE xy-COORDINATE SYSTEM. SEE THAT THERE IS A LINE THAT PASSES THROUGH THEM.

IN GENERAL, SINCE y CAN HAVE ANY VALUE, THERE ARE INFINITE ORDERED PAIRS THAT MAKE THE EQUATION $y = ax + b$ TRUE. THE PLOT OF THESE ORDERED PAIRS MAKES A STRAIGHT LINE.

System of linear equations and their solutions

YOU HAVE DISCUSSED SOLUTIONS TO A LINEAR EQUATION IN TWO VARIABLES AND FOUND THAT THERE ARE INFINITE SOLUTIONS. NOW YOU WILL SEE THE JOINT CONSIDERATION OF LINEAR EQUATIONS IN TWO VARIABLES.

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ACTIVITY 2.5

CONSIDER THE EQUA $y = x + 1$ AND $y = -x + 1$.

1 DETERMINE THE VA y FOR EACH EQUATION WHEN THE x IS V –2, –1, 0, 1 AND 2.

2 PLOT THE ORDERED I THE xy-COORDINATE SYSTEM.

3 WHAT DO YOU OBSERVE FROM THE PLOT

4 DISCUSS WHAT THE PAIR ((.

Definition 2.2

A set of two or more linear equations is called a system of linear equations. Systems of two linear equations in two variables are equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$, where a_1, a_2, b_1, b_2, c_1 AND c_2 are the parameters of the system whose specific values characterize the system and $a_1 \neq 0$ or $b_1 \neq 0$, $a_2 \neq 0$ or $b_2 \neq 0$.

EXAMPLE 2 THE FOLLOWING ARE EXAMPLES OF SYSTEMS OF LINEAR VARIABLES.

A $\begin{cases} 2x + 3y = 1 \\ x - 2y = 3 \end{cases}$ **B** $\begin{cases} 3x - 2y = 2 \\ 9x - 6y = 5 \end{cases}$ **C** $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$

WE NOW DISCUSS HOW TO SOLVE SYSTEMS OF LINEAR EQUATIONS.

Definition 2.3

A solution to a system of linear equations in two variables means the set of ordered pairs (x, y) that satisfy both equations.

EXAMPLE 3 DETERMINE THE SOLUTION OWING SYSTEM OF LINEAR I

$$\begin{cases} 2x + 3y = 8 \\ 5x - 2y = 1 \end{cases}$$

SOLUTION: THE SET $\left\{\left(0, \frac{8}{3}\right), (1, 2), \left(2, \frac{4}{3}\right), \left(3, \frac{2}{3}\right), (4, 0)\right\}$ CONTAINS SOME (SOLUTIONS TO THE LINEA $2x + 3y = 8$.

THE SET $\left\{\left(0,-\frac{1}{2}\right),(1,2),\left(2,\frac{9}{2}\right),(3,7)\left(4,\frac{19}{2}\right)\right\}$ CONTAINS SOME OF THE SOLUTIONS TO THE LINEAR EQUATION $5x - 2y = 1$.

FROM THE DEFINITION GIVEN ABOVE, THE SOLUTION TO THE GIVEN SYSTEM EQUATIONS SHOULD SATISFY BOTH EQUATIONS $x + 3y = 7$ AND $5x - 2y = 1$.

THEREFORE, THE SOLUTION IS (1, 2) AND IT SATISFIES BOTH EQUATIONS.

Solution to a system of linear equations in two variables

YOU SAW IN EXAMPLE 3 ABOVE THAT A SOLUTION TO A SYSTEM OF LINEAR EQUATIONS IS AN ORDERED PAIR THAT SATISFIES BOTH EQUATIONS IN THE SYSTEM. WE OBTAINED IT BY ORDERED PAIRS THAT SATISFY EACH OF THE COMPONENT EQUATIONS AND SELECTIN ONE. BUT IT IS NOT EASY TO LIST SUCH SOLUTIONS. SO WE NEED TO LOOK FOR ANOTH TO SOLVING SYSTEMS OF LINEAR EQUATIONS. THESE INCLUDE THE graphical method, substitution method AND elimination method.

Group Work 2.2

1 DRAW THE LINE OF EACH COMPONENT EQUATION OF THE FOLLOWING SYSTEMS.

A $\begin{cases} x+y=1 \\ 2x-2y=4 \end{cases}$ B $\begin{cases} 2x-y=2 \\ 4x-2y=5 \end{cases}$ C $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$

2 DO EACH PAIR OF LINES INTERSECT?

3 WHAT CAN YOU CONCLUDE FROM THESE LINES AND THE SOLUTIONS OF EACH SYST

4 IN A CERTAIN AREA, THE UNDERAGE MARRIAGE RATE DECREASES FROM 15% TO 0.5% IN 12 YEARS. BY CONSIDERING THE YEAR 1990 AS 0, THE LINEAR EQUATION $y = mx + b$ IS USED TO MODEL THE UNDERAGE MARRIAGE RATE.

A WRITE THE EQUATION OF THE STRAIGHT LINE AND DETERMINE THE YEAR IN WHICH UNDER AGE MARRIAGE RATE IN THAT AREA IS 0.001% OR BELOW.

B DISCUSS HOW TO MODEL SUCH CASES IN YOUR KEBELE.

When we draw the lines of each of the component equations in a system of two linear equations, we can observe three possibilities.

1 THE TWO LINES INTERSECT AT ONE POINT, IN WHICH CASE THE SYSTEM HAS ONE SO

2 THE TWO LINES ARE PARALLEL AND NEVER INTERSECT. IN THIS CASE, WE SAY THE SYSTEM DOES NOT HAVE A SOLUTION.

3 THE TWO LINES COINCIDE (FIT ONE OVER THE OTHER). IN THIS CASE, THERE ARE INFINITE SOLUTIONS.

WE NOW DISCUSS A FEW GRAPHICAL AND METHODS TO SOLVE A SYSTEM EQUATIONS IN TWO VARIABLES: a graphical method, the substitution method, AND the elimination method.

Solving system of linear equations by a graphical method

IN THIS METHOD, WE NEED TO DRAW OF EACH COMPONENT EQUSING THE SAME COORDINATE SYSTEM. IF THE LINES INTERSECT, THERE IS (THAT IS THE POINT THEIR INTERSECTION. IF THE LINES ARE PARALLEL, no solution. IF THE LINES CO, THEN THERE ARE INFINITE SOLUTIONS TO THE SYSTEM, SINCE EVERY POINT (ORDERED P LINE SATISFIES BOTH EQUATIONS I

ACTIVITY 2.6

SOLVE EACH SYSTEM BY DRAWING THE GRAPH OF EACH EQ

A $\begin{cases} y = x+1 \\ y = x+2 \end{cases}$ **B** $\begin{cases} y = x+2 \\ y = -x-2 \end{cases}$ **C** $\begin{cases} x+y = 2 \\ 2x+2y = 4 \end{cases}$

EXAMPLE 4 SOLVE EACH OF THE FOLLOWING SYSTEMS OF.

A $\begin{cases} 2x-2y = 4 \\ 3x+4y = 6 \end{cases}$ **B** $\begin{cases} x+2y = 4 \\ 3x+6y = 6 \end{cases}$ **C** $\begin{cases} 3x-y = 5 \\ 6x-2y = 10 \end{cases}$

SOLUTION:

A FIRST, DRAW THE GRAPH OF EAC

IN THE GRAPH, OBSERVE THAT THE T INTERSECTING AT (2, 0). THUS, THE SYST SOLUTION WHICH IS

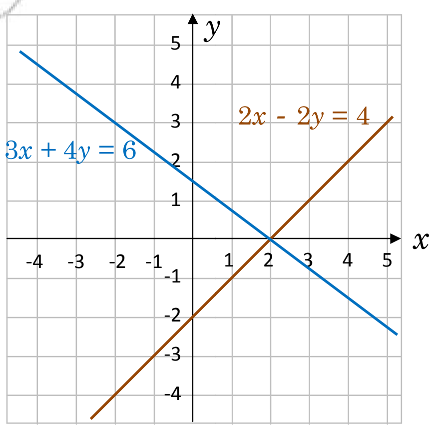

Figure 2.2

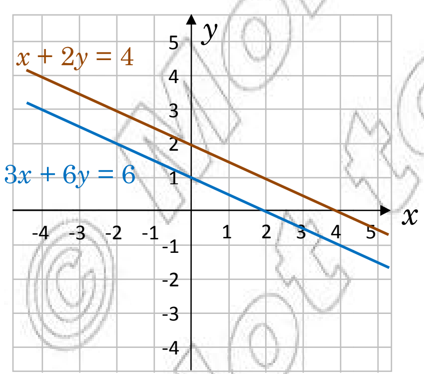

Figure 2.3

B WHEN WE DRAW THE LINE COMPONENT EQUATIONS, WE SEE THAT THE LINES ARE PA MEANS THE LINES DO NOT INTERSECT SYSTEM DOES NOT HAVE A SOLUTION.

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C WHEN WE DRAW THE LINE OF EACH COMPONENT EQUATION, WE SEE THAT THE LINES COINCIDE ONE OVER THE OTHER, WHICH SHOWS THAT THE SYSTEM HAS INFINITE SOLUTIONS. THAT IS, ALL POINTS (ORDERED PAIRS) ON THE LINE ARE SOLUTIONS OF THE SYSTEM.

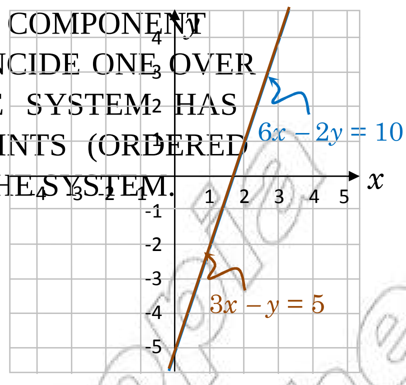

Figure 2.4

Group Work 2.3

FORM A GROUP AND DO THE FOLLOWING.

CONSIDER THE FOLLOWING SYSTEMS OF LINEAR EQUATI… ARIABLES.

A $\begin{cases} x+4y=2 \\ 3x-4y=6 \end{cases}$ **B** $\begin{cases} -x+2y=4 \\ 3x-y=3 \end{cases}$

1 SOLVE EACH BY USING SUBSTITUTION METHOD.

2 SOLVE EACH BY USING ELIMINATION METHOD.

Solving systems of linear equations by the substitution method

To solve a system of two linear equations by the substitution method, you follow the following steps.

1 TAKE ONE OF THE LINEAR EQUATIONS FROM THE SYSTEM AND WRITE ONE OF THE VARIABLES IN TERMS OF THE OTHER.

2 SUBSTITUTE YOUR RESULT INTO THE OTHER EQUATION AND SOLVE FOR THE SECO

3 SUBSTITUTE THIS RESULT INTO ONE OF THE EQUATIONS AND SOLVE FOR THE FIRST

EXAMPLE 5 SOLVE THE SYSTEM OF LINEAR EQUATIONS GIVEN BY $\begin{cases} 2x-3y=5 \\ 5x+3y=9 \end{cases}$

SOLUTION:

Step 1 TAKE $2x - 3y = 5$ AND SOLVE FOR y IN TERMS OF x.

$2x - 3y = 5$ BECOMES $3y = 2x - 5$

HENCE $y = \frac{2}{3}x - \frac{5}{3}$.

Step 2 SUBSTITUTE $y = \frac{2}{3}x - \frac{5}{3}$ IN $5x + 3y = 9$ AND SOLVE FOR x

$$5x + 3\left(\frac{2}{3}x - \frac{5}{3}\right) = 9$$

$$5x + 2x - 5 = 9$$

$$7x - 5 = 9$$

$$7x = 14$$

$$x = 2$$

Step 3 SUBSTITUTE $x = 2$ AGAIN INTO ONE OF THE EQUATIONS AND SOLVE FOR THE REMAINING VARIABLE y

CHOOSING $2x - 3y = 5$, WHEN WE SUBSTITUTE $x = 2$, WE GET $2(2) - 3y = 5$

WHICH BECOMES $4 - 3y = 5$

$$-3y = 1$$

$$y = -\frac{1}{3}$$

THEREFORE THE SOLUTION IS $\left(2, -\frac{1}{3}\right)$.

EXAMPLE 6 SOLVE EACH OF THE FOLLOWING SYSTEMS OF LINEAR EQUATIONS.

A $\begin{cases} 2x - 4y = 5 \\ -6x + 12y = -15 \end{cases}$ **B** $\begin{cases} 2x - y = 1 \\ 3x - 2y = -4 \end{cases}$ **C** $\begin{cases} 4x + 3y = 8 \\ -2x - \frac{3}{2}y = -6 \end{cases}$

SOLUTION:

A $\begin{cases} 2x - 4y = 5 \\ -6x + 12y = -15 \end{cases}$

FROM $2x - 4y = 5$

$$-4y = -2x + 5$$

$$y = \frac{1}{2}x - \frac{5}{4}$$

SUBSTITUTING $y = \frac{1}{2}x - \frac{5}{4}$ IN $-6x + 12y = -15$, WE GET

$$-6x + 12\left(\frac{1}{2}x - \frac{5}{4}\right) = -15$$

$$-6x + 6x - 15 = -15$$

$-15 = -15$ WHICH IS ALWAYS TRUE.

THEREFORE, THE SYSTEM HAS INFINITE SOLUTIONS.

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B $\begin{cases} 2x - y = 1 \\ 3x - 2y = -4 \end{cases}$

FROM $2x - y = 1$, WE FIND $y = 2x - 1$

SUBSTITUTING: $3x - 2(2x - 1) = -4$

$$3x - 4x + 2 = -4$$

$$-x = -6$$

THEREFORE $x = 6$.

SUBSTITUTING $x = 6$ IN $2x - y = 1$ GIVES

$$12 - y = 1$$

$$y = 11$$

SO THE SOLUTION IS (6, 11).

C $\begin{cases} 4x + 3y = 8 \\ -2x - \dfrac{3}{2}y = -6 \end{cases}$

FROM $4x + 3y = 8$

$$3y = -4x + 8$$

$$y = -\frac{4}{3}x + \frac{8}{3}$$

SUBSTITUTING $y = -\frac{4}{3}x + \frac{8}{3}$ IN $-2x - \frac{3}{2}y = -6$ GIVES $-2x - \frac{3}{2}\left(-\frac{4}{3}x + \frac{8}{3}\right) = -6$

$$-2x + 2x - 4 = -6$$

$-4 = -6$ WHICH IS ALWAYS FALSE.

THEREFORE, THE SYSTEM HAS NO SOLUTION.

Solving systems of linear equations by the elimination method

To solve a system of two linear equations by the elimination method, you follow the following steps.

1. SELECT ONE OF THE VARIABLES AND MAKE THE COEFFICIENTS OF THE SELECTED VARIABLE EQUAL BUT OPPOSITE IN SIGN IN THE TWO EQUATIONS.
2. ADD THE TWO EQUATIONS TO ELIMINATE THE SELECTED VARIABLE AND SOLVE FOR THE RESULTING VARIABLE.
3. SUBSTITUTE THIS RESULT AGAIN INTO ONE OF THE EQUATIONS AND SOLVE FOR THE REMAINING VARIABLE.

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EXAMPLE 7 SOLVE THE SYSTEM OF LINEAR EQUATIONS GIVEN BY

$$\begin{cases} 2x - y = 5 \\ 2x + 3y = 9 \end{cases}$$

SOLUTION:

Step 1 SELECT ONE OF THE VARIABLES, SAY y, AND MAKE THE COEFFICIENTS OF y OPPOSITE TO ONE ANOTHER BY MULTIPLYING THE FIRST EQUATION BY 3.

$$\begin{cases} 2x - y = 5 \\ 2x + 3y = 9 \end{cases} \text{ IS EQUIVALENT WITH } \begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$

Step 2 ADD THE TWO EQUATIONS IN THE SYSTEM:

$$\begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$ GIVING $6x - 3y + 2x + 3y = 15 + 9$ WHICH BECOMES

$8x = 24$.

THEREFORE $x = 3$.

Step 3 SUBSTITUTE $x = 3$ INTO ONE OF THE ORIGINAL EQUATIONS AND SOLVE FOR y.

CHOOSING $2x - y = 5$ AND REPLACING $x = 3$, WE GET $2(3) - y = 5$ FROM WHICH

$-y = 5 - 6$

$-y = -1$ WHICH IS THE SAME AS $y = 1$

THEREFORE THE SOLUTION IS (3, 1).

EXAMPLE 8 SOLVE EACH OF THE FOLLOWING SYSTEMS OF LINEAR EQUATIONS.

A $\begin{cases} 7x + 5y = 11 \\ -3x + 3y = -3 \end{cases}$ **B** $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$ **C** $\begin{cases} 2x - 7y = 9 \\ -6x + 21y = 6 \end{cases}$

SOLUTION:

A $\begin{cases} 7x + 5y = 11 \\ -3x + 3y = -3 \end{cases}$

MULTIPLY THE FIRST EQUATION BY 3 AND THE SECOND EQUATION BY 7 TO MAKE THE COEFFICIENTS OF THE VARIABLE x OPPOSITE.

WE GET $\begin{cases} 21x + 15y = 33 \\ -21x + 21y = -21 \end{cases}$

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ADDING THE TWO EQUATIONS

$21x + 15y – 21x + 21y = 33 – 21$

WHICH BECOMES $36y = 12$

$$y = \frac{1}{3}$$

SUBSTITUTING $y = \frac{1}{3}$ IN ONE OF THE EQUATIONS, SAY $7x + 5y = 11$, WE GET

$$7x + 5\left(\frac{1}{3}\right) = 11$$

$$7x = 11 - \frac{5}{3}$$

$$7x = \frac{28}{3}$$

$$x = \frac{28}{21} = \frac{4}{3}$$

THEREFORE THE SOLUTION IS $\left(\frac{4}{3}, \frac{1}{3}\right)$.

B $\begin{cases} 2x - 4y = 8 \\ x - 2y = 4 \end{cases}$

MULTIPLYING THE SECOND EQUATION BY –2, WE GET,

$$\begin{cases} 2x - 4y = 8 \\ -2x + 4y = -8 \end{cases}$$

ADDING THE TWO EQUATIONS $2x – 4y – 2x + 4y = 8 – 8$

WE GET $0 = 0$ WHICH IS ALWAYS TRUE.

THEREFORE, THE SYSTEM HAS INFINITE SOLUTIONS.

C $\begin{cases} 2x - 7y = 9 \\ -6x + 21y = 6 \end{cases}$

MULTIPLY THE FIRST EQUATION BY 3 TO MAKE THE COEFFICIENTS OF THE VARIABL

WE GET $\begin{cases} 6x - 21y = 27 \\ -6x + 21y = 6 \end{cases}$

ADDING THE TWO EQUATIONS $6x – 21y – 6x + 21y = 27 + 6$, WE GET THAT

$0 = 33$ WHICH IS ALWAYS FALSE.

THEREFORE, THE SYSTEM HAS NO SOLUTION.

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Solutions of a system of linear equations in two variables and quotients of coefficients

ACTIVITY 2.7

1 DISCUSS THE SOLUTION SET TO EACH OF THE F(

A $\begin{cases} 3x+y=2 \\ x-2y=3 \end{cases}$ **B** $\begin{cases} x-2y=3 \\ 2x-4y=5 \end{cases}$ **C** $\begin{cases} 2x+3y=1 \\ 4x+6y=2 \end{cases}$

2 DIVIDE EACH PAI CORRESPONDING COEFFICIENTS AND (SAY $\frac{3}{1}, \frac{1}{-2}$ AND $\frac{2}{3}$ FOR 1A) FOR EACH SYSTEM.

3 DISCUSS THE RELATIONSHIP BETWEEN THE NUMBER OF SOLUTION COEFFICIENTS.

4 SOLVE THE GIVEN SYSTEM OF TWO LIN

$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$; $a_2, b_2, c_2 \neq 0$ IN TERMS OF THE GIVEN COEF

FROM QUESTION 4 OF THE AB ACTIVITY, YOU CAN REACH THE FOLLOWING CON

1 IF $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ THE SYSTEM HAS INFINITE SOLUTIONS. IN THIS CASE, THAT SATISFIES ONE OF THE COMPONENT EQUATIONS ALSO SATISFIES A SYSTEM IS SAID TO BE dependent.

2 IF $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ THE SYSTEM HAS NO S. THIS MEANS THE TWO COMP EQUATIONS DO NOT A COMMON SOLUTION. IN THIS CASE, THE SYSTEM IS SAID inconsistent.

3 IF $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ THE SYSTEM HAS ONE SOLUTION. THIS MEANS THERE IS ON THAT SATISFIES BOTH EQUATIONS. IN T, THE SYSTEM IS SAID TO BE independent.

EXAMPLE 9 CONSIDER THE FOLLOWING SYSTEMS OF L

A $\begin{cases} 2x+3y=1 \\ x-2y=3 \end{cases}$ **B** $\begin{cases} 3x-2y=2 \\ 9x-6y=5 \end{cases}$ **C** $\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$

BY CONSIDERING THE RATIO OF THE YOU CAN DETERMINE WHETH SYSTEM HAS A SOLUTIC

A THE RATIO OF THE COEFFICIENTS GIVES $\frac{2}{1} \neq \frac{3}{-2}$.

THEREFORE, THE SYSTEM HAS ONE SOLUTION.

B THE RATIO OF THE COEFFICIENTS GIVES $\frac{3}{9} = \frac{-2}{-6} \neq \frac{2}{5}$

THEREFORE, THE SYSTEM HAS NO SOLUTION.

C THE RATIO OF THE COEFFICIENTS GIVES $\frac{1}{2} = \frac{1}{2} = \frac{3}{6}$

THEREFORE, THE SYSTEM HAS INFINITE SOLUTIONS.

Remark: BEFORE TRYING TO SOLVE A SYSTEM OF LINEAR EQUATIONS, IT IS A GOOD IDE WHETHER THE SYSTEM HAS A SOLUTION OR NOT.

Word problems leading to a system of linear equations

SYSTEMS OF LINEAR EQUATIONS HAVE MANY REAL LIFE APPLICATIONS. THE PROBLEMS NEED TO BE CONSTRUCTED IN A MATHEMATICAL FORM AS A SYSTEM OF LINEAR EQU WILL BE SOLVED BY THE TECHNIQUES DISCUSSED EARLIER. HERE ARE SOME EXAMPLES

Group Work 2.4

1 TESHOME BOUGHT 6 PENCILS AND 2 RUBBER ER SHOP AND PAID A TOTAL OF BIRR 3. MESKEREM AL OF BIRR 3 FOR 4 PENCILS AND 3 RUBBER ERASERS.

2 A COMPANY HAS TWO BRANDS OF FERTILIZERS A AND B FOR SELL. CHALTU COOPER 10 QUINTALS OF BRAND A AND 27 QUINTALS OF BRAND B FERTILIZERS AND PAID BIRR 20,000.

TOLOSA A SUCCESSFUL FARM OWNER, BOUGHT 15 QUINTALS OF BRAND A AND 9 BRAND B FERTILIZERS FROM THE SAME COMPANY AND PAID A TOTAL OF BIRR 14,25

I REPRESENT VARIABLES FOR THE COST OF:

- **A** EACH PENCIL AND EACH RUBBER ERASER IN QUESTION 1
- **B** EACH QUINTAL OF FERTILIZER OF BRAND A AND EACH QUINTAL OF FEI BRAND B IN QUESTION 2.

II FORMULATE THE MATHEMATICAL EQUATIONS REPRESENTING THE SITUATIONS IN EACH OF THE S QUESTIONS 1 AND 2 AS A SYSTEM OF TWO LINEAR EQUATIONS.

III SOLVE EACH SYSTEM AND DETERMINE THE COST OF,

- **A** EACH PENCIL AND EACH RUBBER ERASER IN QUESTION 1
- **B** EACH QUINTAL OF FERTILIZER OF BRAND A AND EACH QUINTAL OF BR QUESTION 2.

EXAMPLE 10 A FARMER COLLECTED A TOTAL OF BIRR 11,000 BY SELLING 3 COWS AND 5 ANOTHER FARMER COLLECTED BIRR 7,000 BY SELLING ONE COW AND 10 WHAT IS THE PRICE FOR A COW AND A SHEEP? (ASSUME ALL COWS HAVE TH PRICE AND ALSO THE PRICE OF EVERY SHEEP IS THE SAME).

SOLUTION: LET x REPRESENT THE PRICE OF A COW AND y THE PRICE OF A SHEEP.

FARMER I SOLD 3 COWS FOR $3x$ AND 5 SHEEP FOR $5y$ COLLECTING A TOTAL OF BIRR 11,000.

WHICH MEANS, $3x + 5y = 11,000$

FARMER II SOLD 1 COW FOR x AND 10 SHEEP FOR $10y$ COLLECTING A TOTAL OF BIRR 7,000.

WHICH MEANS, $x + 10y = 7,000$

WHEN WE CONSIDER THESE EQUATIONS SIMULTANEOUSLY, WE GET THE FOLLOWIN EQUATIONS.

$$\begin{cases} 3x + 5y = 11,000 \\ x + 10y = 7,000 \end{cases}$$

MULTIPLYING THE FIRST EQUATION BY –2 TO MAKE THE COEFFICIENTS OF y OPPOSITE

$$\begin{cases} -6x - 10y = -22,000 \\ \quad x + 10y = 7,000 \end{cases}$$

ADDING THE EQUATIONS WE GET $-6x + x - 10y + 10y = -22,000 + 7,000$

$$-5x = -22,000 + 7,000$$

$$-5x = -15,000$$

$$x = 3,000$$

SUBSTITUTING $x = 3,000$ IN ONE OF THE EQUATIONS, SAY $x + 10y = 7,000$, WE GET,

$$3,000 + 10y = 7,000$$

$$10y = 4,000$$

$$y = 400$$

THEREFORE THE SOLUTION IS (3000, 400) SHOWING THAT THE PRICE FOR A COW IS B 3,000 AND THE PRICE FOR A SHEEP IS BIRR 400.

EXAMPLE 11 SIMON HAS TWIN YOUNGER BROTHERS. THE SUM OF THE AGES OF THE BROTHERS IS 48 AND THE DIFFERENCE BETWEEN HIS AGE AND THE AGE OF HIS YOUNGER BROTHERS IS 3. HOW OLD IS SIMON?

SOLUTION: LET x BE THE AGE OF SIMON AND y BE THE AGE OF EACH OF HIS YOUNGER BROTHERS

THE SUM OF THE AGES OF THE THREE BROTHERS IS 48.

SO $x + y + y = 48$

$x + 2y = 48.$

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THE DIFFERENCE BETWEEN HIS AGE AND THE AGE OF ONE OF HIS YOUNGER BROTH
IMPLYING

$x - y = 3.$

TO FIND SIMON'S AGE, WE NEED TO SOLVE THE SYSTEM $\begin{cases} x + 2y = 48 \\ x - y = 3 \end{cases}$

MULTIPLYING THE SECOND EQUATION BY 2 TO MAKE THE COEFFICIENTS OF y OPPOSITE

$$\begin{cases} x + 2y = 48 \\ 2x - 2y = 6 \end{cases}$$

ADDING THE EQUATIONS, WE GET

$$x + 2x + 2y - 2Y = 48 + 6$$

$$3x = 54$$

$$x = \frac{54}{3} = 18$$

THEREFORE, SIMON IS 18 YEARS OLD.

Exercise 2.2

1 WHICH OF THE FOLLOWING ARE LINEAR EQUATIONS IN TWO VARIABLES?

A $5x + 5y = 7$ **B** $x + 3xy + 2y = 1$ **C** $x = 2y - 7$

D $y = x^2$ **E** $\frac{4}{x} - \frac{3}{y} = 2$

2 THE SUM OF TWO NUMBERS IS 64. TWICE THE LARGER NUMBER PLUS FIVE TIMES T
SMALLER NUMBER IS 20. FIND THE TWO NUMBERS.

3 IN A TWO-DIGIT NUMBER, THE SUM OF THE DIGITS IS 14. TWICE THE TENS DIGIT EXC
THE UNITS DIGIT BY ONE. FIND THE NUMBERS.

4 DETERMINE WHETHER EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS HAS ONE SOLUTION, INFINITE SOLUTIONS OR NO SOLUTION.

A $\begin{cases} 3x - y = 7 \\ -3x + 3y = -1 \end{cases}$ **B** $\begin{cases} 2x + 5y = 12 \\ x - \frac{5}{2}y = 4 \end{cases}$

C $\begin{cases} 3x - y = 7 \\ 2x + 3y = 12 \end{cases}$ **D** $\begin{cases} 4x - 3y = 6 \\ 2x + 3y = 12 \end{cases}$

5 SOLVE EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS BY USING A GRAPHICAL

A $\begin{cases} 3x+5y-11=0 \\ 4x-2y=4 \end{cases}$ **B** $\begin{cases} -3x+y=5 \\ 3x-y=5 \end{cases}$ **C** $\begin{cases} \frac{2}{3}x+y=6 \\ -x-\frac{3}{2}y=12 \end{cases}$

D $\begin{cases} x-2y=1 \\ 7x+4y=16 \end{cases}$ **E** $\begin{cases} 0.5x+0.25y=1 \\ x+y=2 \end{cases}$

6 SOLVE EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS BY THE SUBSTITUTION M

A $\begin{cases} 2x+7y=14 \\ x+\frac{7}{2}y=4 \end{cases}$ **B** $\begin{cases} y=x-5 \\ x=y \end{cases}$ **C** $\begin{cases} \frac{2}{3}x-\frac{1}{3}y=2 \\ -x+\frac{1}{2}y=-3 \end{cases}$

D $\begin{cases} -2x+2y=3 \\ 7x+4y=17 \end{cases}$ **E** $\begin{cases} x+3y=1 \\ 2x+5y=2 \end{cases}$

7 SOLVE EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS BY THE ELIMINATION M

A $\begin{cases} -3x+y=5 \\ 3x+y=5 \end{cases}$ **B** $\begin{cases} 4x-3y=6 \\ 2x+3y=12 \end{cases}$ **C** $\begin{cases} \frac{2}{3}x-\frac{1}{3}y=2 \\ -x+\frac{1}{3}y=-3 \end{cases}$

D $\begin{cases} \frac{1}{2}x-2y=5 \\ 7x+4y=6 \end{cases}$ **E** $\begin{cases} x+3y=1 \\ 2x+5y=2 \end{cases}$

8 SOLVE

A $\begin{cases} 3x-0.5y=6 \\ -2x+y=4+2y \end{cases}$ **B** $\begin{cases} \frac{2}{x}+\frac{3}{y}=-2 \\ \frac{4}{x}-\frac{5}{y}=1 \end{cases}$

Hint: LET $a=\frac{1}{x}$ AND $b=\frac{1}{y}$

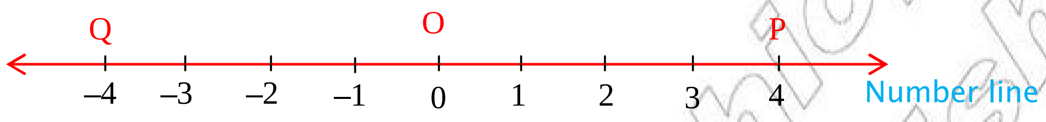
9 FIND b AND c GIVEN THAT THE GRAPH OF $y = x^2 + bx + c$ PASSES THROUGH (3, 14) AND (–4, 7).

10 A STUDENT IN A CHEMISTRY LABORATORY HAS ACCESS TO TWO ACID SOLUTI SOLUTION IS 20% ACID AND THE SECOND SOLUTION IS 45% ACID. (THE PERCENTAG VOLUME). HOW MANY MILLILITRES OF EACH SOLUTION SHOULD THE STUDENT MIX OBTAIN 100 ML OF A 30% ACID SOLUTION?

2.3 EQUATIONS INVOLVING ABSOLUTE VALUE

IN PREVIOUS SECTIONS, YOU WORKED WITH EQUATIONS HAVING SOLUTIONS THAT CAN ASSUME ANY VALUE. BUT SOMETIMES IT BECOMES NECESSARY TO CONSIDER ONLY NON-NEGA FOR EXAMPLE, IF YOU CONSIDER DISTANCE, IT IS ALWAYS NON-NEGATIVE. THE DISTAN x IS LOCATED ON THE REAL LINE FROM THE ORIGIN IS A POSITIVE NUMBER.

FROM UNIT ONE, RECALL THAT THE SET OF REAL NUMBERS CAN BE REPRESENTED ON A LI



FROM THIS, IT IS POSSIBLE TO DETERMINE THE DISTANCE OF EACH POINT, REPRESENTI LOCATED FAR AWAY FROM THE ORIGIN OR THE POINT REPRESENTING 0.

EXAMPLE 1 LET P AND Q BE POINTS ON A NUMBER LINE WITH COORDINATES 4 AN RESPECTIVELY. HOW FAR ARE THE POINTS P AND Q FROM THE ORIGIN?

SOLUTION: THE DISTANCE OF P AND Q FROM THE ORIGIN IS THE SAME ON THE REAL LIN

Note: IF X IS A POINT ON A NUMBER LINE WITH COORDINATE x, THEN THE DISTANCE OF X FROM THE ORIGIN IS CALLED THE absolute value OF x AND IS DENOTED BY $|x|$

EXAMPLE 2 THE POINTS REPRESENTED BY NUMBERS 2 AND –2 ARE LOCATED ON THE LINE AT AN EQUAL DISTANCE FROM THE ORIGIN. HENCE, $|2| = |-2| = 2$.

EXAMPLE 3 FIND THE ABSOLUTE VALUE OF EACH OF THE FOLLOWING.

A -5 **B** 7 **C** -0.5

SOLUTION:

A $|-5| = 5$ **B** $|7| = 7$ **C** $|-0.5| = 0.5$

IN GENERAL, THE DEFINITION OF AN ABSOLUTE VALUE IS GIVEN AS FOLLOWS.

Definition 2.4

The absolute value of a number x, denoted by $|x|$, is defined as follows.

$$|x| = \begin{cases} x \text{ IF } x \geq 0 \\ -x \text{ IF } x < 0 \end{cases}$$

EXAMPLE 4 USING THE DEFINITION, DETERMINE THE ABSOLUTE VALUE OF THE FOLLOWING.

A 3 **B** -2 **C** -0.4

SOLUTION:

A SINCE $3 > 0$, $|3| = 3$ **B** SINCE $-2 < 0$, $|-2| = -(-2) = 2$

C $-0.4 < 0$, AND THUS $|-0.4| = -(-0.4) = 0.4$

Note: **1** FOR ANY REAL NUMBER x, $|x| = |-x|$.

2 FOR ANY REAL NUMBER x, $|x|$ IS ALWAYS NON-NEGATIVE.

WE CONSIDERED ABSOLUTE VALUE AS A DISTANCE (OF THE POINT REPRESENTING A NUMBER) FROM THE ORIGIN, OR THE DISTANCE BETWEEN THE LOCATION OF THE NUMBER AND THE ORIGIN. ALSO POSSIBLE TO CONSIDER THE DISTANCE BETWEEN ANY OTHER TWO POINTS ON THE

EXAMPLE 5 FIND THE DISTANCE BETWEEN THE POINTS REPRESENTED BY THE NUMBERS 3 AND 9.

SOLUTION: THE DISTANCE BETWEEN THE POINTS REPRESENTED BY 3 AND 9 IS GIVEN AS

$$|3-9| = |-6| = 6 \text{ OR } |9-3| = |6| = 6$$

THE DISTANCE BETWEEN THE LOCATION OF ANY TWO REAL NUMBERS x AND y IS $|x-y|$ OR $|y-x|$.

NOTE THAT $|x-y| = |y-x|$.

EXAMPLE 6 $|5-3| = |2| = 2$ OR $|3-5| = |-2| = 2$.

EXAMPLE 7 EVALUATE EACH OF THE FOLLOWING.

A $|2-5|$ **B** $|-3-4|$ **C** $|8-3|$ **D** $|2-(-5)|$

SOLUTION:

A $|2-5| = |-3| = 3$ **B** $|-3-4| = |-7| = 7$

C $|8-3| = |5| = 5$ **D** $|2-(-5)| = |2+5| = |7| = 7$

NEXT, WE WILL DISCUSS EQUATIONS THAT INVOLVE ABSOLUTE VALUES AND THE PREVIOUSLY, WE SAW $|3| = |-3| = 3$. SO FOR THE EQUATION $|x| = 3$, IT IS APPARENT THAT $x = 3$ OR $x = -3$.

Note: FOR ANY NON-NEGATIVE NUMBER a,

$$|x| = a \text{ MEANS } x = a \text{ OR } x = -a.$$

EXAMPLE 8

A $|x-2| = 3$ MEANS $x - 2 = 3$ OR $x - 2 = -3$

$x = 5$ OR $x = -1$

B $|x+4| = 5$ MEANS $x + 4 = 5$ OR $x + 4 = -5$

$x = 1$ OR $x = -9$

THIS CONCEPT OF ABSOLUTE VALUE IS ESSENTIAL IN SOLVING VARIOUS PROBLEMS. HE HOW WE CAN SOLVE EQUATIONS INVOLVING ABSOLUTE VALUES.

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EXAMPLE 9 SOLVE $|2x - 3| = 5$

SOLUTION: FOLLOWING THE DEFINITION $|2x - 3| = 5$ MEANS $2x - 3 = 5$ OR $2x - 3 = -5$,

SOLVING THESE LINEAR EQUATIONS, $x = 4$ OR $x = -1$.

EXAMPLE 10 DETERMINE THE VALUE OF THE VARIABLE x IN EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS.

A $|x| = 4$ **B** $|x - 1| = 5$ **C** $|-2x + 3| = 4$

D $|x| = -5$ **E** $|2x + 3| = -3$

SOLUTION:

A $|x| = 4$ MEANS $x = 4$ OR $x = -4$

B $|x - 1| = 5$ MEANS $x - 1 = 5$ OR $x - 1 = -5$

THEREFORE $x = 6$ OR $x = -4$.

C $|-2x + 3| = 4$ MEANS $-2x + 3 = 4$ OR $-2x + 3 = -4$

$-2x = 1$ OR $-2x = -7$

THEREFORE $x = \frac{-1}{2}$ OR $x = \frac{7}{2}$

D SINCE $|x|$ IS ALWAYS NON-NEGATIVE, $|x| = -5$ HAS NO SOLUTION.

E SINCE $|x|$ IS ALWAYS NON-NEGATIVE, $|2x + 3| = -3$ HAS NO SOLUTION.

Note: FOR ANY REAL NUMBER a, $|x| = |a|$ MEANS $x = a$ OR $x = -a$.

EXAMPLE 11 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $|x - 1| = |2x + 1|$ **B** $|3x + 2| = |2x - 1|$

SOLUTION: A $|x - 1| = |2x + 1|$ MEANS $x - 1 = 2x + 1$ OR $x - 1 = -(2x + 1)$

$x - 2x = 1 + 1$ OR $x + 2x = -1 + 1$

$-x = 2$ OR $3x = 0$

THEREFORE $x = -2$ OR $x = 0$.

B $|3x + 2| = |2x - 1|$ MEANS $3x + 2 = 2x - 1$ OR $3x + 2 = -(2x - 1)$

$3x - 2x = -1 - 2$ OR $3x + 2x = 1 - 2$

$x = -3$ OR $5x = -1$

THEREFORE $x = -3$ OR $x = -\frac{1}{5}$

EXAMPLE 12 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $|x - 1| = |x + 1|$ **B** $|2x + 2| = |2x - 1|$

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SOLUTION:

A $|x-1| = |x+1|$ MEANS $x-1 = x+1$ OR $x-1 = -(x+1)$

$x - x = 1 + 1$ OR $x + x = -1 + 1$

$0 = 2$ OR $2x = 0$

BUT $0 = 2$ IS IMPOSSIBLE.

THEREFORE, $x = 0$.

B $|2x+2| = |2x-1|$ MEANS $2x + 2 = 2x - 1$ OR $2x + 2 = -(2x - 1)$

$2x - 2x = -1 - 2$ OR $2x + 2x = 1 - 2$

$0 = -3$, OR $4x = -1$.

BUT $0 = -3$ IS NOT POSSIBLE.

THEREFORE, $x = -\dfrac{1}{4}$.

Properties of absolute value

FOR ANY REAL NUMBERS x AND y,

1 $x \le |x|$.

2 $|xy| = |x||y|$.

3 $\sqrt{x^2} = |x|$.

4 $|x+y| \le |x| + |y|$ (THIS IS CALLED THE triangle inequality).

A IF x AND y ARE BOTH NON-POSITIVE OR BOTH NON-NEGATIVE, $|x+y| = |x| + |y|$.

B IF ONE OF x OR y IS POSITIVE AND THE OTHER IS NEGATIVE, $|x+y| < |x| + |y|$

5 IF $y \neq 0$ THEN $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

6 $-|x| \le x \le |x|$.

Exercise 2.3

1 EVALUATE EACH OF THE FOLLOWING.

A $|2-(-3)|$ **B** $|-4+9|$ **C** $|-5-2|$ **D** $|8| - |3-7|$

2 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $|x-5| = -5$ **B** $|x-5| = 5$ **C** $|-(2x-3)| = 7$

D $|3-4x| = 8$ **E** $|x-(3+2x)| = 6$ **F** $|12-(x+7)| = 3$

3 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $|5 - x| = |3x - 7|$ **B** $|3x - 2| = |3x - 7|$

C $|5 - 4x| = |7 + 3x|$ **D** $|3x + 4| - |x + 7| = 0$

E $|7 - (x + 3)| + |3x - 3| = 0$

4 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $|x - 3| + |x - 3| = 9$ **B** $|3x + 2| - |x - 3| = 5$

C $|-(2x - 3)| + |x| = 12$ **D** $|4x - 2| = 8 + |x - 3|$

E $|5x - (1 - 2x)| - |3 - 2x| = 8$ **F** $|12 - (x + 7)| + |x - 3| = 3$

Hint: HERE, FOR $|x + a| + |x + b| = c$, NOTICE THAT $|x + a|$ TAKES EITHER $x + a$ OR $-(x + a)$ AND ALSO $|x + b|$ TAKES EITHER $x + b$ OR $-(x + b)$, DEPENDING ON WHETHER THEY ARE GREATER THAN 0 OR LESS THAN 0. THEREFORE, YOU NEED TO CONSIDER FOUR CASES TO SOLVE SUCH PROBLEMS!

5 VERIFY EACH OF THE FOLLOWING.

A $|y - x| \le |x| + |y|$ WHEN $x = -2$ AND $y = 3$.

B $\sqrt{(3x-7)^2} = |3x-7|$, WHEN $x = 5$.

2.4 QUADRATIC EQUATIONS

RECALL THAT FOR REAL NUMBERS a AND b, ANY EQUATION THAT CAN BE REDUCED TO THE FORM

$ax + b = 0$, WHERE $a \neq 0$ IS CALLED A **linear equation**.

FOLLOWING THE SAME ANALOGY, FOR REAL NUMBERS a, b AND c, ANY EQUATION THAT CAN BE REDUCED TO THE FORM

$ax^2 + bx + c = 0$, WHERE $a \neq 0$ IS CALLED A **quadratic equation**.

$x^2 + 3x - 2 = 0$, $2x^2 - 5x = 3$, $3x^2 - 6x = 0$, $(x + 3)(x + 2) = 7$ ETC, ARE EXAMPLES OF QUADRATIC EQUATIONS.

IN THIS SECTION, YOU WILL STUDY SOLVING QUADRATIC EQUATIONS. YOU WILL DISCUSS APPROACHES TO SOLVE QUADRATIC EQUATIONS, NAMELY, THE **method of factorization**, THE **method of completing the square**, AND THE **general formula**. BEFORE YOU PROCEED TO SOLVE QUADRATIC EQUATIONS, YOU WILL FIRST DISCUSS THE CONCEPT OF FACTORIZATION.

Expressions

EXPRESSIONS ARE COMBINATIONS OF VARIOUS TERMS THAT ARE REPRESENTED AS A PRODUCT OF VARIABLES OR NUMBERS AND VARIABLES.

EXAMPLE 1 $x^2 + 2x$, $2x^2 + 4x + 2$, $(x + 1)x^2 + 6x$, ETC. ARE EXPRESSIONS.

x^2 AND $2x$ ARE THE TERMS IN $x^2 + 2x$ AND $2x^2$, $4x$, AND 2 ARE THE TERMS IN $2x^2 + 4x + 2$.

Factorizing expressions

ACTIVITY 2.8

1 MULTIPLY EACH OF THE FOLLO.

A $x(x+9)$ B $(x+3)(x-3)$ C $(x+2)(x+3)$

2 HOW WOULD IT BE POSSIBLE TO GO BACK FROM PRC FACTOR EACH OF THE FOLLOWING.

A x^2-9 B x^2+9x C x^2+5x+6

FACTORIZING AN EXPRESSION IS IT AS A PRODUCT OF ITS SIMPLI

EXAMPLE 2 FACTORIZE $2x^2-9x$.

SOLUTION: THE TWO TERMS IN THIS EX, $2x^2$ AND $-9x$ HAVE x AS A COMM FACTOR. HENCE $2x^2-9x$ CAN BE FACTORIZED AS $x(2x-9)$.

SO $2x^2-9x=x(2x-9)$.

EXAMPLE 3 FACTORIZE $4x^2+12x$.

SOLUTION: $4x^2+12x=(4x)x+3(4x)=(4x)(x+3)$

EXAMPLE 4 FACTORIZE $(2x-1)(3x)+2(2x-1)$.

SOLUTION: $(2x-1)(3x)+2(2x-1)=(2x-1)(3x+2)$ SINCE $(2x-1)$ IS A COMM FACTOR.

Factorizing the difference of two squares

IF WE MULTIPLY $(x+2)$ AND $(x-2)$, WE SEE THAT $(x+2)(x-2)=x^2-4=x^2-2^2$.

ACTIVITY 2.9

1 WHAT IS 75^2-25^2? HOW WOULD YOU COMPU

2 WHAT IS 200^2-100^2?

IN GENERAL,

$$x^2-a^2=(x-a)(x+a).$$

EXAMPLE 5 FACTORIZE x^2-9.

SOLUTION: $x^2-9=x^2-3^2=(x-3)(x+3)$

EXAMPLE 6 FACTORIZE $4x^2-16$.

SOLUTION: $4x^2-16=(2x)^2-16=(2x)^2-4^2=(2x-4)(2x+4)$

Factorizing trinomials

YOU SAW HOW TO FACTORIZE EXPRESSIONS THAT HAVE A COMMON FACTOR. YOU ALSO SAW FACTORIZING THE DIFFERENCE OF TWO SQUARES. NOW YOU WILL SEE HOW TO FACTORIZE $ax^2 + bx + c$ BY GROUPING TERMS, IF YOU ARE ABLE TO FIND TWO NUMBERS p AND q SUCH THAT $p + q = b$ AND $pq = ac$.

EXAMPLE 7 FACTORIZE $x^2 + 5x + 6$.

SOLUTION: TWO NUMBERS WHOSE SUM IS 5 AND PRODUCT 6 ARE 2 AND 3

SO, IN THE EXPRESSION, WE WRITE $2x + 3x$ INSTEAD OF $5x$

$$x^2 + 5x + 6 = x^2 + (2x + 3x) + 6 \text{ BECAUSE } 2x + 3x = 5x.$$

$$= (x^2 + 2x) + (3x + 6) \quad (\textit{grouping into two parts})$$

$$= x(x+2) + 3(x+2) \ldots \quad (\textit{factorizing each part})$$

$$= (x+2)(x+3) \text{ BECAUSE } (x+2) \text{ IS A COMMON FACTOR.}$$

EXAMPLE 8 FACTORIZE $x^2 + 4x + 4$.

SOLUTION: TWO NUMBERS WHOSE SUM IS 4 AND PRODUCT 4 ARE 2 AND 2. TAKE $2x + 2x$ INSTEAD OF $4x$

$$x^2 + 4x + 4 = x^2 + (2x + 2x) + 4 \text{ BECAUSE } 2x + 2x = 4x$$

$$= \left(x^2 + 2x\right) + \left(2x + 4\right) \ldots. (\textit{grouping})$$

$$= x\left(x + 2\right) + 2\left(x + 2\right) \ldots\ldots (\textit{take out the common factor for each group})$$

$$= \left(x + 2\right)\left(x + 2\right) = \left(x + 2\right)^2.$$

SUCH EXPRESSIONS ARE CALLED **perfect squares**.

EXAMPLE 9 FACTORIZE $3x^2 - 14x - 5$.

SOLUTION: DO YOU HAVE NUMBERS WHOSE SUM IS –14 AND WHOSE PRODUCT IS $3 \times -5 = -15$?

$-15 + 1 = -14$ AND $-15 \times 1 = -15$. THIS MEANS YOU CAN USE –15 AND 1 FOR GROUPING, GIVING

$$3x^2 - 14x - 5 = 3x^2 - 15x + x - 5$$

$$= (3x^2 - 15x) + (x - 5)$$

$$= 3x(x - 5) + 1(x - 5)$$

$$= (3x + 1)(x - 5)$$

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SO $3x^2 - 14x - 5 = (3x+1)(x-5)$.

ACTIVITY 2.10

FACTORIZE EACH OF THE FOLLO.

A $2x^2 + 10x + 12$ **B** $2x^2 - x - 21$ **C** $5x^2 + 14x + 9$

Solving quadratic equations using the method of factorization

LET $ax^2 + bx + c = 0$ BE A QUADRATIC EQUATION AND LET THE QUA $ax^2 + bx + c$ BE EXPRESSIBLE AS A PRODUCT OF TWO LIN $(dx + e)$ AND $(fx + g)$ WHERE d, e, f, g ARE REAL NUMBERS SUCH THAT $d \neq 0$ AND $f \neq 0$.

THEN $ax^2 + bx + c = 0$ BECOM

$$(dx + e)(fx + g) = 0$$

SO, $dx + e = 0$ OR $fx + g = 0$ WHICH GIVES $x = \frac{-e}{d}$ OR $x = \frac{-g}{f}$.

THEREFORE $x = \frac{-e}{d}$ AND $x = \frac{-g}{f}$ ARE POSSIBLE ROOTS OF THE QUAD $ax^2 + bx + c = 0$.

FOR EXAMPLE, THE EQUATION $x^2 - 5x + 6 = 0$ CAN BE EXPRESSED AS:

$$(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \text{ OR } x - 3 = 0$$

$$x = 2 \text{ OR } x = 3$$

THEREFORE THE SOLUTIONS OF THE $x^2 - 5x + 6 = 0$ ARE $x = 2$ AND $x = 3$.

In order to solve a quadratic equation by factorization, go through the following steps:

I CLEAR ALL FRACTIONS AND SQUAF

II WRITE THE EQUATION IN $p(x) = 0$.

III FACTORIZE THE LEFT HAND SIDE INTO A PRODUCT (

IV USE THE *zero-product rule* TO SOLVE THE RESULTING EQUATIC

Zero-product rule: IF a AND b ARE TWO NUMBERS OR EXPRESS $ab = 0$, THEN EITHER $a = 0$ OR $b = 0$ OR BOTH $a = 0$ AND $b = 0$.

EXAMPLE 10 SOLVE EACH OF THE FOLLOWING QUADRATIC EQUATIONS.

A $4x^2 - 16 = 0$ **B** $x^2 + 9x + 8 = 0$ **C** $2x^2 - 6x + 7 = 3$

SOLUTION:

A $4x^2 - 16 = 0$ IS THE SAME AS $(2x)^2 - 4^2 = 0$

$$(2x - 4)(2x + 4) = 0$$

$$(2x - 4) = 0 \text{ OR } (2x + 4) = 0$$

THEREFORE, $x = 2$ OR $x = -2$.

B $x^2 + 9x + 8 = 0$

$$x^2 + x + 8x + 8 = 0$$

$$(x^2 + x) + (8x + 8) = 0$$

$$x(x + 1) + 8(x + 1) = 0$$

$$(x + 1)(x + 8) = 0$$

$$(x + 1) = 0 \text{ OR } (x + 8) = 0$$

THEREFORE, $x = -1$ OR $x = -8$.

C $2x^2 - 6x + 7 = 3$ IS THE SAME AS $2x^2 - 6x + 4 = 0$

$2x^2 - 6x + 4 = 0$ CAN BE EXPRESSED AS

$2x^2 - 2x - 4x + 4 = 0$; (-2 AND -4 HAVE SUM $= -6$ AND PRODUCT $= 8$).

$$(2x^2 - 2x) - (4x - 4) = 0$$

$$2x(x - 1) - 4(x - 1) = 0$$

$$(2x - 4)(x - 1) = 0$$

$$(2x - 4) = 0 \text{ OR } (x - 1) = 0$$

THEREFORE, $x = 2$ OR $x = 1$.

Exercise 2.4

1 SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $(x - 3)(x + 4) = 0$ **B** $2x^2 - 6x = 0$ **C** $x^2 - 3x + 4 = 4$

D $2x^2 - 8 = 0$ **E** $5x^2 = 6x$ **F** $x^2 - 2x - 12 = 7x - 12$

G $-x^2 - 4 = 0$ **H** $2x^2 + 8 = 0$

2 SOLVE EACH OF THE FOLLOWING EQUATIONS.

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A $x^2 - 6x + 5 = 0$ **B** $3x^2 - 2x - 5 = 0$ **C** $x^2 + 7x = 18$

D $-x^2 = 8x - 9$ **E** $5y^2 - 6y + 1 = 0$ **F** $3z^2 + 10z = 8$

3 FIND THE SOLUTION SET OF EACH OF THE FOLLOWING.

A $2x^2 + \frac{3}{2}x + \frac{1}{4} = 0$ **B** $x^2 = -2.5x + \frac{25}{16}$

C $-(6 + 2x^2) + 8x = 0$

Solving quadratic equations by completing the square

Group Work 2.5

CONSIDERING $2x^2 + 5x - 4 = 0$, FORM A GROUP AND DO THE FC

1 DIVIDE EACH COEFFICIENT BY 2.

2 SHIFT THE CONSTANT TERM TO THE RIGHT HAND SIDE (RHS).

3 ADD THE SQUARE OF HALF OF THE MIDDLE TERM TO BOTH S

4 DO WE HAVE ANY PERFECT SQUARE? WHY OR WHY NOT?

5 DO YOU OBSERVE THAT $\left(x + \frac{5}{4}\right)^2 = \frac{57}{16}$?

6 DISCUSS THE SOLUTION.

IN MANY CASES, IT IS NOT CONVENIENT TO SOLVE A QUADRATIC EQUATION BY F METHOD. FOR EXAMPLE, CONSIDER THE EQUATION $x^2 + 8x + 4 = 0$. IF YOU WANT TO FACTORIZE THE LEFT HAND SIDE OF THE EQUATION, I.E., THE POLYNOMIAL $x^2 + 8x + 4$, USING THE METHOD OF SPLITTING THE MIDDLE TERM, YOU NEED TO FIND TWO INTEGERS WHOSE SUM IS 8 AND BUT THIS IS NOT POSSIBLE. IN SUCH CASES, AN ALTERNATIVE METHOD AS DEMONSTR CONVENIENT.

$$x^2 + 8x + 4 = 0$$

$$x^2 + 8x = -4$$

$$x^2 + 8x + (4)^2 = -4 + (4)^2 \quad \left(\text{ADDING } \left(\tfrac{1}{2} \text{ COEFFICIENT OF } x\right)^2 \text{ ON BO}\right)$$

$$(x + 4)^2 = -4 + 16 = 12 \quad (x^2 + 8x + 16 = (x + 4)^2)$$

$$x + 4 = \pm\sqrt{12} \quad \text{(TAKING SQUARE ROOT OF BOTH SIDES)}$$

THEREFORE $x = -4 + \sqrt{12}$ AND $x = -4 - \sqrt{12}$ ARE THE REQUIRED SOLUTIONS.

THIS METHOD IS KNOWN AS THE METHOD OF COMPLETING THE SQUARE.

In general, go through the following steps in order to solve a quadratic equation by the method of completing the square:

I WRITE THE GIVEN QUADRATIC EQUATION IN THE STANDARD FORM.

II MAKE THE COEFFICIENT OF x^2 UNITY, IF IT IS NOT.

III SHIFT THE CONSTANT TERM TO R.H.S.(RIGHT HAND SIDE)

IV ADD $\left(\frac{1}{2}\text{ COEFFICIENT OF } x\right)^2$ ON BOTH SIDES.

V EXPRESS L.H.S.(LEFT HAND SIDE) AS THE PERFECT SQUARE OF A SUITABLE BINOMIAL EXPRESSION AND SIMPLIFY THE R.H.S.

VI TAKE SQUARE ROOT OF BOTH THE SIDES.

VII OBTAIN THE VALUES OF x BY SHIFTING THE CONSTANT TERM FROM L.H.S. TO R.H.S.

Note: THE NUMBER WE NEED TO ADD (OR SUBTRACT) TO FORM A PERFECT SQUARE IS DETERMINED BY USING THE FOLLOWING PRODUCT FORMULAS:

$$x^2 + 2ax + a^2 = (x + a)^2$$

$$x^2 - 2ax + a^2 = (x - a)^2$$

NOTE THAT THE LAST TERM, a^2, ON THE LEFT SIDE OF THE FORMULAE IS THE square of one-half of the coefficient of x AND THE COEFFICIENT OF x^2 IS 1. SO, WE SHOULD ADD (OR SUBTRACT) A SUITABLE NUMBER TO GET THIS FORM.

EXAMPLE 11 SOLVE $x^2 + 5x - 3 = 0$.

SOLUTION: NOTE THAT $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

HENCE, WE ADD THIS NUMBER TO GET A PERFECT SQUARE.

$$x^2 + 5x - 3 = 0$$

$$x^2 + 5x = 3$$

$$x^2 + 5x + \frac{25}{4} = 3 + \frac{25}{4}$$

$$x^2 + 5x + \frac{25}{4} = \frac{37}{4}; \quad \left(x^2 + 5x + \frac{25}{4} \text{ IS A PERFET SQ}\right)$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{37}{4}$$

$$\left(x+\frac{5}{2}\right)=\sqrt{\frac{37}{4}} \quad \text{OR} \quad \left(x+\frac{5}{2}\right)=-\sqrt{\frac{37}{4}}$$

$$x=-\frac{5}{2}+\sqrt{\frac{37}{4}} \quad \text{OR} \quad x=-\frac{5}{2}-\sqrt{\frac{37}{4}}$$

THEREFORE $x=\dfrac{-5+\sqrt{37}}{2}$ OR $x=\dfrac{-5-\sqrt{37}}{2}$.

EXAMPLE 12 SOLVE $3x^2 + 12x + 6 = 0$.

SOLUTION: FIRST DIVIDE ALL TERMS BY 3 SO THAT THE COEFFICIENT OF x^2 IS 1.

$3x^2 + 12x + 6 = 0$ BECOMES $x^2 + 4x + 2 = 0$

$x^2 + 4x = -2$ (*Shifting the constant term to the right side*)

$x^2 + 4x + 4 = -2 + 4$ (*half of 4 is 2 and its square is 4*)

$(x + 2)^2 = 2$ (*$x^2 + 4x + 4 = (x + 2)^2$, a perfect square*)

$(x + 2) = \pm\sqrt{2}$

$x = -2 \pm\sqrt{2}$

THEREFORE $x = -2 - \sqrt{2}$ OR $x = -2 + \sqrt{2}$

EXAMPLE 13 SOLVE $3x^2 + 12x + 15 = 0$.

SOLUTION: FIRST DIVIDE ALL TERMS BY 3 SO THAT THE COEFFICIENT OF x^2 IS 1.

$3x^2 + 12x + 15 = 0$ BECOMES $x^2 + 4x + 5 = 0$

$x^2 + 4x = -5$ (*Shifting the constant term to the right side*)

$x^2 + 4x + 4 = -5 + 4$ (*half of 4 is 2 and its square is 4*)

$(x + 2)^2 = -1$ (*$x^2 + 4x + 4 = (x + 2)^2$, a perfect square*)

$(x+2) = \pm\sqrt{-1}$

SINCE $\sqrt{-1}$ IS NOT A REAL NUMBER, WE CONCLUDE THAT THE QUADRATIC EQUATION HAVE A REAL SOLUTION.

EXAMPLE 14 SOLVE $2x^2 + 4x + 2 = 0$.

SOLUTION: $2x^2 + 4x + 2 = 0$ BECOMES

$x^2 + 2x + 1 = 0$ (*Dividing all terms by 2*)

$(x + 1)^2 = 0$ (*$x^2 + 2x + 1 = (x + 1)^2$ is a perfect square*)

$(x + 1) = 0$

THEREFORE $x = -1$ IS THE ONLY SOLUTION.

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Exercise 2.5

1 SOLVE EACH OF THE FOLLOWING QUADRATIC EQUATIONS BY USING THE METHOD OF COMPLETING THE SQUARE.

A $x^2 - 6x + 10 = 0$ **B** $x^2 - 12x + 20 = 0$ **C** $2x^2 - x - 6 = 0$

D $2x^2 + 3x - 2 = 0$ **E** $3x^2 - 6x + 12 = 0$ **F** $x^2 - x + 1 = 0$

2 FIND THE SOLUTION SET FOR EACH OF THE FOLLOWING EQUATIONS.

A $20x^2 + 10x - 8 = 0$ **B** $x^2 - 8x + 15 = 0$ **C** $6x^2 - x - 2 = 0$

D $14x^2 + 43x + 20 = 0$ **E** $x^2 + 11x + 30 = 0$ **F** $2x^2 + 8x - 1 = 0$

3 REDUCE THESE EQUATIONS INTO THE FORM $ax^2 + bx + c = 0$ AND SOLVE.

A $x^2 = 5x + 7$ **B** $3x^2 - 8x = 15 - 2x + 2x^2$

C $x(x - 6) = 6x^2 - x - 2$ **D** $8x^2 + 9x + 2 = 3(2x^2 + 6x) + 2(x - 1)$

E $x^2 + 11x + 30 = 2 + 11x(x + 3)$

Solving quadratic equations using the quadratic formula

FOLLOWING THE METHOD OF COMPLETING THE SQUARE, YOU CAN DEVELOP A GENERAL FORMULA THAT CAN SERVE FOR CHECKING THE EXISTENCE OF A SOLUTION TO A QUADRATIC EQUA
SOLVING QUADRATIC EQUATIONS.

TO DERIVE THE GENERAL FORMULA FOR SOLVING $ax^2 + bx + c = 0$, $a \neq 0$, WE PROCEED USING THE METHOD OF COMPLETING THE SQUARE.

THE FOLLOWING GROUP WORK WILL HELP YOU TO FIND THE SOLUTION FORMULA OF THE QU
EQUATION $ax^2 + bx + c = 0$, $a \neq 0$, BY USING THE COMPLETING THE SQUARE METHOD.

Group Work 2.6

CONSIDER $ax^2 + bx + c = 0$, $a \neq 0$

1 FIRST DIVIDE EACH TERM BY a.

2 SHIFT THE CONSTANT TERM $\frac{c}{a}$ TO THE RIGHT.

3 ADD THE SQUARE OF HALF OF THE MIDDLE TERM TO BOTH S

4 DO YOU HAVE A PERFECT SQUARE?

5 SOLVE FOR x BY USING COMPLETING THE SQUARE.

6 DO YOU OBSERVE THAT $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$?

7 WHAT WILL BE THE ROOTS OF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$?

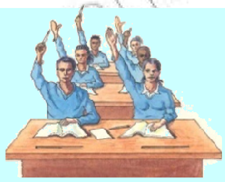
FOR A GENERAL QUADRATIC EQU $ax^2 + bx + c = 0$, $a \neq 0$, BY APPLYING THE MI OF COMPLETING THE SQUARE, YOU CAN CONCLUDE THAT THE ROOTS ARE $r_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ AND

$r_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$.

THEREFORE, THE SOLUTION $\left\{\dfrac{-b-\sqrt{b^2-4ac}}{2a}, \dfrac{-b+\sqrt{b^2-4ac}}{2a}\right\}$.

FROM THE ABOVE DISCU WHAT DO YOU OBSERVE ABOUT $b^2 - 4ac$ IN $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$?

ACTIVITY 2.11



IN $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, DISCUSS THE POSSIBLE C FOR x WHEN,

A $b^2 - 4ac > 0$ **B** $b^2 - 4ac = 0$ **C** $b^2 - 4ac < 0$

Note: IF ANY QUADRATIC E $ax^2 + bx + c = 0$, $a \neq 0$ HAS A SOLUTION, TH SOLUTION IS DETERM $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ AND

1 IF $b^2 - 4ac > 0$, THEN $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ REPRESENTS TWO N, NAMELY

$x = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ AND $x = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$.

THEREFORE, THE EQUATION HAS TWO

2 IF $b^2 - 4ac = 0$ THEN $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{-b}{2a}$ IS THE ONLY SOL

THEREFORE, THE EQUATION HAS ONLY ON

3 IF $b^2 - 4ac < 0$, THEN $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ IS NOT DEFINED IN ℝ

THEREFORE, THE EQUATION DOES NOT HAVE ANY

THE EXPRESSION $b^2 - 4ac$ IS CALLED THE discriminant OR discriminator. IT HELPS TO DETERMINE THE EXISTENCE OF SOL

EXAMPLE 15 USING THE DISCRIM, CHECK TO SEE IF THE FOLLOWING EQUAT SOLUTION(S) AND SOLVE IF THERE IS A.

A $3x^2 - 5x + 2 = 0$ **B** $x^2 - 8x + 16 = 0$ **C** $-2x^2 - 4x - 9 = 0$

SOLUTION:

A $3x^2 - 5x + 2 = 0$; $a = 3$, $b = -5$ AND $c = 2$.

SO $b^2 - 4ac = (-5)^2 - 4(3)(2) = 1 > 0$

THEREFORE, THE EQUATION $3x^2 - 5x + 2 = 0$ HAS TWO SOLUTIONS.

USING THE QUADRATIC FORMULA, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-5) - \sqrt{(-5)^2 - 4(3)(2)}}{2(3)} \text{ OR } x = \frac{-(-5) + \sqrt{(-5)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{5 - \sqrt{25 - 24}}{6} \text{ OR } x = \frac{5 + \sqrt{25 - 24}}{6}$$

$$x = \frac{5 - \sqrt{1}}{6} \text{ OR } x = \frac{5 + \sqrt{1}}{6}$$

$$x = \frac{5 - 1}{6} \text{ OR } x = \frac{5 + 1}{6}$$

$$x = \frac{4}{6} \text{ OR } x = \frac{6}{6}$$

THEREFORE $x = \dfrac{2}{3}$ OR $x = 1$

B IN $x^2 - 8x + 16 = 0$, $a = 1$, $b = -8$ AND $c = 16$

SO $b^2 - 4ac = (-8)^2 - 4(1)(16) = 0$

THEREFORE, THE EQUATION $x^2 - 8x + 16 = 0$ HAS ONLY ONE SOLUTION.

USING THE QUADRATIC SOLUTION FORMULA, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b}{2a}$

$$x = \frac{-(-8)}{2(1)} = 4$$

THEREFORE THE SOLUTION IS $x = 4$.

C IN $-2x^2 - 4x - 9 = 0$, $a = -2$, $b = -4$ AND $c = -9$

SO $b^2 - 4ac = (-4)^2 - 4(-2)(-9) = -56 < 0$

THEREFORE THE EQUATION $-2x^2 - 4x - 9 = 0$ DOES NOT HAVE ANY REAL SOLUTION.

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Exercise 2.6

1 SOLVE EACH OF THE FOLLOWING QUADRATIC EQUA QUADRATIC SOL FORMULA.

A $x^2+8x+15=0$ **B** $3x^2-12x+2=0$ **C** $4x^2-4x-1=0$

D $x^2+3x-2=0$ **E** $5x^2+15x+45=0$ **F** $3x^2-4x-2=0$

2 FIND THE SOLUTION SET FOR EACH OF THE FOL

A $x^2+6x+8=0$ **B** $9+30x+25x^2=0$ **C** $9x^2+15-3x=0$

D $4x^2-36x+81=0$ **E** $x^2+2x+8=0$ **F** $2x^2+8x+1=0$

3 REDUCE THE EQUATIONS INT $ax^2+bx+c=0$ AND SOLVE.

A $3x^2=5x+7-x^2$ **B** $x^2=8+2x+2x^2$

C $x^2-2(x-6)=6-x$ **D** $x^2-4+x(1+6x)+2(x-1)=4x-3$

E $4-8x^2+6x=2x(x+3)+2x$

4 A SCHOOL COMMUNITY HAD PLANNED TO REDUC GRADE STUDENTS PER CLASS ROOM BY CONSTRUCTING ADDITIONAL CLASS ROOMS. HOW LESS ROOMS THAN THEY PL THE RESULT, THE NUMBER OF STUDENTS PER 10 MORE THAN THEY PLANNED. IF THERE ARE 1200 GRADE 9 STU DETERMINE THE CURRENT NUMBER OF CLASS NUMBER OF STUDE PER CLASS.

The relationship between the coefficients of a quadratic equation and its roots

YOU HAVE LEARNED HOW TO SOLVE QUADRATIC EQUATIONS. THE SOLUTIONS TO A Q EQUATION ARE SOMETIM **roots**. THE GENERAL QUADRATIC EQUATION

$ax^2+bx+c=0$, $a\neq 0$ HAS ROOTS (SOL

$$r_1=\frac{-b-\sqrt{b^2-4ac}}{2a} \text{ AND } r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}.$$

ACTIVITY 2.12

1 IF $r_1=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ AND $r_2=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ARE ROOTS O QUADRATIC EQU $ax^2+bx+c=0$, $a\neq 0$ THEN

A FIND THE SUM OF TH (r_1+r_2).

B FIND THE PRODUCT OF T (r_1r_2).

2 WHAT RELATIONSHIP DO YOU OBSERVE BETWEEN THE SUM AND THE PRODUCT OF THE ROOTS WITH RESPECT TO THE QUOTIENTS OF THE COEFFICIENTS OF $ax^2 + bx + c = 0$, NAMELY a, b AND c?

3 TEST YOUR ANSWER ON THE QUADRATIC EQUATION $2x^2 - 7x + 5 = 0$.

THE RELATIONSHIP BETWEEN THE SUM AND PRODUCT OF THE ROOTS OF A QUADRATI ITS COEFFICIENTS IS STATED BELOW AND IT IS CALLED **Viete's theorem.**

Theorem 2.1 Viete's theorem

If the roots of $ax^2 + bx + c = 0$, $a \neq 0$ are $r_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and

$r_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, then $r_1 + r_2 = \dfrac{-b}{a}$ and $r_1 \times r_2 = \dfrac{c}{a}$

YOU CAN CHECK VIETE'S THEOREM AS FOLLOWS:

THE ROOTS OF $ax^2 + bx + c = 0$ ARE

$$r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ AND } r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

THEIR SUM IS $r_1 + r_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a} + \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

$$= \frac{\left(-b-\sqrt{b^2-4ac}\right)+\left(-b+\sqrt{b^2-4ac}\right)}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

AND THEIR PRODUCT IS $r_1 \times r_2 = \left(\dfrac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(\dfrac{-b+\sqrt{b^2-4ac}}{2a}\right)$

$$= \left(\frac{b^2-(b^2-4ac)}{(2a)^2}\right) = \left(\frac{4ac}{4a^2}\right) = \frac{c}{a}$$

SO THE SUM OF THE ROOTS IS $\dfrac{-b}{a}$ AND THE PRODUCT OF THE ROOTS IS $\dfrac{c}{a}$.

EXAMPLE 16 IF $3x^2 + 8x + 5 = 0$, THEN FIND

A THE SUM OF ITS ROOTS. **B** THE PRODUCT OF ITS ROOTS.

SOLUTION: IN $3x^2 + 8x + 5 = 0$, $a = 3$, $b = 8$ AND $c = 5$.

SUM OF THE ROOTS $= \dfrac{-b}{a} = \dfrac{-8}{3}$ AND THE PRODUCT OF THE ROOTS IS $\dfrac{c}{a} = \dfrac{5}{3}$

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Exercise 2.7

1 DETERMINE THE SUM OF THE ROOTS OF THE FOLLOWING EQUATIONS WITHOUT SOLVING THEM.

A $x^2 - 9x + 1 = 0$ B $4x^2 + 11x - 4 = 0$ C $-3x^2 - 9x - 16 = 0$

2 DETERMINE THE PRODUCT OF THE ROOTS OF THE FOLLOWING EQUATIONS WITHOUT SOLVING THEM.

A $-x^2 + 2x + 9 = 0$ B $2x^2 + 7x - 3 = 0$ C $-3x^2 + 8x + 1 = 0$

3 IF THE SUM OF THE ROOTS OF THE EQUATION $x^2 - kx + 1 = 0$ IS 7, THEN WHAT IS THE VALUE OF k?

4 IF THE PRODUCT OF THE ROOTS OF THE EQUATION $kx^2 + 8x + 3 = 0$ IS 1, THEN WHAT IS THE VALUE OF k?

5 IF ONE OF THE ROOTS OF THE EQUATION $x^2 - 4x + k = 0$ EXCEEDS THE OTHER BY 2, THEN FIND THE ROOTS AND DETERMINE THE VALUE OF k.

6 DETERMINE THE VALUE OF k SO THAT THE EQUATION $x^2 + kx + k - 1 = 0$ HAS EXACTLY ONE REAL ROOT.

Word problems leading to quadratic equations

QUADRATIC EQUATIONS CAN BE SUCCESSFULLY USED FOR SOLVING A NUMBER OF PROBLEMS RELATED TO OUR DAY-TO-DAY ACTIVITIES.

The following working rule could be useful in solving such problems.

Step 1 READ THE GIVEN PROBLEM CAREFULLY AND IDENTIFY THE UNKNOWN QUANTITY.

Step 2 DEFINE THE UNKNOWN QUANTITY AS THE VARIABLE x (SAY).

Step 3 USING THE VARIABLE x, TRANSLATE THE GIVEN PROBLEM INTO A MATHEMATICAL STATEMENT, I.E., A QUADRATIC EQUATION.

Step 4 SOLVE THE QUADRATIC EQUATION THUS FORMED.

Step 5 INTERPRET THE SOLUTION OF THE QUADRATIC EQUATION, I.E., TRANSLATE THE RESULT INTO THE LANGUAGE OF THE GIVEN PROBLEM.

Remark:

I AT TIMES IT MAY HAPPEN THAT, OUT OF THE TWO ROOTS OF THE QUADRATIC EQUATION, ONLY ONE HAS A MEANING FOR THE PROBLEM. IN SUCH CASES, THE OTHER ROOT, WHICH DOES NOT SATISFY THE CONDITIONS OF THE GIVEN PROBLEM, MUST BE REJECTED.

II IN CASE THERE IS A PROBLEM INVOLVING TWO OR MORE UNKNOWN QUANTITIES, WE DEFINE ONLY ONE OF THEM AS THE VARIABLE x. THE REMAINING ONES CAN ALWAYS BE EXPRESSED IN TERMS OF x, USING THE CONDITION(S) GIVEN IN THE PROBLEM.

EXAMPLE 17 THE SUM OF TWO NUMBERS IS 11 AND THEIR PRODUCT IS 28. FIND THE NUMBERS.

SOLUTION: LET x AND y BE THE NUMBERS.

YOU ARE GIVEN TWO CONDITIONS, $x + y = 11$ AND $xy = 28$

FROM $xy = 28$ YOU CAN EXPRESS y IN TERMS OF x, GIVING $y = \frac{28}{x}$

REPLACE $y = \frac{28}{x}$ IN $x + y = 11$ TO GET $x + \frac{28}{x} = 11$

NOW PROCEED TO SOLVE FOR x FROM $x + \frac{28}{x} = 11$ WHICH BECOMES

$$\frac{x^2 + 28}{x} = 11$$

$$x^2 + 28 = 11x$$

$x^2 - 11x + 28 = 0$, WHICH IS A QUADRATIC EQUATION.

THEN SOLVING THIS QUADRATIC EQUATION, YOU GET $x = 4$ OR $x = 7$.

IF $x = 4$ THEN FROM $x + y = 11$ YOU GET $4 + y = 11 \Rightarrow y = 7$

IF $x = 7$ THEN FROM $x + y = 11$ YOU GET $7 + y = 11 \Rightarrow y = 4$

THEREFORE, THE NUMBERS ARE 4 AND 7.

EXAMPLE 18 TWO DIFFERENT SQUARES HAVE A TOTAL AREA OF 274 CM^2 AND THE SUM OF THEIR PERIMETERS IS 88 CM. FIND THE LENGTHS OF THE SIDES OF THE SQUARES.

SOLUTION: LET THE SQUARES BE AS GIVEN BELOW.

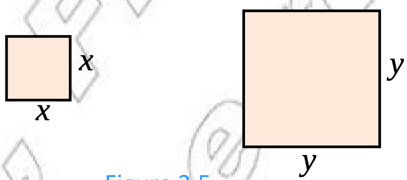

Figure 2.5

RECALL, THE AREA OF THE SMALLER SQUARE IS x^2 AND AREA OF THE BIGGER SQUARE IS y^2.

THE PERIMETER OF THE SMALLER SQUARE IS $4x$ AND THAT OF THE BIGGER SQUARE IS $4y$.

SO THE TOTAL AREA IS $x^2 + y^2 = 274$ AND THE SUM OF THEIR PERIMETERS IS $4x + 4y = 88$.

FROM $4x + 4y = 88$ YOU SOLVE FOR y AND GET $y = 22 - x$.

SUBSTITUTE $y = 22 - x$ IN $x^2 + y^2 = 274$ AND GET $x^2 + (22 - x)^2 = 274$.

THIS EQUATION IS $x^2 + 484 - 44x + x^2 = 274$ WHICH BECOMES THE QUADRATIC EQUATION $2x^2 - 44x + 210 = 0$.

SOLVING THIS QUADRATIC EQUATION, YOU GET $x = 15$ OR $x = 7$.

THEREFORE, THE SIDE OF THE SMALLER SQUARE IS 7 CM AND THE SIDE OF THE BIG IS 15 CM.

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Exercise 2.8

1 THE AREA OF A RECTAN CM^2. IF ONE SIDE EXCEEDS THE OTHER BY, FIND THE DIMENSIONS OF THE RE

2 THE PERIMETER OF AN EQUILATER NUMERICALLY EQUAL TO ITS AREA. F LENGTH OF THE SIDE OF THE EQUILATERA

3 DIVIDE 29 INTO TWO PARTS SO THAT THE SUM OF THE SQUAR FIND THE VALUE OF EACH F

4 THE SUM OF THE SQUARES OF TWO CONSECUTIVE IS 313. FIND TI NUMBERS.

5 A PIECE OF CLOTH COSTS BIRR 200. I WAS 5 M LONGER, AND THE COST METRE OF CLOTH WAS BIRR 2 LESS, THE COST OF THE PIECE W UNCHANGED. HOW LONG IS THE PIECE AND WHAT IS ITS OTRE?

6 BIRR 6,500 WERE DIVIDED EQUALLY AMONG A CERTAIN NUMBER C BEEN 15 MORE PERSONS, EACH WOULD HAVE GOT BIRR 30 LESS. FIND OF PERSONS.

7 A PERSON ON TOUR HAS BIRR 360 FOR HIS DAILY EXPENSES. IF FOR 4 DAYS, HE HAS TO CUT DOWN HIS DAILY EXPENSE BY BIRR 3. FIND TH THE TOUR.

8 IN A FLIGHT OF 600 KM, AN AIRCRAFT WAS SLOWED DOWN DUE TO BA SPEED FOR THE TRIP WAS REDUCED TO 200 KM/HR AND THE 0 MINUTES. FIND THE DURATION OF T

9 AN EXPRESS TRAIN MAKES A RUN OF 240 KM AT A CERTAIN SPEED. A SPEED IS 12 KM/HR LESS TAKES AN HOUR LONGER TO COVER THE SA SPEED OF THE EXPRESS TRAI

Key Terms

absolute value	exponents	quadratic equations
completing the square	factorization	quadratic formula
discriminant	graphical method	radicals
elimination method	linear equations	substitution method

Summary

1 EQUATIONS ARE EQUALITY OF

2 FOR $a > 0$, $a^x = a^y$, IF AND ONLY $x = y$.

3 AN EQUATION OF T $cx + dy = e$, WHERE c AND d ARE ARBITRARY CONS $d \neq 0$, $c \neq 0$ IS CALLED linear equation IN TWO VARIABLES AND ITS SOLUTION (INFINITE POINTS)

4 A SYSTEM OF LINEAR EQUATIONS IS A SET OF TWO OR MORE LINEAR OF TWO LINEAR EQUATIONS IN TWO VARIABLES CAN BE REPRESENTE

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

5 A SOLUTION TO A SYSTEM OF LINEAR EQUATION IN TWO VARIABLES PAIRS (x, y) THAT SATISFY BOTH THE LINE

A $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ IMPLIES THE SYSTEM HAS INFINIT

B $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ IMPLIES THE SYSTEM HAS NO

C $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ IMPLIES THE SYSTEM HAS ON

6 GEOMETRICALLY

A IF TWO LINES INTERSECT AT ONE POINT, THE SYSTEM H

B IF TWO LINES ARE PARALLEL, AND NEVER INTERSES NOT HAVE A SOLUTION.

C IF THE TWO LINES COINCIDE (FIT ONE HER), THE SYSTEM HAS SOLUTIONS.

7 A SYSTEM OF LINEAR EQUATION IN TWO VARIABLE ANY OF THE FOLLC WAYS: graphically, BY substitution OR BY elimination.

8 FOR ANY REAL NU x, $|x| = |-x|$.

9 FOR ANY REAL NU x, $|x|$ IS ALWAYS NON-NEGATIVE.

10 FOR ANY NON-NEGATIVE NU a $(a \geq 0)$; $|x| = a$ MEANS $x = a$ OR $x = -a$.

11 FOR ANY NON-NEGATIVE NU a $(a \geq 0)$; $|x| = |a|$ MEANS $x = a$ OR $x = -a$.

12 FOR REAL NUM a, b AND c, ANY EQUATION THAT CAN BE REDUC $ax^2 + bx + c$, WHER $a \neq 0$ IS CALLED A quadratic equation.

13 WRITING AN EXPRESSION AS A PRODUCT OF ITS SIMPLEST factorizing.

14 FOR REAL NUMBERS a, b AND c, TO SOLVE $ax^2 + bx + c$, WHERE $a \neq 0$, THE FOLLOWING METHODS CAN BE USED: factorization, completing the square, OR THE quadratic formula.

15 IF THE ROOTS OF $ax^2 + bx + c$ ARE $x_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ AND $x_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

THEN $x_1 + x_2 = -\dfrac{b}{a}$ AND $x_1 \times x_2 = \dfrac{c}{a}$.

Review Exercises on Unit 2

1 SOLVE EACH OF THE FOLLOWING.

A $(x - 3)^3 = 27$ **B** $(2x + 1)^2 = 16$ **C** $9^{3x} = 81$

D $\sqrt[3]{(2x)^3} = 14$ **E** $(x - 3)^3 = 27(2x - 1)^3$

2 SOLVE EACH OF THE FOLLOWING LINEAR EQUATIONS.

A $2(3x - 2) = 3 - x$ **B** $4(3 - 2x) = 2(3x - 2)$

C $(3x - 2) - 3(2x + 1) = 4(4x - 3)$ **D** $4 - 3x = 2\left(1 - \dfrac{3}{2}x\right)$

E $2(1 - 4x) = -4\left(-\dfrac{1}{2} + 2x\right)$

3 WITHOUT SOLVING, DETERMINE THE NUMBER OF SOLUTIONS TO EACH OF THE FOLLOWING SYSTEMS OF LINEAR EQUATIONS.

A $\begin{cases} 3x - 4y = 5 \\ 2x + 3y = 3 \end{cases}$ **B** $\begin{cases} 6x + 9y = 7 \\ 2x + 3y = 13 \end{cases}$ **C** $\begin{cases} -x + 4y = 7 \\ 2x - 8y = -14 \end{cases}$

4 APPLYING ALL THE METHODS FOR SOLVING SYSTEMS OF LINEAR EQUATIONS, SOLVE EACH OF THE FOLLOWING.

A $\begin{cases} -2x - 3y = 5 \\ 2x + 3y = -5 \end{cases}$ **B** $\begin{cases} \dfrac{3}{2}x = 5 - 2y \\ x - 3y = 5 \end{cases}$ **C** $\begin{cases} 0.3x - 0.4y = 1 \\ 0.2x + y = 3 \end{cases}$

5 SOLVE EACH OF THE FOLLOWING EQUATIONS THAT INVOLVE ABSOLUTE VALUE.

A $|2x - 3| = 3$ **B** $3|x - 1| = 7$ **C** $\left|\dfrac{1}{2} - 3x\right| = \dfrac{7}{2}$

D $|x + 7| = -1$ **E** $|2 - 0.2x| = 5$ **F** $|2x - 3| = 3|1 - 2x|$

G $|x - 5| = |3 + 2x|$ **H** $|2x - 4| = 2|2 - x|$ **I** $|x + 12| - 2|3x - 1| = 0$

J $|5x - 12| + |x + 2| = 8$ **K** $3|x - 7| + 2|1 - 3x| = 5$

6 FACTORIZE THE FOLLOWING EXPRESSIONS.

A $x^2 - 16x$ **B** $4x^2 + 16x + 12$ **C** $1 - 4x^2$

D $12x + 48x^2$ **E** $x^2 + 11x - 42$

7 SOLVE THE FOLLOWING QUADRATIC EQUATIONS.

A $x^2 - 16x = -64$ **B** $2x^2 + 8x - 8 = 0$

C $4x - 3x^2 - 9 = 10x$ **D** $x^2 + 15x + 31 = 2x - 11$

E $7x^2 + x - 5 = 0$

8 BY COMPUTING THE DISCRIMINANT $b^2 - 4ac$ FOR EACH OF THE FOLLOWING, DETERMINE HOW MANY SOLUTIONS THE EQUATION HAS.

A $x^2 - 16x + 24 = 0$ **B** $2x^2 + 8x - 12 = 0$

C $-4x^2 - x - 2 = 0$ **D** $3x^2 - 6x + 3 = 0$

9 IF TWO ROOTS OF A QUADRATIC EQUATION ARE –2 AND 3, DETERMINE THE QUADRATIC EQUATION.

10 IF THE SUM OF TWO NUMBERS IS 13 AND THEIR PRODUCT IS 42, DETERMINE THE NUMBERS.

11 ALMAZ HAS TAKEN TWO TESTS. HER AVERAGE SCORE IS 7 (OUT OF 10) AND THE PRODUCT OF HER SCORES IS 45. WHAT DID SHE SCORE IN EACH TEST?

12 IF a AND b ARE ROOTS OF $x^2 - 3x + 2 = 0$, THEN FIND

A $a + b$ **B** ab **C** $\frac{1}{a} + \frac{1}{b}$

D $\frac{1}{a+2} + \frac{1}{b+2}$ **E** $a^2 + b^2$ **F** $a^3 - b^3$

13 DETERMINE THE VALUES OF p AND q FOR WHICH (–4, –3) WILL BE SOLUTION OF THE SYSTEM

$$\begin{cases} px + qy = -26 \\ qx - py = 7 \end{cases}.$$

14 AN OBJECT IS THROWN VERTICALLY UPWARD FROM A CLIFF h_o FT HIGH WITH AN INITIAL SPEED OF v_o FT/SEC. ITS HEIGHT h (IN FEET) AFTER t SECONDS IS GIVEN BY $h = -16t^2 + v_o t + h_o$. GIVEN THIS, IF IT IS THROWN VERTICALLY UPWARD FROM THE G WITH AN INITIAL SPEED OF 64 FT/SEC,

A AT WHAT TIME WILL THE HEIGHT OF THE BALL BE 15 FT? (TWO ANSWERS)

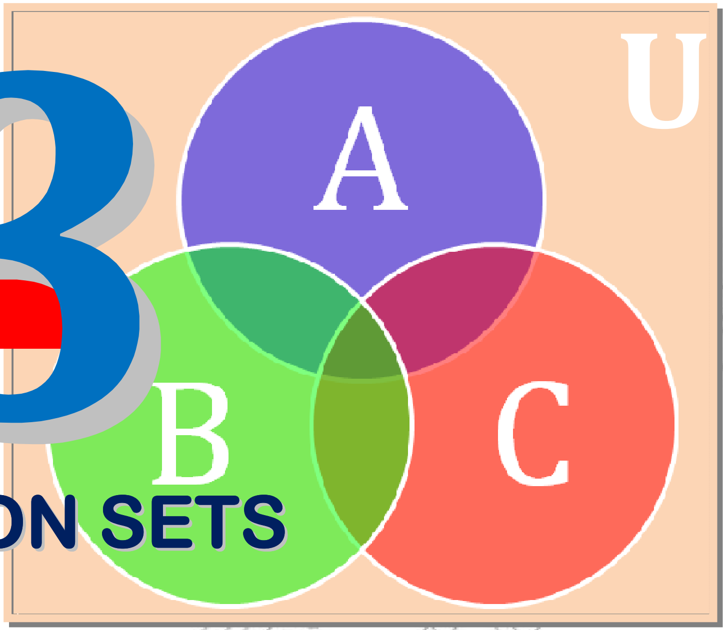
B HOW LONG WILL IT TAKE FOR THE BALL TO REACH 63 FT?

15 DETERMINE THE VALUE OF k SO THAT THE QUADRATIC EQUATION $x^2 - 2x + k^2 - 2k + 1 = 0$ CAN HAVE EXACTLY ONE SOLUTION.

16 THE SPEED OF A BOAT IN STILL WATER IS 15 KM/HR. IT NEEDS FOUR MORE HOURS TO TRAVEL 63 KM AGAINST THE CURRENT OF A RIVER THAN IT NEEDS TO TRAVEL DOWN DETERMINE THE SPEED OF THE CURRENT OF THE RIVER.

Unit 3

FURTHER ON SETS



Unit Outcomes:

After completing this unit, you should be able to:

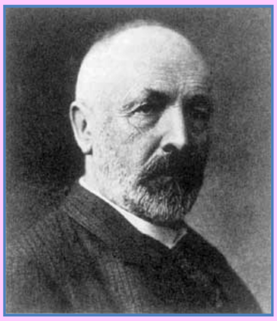
- *understand additional facts and principles about sets.*
- *apply rules of operations on sets and find the result.*
- *demonstrate correct usage of Venn diagrams in set operations.*
- *apply rules and principles of set theory to practical situations.*

Main Contents

INTRODUCTION

IN THE PRESENT UNIT, YOU WILL LEARN MORE ABOUT SETS. PARTICULARLY, YOU WILL DISCUSS THE DIFFERENT WAYS TO DESCRIBE SETS AND THEIR REPRESENTATION THROUGH VENN DIAGRAMS. ALSO, YOU WILL DISCUSS SOME OPERATIONS ON SETS THAT, WHEN PERFORMED ON TWO OR MORE SETS, GIVE RISE TO AN SET. FINALLY, YOU WILL GO THROUGH SOME PRACTICAL PROBLEMS RELATED TO SETS AND TRY TO SOLVE THEM, USING THE NOTION OF UNION AND INTERSECTION OF SETS.

HISTORICAL NOTE:

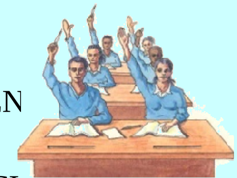


George Cantor (1845-1918)

During the latter part of the 19th century, while working with mathematical entities called infinite series, George Cantor found it helpful to borrow a word from common usage to describe a mathematical idea. The word he borrowed was set. Born in Russia, Cantor moved to Germany at the age of 11 and lived there for the rest of his life. He is known today as the originator of set theory.

3.1 WAYS TO DESCRIBE SETS

ACTIVITY 3.1



1. WHAT IS A SET? WHAT DO WE MEAN WHEN WE SAY AN ELEMENT BELONGS TO A SET?
2. GIVE TWO MEMBERS OR ELEMENTS THAT BELONG TO EACH OF THE FOLLOWING SETS
   - A THE SET OF COMPOSITE NUMBERS LESS THAN 10.
   - B THE SET OF NATURAL NUMBERS LESS THAN 50 AND DIVISIBLE BY 5.
   - C THE SET OF WHOLE NUMBERS BETWEEN ...
   - D THE SET OF REAL NUMBERS BETWEEN ...
   - E THE SET OF NEGATIVE INTEGERS.
   - F THE SET OF INTEGERS THAT SATISFY $(x-2)(2x+1) = 2x^2 - 3x - 2$.
3. A DESCRIBE EACH OF THE SETS IN QUESTION 2 BY ANOTHER METHOD.
   - B STATE THE NUMBER OF ELEMENTS THAT BELONG TO EACH SET IN QUESTION 2.
   - C IN HOW MANY WAYS CAN YOU DESCRIBE THE SETS IN QUESTION 2?
4. WHICH OF THE SETS IN QUESTION 2 HAVE
   - A NO ELEMENTS? B A FINITE NUMBER OF ELEMENTS?
   - C INFINITELY MANY ELEMENTS?

3.1.1 Sets and Elements

Set: A SET IS A well-defined COLLECTION OF OBJECTS.

WHEN WE SAY THAT A SET IS WELL-DEFINED, WE MEAN THAT, GIVEN AN OBJECT, WE ARE ABLE TO DETERMINE WHETHER THE OBJECT IS IN THE SET OR NOT. FOR INSTANCE, "*the collection of all intelligent people in Africa*" CANNOT FORM A WELL-DEFINED SET, SINCE WE MAY NOT AGREE ON WHO IS AN "*intelligent person*" AND WHO IS NOT.

THE INDIVIDUAL OBJECTS IN A SET ARE CALLED THE elements OR THE members. REPEATING ELEMENTS IN A SET DOES NOT ADD NEW ELEMENTS TO THE SET.

FOR EXAMPLE, THE SET $\{a, a\}$ IS THE SAME AS $\{a\}$.

Notation: GENERALLY, WE USE CAPITAL LETTERS TO NAME SETS AND SMALL LETTERS TO REPRESENT ELEMENTS. THE SYMBOL $\in$ STANDS FOR THE PHRASE 'IS AN ELEMENT OF' (OR 'BELONGS TO'). SO, $x \in A$ IS READ AS 'x IS AN ELEMENT OF A' OR 'x BELONGS TO A'. WE WRITE THE STATEMENT 'x DOES NOT BELONG TO A' AS $x \notin A$.

SINCE THE PHRASE '*the set of*' OCCURS SO OFTEN, WE USE THE SYMBOL BRACE (OR CURLY BRACKET) $\{\ \}$.

FOR INSTANCE, '*the set of all vowels in the English alphabet*' IS WRITTEN AS

{ALL VOWELS IN THE ENGLISH ALPHABET} OR $\{a, e, i, o, u\}$.

3.1.2 Description of Sets

A SET MAY BE DESCRIBED BY THREE METHODS:

I Verbal method

WE MAY DESCRIBE A SET IN WORDS. FOR INSTANCE,

- **A** THE SET OF ALL WHOLE NUMBERS LESS THAN TEN OR {WHOLE NUMBERS LESS THAN TEN}.
- **B** THE SET OF ALL NATURAL NUMBERS. THIS CAN ALSO BE WRITTEN AS {NATURAL NUMBERS}.

II The listing method (ALSO CALLED THE *roster* OR *enumeration method*)

IF THE ELEMENTS OF A SET CAN BE LISTED, THEN WE CAN DESCRIBE THE SET BY LISTING. THE ELEMENTS CAN BE LISTED COMPLETELY OR PARTIALLY AS ILLUSTRATED IN THE EXAMPLE:

EXAMPLE 1 DESCRIBE (EXPRESS) EACH OF THE FOLLOWING SETS USING THE LISTING METHOD.

- **A** THE SET WHOSE ELEMENTS ARE 0, 7, a, 2 AND
- **B** THE SET OF NATURAL NUMBERS LESS THAN 51.
- **C** THE SET OF WHOLE NUMBERS.
- **D** THE SET OF NON-POSITIVE INTEGERS.
- **E** THE SET OF INTEGERS.

SOLUTION:

A FIRST LET US NAME THE SET BY A. THEN WE CAN DESCRIBE THE SET AS

$$A = \{a, 2, 7\}$$

B THE NATURAL NUMBERS LESS THAN 51 ARE 1, 2, 3, . . ., 50. SO, NAMING THE SET WE CAN EXPRESS B BY THE LISTING METHOD AS

$$B = \{ 1, 2, 3, . . ., 50\}$$

THE THREE DOTS AFTER THE ELEMENT 3 (CALLED AN ELLIPSIS) INDICATE ELEMENTS IN THE SET CONTINUE IN THAT MANNER UP TO AND INCLUDING ELEMENT 50.

C NAMING THE SET OF WHOLE NUMBERS BY $\mathbb{W}$, WE CAN DESCRIBE IT AS

$$\mathbb{W} = \{0, 1, 2, 3, . . . \}$$

THE THREE DOTS INDICATE THAT THE ELEMENTS CONTINUE IN THE GIVEN P THERE IS NO LAST OR FINAL ELEMENT.

D IF WE NAME THE SET BY L, THEN WE DESCRIBE THE SET AS

$$L = \{ . . ., -3, -2, -1, 0\}$$

THE THREE DOTS THAT PRECEDE THE NUMBERS INDICATE THAT ELEMENT FROM THE RIGHT TO THE LEFT IN THAT PATTERN AND THERE IS NO BEGINNIN

E YOU KNOW THAT THE SET OF INTEGERS IS DENOTED BY $\mathbb{Z}$ AND IS DESCRIBED BY

$$\mathbb{Z} = \{ . . ., -3, -2, -1, 0, 1, 2, 3, . . .\}$$

WE USE THE **partial listing method**, IF LISTING ALL ELEMENTS OF A SET IS DIFFICULT O IMPOSSIBLE BUT THE ELEMENTS CAN BE INDICATED UNAMBIGUOUSLY BY LISTING A FE

Exercise 3.1

1 DESCRIBE EACH OF THE FOLLOWING SETS USING A VERBAL METHOD:

A $A = \{ 5, 6, 7, 8, 9\}$ **B** $M = \{2, 3, 5, 7, 11, 13\}$

C $G = \{8, 9, 10, . . .\}$ **D** $E = \{1, 3, 5, . . ., 99\}$

2 DESCRIBE EACH OF THE FOLLOWING SETS USING THE LISTING METHOD (IF POSSIBL

A THE SET OF PRIME FACTORS OF 72.

B THE SET OF NATURAL NUMBERS THAT ARE LESS THAN 113 AND DIVISIBLE BY 5

C THE SET OF NON-NEGATIVE INTEGERS.

D THE SET OF RATIONAL NUMBERS BETWEEN $\sqrt{2}$ AND $\sqrt{5}$.

E THE SET OF EVEN NATURAL NUMBERS.

F THE SET OF INTEGERS DIVISIBLE BY 3.

G THE SET OF REAL NUMBERS BETWEEN 1 AND 3.

III The set-builder method (*also known as method of defining property*)

ACTIVITY 3.2

TO EACH DESCRIPTION GIVEN IN COLUMN A, MATCH A SET THAT SA
FROM COLUMN B.

	A		B
1	$2 < x < 10$ AND $x \in \mathbb{N}$	**A**	$\{1, 2, 3, 4, 6, 12\}$
2	$x = 2n$ AND $n \in \mathbb{N}$	**B**	$\{-2\}$
3	$2x + 4 = 0$ AND x IS AN INTEGER	**C**	$\{2, 4, 6, . . .\}$
4	$x \in \mathbb{N}$ AND 12 IS A MULTIPLE OF x	**D**	$\{3, 4, 5, 6, 7, 8, 9\}$

THE ABOVE ACTIVITY LEADS YOU TO THE THIRD USEFUL METHOD OF DESCRIBING SETS, KNOWN AS the set-builder method.

FOR EXAMPLE, A = {3, 4, 5, 6, 7, 8, 9} CAN BE DESCRIBED AS

$$A = \{ x \mid x \in \mathbb{N} \text{ AND } 2 < x < 10\} \text{ OR } A = \{x \mid x \in \mathbb{N} \text{ AND } 2 < x < 10\}$$

Set A | is | the set of | all *x* such that

NOTE THAT "SUCH THAT" MAY BE WRITTEN AS "$x \mid$" OR "$x:$".

HENCE WE READ THE ABOVE AS "SET A IS THE SET OF ALL ELEMENTS x SUCH THAT x IS A NATURAL NUMBER BETWEEN 2 AND 10".

NOTE THAT IN THE ABOVE SET A THE PROPERTIES THAT CHARACTERIZE THE ELEMENTS ARE $x \in \mathbb{N}$ AND $2 < x < 10$.

EXAMPLE 2 EXPRESS EACH OF THE FOLLOWING SETS BY THE SET-BUILDER METHOD:

A $\mathbb{N} = \{1, 2, 3, . . .\}$ **B** A = {REAL NUMBERS BETWEEN

C B = {INTEGERS DIVISIBL **D** THE REAL SOLUTION SET OF $|x - 1| = 2$

SOLUTION:

A $\mathbb{N} = \{x \mid x \in \mathbb{N}\}$

B $A = \{x \mid x \in \mathbb{R} \text{ AND } 0 < x < 1\}$

NOTE THAT THIS SET CAN BE EXPRESSED AS $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$

C $B = \{x \mid x = 3n, \text{ FOR SOME INTEGER } n\}$ OR $B = \{3n \mid n \in \mathbb{Z}\}$

D NAMING THE SET, WE WRITE $S = \{x \mid x \in \mathbb{R} \text{ AND } |x - 1| = 2\}$

Exercise 3.2

1 WHICH OF THE FOLLOWING (ARE WELL DEFINED? JUSTIFY YOUR ANS.

A $\{x \mid x$ IS AN INTERESTIN(B $\{x \mid x$ IS A GOOD STU.

C THE SET OF NATURAL NUMBERS L D $\{y \mid y$ IS A FACTOR O

2 WHICH OF THE FOLLARE TRUE AND WHICH ARE FALSE?

A $2\in \{-1, 0, 1\}$ B $a\notin \{\{a, c\}\}$ C $6\in$ {FACTORS O

3 DESCRIBE EACH OF THE FOLLOV

I THE LISTING MI II THE SET–BUILDER METH(

A THE SET OF LETTERS IN THE WORD

B THE SET REGIONAL STATES IN ETHIOPIA.

C THE SET OF WHOLE NUMBERS BETWE

D THE SET OF EVEN NUMBERS LE:

E THE SET OF STUDENTS IN

F THE SET OF ALL ODD NATUR.

4 DESCRIBE EACH OF THE FOLL BY

I A VERBAL ME. II THE SET–BUILDER METH(

A $\{1, 2, 3,\ldots, 10\}$ B $\{1, 3, 5, 7,\ldots\}$ C $\{5, 10, 15, 20,\ldots\}$

D {TUESDAY, THURS E $\{2, 3, 5, 7, 11,\ldots\}$

3.2 THE NOTION OF SETS

3.2.1 Empty Set, Finite Set, Infinite Set, Subset, Proper Subset

ACTIVITY 3.3

1 HOW MANY ELEMENTEACH OF THE FOLLOWING SETS H

A $A = \{x \mid x$ IS A REAL NUMBER WHOSE SQUARE IS I

B $C = \{x \mid x\in \mathbb{N}$ AND $2x < 11\}$ C $D = \{x \mid x \in \{1, 2, 3\}\}$

D $E = \{x \mid x$ IS AN INTE(E $X = \{2, 4, 6, \ldots\}$

2 COMPARE THE SETS C, D, E AND IN QUESTION 1 ABOVE. WHICH SET IS CONTAINED IN AN?

OBSERVE FROM THE ABOVE ACTIVITY THAT A SET MAY HAVE NO ELEMENTS, A LIMIT ELEMENTS OR AN UNLIMITED NUMBER

A Empty set

Definition 3.1

A set that contains no elements is called an empty set, or null set.

An empty set is denoted by either $\varnothing$ or { }.

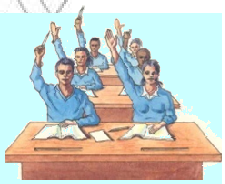
EXAMPLE 1

A IF A = $\{x \mid x$ IS A REAL NUMBER, $x^2 = -1\}$, A = $\varnothing$ (WHY?)

B IF B = $\{x \mid x \neq x\}$, B = $\varnothing$. (WHY?)

B Finite and infinite sets

ACTIVITY 3.4



WHICH OF THE FOLLOWING HAVE A FINITE AND WHICH HAVE NUMBER OF ELEMENTS

1 A = $\{x \mid x \in \mathbb{R}$ AND $0\ x < 3\}$

2 C = $\{x \in \mathbb{N} \mid 7 < x < 7^{100}\}$

3 D = $\{x \in \mathbb{N} \mid x$ IS A MULTIPLE

4 E = $\{x \in \mathbb{Z} \mid 2 < x < 3\}$

5 M = $\{x \in \mathbb{N} \mid x$ IS DIVISIBLE BY 5, $x < 101^4\}$

YOUR OBSERVATIONS FROM THE ABOVE ACTIVITY LEAD TO THE FOLLOWING DEFINITION

Definition 3.2

I A SET S IS CALLED finite, IF IT CONTAINS n ELEMENTS WHERE n IS SOME NON-NEGATIVE INTEGER.

II A SET S IS CALLED infinite, IF IT IS NOT FINITE.

Notation: IF A SET S IS FINITE, THEN WE DENOTE THE NUMBER OF ELEMENTS BY n (S).

EXAMPLE 2 IF S = {–1, 0, 1}, THEN n(S) = 3

USING THIS NOTATION, WE CAN SAY THAT A SET S IS FINITE IF n (S) = 0 OR n (S) IS A NATURAL NUMBER.

EXAMPLE 3 FIND n(S) IF:

A S = $\{x \in \mathbb{R} \mid x^2 = -1\}$ **B** S = $\{x \in \mathbb{N} \mid x$ IS A FACTOR OF 108$\}$

SOLUTION:

A n (S) = 0 **B** n (S) = 12

EXAMPLE 4

A LET E = {2, 4, 6, …}. E IS INFINITE.

B LET T = {x | x IS A REAL NUMBER AND x < 1}. T IS INFINITE.

C Subsets

ACTIVITY 3.5

WHAT IS THE RELATIONSHIP BETWEEN EACH OF THE FOLLOWING PAIRS OF

1 M = {ALL STUDENTS IN YOUR CLASS WHOSE NAMES BEGIN WITH A VOWEL};

N = {ALL STUDENTS IN CLASS WHOSE NAMES BEGIN WITH

2 A = {1, 3, 5, 7}; B = {1, 2, 3, 4, 5, 6, 7, 8}

3 E = {$x \in \mathbb{R}$ | $(x-2)(x-3) = 0$}; F = {$x \in \mathbb{N}$ | $1 < x < 4$}

Definition 3.3

Set A is a subset of set B, denoted by A ⊆ B, if each element of A is an element of B.

Note: IF A IS NOT A SUBSET OF B, THEN WE DENOTE A ⊄ B.

EXAMPLE 5 LET $\mathbb{Z}$ = {x | x IS AN INTEGER}; $\mathbb{Q}$ = {x | x IS A RATIONAL NU

SINCE EACH ELEMENT OF $\mathbb{Z}$ IS ALSO AN ELEMENT OF $\mathbb{Q}$, THEN $\mathbb{Z} \subseteq \mathbb{Q}$

EXAMPLE 6 LET G = {–1, 0, 1, 2, 3} AND H = {0, 1, 2, 3, 4, 5

–1 ∈ G BUT –1 ∉ H, HENCE G ⊄ H.

Note: FOR ANY SET A

I ∅ ⊆ A **II** A ⊆ A

Group Work 3.1

GIVEN A = {a, b, c}

1 LIST ALL THE SUBSE

2 HOW MANY SUBSETS HAVE YO

FROM GROUP WORK 3.1, YOU CAN MAKE THE FOLLOWING DEFINITION.

Definition 3.4

Let A be any set. The power set of A, denoted by P(A), is the set of all subsets of A. That is, $P(A) = \{S \mid S \subseteq A\}$

EXAMPLE 7 LET $M = \{-1, 1\}$. THEN SUBSETS OF M: $\varnothing$, $\{-1\}$, $\{1\}$ AND M.

THEREFORE $P(M) = \{\varnothing, \{-1\}, \{1\}, M\}$

D Proper subset

LET $A = \{-1, 0, 1\}$ AND $B = \{-2, -1, 0, 1\}$. FROM THESE SETS, WE SEE THAT $A \subseteq B$ BUT $B \not\subseteq A$. THIS SUGGESTS THE DEFINITION OF A PROPER SUBSET STATED BELOW.

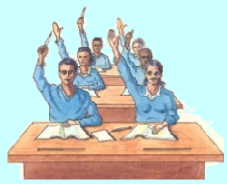
Definition 3.5

Set A is said to be a proper subset of a set B, denoted by $A \subset B$, if A is a subset of B and B is not a subset of A.

THAT IS, $A \subset B$ MEANS $A \subseteq B$ BUT $B \not\subseteq A$.

Note: FOR ANY SET A, A IS NOT A PROPER SUBSET OF ITSELF.

ACTIVITY 3.6



GIVEN $A = \{-1, 0, 1\}$.

i LIST ALL PROPER SUBS

ii HOW MANY PROPER SUB A HAVE YOU FOUND?

YOU WILL NOW INVESTIGATE THE RELATIONSHIP BETWEEN THE NUMBER (AND THE NUMBER OF ITS SUBSETS AND P

ACTIVITY 3.7

1 FIND THE NUMBER OF SUBSETS AND PROPER SUBSET FOLLOWING SETS

A $A = \varnothing$ B $B = \{0\}$ C $C = \{-1, 0\}$ D $D = \{-1, 0, 1\}$

2 COPY AND COMPLETE THE FOLLOWING TABLE:

	Set	No. of elements	Subsets	No. of subsets	Proper subsets	No. of proper subsets
A	∅	0	∅	$1 = 2^0$	-	$0 = 2^0 – 1$
B	{0}	1	∅, {0}	$2 = 2^1$	∅	$1 = 2^1 – 1$
C	{–1, 0}					
D	{–1, 0, 1}		∅, {–1}, {0}, {1}, {–1, 0}, {0, 1}, {–1, 1}, {–1, 0, 1}			$7 = 2^3 – 1$

YOU GENERALIZE THE RESULT OF THE ABOVE ACTIVITY IN THE FORM OF THE FOLLOWING FACT.

Fact: IF A SET A IS FINITE WITH n ELEMENTS, THEN

I THE NUMBER OF SUBSETS OF A IS 2^n AND

II THE NUMBER OF PROPER SUBSETS OF A IS $2^n - 1$

Exercise 3.3

1 FOR EACH SET IN THE LEFT COLUMN, CHOOSE THE SETS FROM THE RIGHT COLUMN
SUBSETS OF IT:

I	$\{a, b, c, d\}$	**A**	{ }
II	$\{o, p, k\}$	**B**	{1, 4, 8, 9}
III	SET OF LETTERS IN THE WOI	**C**	$\{o, k\}$
IV	{2, 4, 6, 8, 10, 12}	**D**	{12}
		E	{6}

2 **A** IF B = {0, 1, 2}, FIND ALL SUBSETS OF B.

B IF B = {0, {1, 2}}, FIND ALL SUBSETS OF B.

3 STATE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE. IF IT IS
YOUR ANSWER.

A $\{1, 4, 3\} \subseteq \{3, 4, 1\}$ **B** $\{1, 3, 1, 2, 3, 2\} \not\subseteq \{1, 2, 3\}$

C $\{4\} \subseteq \{\{4\}\}$ **D** $\varnothing \subseteq \{\{4\}\}$

3.2.2 Venn Diagrams, Universal Sets, Equal and Equivalent Sets

A Venn diagrams

ACTIVITY 3.8

1. WHAT IS THE RELATION BETWEEN THE FOLLOWING PAIRS (

   A $\mathbb{W} = \{0, 1, 2, \ldots\}$ AND $\mathbb{N} = \{1, 2, 3, \ldots\}$.

   B $\mathbb{W} = \{0, 1, 2, \ldots\}$ AND $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$.

   C $\mathbb{N} = \{1, 2, 3, \ldots\}$ AND $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$.

   D $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ AND $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\right\}$.

2. EXPRESS THE RELATION BETWEEN EACH PAIR USING A DIA

3. EXPRESS THE RELATION OF ALL THE SETS $\mathbb{W}$, $\mathbb{N}$, $\mathbb{Z}$ AND $\mathbb{Q}$ USING ONE DIA(

   COMPARE YOUR DIAGRAM WITH THE ACTIVITY 1.1 OF UNIT 1.

TO ILLUSTRATE VARIOUS RELATIONS THAT ARISE BETWEEN SETS, IT IS OFTEN HELPFUL TO USE A PICTORIAL REPRESENTATION CALLED A DIAGRAM NAMED AFTER JOHN VENN (1834 – 1883). THESE DIAGRAMS CONSIST OF RECTANGLES, CLOSED CURVES, USUALLY CIRCLES. THE ELEMENTS OF THE SETS ARE WRITTEN IN THEIR RESPECTIVE CIRCLES.

FOR EXAMPLE, THE RELATION '$A \subset B$' CAN BE ILLUSTRATED BY THE FOLLOWING DIAGRAM.

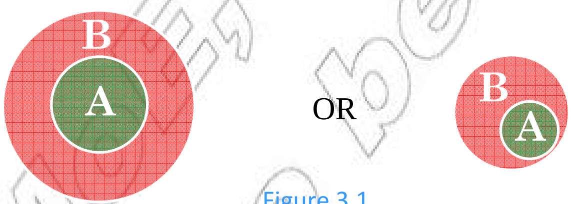


Figure 3.1

EXAMPLE 1 REPRESENT THE FOLLOWING PAIRS OF SETS USING VENN DIAGRAMS:

A $A = \{a, b, c, d\}$; $B = \{a, d\}$

B $C = \{2, 4, 6, 8, \ldots\}$; $D = \{1, 3, 5, 7, \ldots\}$

C $E = \{2^n \mid n \in \mathbb{N}\}$; $F = \{2n \mid n \in \mathbb{N}\}$

D $A = \{1, 3, 5, 7, 9\}$; $B = \{2, 3, 5, 8\}$; $C = \{1, 5, 7\}$

SOLUTION:

A

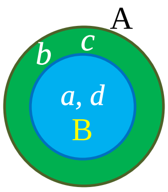


Figure 3.2

B

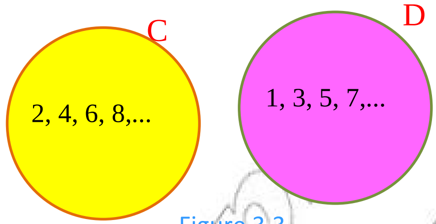


Figure 3.3

C

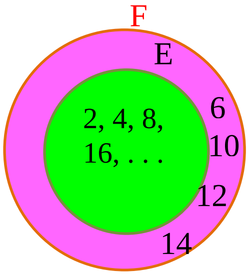


Figure 3.4

D

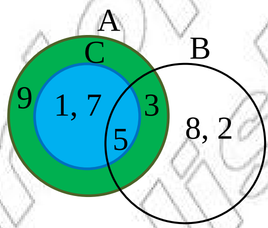


Figure 3.5

B Universal set

SUPPOSE AT A SCHOOL ASSEMBLY, THE FOLLOWING STUDENTS ARE ASKED TO STAY BEF

G = {ALL GRADE 9 STUDENTS}.

I = {ALL STUDENTS INTERESTED IN A SCHOOL PLAY}.

R = {ALL CLASS REPRESENTATIVES OF EACH CLASS}.

EACH SET G, I AND R IS A SUBSET OF S = {ALL STUDENTS IN THE SCHOOL}

IN THIS PARTICULAR EXAMPLE, S IS CALLED THE universal set.

SIMILARLY, A DISCUSSION IS LIMITED TO A FIXED SET OF OBJECTS AND IF ALL THE EL DISCUSSED ARE CONTAINED IN THIS SET, THEN THIS "OVERALL" SET IS CALLED THE universal set. WE USUALLY DENOTE THE UNIVERSAL SET BY U. DIFFERENT PEOPLE MAY CHOOS UNIVERSAL SETS FOR THE SAME PROBLEM.

EXAMPLE 2 LET R = {ALL RED COLOURED CARS IN EAST AFRICA}, T = {ALL TOYOTA CARS IN EAST AFRICA}

I CHOOSE A UNIVERSAL SET FOR R AND T.

II DRAW A VENN DIAGRAM TO REPRESENT THE SETS U, R AND T.

SOLUTION:

I THERE ARE DIFFERENT POSSIBILITIES FOR U. TWO OF THESE ARE:

U = {ALL CARS} OR U = {ALL WHEELED VEHICLES}

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II IN BOTH CASES, THE VENN DIAGRAM OF THE SETS U, R AND T IS

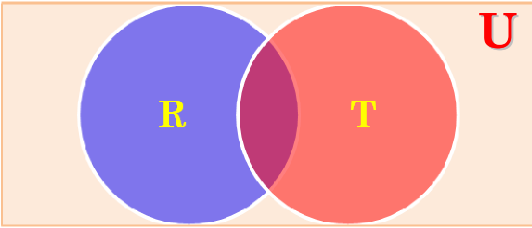

Figure 3.6

Exercise 3.4

1 DRAW VENN DIAGRAMS TO ILLUSTRATE THE RELATIONSHIPS BETWEEN THE FOLLOWING SETS:

A A = {1, 9, 2, 7, 4}; L = {4, 9, 8, 2}

B B = {THE VOWELS IN ENGLISH ALPHABET}

M = {THE FIRST LETTERS OF THE ENGLISH ALPHAB

C C = {1, 2, 3, 4, 5}; M = {6, 9, 10, 8, 7}

D F = {3, 7, 11, 5, 9}; O = {ALL ODD NUMBERS BETWEEN

2 FOR EACH OF THE FO, DRAW A VENN DIAGRAM TO ILLUSTRATE THE RELATIONSHIP BETWEEN THE SE

A U = {ALL ANIMAL; C = {ALL COWS}; G = {ALL GO

B U = {ALL PEOP; M = {ALL MALES}; B = {ALL BO

C Equal and equivalent sets

ACTIVITY 3.9

FROM THE FOLLOWING PAIR IDENTIFY THOSE:

1 THAT ARE THE SAME NUMBER OF

2 THAT HAVE EXACTLY THE SAM

A A = {1, 2}; B = $\{x \in \mathbb{N} \mid x < 3\}$

B E = {–1, 3}; F = $\left\{\frac{1}{2}, \frac{1}{3}\right\}$

C R = {1, 2, 3}; S = $\{a, b, c\}$

D G = $\{x \in \mathbb{N} \mid x$ IS A FACTOR ; H = $\{x \in \mathbb{N} \mid$ 6 IS A MULTIPL $x\}$

E X = {1, 1, 3, 2, 3, 1}; Y = {1, 2, 3}

I Equality of sets

LET US INVESTIGATE THE RELATIONSHIP BETWEEN THE FOLLOWING TWO SETS;

$$E = \{x \in \mathbb{R} \mid (x-2)(x-3) = 0\} \text{ AND } F = \{x \in \mathbb{N} \mid 1 < x < 4\}.$$

BY LISTING COMPLETELY THE ELEMENTS OF EACH SET, WE HAVE E = {2, 3} AND F = {2, 3

WE SEE THAT E AND F HAVE EXACTLY THE SAME ELEMENTS. SO THEY ARE EQUAL.

IS $E \subseteq F$? Is $F \subseteq E$?

Definition 3.6

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. We write this as A = B.

$\therefore$ A = B, if and only if $A \subseteq B$ and $B \subseteq A$.

EXAMPLE 3 LET A = {1, 2, 3, 4} AND B = {1, 4, 2, 3}.

A = B, SINCE THESE SETS CONTAIN EXACTLY THE SAME ELEMENTS.

Note: IF A AND B ARE NOT EQUAL, WE WRITE $A \neq B$.

EXAMPLE 4 LET C = {–1, 3, 1} AND D = {–1, 0, 1, 2}.

$C \neq D$, BECAUSE $2 \in D$, BUT $2 \notin C$.

II Equivalence of sets

CONSIDER THE SETS A = {*a*, *b*, *c*} AND B = {2, 3, 4}. EVEN THOUGH THESE TWO SETS ARE NOT EQUAL, THEY HAVE THE SAME NUMBER OF ELEMENTS. SO, FOR EACH MEMBER OF SE FIND A PARTNER IN SET A.

A = {*a*, *b*, *c*}

B = {2, 3, 4}

THE DOUBLE ARROW SHOWS HOW EACH ELEMENT OF A SET IS MATCHED WITH AN ANOTHER SET. THIS MATCHING COULD BE DONE IN DIFFERENT WAYS, FOR EXAMPLE:

A = {*a*, *b*, *c*}

B = {2, 3, 4}

NO MATTER WHICH WAY WE MATCH THE SETS, EACH ELEMENT OF A IS MATCHED WITH ELEMENT OF B AND EACH ELEMENT OF B IS MATCHED WITH EXACTLY ONE ELEMENT C THAT THERE IS A ONE–TO–ONE CORRESPONDENCE BETWEEN A AND B.

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Definition 3.7

Two sets A and B are said to be equivalent, written as A↔B (or A~B), if there is a one-to-one correspondence between them.

Observe that two finite sets A and B are equivalent, if and only if

n (A) = n (B)

EXAMPLE 5 LET A = {$\sqrt{2}$, e, } AND B = {1, 2, 3}.

SINCE n(A) = n(B), A AND B ARE EQUIVALENT SETS AND WE WRITE

A ↔ B.

NOTE THAT EQUAL SETS ARE ALWAYS EQUIVALENT SINCE EACH ELEMENT CAN BE ITSELF, BUT EQUIVALENT SETS ARE NOT NECESSARILY EQUAL. FOR EXAMPLE,

{1, 2} ↔ {a, b} BUT {1, 2} ≠ {a, b}.

Exercise 3.5

WHICH OF THE FOLLOWING PAIRS REPRESENT EQUAL SETS AND WHICH OF THEM REPRE EQUIVALENT SETS?

1. {a, b} AND {2, 4}
2. {∅} AND ∅
3. {$x \in \mathbb{N} \mid x < 5$} AND {2, 3, 4, 5}
4. {1, {2, 4}} AND {1, 2, 4}
5. {$x \mid x < x$} AND {$x \in \mathbb{N} \mid x < 1$}

3.3 OPERATIONS ON SETS

THERE ARE OPERATIONS ON SETS AS THERE ARE OPERATIONS ON NUMBERS. LIKE THE (ADDITION AND MULTIPLICATION ON NUMBERS, INTERSECTION AND UNION ARE OPERAT

3.3.1 Union, Intersection and Difference of Sets

A Union of sets

Definition 3.8

The union of two sets A and B, denoted by A∪B and read "A union B" is the set of all elements that are members of set A or set B or both of the sets. That is, A∪B = {$x \mid x \in A$ or $x \in B$}

THE *red* SHADED REGION OF THE DIAGRAM ON THE RIGHT REPRES $\cup$ B.

AN ELEMENT COMMON TO BOTH SETS IS LISTE THE UNION. FOR EXAMPLE, IF A = $\{a, b, c, d, e\}$ AND B = $\{c, d, e, f, g\}$, THEN

$A\cup B = \{a, b, c, d, e, f, g\}$.

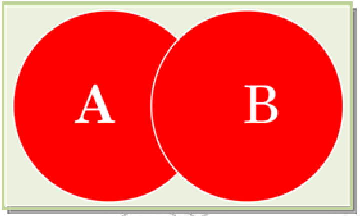


Figure 3.7

EXAMPLE 1

A $\{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\}$

B $\{1, 2, 3, 4, 5\} \cup \varnothing = \{1, 2, 3, 4, 5\}$

Properties of the union of sets

ACTIVITY 3.10

LET A = {1, 2, 3, 4}, B = {2, 4, 6, 8} AND C = {3, 4, 5, 6}.

1 FIND **A** $A\cup B$ **B** $B\cup A$

WHAT IS THE RELATIONSHIP BETWEEN $A\cup B$ AND $B\cup A$?

2 FIND **A** $A\cup B$ **B** $(A\cup B)\cup C$ **C** $B\cup C$ **D** $A\cup(B\cup C)$

WHAT IS THE RELATIONSHIP BETWEEN $(A\cup B)\cup C$ AND $A\cup(B\cup C)$?

3 FIND $A\cup\varnothing$, WHAT IS THE RELATIONSHIP BETWEEN $A\cup\varnothing$ AND A?

THE ABOVE ACTIVITY LEADS YOU TO THE FOLLOWING PROPERTIES:

FOR ANY SETS A, B AND C

1 COMMUTATIVE PRC $A\cup B = B\cup A$

2 ASSOCIATIVE PRC $(A\cup B)\cup C = A\cup(B\cup C)$

3 IDENTITY PROPER $A\cup\varnothing = A$

Exercise 3.6

1 GIVEN A = {1, 2, {3}} B = {2, 3} AND C = {{3}, 4}, FIND:

A $A\cup B$ **B** $B\cup C$ **C** $A\cup C$

D $A\cup(B\cup C)$ **E** $(A\cup B)\cup C$

2 STATE WHETHER EACH OF THE FOLLOWING STATEMENTS:

A IF $x \in A$ AND $x \notin B$, THEN $x \notin (A\cup B)$.

B IF $x \in (A\cup B)$ AND $x \notin A$, THEN $x \in B$.

C IF $x \notin A$ AND $x \notin B$, THEN $x \notin (A\cup B)$.

D FOR ANY SET A, $A\cup A = A$.

E FOR ANY SET A, $A\cup\varnothing = A$.

F IF $A \subseteq B$, THEN $A\cup B = B$.

g FOR ANY TWO SETS A AND B, A ⊆ (A∪B) AND B ⊆ (A∪B).

h FOR ANY THREE SETS A, B AND C, IF A ⊆ B, AND B ⊆ C, THEN A ∪ B = C.

i FOR ANY THREE SETS A, B AND C, IF A∪B = C, THEN B ⊆ C.

j IF A∪B = ∅, THEN A = ∅ AND B = ∅.

3 USING COPIES OF THE VENN DIAGRAMS BELOW, SHADE A∪B.

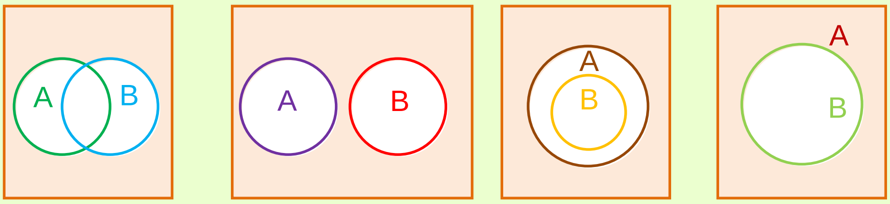


Figure 3.8

B Intersection of sets

ACTIVITY 3.11

CONSIDER THE TWO SETS G = {2, 4, 6, 8, 10, 12} AND H = {1, 2, 3, 4, 5}.

a DRAW A VENN DIAGRAM THAT SHOWS THE RELATIONSHIP BETWEEN THE TWO SETS.

b SHADE THE REGION COMMON TO BOTH SETS AND FIND THEIR COMMON ELEMENTS.

Definition 3.9

The intersection of two sets A and B, denoted by A∩B and read as "*A intersection B*", is the set of all elements common to both set A and set B. That is, A∩B = {x | x∈A and x∈ B}.

USING THE VENN DIAGRAM, A ∩ B IS REPRESENTED BY THE *blue* SHADED REGION:

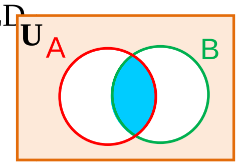


Figure 3.9

Example 2 LET S = {*a*, *b*, *c*, *d*} AND T = {*b*, *d*, *g*}. THEN S ∩ T = {*b*, *d*}.

Example 3 LET V = {2, 4, 6,...} (MULTIPLES OF 2) AND

W = {3, 6, 9,...} (MULTIPLES OF 3).

THEN V ∩ W = {6, 12, 18,...}, THAT IS, MULTIPLES OF 6.

EXAMPLE 4 LET A = {1, 2, 3} AND B = {5 7, 8}, THEN A ∩B =∅.

Definition 3.10

Two or more sets are disjoint if they have no common element.

A and B are disjoint, if and only if A ∩ B = ∅.

IN THE VENN DIAGRAM, THE SETS A AND I

HEREA ∩ B = ∅

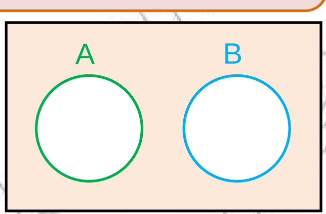


Figure 3.10

Properties of the intersection of sets

ACTIVITY 3.12

LET U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} BETHE UNIVERSAL SET AND I

A = {0, 2, 3, 5, 7}, B = {0, 2, 4, 6, 8} ANI

C = {x | x IS A FACTOR (

1 FIND **A** A∩C **B** C∩A

WHAT IS THE RESHI BETWEENA∩C AND C∩A?

2 FIND **A** A∩B **B** (A∩B) ∩C **C** B∩C **D** A∩(B∩C)

WHAT IS THE RESHI BETWEEN(A∩B) ∩C AND A∩(B∩C)?

3 FIND A∩U. WHAT IS THE RESHIP BETWEENA∩U AND A?

THE ABOVE ACTIVITY LEADS TO THE FOLLOWING PROPERTIES:

FOR ANY SETS A, B AND C AND THE UNIVERSAL SET U

1 COMMUTATIVE PRC A∩B = B∩A.

2 ASSOCIATIVE PROI (A∩B)∩C = A∩(B∩C).

3 IDENTITY PROPEF A∩U = A.

Exercise 3.7

1 GIVEN A = {a, b, {c}}, B = {b, c} AND C = {{c}, d}, FIND:

A A∩B **B** A∩C **C** B∩C **D** A∩(B∩C)

2 STATE WHETHER OF THE FOLLOWING ST IS TRUE OR FAL

A IF x∈ A AND x∉B, THEN x∈ (A∩B). **B** IF x∈ (A∩B), THEN x∈ A AND x∈B.

C IF x∉ A AND x∈B, THEN x∈ (A∩B). **D** FOR ANY SET A, A∩A = A.

E IF A ⊆B, THEN A∩B = A.

F FOR ANY TWO SETS A AND B, $A \cap B \subseteq A$ AND $A \cap B \subseteq B$.

G IF $A \cap B = \varnothing$, THEN $A = \varnothing$ OR $B = \varnothing$.

H IF $(A \cup B) \subseteq A$, then $B \subseteq A$.

I IF $A \subseteq B$, THEN $A \cap B = B$.

J IF $A \subseteq B$, THEN $A \cap B = \varnothing$.

K IF $A \subseteq B$, THEN $B' \subseteq A'$.

3 IN EACH VENN DIAGRAM BELOW, SHADE $(A \cap B) \cap C$.

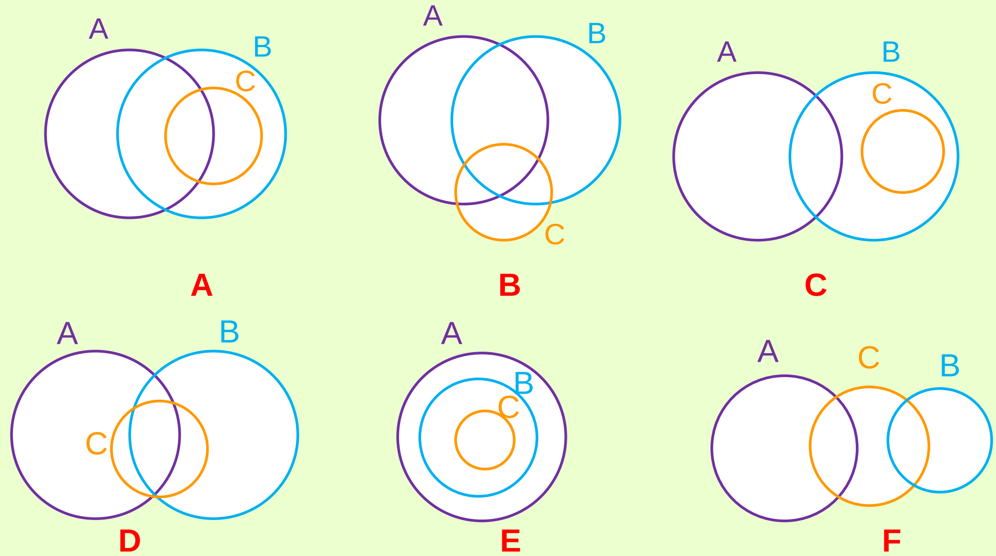


Figure 3.11

C Difference and symmetric difference of sets

I The relative complement (or difference) of two sets

GIVEN TWO SETS A AND B, THE COMPLEMENT OF B RELATIVE TO A (OR THE DIFFERENCE AND B) IS DEFINED AS FOLLOWS.

Definition 3.11

The relative complement of a set B with respect to a set A (or the difference between A and B), denoted by A – B, read as "A difference B", is the set of all elements in A that are not in B.

That is, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Note: A – B IS SOMETIMES DENOTED BY A\B (READ AS 'A LESS B')

A – B AND A\B ARE USED INTERCHANGEABLY.

USING A VENN DIAGRAM, A\B CAN BE REPRESENTED BY SHADING THE REGION IN A WHICH IS N.

A\B IS SHADED IN *light green*

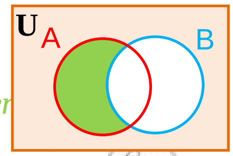


Figure 3.12

EXAMPLE 5 IF A = {*x*, *y*, *z*, *w*} AND B = { *a*, *b*, *x*, *y*}, THEN FIND:

A THE COMPLEMENT OF B REL **B** B\ A **C** B\B

SOLUTION:

A NOTE THAT FINDING "THE COMPLEMENT OF B RELATIVE TO A" I "THE RELATIVE COMPLEMENT OF B WITH RESPEC\B.

SO, A\B = {*z*, *w*}.

B B\A = { *a*, *b*}.

C B\B = ∅.

ACTIVITY 3.13

LET A = {0, 2, 3, 5, 7}, B = {0, 2, 4, 6, 8} AND C = {1, 2, 3, 6}. FIND:

A A\B **B** B\A **C** (A\B)\C **D** A\(B\C)

FROM THE RESULTS OF THE ACTIVITY, WE CAN CONCLUDE THAT THE RELATIVE COMI SETS IS NEITHER COMMUTATIVE NO

II The complement of a set

LET U = {ALL HUMAN BEINGS} AND F = {A

THE VENN DIAGRAM OF THESE TWO SHOWN. TH YELLOW SHADED REGION (U BUT OUTSIDE F) IS CALL complement OF F, DENOTED F'.

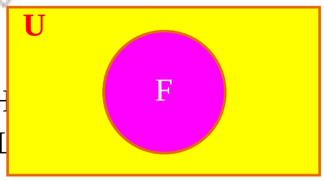


Figure 3.13

IT REPRESENTS ALL HUMAN B ARE NOT FEMALE. THE MEMB ' ARE ALL THOSE MEMBERS OF U THAT ARE NOT ME

Definition 3.12

Let A be a subset of a universal set U. The complement (or absolute complement) of A, denoted by A', is defined to be the set of all elements of U that are not in A.

i.e., A' = {*x* | *x* ∈ U and *x* ∉ A}.

USING A VENN DIAGRAM, WE CAN REPRESENT A' BY THE SHA AS SHOWN IN FIGURE 3.14.

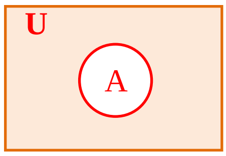


Figure 3.14

NOTE THAT FOR ANY SET A AND UN,

A′ = U\A

EXAMPLE 6 IN COPIES OF THE VENN DIAGRAM ON THE RIGHT, SHADE

A A\B **B** (A∩B)'

C A∩B' **D** A'∪B'

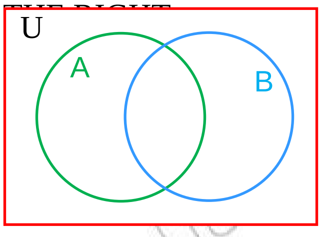


Figure 3.15

SOLUTION:

A SINCE A\B IS THE SET OF ALL ELEMENTS IN A THAT ARE NOT IN B, WE SHADE THE REGION IN A THAT IS NOT PART OF B (*shaded in green in* FIGURE 3.16).

A – B IS SHADED

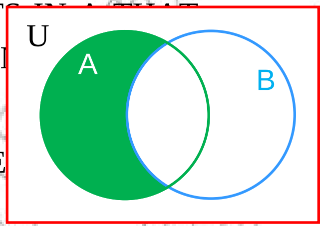
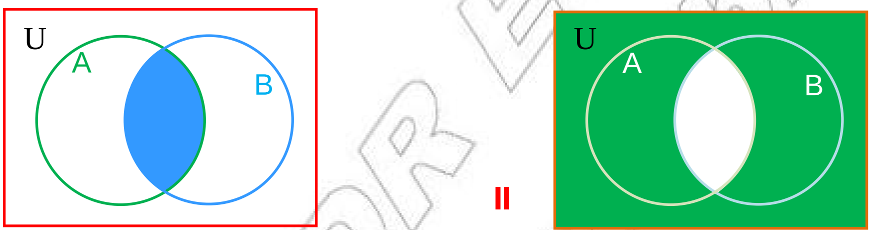


Figure 3.16

B FIRST WE SHADE THE REGION A∩B; THEN (A∩B)' IS THE REGION OUTSIDE A∩B.

I



II

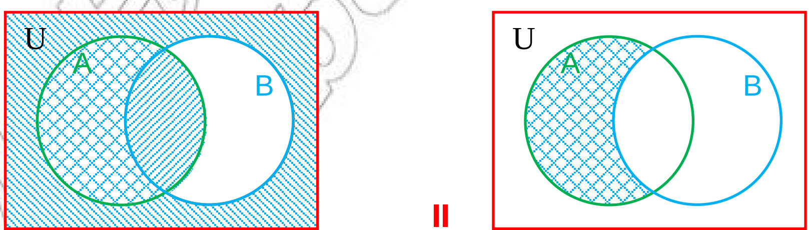
Figure 3.17

A∩B IS THE SHADED (*blue*) REGION. (A∩B)' IS THE *green* SHADED REGION.

C FIRST WE SHADE A WITH STROKES THAT SLANT UPWARD TO THE RIGHT (///) AND THEN SHADE B' WITH STROKES THAT SLANT DOWNWARD TO THE RIGHT (\\\).

THEN A∩B' IS THE CROSS-HATCHED REGION.

I



II

Figure 3.18

A AND B' ARE SHADED A∩B' IS SHADED

NOTE THAT THE REGION OF A\B IS THE SAME AS THE REGION OF A∩B'.

D FIRST WE SHADE A', THE REGION OUTSIDE A, WITH STROKES THAT SLANT UPWARD TO THE RIGHT (///) AND THEN SHADE B' WITH STROKES THAT SLANT DOWNWARD TO THE RIGHT (\\\).

THEN A'∪B' IS THE TOTAL SHADED

I

II

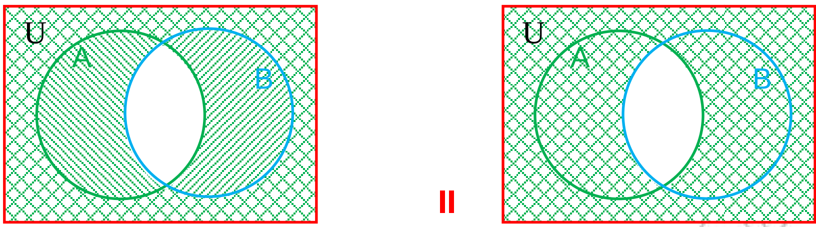


Figure 3.19

A' OR B' ARE SHADED   A'∪B' IS SHADED

NOTE THAT THE REGION (A∩B)' IS THE SAME AS THE REGION A'∪B'.

Note: WHEN WE DRAW TWO OVERLAPPING CIRCLES IN A UNIVERSAL SET, FOUR REGIONS ARE FORMED. EVERY ELEMENT OF THE UNIVERSAL SET U IS IN EXACTLY ONE OF THE FOLLOWING REGIONS.

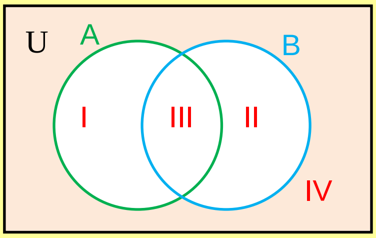


Figure 3.20

I IN A AND NOT IN B (A\B)

II IN B AND NOT IN A (B\A)

III IN BOTH A AND B (A∩B)

IV IN NEITHER A NOR B ((A∪B)')

FROM ACTIVITY 3.13 AND THE ABOVE EXAMPLES, YOU GENERALIZE AS FOLLOWS:

FOR ANY TWO SETS A AND B, THE FOLLOWING PROPERTIES HOLD:

1 A\B = A ∩ B'   **2** (A ∩ B)' = A' ∪ B'   **3** (A\B)∪B = A∪B

ACTIVITY 3.14

1 IN COPIES OF THE SAME VENN DIAGRAM USED IN EXAMPLE 6, SHADE

A (A∪B)'   **B** A'∩B'

2 GENERALIZE THE RESULT YOU GET IN QUESTION 1.

HISTORICAL NOTE:

Augustus De Morgan (1806-1871)

Augustus De Morgan was the first professor of mathematics at University College London and a cofounder of the London Mathematical Society.

De Morgan formulated his laws during his study of symbolic logic. De Morgan's laws have applications in the areas of set theory, mathematical logic and the design of electrical circuits.

Group Work 3.2

1 COPY FIGURE 3.21 AND SHADE THE REGION THAT REPR
OF THE FOLLOWING

A (A∪B)'

B A'∪B'

C (A∩B)'

D A'∩B'

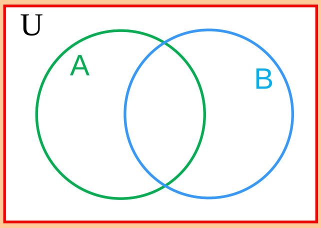


Figure 3.21

2 DISCUSS WHAT YOU HAVE OBSERVED FROM QUESTION 1

THE ABOVE GROUP WORK LEADS YOU TO THE FOLLOWING LAW CALLED De Morgan's law.

Theorem 3.1 **De Morgan's law**

For any two sets A and B

1 $(A\cap B)' = A'\cup B'$ **2** $(A\cup B)' = A'\cap B'$

Exercise 3.8

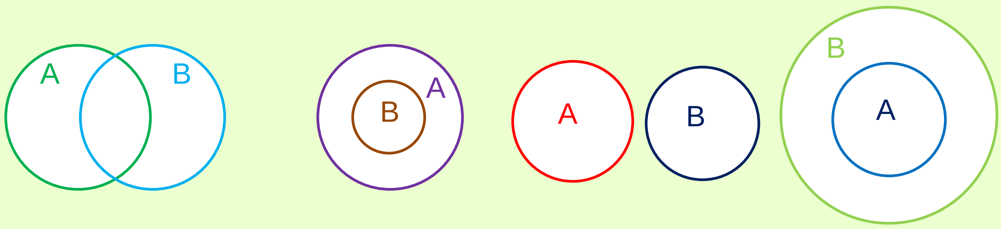
1 GIVEN A = {a, b, c} AND B = {b, c, d, e} FIND:

A THE RELATIVE COMPLEMENT OF A WITH RESPECT TO B.

B THE COMPLEMENT OF B RELATIVE TO A.

C THE COMPLEMENT OF A RELATIVE TO B.

2 IN EACH OF THE VENN DIAGRAMS GIVEN BELOW, SHADE A\ B.



A B C D

Figure 3.22

3 DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE:

A IF $x \in$ A AND $x \notin$ B THEN $x \in$ (B\A)

B IF $x \in$ (A\B) THEN $x \in$ A

C B\A $\subseteq$ B, FOR ANY TWO SETS A AND B

D (A\B) ∩ (A∩B) ∩ (B\A) = ∅, FOR ANY TWO SETS A A

E IF A\B = ∅ THEN A ∅ AND B = ∅

F IF A ⊆ B THEN \B = ∅

G IF A∩B = ∅ THEN (\B) = A

H (A\B) ∪ B = A∪B, FOR ANY TWO SETS A AND B

I A∩A' = ∅

4 LET U = {1, 2, 3, . . ., 8, 9} B THE UNIVERSAL SET AND A = {, B = {2, 4, 6, 8} AND C = { 3, 4, 5, 6} LIST THE ELEME EACH OF THE FOLLO

A A'	**B** B'	**C** (A∪C)'	**D** (A\B)'
E A'∩B'	**F** (A∪B)'	**G** (A')'	**H** B\C

I B∩C'

III The symmetric difference between two sets

ACTIVITY 3.15

LET A = {*a*, *b*, *c*, *d*} AND B = {*b*, *d*, *e*}. THEN FIND:

A A∩B	**B** A∪B	**C** A\B
D B\A	**E** (A∪B)\(A∩B)	**F** (A\B) ∪ (B\A)

COMPARE THE RESULT AND F.

WHAT CAN YOU CONCLU YOUR ANSWERS?

THE RESULT OF THE ACTIVITY LEADS YOU TO STATE THE FOLLOWING

Definition 3.13

Let A and B be any two sets. The symmetric difference between A and B, denoted by AΔB, is the set of all elements in A∪B that are not in A∩B. That is AΔB = {x | x ∈ (A∪B) and x ∉ (A∩B)}

or AΔB = (A∪B)\(A∩B).

USING A VENN DIAGRAM, AΔB IS ILLUSTRATED BY SHADI REGION IN A∪B THAT IS NOT PA A∩B AS SHOWN.

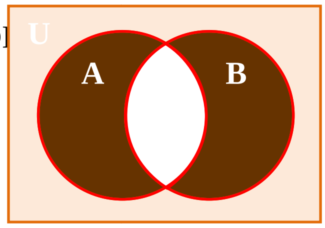


Figure 3.23

AΔB IS THE SHA *dark brown region.*

FROM ACTIVITY 3.15 AND THE ABOVE VENN DIAGRAM, YOU OBSERVE THAT

AΔB = (A∪B)\ (A∩B) = (A\B) ∪ (B\A).

Note: IF A ∩ B = ∅ THEN AΔB = A ∪ B.

EXAMPLE 7 LET $A = \{-1, 0, 1\}$ AND $B = \{1, 2\}$. FIND $A \Delta B$.

SOLUTION: $A \cup B = \{-1, 0, 1, 2\}$; $A \cap B = \{1\}$

$\therefore A \Delta B = (A \cup B) \setminus (A \cap B) = \{-1, 0, 2\}$

EXAMPLE 8 LET $A = \{a, b, c\}$ AND $B = \{d, e\}$. FIND $A \Delta B$.

SOLUTION: $A \cup B = \{a, b, c, d, e\}$; $A \cap B = \varnothing$

$\therefore A \Delta B = (A \cup B) \setminus \varnothing = A \cup B = \{a, b, c, d, e\}$

Distributivity

Group Work 3.3

1 GIVEN SETS A, B AND C, SHADE THE REGION THAT EACH OF THE FOLLOWING

- **A** $A \cup (B \cap C)$
- **B** $(A \cup B) \cap (A \cup C)$
- **C** $A \cap (B \cup C)$
- **D** $(A \cap B) \cup (A \cap C)$

2 DISCUSS WHAT YOU HAVE OB FROM QUESTION 1

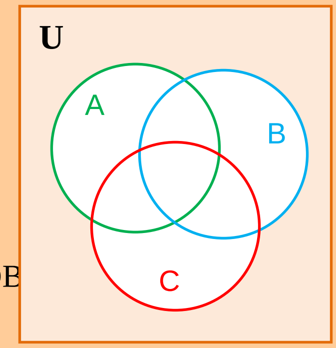


Figure 3.24

AS YOU MAY HAVE NOTICED FROM THE ABOVE GROUP WORK, THE FOLLOWING DISTRIBUTIVE PROPERTIES ARE TRUE:

Distributive properties

FOR ANY SETS A, B AND C

1 UNION IS DISTRIBUTIVE OVER THE INTERSECTION OF SETS.

I.E., $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

2 INTERSECTION IS DISTRIBUTIVE OVER THE UNION OF SETS.

I.E., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 3.9

1 IF $A \cap B = \{1, 0, -1\}$ AND $A \cap C = \{0, -1, 2, 3\}$, THEN FIND $A \cap (B \cup C)$.

2 SIMPLIFY EACH OF THE FOLLOWING BY USING VENN DIAGRAM OR ANY OTHER PRO

- **A** $A \cap (A \cup B)$
- **B** $P' \cap (P \cup Q)$
- **C** $A \cap (A' \cup B)$
- **D** $P \cup (P \cap Q)$

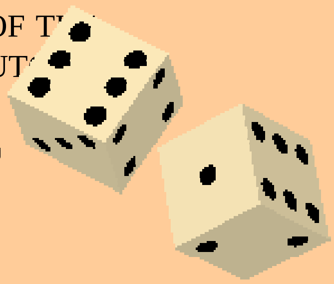
3.3.2 Cartesian Product of Sets

IN THIS SUBSECTION, YOU WILL LEARN HOW TO FORM SETS OF ORDERED PAIRS FROM TWO GIVEN SETS BY TAKING THE CARTESIAN PRODUCT OF THE SETS (NAMED AFTER THE MATHEMATICIAN Rene Descartes).

Group Work 3.4

A SIX-SIDED DIE (A CUBE) HAS FACES MARKED WITH NUMBERS 1,2,3,4,5 AND 6 RESPECTIVELY.

TWO SUCH DICE ARE THROWN AND THE NUMBERS ON THE UPPER FACES ARE RECORDED. FOR EXAMPLE, (6, 1) MEANS THAT THE NUMBER ON THE UPPER FACE OF THE FIRST DIE IS 6 AND THAT OF THE SECOND DIE IS 1. WE CALL THESE ORDERED PAIRS, THE OUTCOMES OF THE THROW OF OUR DICE.



LIST THE SET OF ALL POSSIBLE OUTCOMES OF THROWING TWO DICE SUCH THAT THE TWO NUMBERS:

- **I** A: ARE BOTH EVEN.
- **II** B: ARE BOTH ODD.
- **III** C: ARE EQUAL.
- **IV** D: HAVE A SUM EQUAL TO 7.
- **V** E: HAVE A SUM EQUAL TO 1.
- **VI** F: HAVE AN EVEN SUM.
- **VII** G: HAVE THE FIRST NUMBER 1 AND THE SECOND NUMBER ODD.
- **VIII** H: HAVE A SUM LESS THAN 12.

FOR EXAMPLE, A= {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}.

THE ACTIVITY OF THIS GROUP WORK LEADS YOU TO LEARN ABOUT THE SETS WHOSE ELEMENTS ARE ORDERED PAIRS.

Ordered pair

AN *ordered pair* IS AN ELEMENT (x, y) FORMED BY TAKING x FROM ONE SET AND y FROM ANOTHER SET. IN (x, y), WE SAY THAT x IS THE first ELEMENT AND y IS THE second ELEMENT.

SUCH A PAIR IS ORDERED IN THE SENSE THAT (x, y) AND (y, x) ARE NOT EQUAL UNLESS $x = y$.

Equality of ordered pairs

$(a, b) = (c, d)$, IF AND ONLY IF $a = c$ AND $b = d$.

EARLIER ALSO WE HAVE DISCUSSED ORDERED PAIRS WHEN WE REPRESENT POINTS IN THE CARTESIAN COORDINATE SYSTEM. A POINT P IN THE PLANE CORRESPONDS TO AN ORDERED PAIR (a, b) WHERE a IS THE x-COORDINATE AND b IS THE y-COORDINATE OF THE POINT P.

EXAMPLE 1 A WEATHER BUREAU RECORDED HOURLY TEMPERATURES AS SHOWN IN TH FOLLOWING TABLE.

Time	9	10	11	12	1	2	3
Temp	61	62	65	69	68	72	76

THIS DATA ENABLES US TO MAKE SEVEN SENTENCES OF THE FORM:

AT x O'CLOCK THE TEMPERATURE WAS y DEGREES.

THAT IS, USING THE ORDERED PAIR (x, y), THE ORDERED PAIR (9, 61) MEANS.

At 9 *o'clock the temperature was* 61 *degrees.*

SO THE SET OF ORDERED PAIRS {(9, 61), (10, 62), (11, 65), (12, 69), (1, 68), (2, 72), (3, 76)} ARE ANOTHER FORM OF THE DATA IN THE TABLE, WHERE THE FIRST ELEM PAIR IS TIME AND THE SECOND ELEMENT IS THE TEMPERATURE RECORDED AT THAT

Definition 3.14

Given two non-empty sets A and B, the set of all ordered pairs (a, b) where $a \in$ A and $b \in$ B is called the Cartesian product of A and B, denoted by A × B (read "A cross B").

i.e., $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

NOTE THAT THE SETS A AND B IN THE DEFINITION CAN BE THE SAME OR DIFFERENT.

EXAMPLE 2 IF A = { 1, 2, 3} AND B = { 4, 5}, THEN

$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

EXAMPLE 3 LET A = { a, b}, THEN FORM A × A.

SOLUTION: $A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$.

EXAMPLE 4 LET A = {–1, 0} AND B = {–1, 0, 1}.

FIND A × B AND ILLUSTRATE IT BY MEANS OF A DIAGRAM.

SOLUTION: $A \times B = \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1)\}$

THE DIAGRAM IS AS SHOWN IN FIGURE 3.25.

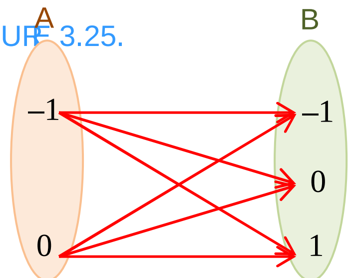


Figure 3.25

Note: $n(A \times B) = n(A) \times n(B)$.

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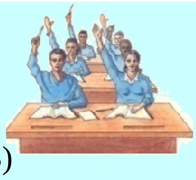
ACTIVITY 3.16



1 LET A = {2, 3} AND B = {0, 1, 2}. FIND:

A $A \times B$ **B** $B \times A$ **C** $n(A \times B)$

2 LET A = {a, b}, B = {c, d, e} AND C = {b, c}. FIND:

A $A \times (B \cap C)$ **B** $A \times (B \cup C)$

C $(A \times B) \cap (A \times C)$ **D** $(A \times B) \cup (A \times C)$

FROM THE RESULT OF ACTIVITY, YOU CONCLUDE THAT:

FOR ANY SETS A, B AND C,

I $A \times B \neq B \times A$, FOR $A \neq B$ *Cartesian product of sets is not commutative.*

II $n(A \times B) = n(A) \times n(B) = n(B \times A)$. *where A and B are finite sets.*

III $A \times (B \cap C) = (A \times B) \cap (A \times C)$. *Cartesian product is distributive over intersection.*

IV $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *Cartesian product is distributive over union.*

Exercise 3.10

1 GIVEN A = {2}, B = {1, 5}, C = {–1, 1}, FIND:

A $A \times B$ **B** $B \times A$ **C** $B \times C$ **D** $A \times (B \cap C)$

E $(A \cup C) \times B$ **F** $(A \times B) \cup (A \times C)$ **G** $B \times B$

2 IF $B \times C$ = {(1, 1), (1, 2), (1, 3), (4, 1), (4, 2), (4, 3)}, FIND:

A B **B** C **C** $C \times B$

3 IF $n(A \times B) = 18$ AND $n(A) = 3$ THEN FIND $n(B)$.

4 LET U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} BE THE UNIVERSAL SET, A = {0, 2, 4, 6, 8, 9}, B = {1, 3, 6, 8} AND C = {0, 2, 3, 4, 5}. FIND:

A $A' \times C'$ **B** $B \times A'$ **C** $B \times (A' \backslash C)$

5 IF $(2x + 3, 7) = (7, 3y + 1)$, FIND THE VALUES OF x AND y.

3.3.3 Problems Involving Sets

IN THIS SUBSECTION, YOU WILL LEARN HOW TO SOLVE PR INVOLVE SETS, IN PARTICULAR NUMBERS OF ELEMENTS IN S

THE NUMBER OF ELEMENTS THAT ARE IN SET A OR SET DENOTED BY $n(A \cup B)$, MAY NOT NECESSARILY $n(A) + n(B)$ AS WE CAN SEE IN THE FIGURE 3.6.

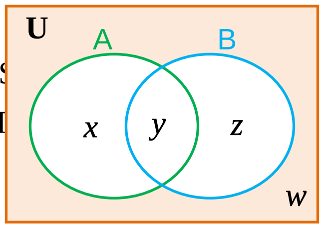


Figure 3.26

IN THIS FIGURE, SUPPOSE THE NUMBER OF ELEMENTS IN THE CLOSED REGIONS OF DIAGRAM ARE DENOTED BY x, y, z AND w.

n (A) = $x + y$ AND n (B) = $y + z$.

SO, n (A) + n (B) = $x + y + y + z$.

n (A ∪B) = $x + y + z$ = n (A) + n (B) – y

I.E., n (A∪B)= n (A) + n (B) – n (A∩B).

Number of elements in (A∪B)

FOR ANY FINITE SETS A AND B, THE NUMBER OF ELEMENTS THAT ARE IN A ∪B IS

n (A∪B) = n (A) + n (B) – n (A∩B).

Note: IF A∩B =∅, THEN n (A∪B) = n (A) + n (B).

EXAMPLE 1 EXPLAIN WHY n (A – B) = n (A) – n (A∩B).

SOLUTION: FROM FIGURE 3.28 ABOVE, n (A) = $x + y$, n (A∩B) = y

n (A) – n (A∩B) = $(x + y) - y = x$,

x IS THE NUMBER OF ELEMENTS IN A THAT ARE NOT IN B. SO, n (A – B) = x.

∴ n (A – B) = x = n (A) – n (A∩B).

FOR ANY FINITE SETS A AND B,

n (A\B) = n (A) – n (A∩B)

EXAMPLE 2 AMONG 1500 STUDENTS IN A SCHOOL, 13 STUDENTS FAILED IN ENGLISH, 12 STUDENTS FAILED IN MATHEMATICS AND 7 STUDENTS FAILED IN BOTH EN MATHEMATICS.

I HOW MANY STUDENTS FAILED IN EITHER ENGLISH OR IN MATHEMATICS?

II HOW MANY STUDENTS PASSED BOTH IN ENGLISH AND IN MATHEMATICS?

SOLUTION: LET E BE THE SET OF STUDENTS WHO FAILED IN ENGLISH, M BE THE SET OF STUDENTS WHO FAILED IN MATHEMATICS AND U BE THE SET OF ALL STUDENTS IN THE S

THEN, n (E) = 13, n (M) = 12, n (E∩M) = 7 AND n (U) = 1500.

I n (E∪M) = n (E) + n (M) – n (E∩M) = 13 + 12 – 7 = 18.

II THE SET OF ALL STUDENTS WHO PASSED IN BOTH SUBJECTS IS U\(E∪M).

n (U\(E∪M)) = n (U) – n (E∪M)= 1500 – 18 = 1482.

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Exercise 3.11

1. FOR A = {2, 3, . . . 6} AND B = {6, 7, . . . 10} SHOW THAT:

   A $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ B $n(A \times B) = n(A) \times n(B)$

   C $n(A \times A) = n(A) \times n(A)$

2. IF $n(C \cap D) = 8$ AND $n(C \backslash D) = 6$ THEN FIND $n(C)$.

3. USING A VENN DIAGRAM, OR A FORMULA, ANSWER EACH OF THE FOLLOWING:

   A GIVEN $n(Q \backslash P) = 4$, $n(P \backslash Q) = 5$ AND $n(P) = 7$ FIND $n(Q)$.

   B IF $n(R' \cap S') + n(R' \cap S) = 3$, $n(R \cap S) = 4$ AND $n(S' \cap R) = 7$, FIND $n(U)$.

4. INDICATE WHETHER THE STATEMENTS BELOW ARE TRUE OR FALSE FOR ALL FINITE SETS A AND B. IF A STATEMENT IS FALSE GIVE A COUNTER EXAMPLE.

   A $n(A \cup B) = n(A) + n(B)$ B $n(A \cap B) = n(A) - n(B)$

   C IF $n(A) = n(B)$ THEN A = B D IF A = B THEN $n(A) = n(B)$

   E $n(A \times B) = n(A) \cdot n(B)$ F $n(A) + n(B) = n(A \cup B) - n(A \cap B)$

   G $n(A' \cup B') = n((A \cup B)')$ H $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$

   I $n(A) + n(A') = n(U)$

5. SUPPOSE A AND B ARE SETS SUCH THAT $n(A) = 10$, $n(B) = 23$ AND $n(A \cap B) = 4$, THEN FIND:

   A $n(A \cup B)$ B $n(A \backslash B)$ C $n(A \Delta B)$ D $n(B \backslash A)$

6. IF A = $\{x \mid x$ IS A NON-NEGATIVE INTEGER AND $x^2 = x\}$, THEN HOW MANY PROPER SUBSETS DOES A HAVE?

7. OF 100 STUDENTS, 65 ARE MEMBERS OF A MATHEMATICS CLUB AND 40 ARE MEMBERS OF A PHYSICS CLUB. IF 10 ARE MEMBERS OF NEITHER CLUB, THEN HOW MANY STUDENTS ARE MEMBERS OF:

   A BOTH CLUBS? B ONLY THE MATHEMATICS CLUB?

   C ONLY THE PHYSICS CLUB?

8. THE FOLLOWING VENN DIAGRAM SHOWS … AND B. IF $n(A) = 13$, $n(B) = 8$, THEN FIND:

   A $n(A \cup B)$ B $n(U)$

   C $n(B \backslash A)$ D $n(A \cap B')$

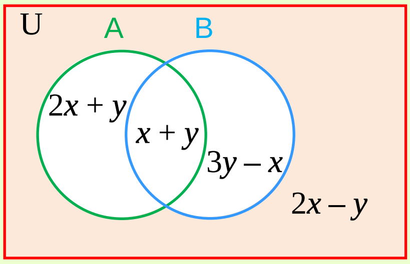


Figure 3.27

Key Terms

complement	infinite set	set
disjoint sets	intersection of sets	subset
element	power set	symmetric difference between sets
empty set	proper subset	union of sets
finite set	relative complement	universal set

Summary

1 A SET IS A WELL-DEFINED COLLECTION OF OBJECTS. THE OBJECTS ITS elements (OR members).

2 SETS ARE DESCRIBED IN THE FOL:

  A VERBAL MET

  B LISTING MET

    I PARTIAL LISTING II COMPLETE LISTING ME

  C SET-BUILDER ME

3 THE universal set IS A SET THAT CONTAINS ALL ELEMENTS CONSIDERE IN A DISCUSSION.

4 THE COMPLEMENT OF A SET A IS THE SET OF ALL ELEMENTS THAT A SET BUT NOT IN A

5 A SET S IS CALLED finite IF AND ONLY IF IT IS THE EMPTY SET OR HAS EXACTLY n ELEMENTS, WHERE n IS A NATURAL NU. OTHERWISE, IT IS CALLED infinite.

6 A SET A IS A SUBSET OF B IF AND ONLY IF EACH E IN SET B.

7 I P (A), THE POWER SET OF A, IS THE SET OF ALL SUBS

  II IF $n(A) = n$, THEN THE NUMBER OF SUBSE ^{n}.

8 TWO SETS A AND B ARE SA equal IF AND ONLY IF A $\subseteq$ B AND B $\subseteq$ A.

9 TWO SETS A AND B ARE SA equivalent IF AND ONLY IF THERE-TO-ONE CORRESPONDENCE BTHEIR ELEMENTS.

10 I A SET A IS A proper subset OF SET B, DENOTED BY A $\subset$ B, IF AND ONLY $\subseteq$ B AND B $\not\subseteq$ A.

  II IF $n(A) = n$, THEN THE NUMBER OF PROPER SUBS $^n - 1$.

11 OPERATIONS ON SETS; FOR ANY SETS A AND B,

I $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$.

II $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$.

III $A - B$ (OR $A \backslash B$) $= \{x \mid x \in A \text{ AND } x \notin B\}$.

IV $A \Delta B = \{x \mid x \in (A \cup B) \text{ AND } x \notin (A \cap B)\}$.

V $A \times B = \{(a, b) \mid a \in A \text{ AND } b \in B\}$.

12 PROPERTIES OF UNION, INTERSECTION, SYMMETRIC DIFFERENCE AND CARTESIAN P

FOR ALL SETS A, B AND C:

I COMMUTATIVE PROPERTIES

A $A \cup B = B \cup A$ **B** $A \cap B = B \cap A$ **C** $A \Delta B = B \Delta A$

II ASSOCIATIVE PROPERTIES

A $A \cup (B \cup C) = (A \cup B) \cup C$ **C** $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

B $A \cap (B \cap C) = (A \cap B) \cap C$

III IDENTITY PROPERTIES

A $A \cup \varnothing = A$ **B** $A \cap U = A$ (U is a universal set)

IV DISTRIBUTIVE PROPERTIES

A $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

B $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

C $A \times (B \cup C) = (A \times B) \cup (A \times C)$

D $A \times (B \cap C) = (A \times B) \cap (A \times C)$

V DE MORGAN'S LAW

A $(A \cup B)' = A' \cap B'$ **B** $(A \cap B)' = A' \cup B'$

VI FOR ANY SET A

A $A \cup A' = U$ **B** $(A')' = A$

C $A \cap A' = \varnothing$ **D** $A \times \varnothing = \varnothing$

Review Exercises on Unit 3

1 WHICH OF THE FOLLOWING ARE SETS?

A THE COLLECTION OF ALL TALL STUDENTS IN YOUR CLASS.

B THE COLLECTION OF ALL NATURAL NUMBERS DIVISIBLE BY 3.

C THE COLLECTION OF ALL STUDENTS IN YOUR SCHOOL.

D THE COLLECTION OF ALL INTELLIGENT STUDENTS IN ETHIOPIA.

E THE COLLECTION OF ALL SUBSETS OF THE SET {1, 2, 3, 4, 5}.

2 REWRITE THE FOLLOWING STATEMENTS, USING THE CORRECT NOTATION:

A B IS A SET WHOSE ELEMENTS ARE x, y, z AND w

B 3 IS NOT AN ELEMENT OF SET B.

C D IS THE SET OF ALL RATIONAL NUMBERS BETWEEN $\sqrt{2}$ AND $\sqrt{5}$.

D H IS THE SET OF ALL POSITIVE MULTIPLES OF 3.

3 WHICH OF THE FOLLOWING PAIRS OF SETS ARE EQUIVALENT?

A $\{1, 2, 3, 4, 5\}$ AND $\{l, m, o, p, q\}$

B $\{x \mid x$ IS A LETTER IN THE WORD MATHEMATICS$\}$ AND $\{y \mid y \in \mathbb{N}, 1 \leq y \leq 11\}$

C $\{a, b, c, d, e, f, \ldots m\}$ AND $\{1, 2, 3, 4, 5, \ldots 13\}$

4 WHICH OF THE FOLLOWING REPRESENT EQUAL SETS?

A = $\{a, b, c, d\}$ B = $\{x, y, z, w\}$

C = $\{x \mid x$ IS ONE OF THE FIRST FOUR LETTERS IN THE ENGLISH ALPHABET$\}$

D = $\varnothing$ E = $\{0\}$ F = $\{x \mid x \neq x\}$ G = $\{x \in \mathbb{Z} \mid -1 < x < 1\}$

5 IF U = $\{a, b, c, d, e, f, g, h\}$, A = $\{b, d, f, h\}$ AND B = $\{a, b, e, f, g, h\}$, FIND THE FOLLOWING:

A A' **B** B' **C** $A \cap B$ **D** $(A \cap B)'$ **E** $A' \cap B'$

6 IN THE VENN DIAGRAM GIVEN BELOW, WRITE THE REGION LABELLED BY I, II, III AND IV IN TERMS OF A AND B.

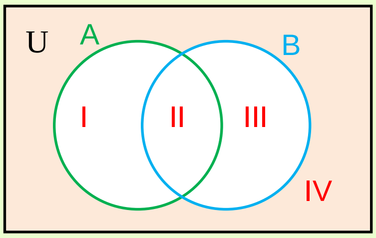


Figure 3.28

7 FOR EACH OF QUESTIONS A, B AND C, COPY THE FOLL
VENN DIAGRAM AND SHADE THE REGIONS TH

A $A \cap (B \cap C)$. **B** $A \backslash (B \cap C)$.

C $A \cup (B \backslash C)$.

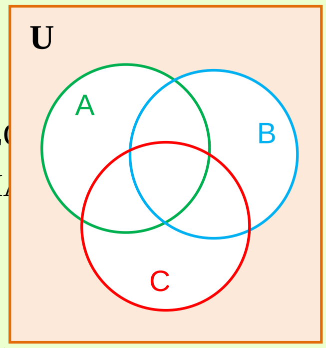


Figure 3.29

8 LET U = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, A = $\{1, 2, 3, 4, 5\}$,
B = $\{0, 2, 1, 6, 8\}$ AND C = $\{3, 6, 9\}$. THEN FIND:

A A' **B** $B \backslash A$ **C** $A \cap C'$

D $C \times (A \cap B)$ **E** $(B \backslash A) \times C$

9 SUPPOSE B IS A PROPER SUBSET OF C,

A IF n (C) = 8, WHAT IS THE MAXIMUM NUMBER OF ELEMENTS IN B?

B WHAT IS THE LEAST POSSIBLE NUMBER OF ELEMENTS IN B?

10 IF n (U) = 16, n (A) = 7 AND n (B) = 12, FIND:

A n (A') **B** n (B')

C GREATEST n (A ∩ B) **D** LEAST n (A ∪ B)

11 IN A CLASS OF 31 STUDENTS, 22 STUDENTS STUDY PHYSICS, 20 STUDENTS STUDY CH AND 5 STUDENTS STUDY NEITHER. CALCULATE THE NUMBER OF STUDENTS WHO ST SUBJECTS.

12 SUPPOSE A AND B ARE SETS SUCH THAT A ∪ B HAS 20 ELEMENTS, A ∩ B HAS 7 ELEMENTS, AND THE NUMBER OF ELEMENTS IN B IS TWICE THAT OF A. WHAT IS THE NUMBER O ELEMENTS IN:

A A? **B** B?

13 STATE WHETHER EACH OF THE FOLLOWING IS: infinite finite

A $\{x \mid x$ IS AN INTEGER LESS THAN 5$\}$

B $\{x \mid x$ IS A RATIONAL NUMBER BETWEEN 0 AND 1$\}$

C $\{x \mid x$ IS THE NUMBER OF POINTS ON A 1 CM-LONG LINE SEGMENT$\}$

D THE SET OF TREES FOUND IN ADDIS ABABA.

E THE SET OF “TEFF” IN 1,000 QUINTALS.

F THE SET OF STUDENTS IN THIS CLASS WHO ARE 10 YEARS OLD.

14 HOW MANY LETTERS IN THE ENGLISH ALPHABET PRECEDE K? (THINK OF A SHORTCUT METHOD).

15 OF 100 STAFF MEMBERS OF A SCHOOL, 48 DRINK COFFEE, 25 DRINK BOTH TEA AND CO AND EVERYONE DRINKS EITHER COFFEE OR TEA. HOW MANY STAFF MEMBERS DRIN

16 GIVEN THAT SET A HAS 15 ELEMENTS AND SET B HAS 12 ELEMENTS, DETERMINE EA FOLLOWING:

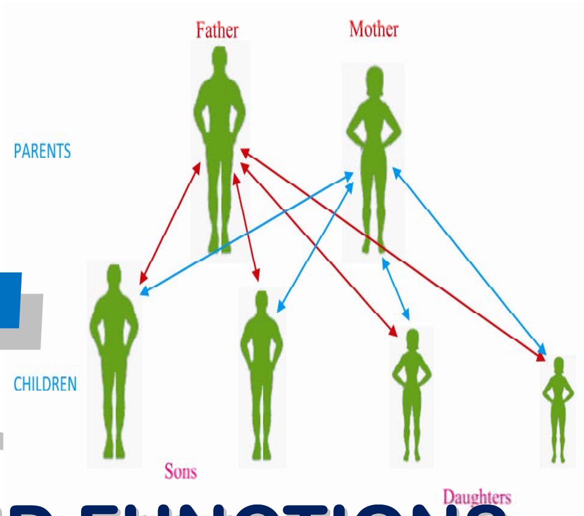
A THE MAXIMUM POSSIBLE NUMBER OF ELEMENTS IN A ∪ B

B THE MINIMUM POSSIBLE NUMBER OF ELEMENTS IN A ∪ B

C THE MAXIMUM POSSIBLE NUMBER OF ELEMENTS IN A ∩ B

D THE MINIMUM POSSIBLE NUMBER OF ELEMENTS IN A ∩ B

Unit 4



RELATIONS AND FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know specific facts about relation and function.*
- *understand the basic concepts and principles about combination of functions.*
- *sketch graphs of relations and functions (i.e. of linear and quadratic functions).*

Main Contents

4.1 Relations

4.2 Functions

4.3 Graphs of functions

Key Terms

Summary

Review Exercises

INTRODUCTION

IN OUR DAILY LIFE, WE COME ACROSS MANY PATTERNS THAT CHARACTERIZ BROTHER AND SISTER, TEACHER AND STUDENT, ETC. SIMILARLY, IN MA ACROSS DIFFERENT RELATIONS SU a IS LESS THAN NUMBER b, ANGLE IS GREATER THAN ANGLE, SET A IS SUBSET OF SET B, AND SO ON. IN ALL THESE CASES, WE F INVOLVES PAIRS OF OBJECTS IN SOME SPECIFIC ORDER. YOU WILL LEARN HOW PAIRS OF OBJECTS FROM TWO SETS AND THEN INTRODUCE RELATIONS B THE PAIR. YOU ALSO LEARN HERE ABOUT SPECIAL RELATIONS WHICH WILL

4.1 RELATIONS

Group Work 4.1

FORM A GROUP AND DO THE .

1 EXPLAIN AND DISCUSS THE MEANING OF "RELATION" IN DAILY LIFE.

2 GIVE SOME EXAMPLES OF RE FROM YOUR DAILY LIFE.

3 HOW DO YOU UNDERSTAND RELATIONS IN MATHEMA

IN OUR DAILY LIFE WE USUALLY TALK ABOUT RELATIONS FOR EXAMPLE, SAY SOMEONE IS THE FATHER ANOTHER PERSON, 5 IS GREATER THAN 3, ADDIS AB CAPITAL CITY OF ETHIOPIA, WALLIA IBEX IS ENDEMIC, ETC.

THE CARTESIAN PRODUCT OF SETS IS ONE OF THE USEFUL WAYS MATHEMATICALLY. FOR EXAMPLE, LET A = {ADDIS ABABA, JIMMA, N

B = {ETHIOPIA, KENYA, SUD x AND y IN THE ORDERED PAIR (x, y), WHERE $x \in$ A AND $y \in$ B, ARE RELATED BY THE PHRASE "x IS THE CAPITAL CITY OF y", THEN THE RELATION CAN BE D THE SET OF ORDERED PAIRS; {(ADDIS ABABA, ETHIOPIA), (NAIROBI, KENY RELATION IS A SUBSET OF A × B.

4.1.1 The Notion of a Relation

IN THE PREVIOUS SUBUNIT, YOU SAW RELATIONS IN A GENERAL SENSE AS RELATIONSHIPS BETWEEN ANY TWO THINGS WITH SOME RI THE FOLLOWING ACTIVITY WILL HELP YOU TO REALIZE THE MATHEMATICAL DI RELATION.

ACTIVITY 4.1

1 LET A = { 1, 2, 4, 6, 7} AND B = { 5, 12, 7, 9, 8,

LIST ALL ORDERED (x, y) WHICH SATISFY EACH OF THE FO SENTENCES WHERE $x \in$ A, $y \in$ B.

a x is greater than y **b** y is a multiple of x

c the sum of x and y is odd **d** x is half of y

2 Let A = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

List all ordered pairs (x, y) which satisfy each of the following sentences, where $x, y \in$ A.

a y is a multiple of x **b** x is the square of y

c x is less than y **d** x is a prime factor of y

3 Let U = {x | x is a student in your class}

i In each of the following, list all ordered pairs (x, y) which satisfy the given sentence where $x, y \in$ U.

a x is taller than y **b** x is younger than y

ii Discuss other ways that you can relate the students in your class.

As you have noticed from the above Activity, each sentence involves what is intuitively understood to be a relationship. Expressions of the type "is greater than", "is a multiple of", "is a factor of", "is taller than", etc. which express the relation are called relating phrases.

From Activity 4.1 you might have observed the following:

i In considering relations between objects, order is often important.

ii A relation establishes a pairing between objects.

Therefore, from a mathematical stand point, the meaning of a relation is made precise and defined as follows.

Definition 4.1

Let A and B be non-empty sets. A relation R from A to B is a subset of A × B. In other words, R is a relation from A to B if and only if R ⊆ (A × B).

Example 1 Let A = { 1, 2, 3, 4} and B = { 1, 3, 5}

i R_1 = {(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)} is a relation from A to B because $R_1 \subseteq$ (A × B). Is R_1 a relation from B to A? Justify.

Notice that we can represent R_1 in the set builder method as

$$R_1 = \{ (x, y) \mid x \in A, y \in B, x < y\}$$

ii R_2 = {(1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3)} is a relation from A to B because $R_2 \subseteq$ (A × B).

In the set builder method, R_2 is represented by $R_2 = \{ (x, y) \mid x \in A, y \in B, x \geq y\}$

EXAMPLE 2 LET A = {1, 2, 3} THEN OBSERV

$R_1 = \{(1, 2), (1, 3), (2, 3)\}$, $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

AND $R_3 = \{(x, y) \mid x, y \in A, x + y$ IS ODD$\}$ ARE RELATIONS ON A.

Exercise 4.1

1 FOR EACH OF THE FOLLOWIS, DETERMINE THE RELATI

A $R = \{(x, y): x$ IS TALLER $y\}$

B $R = \{(x, y): y$ IS THE SQUARE R$x\}$

C $R = \{(x, y): y = 2x\}$

2 LET A = {2, 4, 6} AND B = {1, 3,

A R = {(2, 2), (4, 4), (6, 6)} IS A RELATION ON A. EXPRESS THE RESET BUILDER ME.

b IS R = {(2, 1), (2, 3), (2, 5), (1, 2), (3, 4), (5, 6)} A RELATION FROM A GIVE THE REASON FOR YOUR

C IF R IS A RELATION FROM A TO B R = $\{(x, y): y = x - 1\}$, THEN LIST THE ELEMENTS O

3 IF R = $\{(x, y): y = 2x + 1\}$ IS A RELATION ON A, WHERE A = {1, 2, 3, 4, 5, 6} THE ELEMENTS O

4 WRITE SOME ORDPAIRS THAT BELONG TO T GIVEN BY

$R = \{(x, y): y < 2x;\ x \in \mathbb{Z}$ AND $y \in \mathbb{Z}\}$

4.1.2 Domain and Range

ACTIVITY 4.2

LET A = {1, 2, 4, 6, 7} AND B = {5, 12, 7, 9, 8, 3}

LET R_1 AND R_2 BE RELATIONS GI

$R_1 = \{(x, y) \mid x \in A, y \in B, x > y\}$ AND $R_2 = \left\{(x, y): x \in A, y \in B, x = \frac{1}{2}y\right\}$

REPRESENT EACH OF THE FOLLUSING COMPLETE LISTING M

A $D = \{x : (x, y) \in R_1\}$ **B** $D = \{x : (x, y) \in R_2\}$

C $R = \{y : (x, y) \in R_1\}$ **D** $R = \{y : (x, y) \in R_2\}$

OBSERVE THAT IN EACH CASE, THE SETS REPRESENTED BY D CONTAIN THE FIRST CO
SETS REPRESENTED BY R CONTAIN THE SECOND COORDINATES OF THE RESPECTIVE REL

IN THE ABOVE DISCUSSION THE SET OF ALL THE FIRST COORDINATES OF THE ORDE
RELATION R IS CALLED THE **domain** OF R AND THE SET OF ALL SECOND COORDINATES OF THE OR
PAIRS OF R IS CALLED THE **range** OF R.

WE GIVE THE DEFINITION OF DOMAIN AND RANGE FORMALLY AS FOLLOWS.

Definition 4.2

LET R BE A RELATION FROM A SET A TO A SET B. THEN

I DOMAIN OF R = $\{x : (x, y)$ BELONGS TO R FOR SOME $y\}$

II RANGE OF R = $\{y : (x, y)$ BELONGS TO R FOR SOME $x\}$

EXAMPLE 1 GIVEN THE RELATION R = {(1, 3), (2, 5), (7, 1), (4, 3)}, FIND THE DOMAIN AND RANGE OF THE RELATION R.

SOLUTION: SINCE THE DOMAIN CONTAINS THE FIRST COORDINATES, DOMAIN = {1, 2, 7,
THE RANGE CONTAINS THE SECOND COORDINATES, RANGE = {3, 5, 1}

EXAMPLE 2 GIVEN A = {1, 2, 4, 6, 7} AND B = {5, 12, 7, 9, 8, 3}

FIND THE DOMAIN AND RANGE OF THE RELATION R = $\{(x, y): x \in A, y \in B, x > y\}$

SOLUTION: IF WE DESCRIBE R BY COMPLETE LISTING METHOD, WE WILL FIND

R = {(4, 3), (6, 3), (7, 3), (6, 5), (7, 5)}.

THIS SHOWS THAT THE DOMAIN OF R = {4, 6, 7} AND THE RANGE OF R = {3, 5}

Exercise 4.2

1. FOR THE RELATION GIVEN BY THE SET OF ORDERED PAIRS {(−2, 3), (5, 2), (6, 3), (−2, 4), (DETERMINE THE DOMAIN AND THE RANGE.
2. LET A = {1, 2, 3, 4} AND R = $\{(x, y): y = x + 1;\ x, y \in A\}$ LIST THE ORDERED PAIRS THAT SATISFY THE RELATION AND DETERMINE THE DOMAIN AND THE RANGE OF R.
3. FIND THE DOMAIN AND THE RANGE OF EACH OF THE FOLLOWING RELATIONS:

   A R = $\left\{(x, y): y = \sqrt{x}\right\}$ **B** R = $\left\{(x, y) : y = x^2\right\}$

   C R = $\{(x, y) : y$ IS A MATHEMATICS TEACHER OF x IN SECTION 9$\}$
4. LET A = $\{x : 1 \le x < 10\}$ AND B = {2, 4, 6, 8}. IF R IS A RELATION FROM A TO B GIVEN BY R = $\{(x, y): x + y = 12\}$, THEN FIND THE DOMAIN AND THE RANGE OF R.

4.1.3 Graphs of Relations

BY NOW, YOU HAVE UNDERSTOOD WHAT A RELATION IS AND HOW IT CAN BE DESCRIBED USING ... YOU WILL NOW SEE HOW A RELATION CAN BE REPRESENTED THROUGH GRAPHS.

YOU MAY GRAPHICALLY REPRESENT A RELATION R FROM A TO B AS ORDERED PAIRS IN A COORDINATE SYSTEM, BY USING ARROWS IN A DIAGRAM DISPLAYING THE SETS, OR AS A REGION ON THE COORDINATE S.

ACTIVITY 4.3

DISCUSS THE FOLLOWING:

A A COORDINATE S (OR *xy*-COORDINATE SYSTEM).

B A POINT ON COORDINATE S.

C A REGION OF COORDINATE S.

FROM UNIT 3, RECALL THAT $\mathbb{R} \times \mathbb{R} = \{(x, y): x \in \mathbb{R}$ AND $y \in \mathbb{R}\}$ IS REPRESENTED AS A SET OF POINTS IN THE *xy*-COORDINATE S.

EXAMPLE 1 LET A = {2, 3, 5} AND B = {6, 7, 10} AND THE RELATION FROM B BE "x IS A FACTOR OF y".

ELEMENTS OF R = {(2, 6), (2, 10), (3, 6), (5, 10)} WITH DOMAIN = {2, 3, 5} AND RANGE = {6, 10}.

THIS RELATION CAN BE GRAPHICALLY REPRESENTED AS SHOWN IN THE ADJACENT FIGURE.

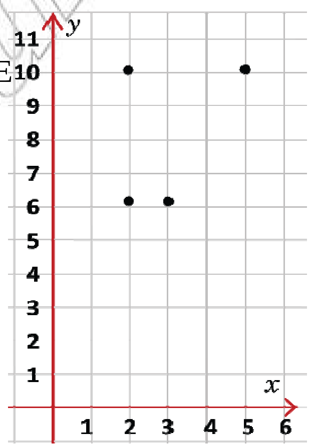


Figure 4.1

ALTERNATIVELY, WE CAN USE ARROWS IN A DIAGRAM DISPLAYING THE RELATION BETWEEN MEMBERS OF BOTH SETS AS SHOWN BELOW.

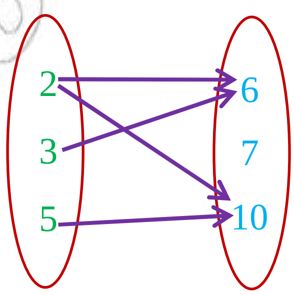


Figure 4.2

Group Work 4.2

FORM A GROUP AND PERFORM EACH OF THE FOLLOWING TO SKETCH THE GRAPH OF THE RELATION R = {(x, y): y > x, WHERE x AND y ARE REAL NUMBERS}

1. DRAW THE GRAPH OF THE LINE $y = x$ IN THE COORDINATE SYSTEM USING A BROKEN LINE.
2. CHOOSE ARBITRARY ORDERED PAIRS, ONE FROM ONE SIDE AND THE OTHER FROM THE OTHER SIDE OF THE LINE(S) AND DETERMINE WHICH OF THE PAIRS SATISFY THE RELATION.
3. WHAT DO YOU THINK WILL THE REGION THAT CONTAINS THE ORDERED PAIR SATISFYING THE RELATION BE?
4. SHADE THE REGION WHICH CONTAINS POINTS REPRESENTING THE ORDERED PAIR SATISFYING THE RELATION.
5. DETERMINE THE DOMAIN AND THE RANGE OF THE RELATION.

In general, to sketch graphs of relations involving inequalities, do the following:

1. DRAW THE GRAPH OF A LINE(S) IN THE RELATION ON THE xy-COORDINATE SYSTEM.
2. IF THE RELATING INEQUALITY IS ≤ OR ≥, USE A SOLID LINE; IF IT IS < OR >, USE A BROKEN LINE.
3. THEN TAKE ARBITRARY ORDERED PAIRS REPRESENTED BY POINTS, ONE FROM ONE SIDE AND THE OTHER FROM ANOTHER SIDE OF THE LINE(S), AND DETERMINE WHICH OF THE PAIRS SATISFY THE RELATION.
4. THE REGION THAT CONTAINS POINTS REPRESENTING THE ORDERED PAIR SATISFYING THE RELATION WILL BE THE GRAPH OF THE RELATION.

Note: A GRAPH OF A RELATION WHEN THE RELATING PHRASE IS AN INEQUALITY IS A REGION IN THE COORDINATE SYSTEM.

EXAMPLE 2 SKETCH THE GRAPH OF THE RELATION

R = {(x, y): $y > x$, WHERE x AND y ARE REAL NUMBERS}.

SOLUTION: TO SKETCH THE GRAPH,

1. DRAW THE GRAPH OF THE LINE $y = x$.
2. SINCE THE RELATION INVOLVES $y > x$, USE A BROKEN LINE.
3. TAKE POINTS REPRESENTING ORDERED PAIRS (0, 4) AND (3, –2) FROM ABOVE AND BELOW THE LINE $y = x$.
4. THE ORDERED PAIR (0, 4) SATISFIES THE RELATION. HENCE, THE REGION ABOVE THE LINE $y = x$, WHERE THE POINT REPRESENTING (0, 4) IS CONTAINED, IS THE GRAPH OF THE RELATION R.

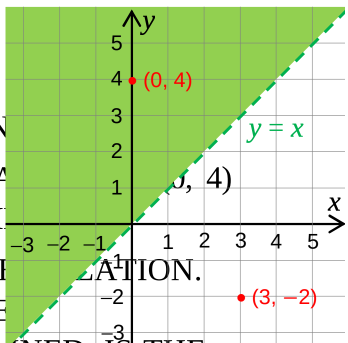


Figure 4.3

ACTIVITY 4.4

SKETCH THE GRAPH OF THE REL $x, y)$: $y \le 2x$; $x \in \mathbb{R}$ AND $y \in \mathbb{R}\}$

EXAMPLE 3 SKETCH THE GRAPH OF THE REL $x, y)$: $y \ge x + 1$; $x \in \mathbb{R}$ AND $y \in \mathbb{R}\}$

SOLUTION:

1 DRAW THE GRAPH OF $y = x + 1$.

2 SINCE THE RELATING INE≥ USE SOLID L

3 SELECT TWO POINTS ONE FROM ONE SIDE AND FROM THE OTHER SIDE LINE. FOR EXAM POINTS WITH COORD (0, 5) AND (2, 0) OBVIOUSLY, (0, 5) SAT THE RELATION

$R = \{(x, y): y \ge x + 1\}$, AS $5 \ge 0 + 1$.

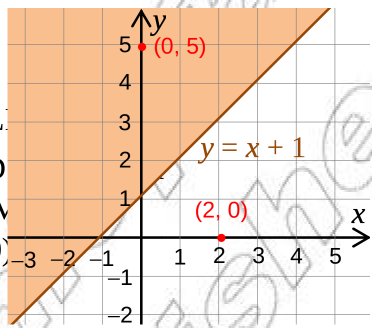


Figure 4.4

4 SHADE THE REGION CO THE POINT WITH COORDINATES (0, 5). SO THE GRAPH OF THE RELATION $x, y)$: $y \ge x + 1\}$ IS AS SHOWN BY THE SHADED R.

EXAMPLE 4 SKETCH THE GRAPH RELATION $R = \{(x, y): y \ge x^2\}$.

1 DRAW THE GRA $y = x^2$ USING SOLID CURVE.

2 SELECT TWO POINTS FROM INSIDE AND O CURVE, SAY THE POINT WITH COOR (0, 2) FROM INSIDE OF THE CURVE AND (3 OUTSIDE OF THE CURVE. CLEARLY, (0, THE RELATION S $2 \ge 0^2$ IS TRUE.

HENCE, THE GRAPH OF THE RELATION IS THE SH FIGURE (CONTAINING THE POINT WITH COOR (0, 2)).

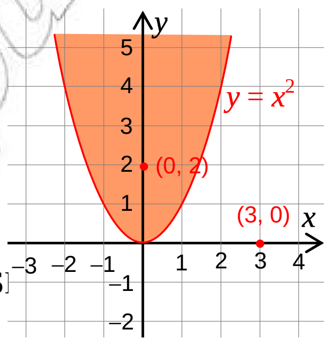


Figure 4.5

WE HAVE DISCUSSED HOW TO SKETCH GRAPHS OF RELATIONS INVOLVING ONE INEQUALITY. IT IS ALSO POSSIBLE TO SKETCH THE GRAPH OF A RELATION WITH TWO OR MORE RELATI APPROACH TO SKETCHING THIS IS SIMILAR, EXCEPT THAT, IN SUCH CASES WE CONSIDER THE INTERSECTION OF REGIONS. IF A RELATION HAS THE CONNECTIVE "OR", WE TAKE UNION INSTEAD OF INTERSECTION.

ACTIVITY 4.5

1 SKETCH THE GRAPH OF EACH OF THE FOL RELATIONS

A $R_1 = \{(x, y): x \ge 0; x, y \in \mathbb{R}\}$ **B** $R_2 = \{(x, y): y \ge 0; x, y \in \mathbb{R}\}$

C $R_3 = \{(x, y): x \ge 0$ AND $y \ge 0; x, y \in \mathbb{R}\}$

2 WHAT RELATION DID YOU OBSERVE AMONG THE GRA R_1, R_2 AND R_3?

To sketch the graph of a relation with two or more inequalities,

I USING THE SAME COORDINATE SYSTEM, SKETCH THE REGIONS OF EACH INEQUALIT

II DETERMINE THE INTERSECTION OF THE REGIONS.

EXAMPLE 5 SKETCH THE GRAPH OF THE RELATION

$R = \{(x, y): y \geq x + 2 \text{ AND } y > -x, x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$.

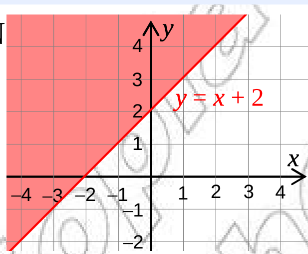


Figure 4.6

SOLUTION:

FIRST SKETCH THE GRAPH OF THE RELATION

$R = \{(x, y): y \geq x + 2, x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$.

NEXT, ON THE SAME DIAGRAM, SKETCH THE

$R = \{(x, y): y > -x, x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$

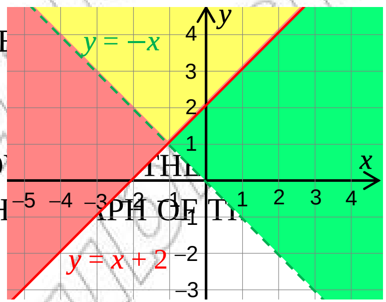


Figure 4.7

THE TWO SHADED REGIONS HAVE SOME O... THE INTERSECTION OF THE TWO REGIONS IS TH... GRAPH OF TH... RELATION.

SO, TAKING ONLY THE COMMON REGION, WE OBTAIN THE GRAPH OF THE RELATION

$R = \{(x, y): y \geq x + 2 \text{ AND } y > -x, x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$

AS SHOWN IN FIGURE 4.8.

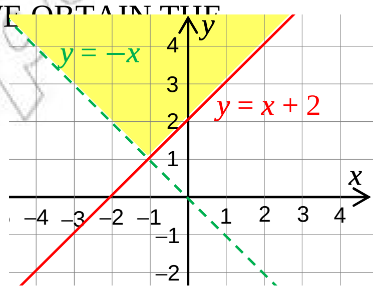


Figure 4.8

Group Work 4.3

1 DISCUSS HOW YOU CAN DETERMINE THE DOMAIN A... RELATION FROM ITS GRAPH.

2 IS THERE ANY SIMPLE WAY OF FINDING THE DOMAIN AND RANGE OF A RELATION FROM THE GRAPH OF THE RELATION?

IT IS POSSIBLE TO DETERMINE THE DOMAIN AND RANGE OF A RELATION FROM ITS DOMAIN OF A RELATION IS THE x-COORDINATE OF THE SET OF POINTS THROUGH WHICH A VER LINE MEETS THE GRAPH OF THE RELATION AND THE RANGE OF A RELATION IS THE y-COORDINATE OF THE SET OF POINTS THROUGH WHICH A HORIZONTAL LINE MEETS THE GRAPH OF THE

EXAMPLE 6 FIND THE DOMAIN AND THE RANGE OF THE RELATION

$$R = \{(x, y): y \geq x + 2 \text{ AND } y \geq -x;\ x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}.$$

FROM THE GRAPH SKETCHED ABOVE, SINCE ANY VERTICAL LINE MEETS THE G DOMAIN OF THE RELATION IS THE SET OF ℝ REAL NUMBERS,

THAT IS, DOMAIN OF R = ℝ BUT NOT ALL HORIZONTAL LINES MEET THE GRAPH, ONLY T THAT PASS THROUGH $y \geq 1$. HENCE, THE RANGE OF THE RELATION IS THE SET $\{y: y \geq 1\}$.

EXAMPLE 7 SKETCH THE GRAPH OF THE FOLLOWING RELATION AND DETERMINE ITS D RANGE.

$$R = \{(x, y): y < 2x \text{ AND } y > -x\}.$$

SOLUTION: SKETCH THE GRAPHS OF $y = 2x$ AND $y = -x$ ON SAME COORDINATE SYSTEM.

NOTE THAT THESE TWO LINES DIVIDE THE COORDINATE SYSTEM INTO FOUR REGIONS.

TAKE ANY POINTS ONE FROM EACH REGION AND CHECK IF THEY SATISFY THE RELATION. SAY, (0, 4), (–1, 0), (3, 0) AND (0,–2).

(3, 0) SATISFIES BOTH INEQUALITIES OF THE RELATION. THE GRAPH OF THE RELATION IS THE REGION THAT CONTAINS (3, 0).

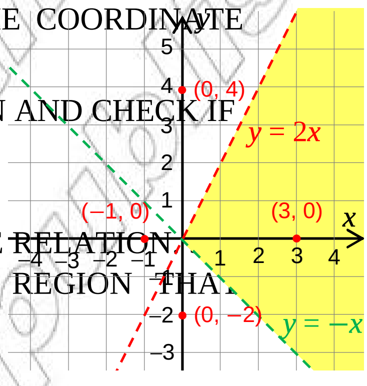


Figure 4.9

HENCE, DOMAIN OF R = $\{x: x \in \mathbb{R}, x > 0\}$

RANGE OF R = $\{y: y \in \mathbb{R}\}$.

Exercise 4.3

1 LET $A = \{2, 3, 5\}$ AND $B = \{6, 10, 15\}$ AND $R: A \to B$

A IF $R = \{(x, y): y = 2x + 5\}$, THEN PLOT THE POINTS OF R ON A COORDINATE SYSTE AND DETERMINE THE DOMAIN AND RANGE OF THE RELATION.

B LET $R = \{(x, y): x$ IS A DIVISOR OF $y\}$. PLOT THE POINTS OF R ON A COORDINATE SYSTEM, AND DETERMINE THE DOMAIN AND RANGE OF THE RELATION.

2 FOR EACH OF THE FOLLOWING RELATIONS, SKETCH THE GRAPH AND DETERMINE THE DOMAIN AND THE RANGE.

A $R = \{(x, y): y \geq 3x - 2\}$ **B** $R = \{(x, y): y \geq 2x - 1 \text{ AND } y \leq -2x + 1\}$

C $R = \{(x, y): y \geq 2x - 1 \text{ AND } y \leq 2x - 1\}$

3 FROM THE GRAPH OF EACH OF THE FOLLOWING RELATIONS, REPRESENTED BY SHADED REGION, SPECIFY THE RELATION AND DETERMINE THE DOMAIN AND THE RANGE:

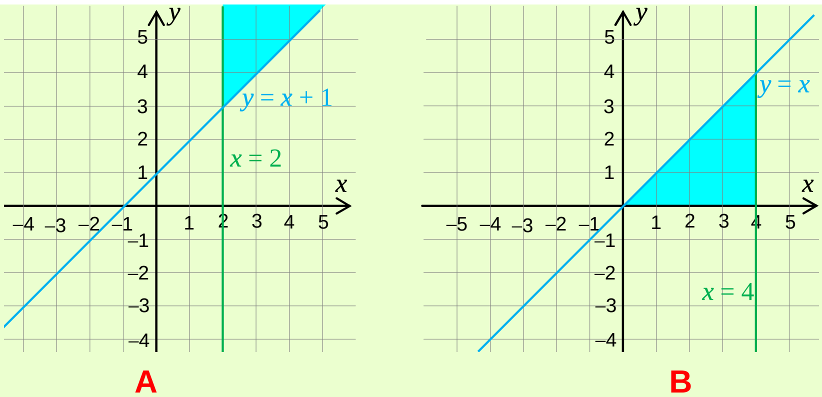


Figure 4.10

4.2 FUNCTIONS

IN THIS SECTION, YOU SHALL LEARN ABOUT PARTICULAR TYPES OF RELATIONS WI FUNCTIONS, THE DOMAIN AND RANGE OF A FUNCTION, AND COMBINATIONS OF REMEMBER THAT THE CONCEPT OF functions IS ONE OF THE MOST IMPORTANT IN MATHEMATICS THERE ARE MANY TERMS SUCH AS 'MAP' OR 'MAPPING' USED TO DENOTE A FUNCTION.

4.2.1 Functions

Group Work 4.4

1 CONSIDER THE FOLLOWING RELATIONS

$R_1 = \{(1, 2), (3, 4), (2, 5), (5, 6), (4, 7)\}$

$R_2 = \{(1, 2), (3, 2), (2, 5), (6, 5), (4, 7)\}$

$R_3 = \{(1, 2), (1, 4), (2, 5), (2, 6), (4, 7)\}$

A WHAT DIFFERENCES DO YOU SEE BETWEEN THESE RELATIONS?

B HOW ARE THE FIRST ELEMENTS OF THE COORDINATES PAIRED WITH THE SECO ELEMENTS OF THE COORDINATES?

C IN EACH RELATION, ARE THERE ORDERED PAIRS WITH THE SAME FIRST COORD

2 LET $R_1 = \{(x, y)$: x AND y ARE PERSONS IN YOUR KEBELE WHERE y IS THE FATHER OF $x\}$

$R_2 = \{(x, y)$: x AND y ARE PERSONS IN YOUR KEBELE WHERE x IS THE FATHER OF $y\}$

DISCUSS THE DIFFERENCE BETWEEN THESE TWO RELATIONS R_1 AND R_2.

Definition 4.3

A function is a relation such that no two ordered pairs have the same *first*-coordinates and different *second*-coordinates.

EXAMPLE 1 CONSIDER THE RELATION R = {(1, 2), (7, 8), (4, 3), (7, 6)}

SINCE 7 IS PAIRED WITH BOTH 8 AND 6 THE RELATION R IS NOT A FUNCTION.

EXAMPLE 2 LET R = {(1, 2), (7, 8), (4, 3)}. THIS RELATION IS A FUNCTION BECAUSE NO *first*-COORDINATE IS PAIRED (MAPPED) WITH MORE THAN ONE ELEMENT OF THE *second*-COORDINATE.

EXAMPLE 3 CONSIDER THE FOLLOWING ARROW DIAGRAMS.

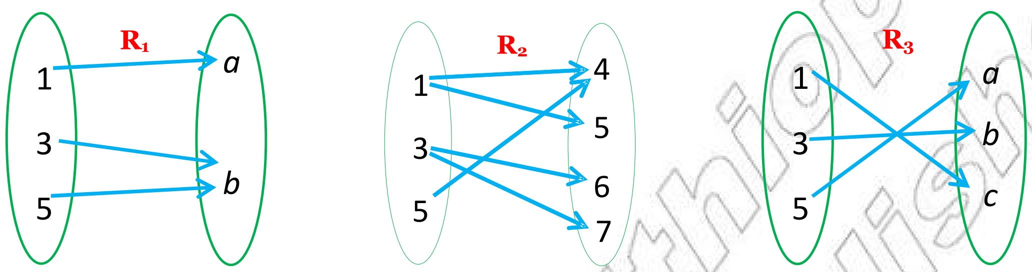


Figure 4.11

WHICH OF THESE RELATIONS ARE FUNCTIONS?

SOLUTION: R_1 IS A FUNCTION. (WHY?)

R_2 IS NOT A FUNCTION BECAUSE 1 AND 3 ARE BOTH MAPPED ONTO TWO NUM

R_3 IS A FUNCTION. (WHY?)

EXAMPLE 4 THE RELATION R = {(x, y): y IS THE FATHER OF x} IS A FUNCTION BECAUSE NO CHILD HAS MORE THAN ONE FATHER.

EXAMPLE 5 CONSIDER THE RELATION R = {(x, y): y IS A GRANDMOTHER OF x}.

THIS RELATION IS NOT A FUNCTION SINCE EVERY BODY (x) HAS TWO GRANDMOTHERS.

Domain and range of a function

IN SECTION 4.1.2 YOU LEARNT ABOUT THE DOMAIN AND RANGE OF A RELATION. AS A FUNC SPECIAL TYPE OF A RELATION, THE DOMAIN AND RANGE OF A FUNCTION ARE DETERMIN THE SAME WAY.

EXAMPLE 6 FOR EACH OF THE FOLLOWING FUNCTIONS, DETERMINE THE DOMAIN AND THE RANGE.

A F = {(2, –1), (4, 3), (0, 1)} **B** F = {(2, –1), (4, 3), (0, –1), (3, 4)}

SOLUTION:

A DOMAIN D = {0, 2, 4} AND RANGE R = {–1, 1, 3 }

B DOMAIN D = {0, 2, 3, 4} AND RANGE R = {–1, 3, 4}

YOU WILL NOW CONSIDER SOME FUNCTIONS THAT ARE DEFINED BY A FORMULA.

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EXAMPLE 7 IS THE RELATION {(x, y): $x = y^2$} A FUNCTION?

SOLUTION: THIS IS NOT A FUNCTION BECAUSE THE x ARE PAIRED WITH MORE THAN ONE NUMBER y. FOR EXAMPLE, (9, –3) AND (9, 3) SATISFY THE RELATION WITH 9 BEING MAPPED TO BOTH –3 AND 3.

EXAMPLE 8 IS R = $\{(x, y) : y = |x|\}$ A FUNCTION?

SOLUTION: SINCE FOR EVERY NUMBER THERE IS UNIQUE ABSOLUTE VALUE, x IS MAPPED TO ONE AND ONLY ONE y, SO THE RELATION {(x, y): $y = |x|$} IS A FUNCTION.

Notation: IF x IS AN ELEMENT IN THE DOMAIN OF f, THEN THE ELEMENT IN THE RANGE THAT IS ASSOCIATED WITH x IS DENOTED BY $f(x)$ AND IS CALLED THE IMAGE OF x UNDER THE FUNCTION f. THIS MEANS $f = \{(x, y) : y = f(x)\}$

THE NOTATION $f(x)$ IS CALLED **function notation**. READ $f(x)$ AS f OF x.

Note: f, g AND h ARE THE MOST COMMON LETTERS USED TO DESIGNATE FUNCTIONS. ANY LETTER OF THE ALPHABET CAN BE USED.

A FUNCTION FROM A TO B CAN SOMETIMES BE DENOTED AS f: A→B, WHERE THE DOMAIN OF f IS A AND THE RANGE OF f IS A SUBSET OF B, IN WHICH CASE B CONTAINS THE IMAGE ELEMENTS OF A UNDER THE FUNCTION f.

EXAMPLE 9 CONSIDER THE FUNCTION f = {(x, y): $y = |x|$}. HERE THE RULE $y = |x|$ CAN BE WRITTEN AS $f(x) = |x|$ AS A RESULT OF WHICH, $f(0) = |0| = 0$, $f(-2) = |-2| = 2$ AND $f(3) = |3| = 3$.

EXAMPLE 10 IF R = {(x, y): y IS TWICE x}, THEN WE CAN DENOTE THIS BY $f(x) = 2x$.

ACTIVITY 4.6

1 CONSIDER THE FOLLOWING ARROW DIAGRAM OF f.

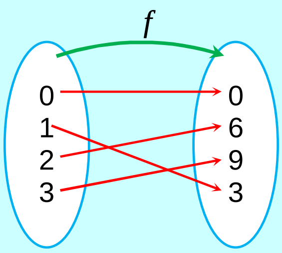

Figure 4.12

FIND AN ALGEBRAIC RULE FOR f.

2 FOR EACH OF THE FOLLOWING FUNCTIONS FIND THE DOMAIN:

A $f(x) = 5x - 1$ **B** $f(x) = x^2$ **C** $f(x) = \sqrt{x^2 - 3}$

OBSERVE THAT THE DOMAIN OF A FUNCTION IS THE SET ON WHICH THE GIVEN FUNCTION

EXAMPLE 11 CONSIDER $f(x) = 2x + 2$.

SINCE $f(x) = 2x + 2$ IS DEFINED FOR ALL $x \in \mathbb{R}$, THE DOMAIN OF THE FUNCTION IS THE SET OF ALL REAL NUMBERS. THE RANGE IS ALSO $\mathbb{R}$ SINCE FOR EVERY REAL NUMBER y THERE IS A REAL NUMBER x SUCH THAT $y = f(x) = 2x + 2$.

EXAMPLE 12 LET $f(x) = \sqrt{x-3}$

SINCE THE EXPRESSION IN THE RADICAL MUST BE NON-NEGATIVE, $x - 3 \geq 0$.

THIS IMPLIES $x \geq 3$. SO THE DOMAIN IS THE SET D = $\{x: x \geq 3\}$.

SINCE THE VALUE OF $\sqrt{x-3}$ IS ALWAYS NON-NEGATIVE, THE RANGE IS THE SET

R = $\{y: y \geq 0\}$.

EXAMPLE 13 LET A = {1, 2, 3, 4} AND B = {3, 4, 5, 7, 9}

IF f : A → B IS THE FUNCTION GIVEN BY $f(x) = 2x + 1$, THEN FIND THE DOMAIN AND THE RANGE OF f

SOLUTION: SINCE $f(1) = 3 \in$ B, $f(2) = 5 \in$ B, $f(3) = 7 \in$ B AND $f(4) = 9 \in$ B, THE DOMAIN OF f IS D = {1, 2, 3, 4} AND THE RANGE OF f IS R = {3, 5, 7, 9}.

Remark: IF f: A → B IS A FUNCTION, THEN, FOR ANY $x \in$ A THE IMAGE OF x UNDER f, $f(x)$ IS CALLED THE **functional value of f at x**. FOR EXAMPLE, IF $f(x) = x - 3$, THEN THE FUNCTIONAL VALUE OF f AT $x = 5$ IS $f(5) = 5 - 3 = 2$. FINDING THE FUNCTIONAL VALUE OF f AT x IS ALSO CALLED **evaluating the function** AT x.

EXAMPLE 14 TAKE $f(x) = \sqrt{x-3}$ AND EVALUATE:

A $f(3)$ **B** $f(12)$

SOLUTION:

A $f(3) = \sqrt{3-3} = \sqrt{0} = 0$ **B** $f(x) = \sqrt{12-3} = \sqrt{9} = 3$

EXAMPLE 15 FOR THE FUNCTION $f(x) = 1 - x^2$

A FIND THE DOMAIN AND THE RANGE. **B** EVALUATE $f(2)$ AND $f(-1)$

SOLUTION:

A THE DOMAIN OF THE FUNCTION IS $\{x: x \in \mathbb{R}\}$, SINCE IT IS DEFINED FOR ALL REAL NUMBERS. THE RANGE IS R = $\{y : y \leq 1\}$.

B $f(2) = 1 - (2)^2 = 1 - 4 = -3$ AND $f(-1) = 1 - (-1)^2 = 1 - 1 = 0$.

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Exercise 4.4

1 DETERMINE WHETHER EACH OF THE FOLLOWING RELATIONS IS A FUNCTION OR NO REASONS FOR THOSE THAT ARE NOT FUNCTIONS.

- **A** R = {(–1, 2), (1, 3), (–1, 3)}
- **B** R = {(1, 1), (1, 3), (–1, 3), (2, 1)}
- **C** R = {(x, y): y IS THE AREA OF TRIANGLE x}
- **D** R = {(x, y): x IS THE AREA OF TRIANGLE y}
- **E** R = {(x, y): y IS A MULTIPLE OF x}
- **F** R = {(x, y): $y = x^2 + 3$}
- **G** R = {(x, y): $y < x$}
- **H** R = {(x, y): x IS THE SON OF y}

2 IS EVERY FUNCTION A RELATION? EXPLAIN YOUR ANSWER.

3 FIND THE DOMAIN AND THE RANGE OF EACH OF THE FOLLOWING FUNCTIONS:

- **A** $f(x) = 3$
- **B** $f(x) = 1 - 3x$
- **C** $f(x) = \sqrt{x+4}$
- **D** $f(x) = |x| - 1$
- **E** $f(x) = \dfrac{1}{2x}$

4 IF $f(x) = 2x + \sqrt{x+4}$, EVALUATE EACH OF THE FOLLOWING:

- **A** $f(-4)$
- **B** $f(5)$

5 MATCH EACH OF THE FUNCTIONS IN COLUMN A WITH ITS CORRESPONDING DOMAIN COLUMN B:

	A		B
1	$f(x) = \sqrt{2-x}$	A	$\{x: x \geq 3\}$
2	$f(x) = 2x - 1$	B	$\{x: x \leq 2\}$
3	$f(x) = \sqrt{x-3}$	C	$\{x: x \in \mathbb{R}\}$

6 MATCH EACH OF THE FUNCTIONS IN COLUMN A WITH ITS CORRESPONDING RANGE IN COLUMN B.

	A		B
1	$f(x) = \sqrt{2-x}$	A	$\{y: y \geq 0\}$
2	$f(x) = 2x - 1$	B	$\{y: y \in \mathbb{R}\}$
3	$f(x) = \sqrt{x-3}$	C	$\{y: y \geq 10\}$

4.2.2 Combinations of Functions

IN THIS SUB-SECTION, YOU WILL LEARN HOW TO FIND THE SUM, DIFFERENCE, PRODUCT
OF TWO FUNCTIONS, ALL KNOWN AS combinations of functions

Group Work 4.5

1 CONSIDER THE FUNCTIONS $f(x) = \sqrt{x-3}$ AND $g(x) = \sqrt{10-x}$

A FIND $f + g$; $f - g$; fg AND $\frac{f}{g}$.

B DETERMINE THE DOMAIN AND THE RANGE OF EACH FUNCTION.

C IS THE DOMAIN OF fg THE SAME AS THE DOMAIN OF $f + g$? IS THIS ALWAYS TRU

A Sum of functions

SUPPOSE f AND g ARE TWO FUNCTIONS. THE SUM OF THESE FUNCTIONS IS A FUNCTION W
DEFINED AS $f + g$, WHERE $(f + g)(x) = f(x) + g(x)$.

EXAMPLE 1 IF $f(x) = 2 - x$ AND $g(x) = 3x + 2$ THEN THE SUM OF THESE FUNCTIONS IS GIVEN BY

$(f + g)(x) = (2 - x) + (3x + 2) = 2x + 4$, WHICH IS ALSO A FUNCTION.

THE DOMAIN OF $f = \mathbb{R}$ AND THE DOMAIN OF $g = \mathbb{R}$.

THE FUNCTION $(f + g)(x) = 2x + 4$ HAS ALSO DOMAIN $= \mathbb{R}$

EXAMPLE 2 LET $f(x) = 2x$ AND $g(x) = \sqrt{2x}$. DETERMINE

A THE SUM $f + g$

B THE DOMAIN OF $(f + g)$

SOLUTION:

A $(f + g)(x) = f(x) + g(x) = 2x + \sqrt{2x}$

B DOMAIN OF $f + g = \{x : x \geq 0\}$.

B Difference of functions

SUPPOSE f AND g ARE TWO FUNCTIONS. THE DIFFERENCE OF THESE FUNCTIONS IS ALSO A
DEFINED AS $f - g$, WHERE $(f - g)(x) = f(x) - g(x)$.

EXAMPLE 3 IF $f(x) = 3x + 2$ AND $g(x) = x - 4$, THEN THE DIFFERENCE OF THESE FUNCTIONS IS

$(f - g)(x) = f(x) - g(x) = (3x + 2) - (x - 4) = 2x + 6$ AND

THE DOMAIN OF $f - g = \mathbb{R}$.

EXAMPLE 4 LET $f(x) = 2x$ AND $g(x) = \sqrt{1-x}$. DETERMINE:

A THE DIFFERENCE $f - g$

B THE DOMAIN OF $f - g$

SOLUTION:

A $(f-g)(x) = f(x) - g(x) = 2x - \sqrt{1-x}$

B DOMAIN OF $f-g = \{x: x \le 1\}$.

C Product of functions

SUPPOSE f AND g ARE TWO FUNCTIONS. THE PRODUCT OF THESE FUNCTIONS IS ALSO A FUNCTION DEFINED AS $(fg)(x) = f(x)g(x)$. AGAIN,

EXAMPLE 5 IF $f(x) = 2x$ AND $g(x) = 3 - x$ THEN THE PRODUCT OF THESE FUNCTIONS

$(fg)(x) = f(x)\, g(x) = (2x)(3-x) = 6x - 2x^2$ AND

THE DOMAIN OF $fg = \mathbb{R}$

Note: THE DOMAIN OF THE SUM, DIFFERENCE AND PRODUCT OF FUNCTIONS f AND g IS THE INTERSECTION OF THE DOMAIN OF f AND THE DOMAIN OF g.

D Quotients of functions

SUPPOSE f AND g ARE TWO FUNCTIONS WITH $g \neq 0$. THE QUOTIENT OF THESE FUNCTIONS IS ALSO A FUNCTION, DEFINED AS $\frac{f}{g}$ WHERE $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

EXAMPLE 6 IF $f(x) = 3$ AND $g(x) = 2 + x$ THEN THE QUOTIENT OF THESE FUNCTIONS

$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3}{2+x}$ AND THE DOMAIN OF $\frac{f}{g} = \mathbb{R} \setminus \{-2\}$.

EXAMPLE 7 LET $f(x) = \frac{x}{x-2}$ AND $g(x) = \frac{x-3}{2x}$.

1 FIND **A** $f+g$ **B** $f-g$ **C** fg **D** $\frac{f}{g}$ AND

2 DETERMINE THE DOMAIN OF EACH FUNCTION.

SOLUTION:

1 **A** $(f+g)(x) = f(x) + g(x) = \frac{x}{x-2} + \frac{x-3}{2x} = \frac{3x^2 - 5x + 6}{2x(x-2)}$

B $(f-g)(x) = f(x) - g(x) = \frac{x}{x-2} - \frac{x-3}{2x} = \frac{x^2 + 5x - 6}{2x(x-2)}$

C $(fg)(x) = f(x)\, g(x) = \left(\frac{x}{x-2}\right)\left(\frac{x-3}{2x}\right) = \frac{x(x-3)}{2x(x-2)} = \frac{x-3}{2(x-2)}$

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D $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{\dfrac{x}{x-2}}{\dfrac{x-3}{2x}} = \left(\dfrac{x}{x-2}\right)\left(\dfrac{2x}{x-3}\right) = \dfrac{2x^2}{x^2-5x+6}$

2 DOMAIN OF $f + g$ = DOMAIN OF $f - g$ = DOMAIN OF fg

$= \mathbb{R}\setminus\{0, 2\}$ OR $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

DOMAIN OF $\dfrac{f}{g} = \mathbb{R}\setminus\{0, 2, 3\}$ OR $(-\infty, 0) \cup (0, 2) \cup (2, 3) \cup (3, \infty)$.

EXAMPLE 8 LET $f(x) = 8 - 3x$ AND $g(x) = -x - 5$. DETERMINE:

A $2f + g$ **B** $3g - 2f$ **C** $(3f)g$ **D** $\dfrac{4g}{3f}$

SOLUTION:

A $2f(x) + g(x) = 2(8 - 3x) + (-x - 5) = 11 - 7x$

B $3g(x) - 2f(x) = 3(-x - 5) - 2(8 - 3x) = -3x - 15 - 16 + 6x = 3x - 31$

C $(3f(x))g(x) = 3(8 - 3x)(-x - 5) = 9x^2 + 21x - 120$

D $\dfrac{4g(x)}{3f(x)} = \dfrac{4(-x-5)}{3(8-3x)} = \dfrac{-4x-20}{24-9x}$

THROUGH THE ABOVE EXAMPLES, YOU HAVE SEEN HOW TO DETERMINE THE COMBI FUNCTIONS. NOW, YOU SHALL DISCUSS HOW TO EVALUATE FUNCTIONAL VALUES O FUNCTIONS FOR GIVEN VALUES IN THE DOMAINS IN THE EXAMPLES THAT FOLLOW.

EXAMPLE 9 LET $f(x) = 2 - 3x$ AND $g(x) = x - 3$. EVALUATE $\dfrac{f}{g}(4)$ AND $(f + g)(4)$

SOLUTION: $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{2-3x}{x-3}$. SO $\dfrac{f}{g}(4) = \dfrac{2-3(4)}{4-3} = -10$

$(f + g)(x) = f(x) + g(x) = -2x - 1$. SO $(f + g)(4) = -2(4) - 1 = -9$.

EXAMPLE 10 LET $f(x) = x - 1$ AND $g(x) = 3x$. DETERMINE:

A $(2f+3g)(1)$ **B** $\dfrac{f}{2g}(3)$

SOLUTION:

A $(2f + 3g)(1) = 2(1 - 1) + 3(3(1)) = 9$ **B** $\dfrac{f}{2g}(3) = \dfrac{3-1}{2(3)(3)} = \dfrac{2}{18} = \dfrac{1}{9}$

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Exercise 4.5

1. IF $f = \{(1, 2), (-3, 2), (2, 5)\}$ AND $g = \{(2, 4), (1, 5), (3, 2)\}$. FIND:
   A $f + g$ AND $f - g$ **B** THE DOMAIN OF $(f + g)$
2. LET $f = \{(2, 3), (4, 9), (3, -8)\}$ AND $g = \{(1, 2), (2, 5), (3, 10), (4, 17)\}$. DETERMINE:
   A $-2f$ **B** fg **C** $fg(2)$ **D** g^2
3. WRITE DOWN THE DOMAIN OF EACH FUNCTION IN QUESTION NUMBER 2.
4. LET $f(x) = \dfrac{2}{x-1}$ AND $g(x) = \dfrac{2x-2}{3x+3}$. FIND:
   A $f + g$ **B** fg **C** DOMAIN OF $(f + g)$ AND fg
5. LET $f(x) = 3x - 3$ AND $g(x) = \dfrac{2}{x-1}$. EVALUATE:
   A $2fg\,(2)$ **B** $\left(\dfrac{f}{g} - 2f\right)(3)$ **C** $(f - g)(4)$
6. IS IT ALWAYS POSSIBLE TO DEDUCE THE DOMAIN OF
   I $f + g$ **II** $f - g$ **III** $f \,.\, g$ **IV** $\dfrac{f}{g}$
   FROM THE DOMAIN OF f AND g? IF YOUR ANSWER IS YES, HOW?

4.3 GRAPHS OF FUNCTIONS

IN THIS SECTION, YOU WILL LEARN HOW TO DRAW GRAPHS OF FUNCTIONS, WITH SPECIA LINEAR AND QUADRATIC FUNCTIONS. YOU WILL ALSO STUDY SOME OF THE IMPORTAN THESE GRAPHS.

4.3.1 Graphs of Linear Functions

Definition 4.4

If a and b are fixed real numbers, $a \neq 0$, then $f(x) = ax + b$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$, then $f(x) = b$ is called a constant function. Sometimes linear functions are written as $y = ax + b$.

EXAMPLE 1 $f(x) = 2x + 1$ IS A LINEAR FUNCTION WITH $a = 2$ AND $b = 1$

EXAMPLE 2 $f(x) = 3$ IS A CONSTANT FUNCTION.

FROM SECTION 4.2.1 RECALL THAT FUNCTIONS ARE SPECIAL TYPES OF RELATIONS. HENCE LINEAR FUNCTION IS ALSO A RELATION. FROM THE DESCRIPTION WE USED FOR RELATIONS, LI CAN ALSO BE DESCRIBED AS

$$R = \{(x, y): y = ax + b;\ x, y \in \mathbb{R}\};\ \text{OR}\ R = \{(x, f(x)): f(x) = ax + b;\ x, y \in \mathbb{R}\}$$

WHAT ARE THE PROPERTIES OF LINEAR FUNCTIONS? WHAT DO a AND b STAND FOR?

DRAWING GRAPHS OF LINEAR FUNCTIONS WILL HELP US TO ANSWER THESE QUESTIONS HOW TO EVALUATE FUNCTIONS:

EXAMPLE 3 IF $f(x) = 3x - 1$, THEN $f(2) = 3(2) - 1 = 6 - 1 = 5$.

YOU WILL NOW EVALUATE FUNCTIONS AT SELECTED POINTS FROM THE DOMAIN A THESE POINTS TO DRAW GRAPHS OF LINEAR FUNCTIONS.

EXAMPLE 4 CONSIDER THE LINEAR FUNCTION $f(x) = 2x + 3$.

EVALUATE THE VALUES OF THE FUNCTION FOR THE x VALUES IN THE TABLE BELOW.

x	–3	–2	–1	0	1	2	3
$f(x)$							

AT $x = -3$, $f(-3) = 2(-3) + 3 = -3$ AND AT $x = -2$, $f(-2) = 2(-2) + 3 = -1$.

IN THE SAME WAY, $f(-1) = 1$; $f(0) = 3$; $f(1) = 5$; $f(2) = 7$; AND $f(3) = 9$. SO THE TABLE BECOMES

x	–3	–2	–1	0	1	2	3
$f(x)$	–3	–1	1	3	5	7	9

THIS TABLE IS PAIRING THE VALUES OF x AND $f(x)$. THIS IS TAKEN AS A REPRESENTATIVE OF

$$R = \{(-3, -3), (-2, -1), (-1, 1), (0, 3), (1, 5), (2, 7), (3, 9)\}$$

NOW YOU CAN PLOT THESE POINTS IN A COORDINATE SYSTEM TO DRAW THE GRAPH FUNCTION.

EXAMPLE 5 DRAW THE GRAPH OF THE LINEAR FUNCTION $f(x) = -2x + 3$.

SOLUTION:

A FIRST YOU CONSTRUCT A TABLE OF VALUES FROM THE DOMAIN.

x	–3	–2	–1	0	1	2	3
$f(x)$	9	7	5	3	1	–1	–3

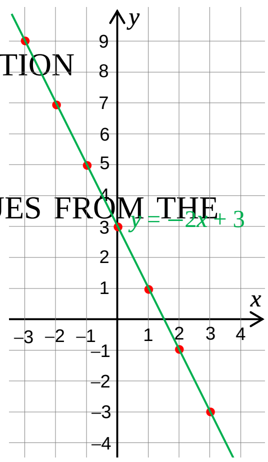


Figure 4.13

B NOW YOU PLOT THESE POINTS ON THE COORDINATE SYSTEM AND DRAW A LINE THROUGH THESE POINTS. THIS LINE IS THE GRAPH OF THE FUNCTION $f(x) = -2x + 3$. (*see* Figure 4.12).

EXAMPLE 6 DRAW THE GRAPH OF THE CONSTANT FUNCTION

$f(x) = 2$.

SOLUTION: YOU CONSTRUCT A TABLE OF VALUES OF x AND $f(x)$, PLOT THE ORDERED PAIRS AND DRAW A LINE THROUGH THE POINTS TO GET THE REQUIRED GRAPH.

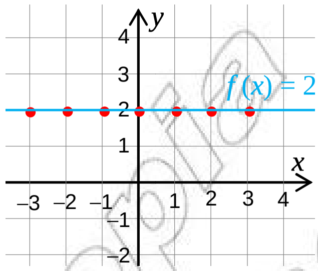
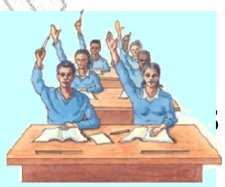


Figure 4.14

x	–3	–2	–1	0	1	2	3
$f(x)$	2	2	2	2	2	2	2

ACTIVITY 4.7



WRITE DOWN WHAT YOU OBSERVE FROM THE GRAPHS OF THE LINEAR FUNCTIONS DRAWN ABOVE.

IN A LINEAR FUNCTION $f(x) = ax + b$, a IS CALLED THE coefficient OF x. THIS a IS ALSO THE SLOPE OF THE GRAPH OF THE LINEAR FUNCTION. FROM THE GRAPHS, YOU SHOULD HAVE NOTICED THAT:

I GRAPHS OF LINEAR FUNCTIONS ARE STRAIGHT LINES.

II IF $a > 0$, THEN THE GRAPH OF THE LINEAR FUNCTION $f(x) = ax + b$ IS INCREASING.

III IF $a < 0$, THEN THE GRAPH OF THE LINEAR FUNCTION $f(x) = ax + b$ IS DECREASING.

IV IF $a = 0$, THEN THE GRAPH OF THE CONSTANT FUNCTION $f(x) = b$ IS A HORIZONTAL LINE.

V IF $x = 0$, THEN $f(0) = b$. THIS MEANS $(0, b)$ LIES ON THE GRAPH OF THE FUNCTION, AND THE GRAPH PASSES THROUGH THE POINT $(0, b)$. THIS POINT IS CALLED THE y-intercept. IT IS THE POINT AT WHICH THE GRAPH INTERSECTS THE y-AXIS.

VI IF $f(x) = 0$, THEN $0 = ax + b \Rightarrow x = \frac{-b}{a}$. THIS MEANS $(\frac{-b}{a}, 0)$ LIES ON THE GRAPH OF THE FUNCTION AND THE GRAPH PASSES THROUGH THE POINT $(\frac{-b}{a}, 0)$. THIS POINT IS CALLED THE x-intercept. IT IS THE POINT AT WHICH THE GRAPH INTERSECTS THE x-AXIS.

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EXAMPLE 7 FOR THE LINEAR FUNCTION $f(x) = 7x + 2$, DETERMINE THE y-INTERCEPT AND THE x-INTERCEPT.

SOLUTION: AT THE y-INTERCEPT $x = 0$ AND $f(0) = 2$. SO THE y-INTERCEPT IS $(0, 2)$.

AT THE x-INTERCEPT $y = 0$ AND $0 = 7x + 2 \Rightarrow x = -\frac{2}{7}$. SO THE x-INTERCEPT IS $\left(-\frac{2}{7}, 0\right)$.

EXAMPLE 8 IS THE GRAPH OF THE LINEAR FUNCTION $f(x) = 2 - 2x$ INCREASING OR DECREASING?

SOLUTION: SINCE $f(x) = 2 - 2x$ IS THE SAME AS $f(x) = -2x + 2$ AND THE COEFFICIENT OF x IS -2, THE GRAPH IS DECREASING.

YOU HAVE LEARNT HOW TO USE TABLE OF VALUES OF A LINEAR FUNCTION TO DRA IS ALSO POSSIBLE TO DRAW THE GRAPH OF A LINEAR FUNCTION BY USING THE x-INTERCEPT AND THE y-INTERCEPT.

EXAMPLE 9 DRAW THE GRAPH OF $f(x) = 4x - 4$.

SOLUTION: THE x-INTERCEPT IS THE ORDERED PAIR WITH $y = 0$. THAT IS, $(1, 0)$.

THE y-INTERCEPT IS THE ORDERED PAIR WITH $x = 0$. THAT IS, $(0, -4)$.

PLOT THESE INTERCEPTS ON A COORDINATE SYSTEM AND DRAW A LINE THAT PASSES THROUGH THEM.

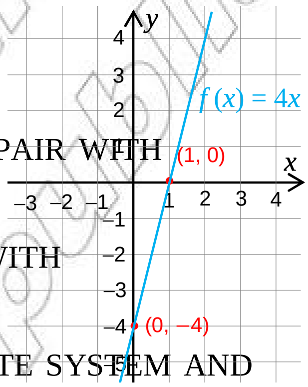

Figure 4.15

YOU CAN ALSO USE THE CONCEPT OF SLOPE FOR DRAWING THE GRAPH OF LINEA FUNCTIONS. TO DRAW THE GRAPH OF A LINEAR FUNCTION $f(x) = ax + b$, FIRST MARK THE y-INTERCEPT. THEN FROM THE y-INTERCEPT MOVE a UNITS UP (IF $a > 0$) OR a UNITS DOWN (IF $a < 0$) AND ONE UNIT TO THE RIGHT, AND LOCATE A POINT. THEN, DRAW THE LINE THAT PAS THE y-INTERCEPT AND THIS POINT. THIS LINE IS THE GRAPH OF THE LINEAR FUNCTIO

EXAMPLE 10 DRAW THE GRAPH OF $f(x) = 2x + 1$.

SOLUTION: THE SLOPE OF THE GRAPH OF THE LINEAR FUNCTION $f(x) = 2x + 1$ IS 2 AND THE y-INTERCEPT IS $(0, 1)$.

IF YOU MOVE 2 UNITS UP FROM THE y-INTERCEPT AND ONE UNIT TO THE RIGHT, YOU WILL GET THE POINT $(1, 3)$. SO THE LINE THAT PASSES THROUGH $(0, 1)$ AND $(1, 3)$ IS THE GRAPH OF THE FUNCTION $f(x) = 2x + 1$.

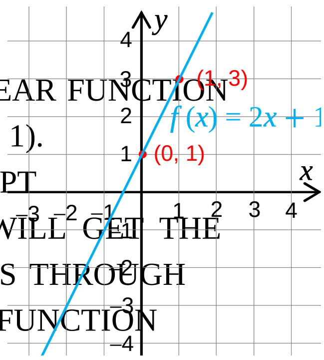

Figure 4.16

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EXAMPLE 11 DRAW THE GRAPH OF THE LINEAR FUNCTION $f(x) = -3x + 1$.

SOLUTION: THE SLOPE OF THE GRAPH OF THE LINEAR FUNCTION $f(x) = -3x + 1$ IS –3 AND THE y-INTERCEPT IS (0, 1).

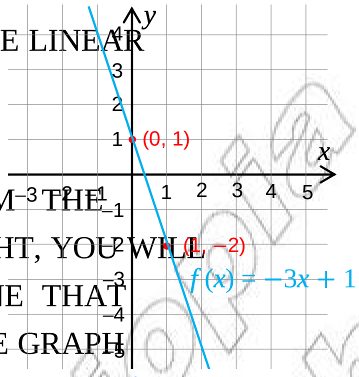

Figure 4.17

IF YOU MOVE 3 UNITS DOWN FROM THE y-INTERCEPT AND ONE UNIT TO THE RIGHT, YOU WILL GET THE POINT (1, –2). THEN THE LINE THAT PASSES THROUGH (0, 1) AND (1, –2) IS THE GRAPH OF THE FUNCTION $f(x) = -3x + 1$.

Exercise 4.6

1 DETERMINE WHETHER EACH OF THE FOLLOWING IS A LINEAR FUNCTION OR NOT.

A $f(x) - 1 = 3x$ **B** $3 = x - 2y$

C $x + y = 1 - 3x$ **D** $2x^2 - 2x = y$

2 CONSTRUCT TABLES OF VALUES OF THE FOLLOWING FUNCTIONS FOR THE GIVEN DC

A $f(x) = 2x - 1$; A = {–1, 1, 2, 3} **B** $y = \frac{x}{3} - 1$; A = { –6, –3, 0, 3, 6}

C $f(x) = 1 - 3x$; A = {–3, –2, –1, 0, 1, 2, 3}

3 DETERMINE THE SLOPE, y-INTERCEPT AND x-INTERCEPT OF EACH OF THE FOLLOWING LINEA FUNCTIONS:

A $x + y - 1 = 0$ **B** $f(x) = 3x - 4$

C $y - 3 = x$ **D** $f(x) - 5 = 3x$

4 STATE IF THE GRAPH OF EACH OF THE FOLLOWING LINEAR FUNCTIONS IS INCREASING OR DECREASING:

A $3x - 2 = 2y$ **B** $y - 2x + 5 = 1$

C $f(x) - 7 = 2$ **D** $f(x) = 4$

5 DRAW THE GRAPH OF EACH OF THE FOLLOWING BY CONSTRUCTING A TABLE OF VA

$-3 \le x \le 3$:

A $y - 3x - 5 = 4$ **B** $4 = 4x - 2y$

C $f(x) = 1 - 7x$ **D** $y = 1$

6 DRAW THE GRAPH OF EACH OF THE FOLLOWING BY USING THE INTERCEPTS:

A $3x - 5 = y$ **B** $4 + 2y = 4x$ **C** $f(x) = 3x - 5$

7 DRAW THE GRAPH OF EACH OF THE FOLLOWING BY USING THE VALUE OF SLOPE:

A $3y - 3x - 5 = 4$ **B** $f(x) = 4x + 2$ **C** $3x - 4 = 5x - 2y$

4.3.2 Graphs of Quadratic Functions

IN THE PREVIOUS SUB-SECTION, YOU HAVE DISCUSSED LINEAR FUNCTIONS, THEIR GRAPHS AND IMPORTANT PROPERTIES. IN THIS SUB-SECTION, YOU WILL LEARN ABOUT QUADRATIC FUNCTIONS, THEIR GRAPHS AND SOME PROPERTIES THAT INCLUDE THE MINIMUM AND MAXIMUM OF FUNCTIONS.

Definition 4.5

A function defined by $f(x) = ax^2 + bx + c$ where a, b, $c \in \mathbb{R}$ and $a \neq 0$ is called a quadratic function. a is called the leading coefficient.

EXAMPLE 1 $f(x) = 2x^2 + 3x + 2$ IS A QUADRATIC FUNCTION WITH $a = 2$, $b = 3$ AND $c = 2$.

Note: ANY FUNCTION THAT CAN BE REDUCED TO $f(x) = ax^2 + bx + c$ IS ALSO CALLED A quadratic function.

EXAMPLE 2 $f(x) = (x - 2)(x + 2)$ CAN BE EXPRESSED AS $f(x) = x^2 - 4$.

SO $f(x) = (x - 2)(x + 2)$ IS A QUADRATIC FUNCTION WITH $a = 1$, $b = 0$, AND $c = -4$.

LET US NOW DRAW GRAPHS OF QUADRATIC FUNCTIONS BY CONSTRUCTING TABLES OF VALUES.

ACTIVITY 4.8

1 CONSTRUCT A TABLE OF VALUES FOR EACH OF THE FOLLOWING FUNCTIONS, FOR $-3 \leq x \leq 3$:

A $f(x) = x^2$ B $f(x) = x^2 + 3x + 2$ AND C $f(x) = -2x^2 + x - 4$

2 USING THE TABLE IN QUESTION 1, PLOT THE POINTS $(x, f(x))$ ON THE xy-COORDINATE SYSTEM. CONNECT THOSE POINTS BY SMOOTH CURVES.

3 DISCUSS THE TYPE OF CURVES YOU OBTAINED.

THE GRAPH OF A QUADRATIC FUNCTION IS A CURVE CALLED A parabola.

EXAMPLE 3 DRAW THE GRAPH OF $f(x) = -x^2$.

SOLUTION: THE TABLE OF VALUES IS:

x	–2	–1	0	1	2
$f(x)$	–4	–1	0	–1	–4

THE GRAPH IS SHOWN IN FIGURE 4.18

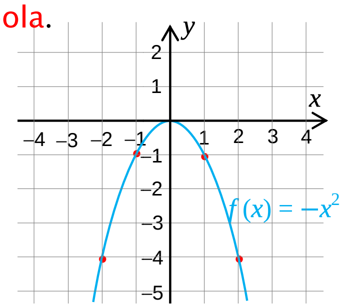

Figure 4.18

ACTIVITY 4.9

WRITE DOWN WHAT YOU OBSERVE FROM THE GRAPHS OF THE QUADRATIC FUNCTIONS DRAWN ABOVE.

You may have noticed that:

- **i** The graph of the parabola is opened either upward or downward depending on the sign of the coefficient of x^2.
- **ii** There is a turning point on the graphs.
- **iii** These graphs are symmetrical.

The turning point of the graph of a quadratic function is called the vertex of the parabola and the vertical line that passes through the vertex is called the axis of the parabola.

EXAMPLE 4 For the quadratic function $f(x) = x^2$, determine the vertex and the axis of the parabola.

SOLUTION: The graph of the quadratic function $f(x) = x^2$ is as given. The vertex of the parabola is (0, 0) and the axis of the parabola is y-axis.

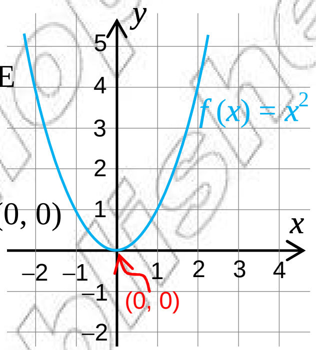

Figure 4.19

Having drawn the graphs of $f(x) = x^2$ and $f(x) = -x^2$, you shall now examine quadratic functions of the type $f(x) = ax^2 + c$ for some $c \in \mathbb{R}$.

Group Work 4.6

1 Using the same coordinate system, sketch the g the following quadratic functions by using table of v

i	**A** $f(x) = 3x^2$		**B** $f(x) = 3x^2 - 1$		**C** $f(x) = 3x^2 + 1$	
ii	**A** $f(x) = -3x^2$		**B** $f(x) = -3x^2 - 1$		**C** $f(x) = -3x^2 + 1$	

2 Write down your observations from the graphs and discuss in groups.

3 Can you sketch these graphs using some other methods? Explain and discuss.

Sketching graphs of quadratic function using a table of values

EXAMPLE 5 Sketch the graph of $f(x) = 2x^2$.

x	–2	–1	0	1	2
$f(x)$	8	2	0	2	8

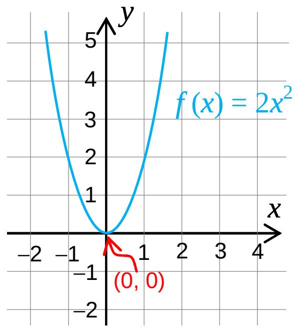

Figure 4.20

EXAMPLE 6 SKETCH THE GRAPH OF $f(x) = 2x^2 - 3$.

x	–2	–1	0	1	2
$f(x)$	5	–1	–3	–1	5

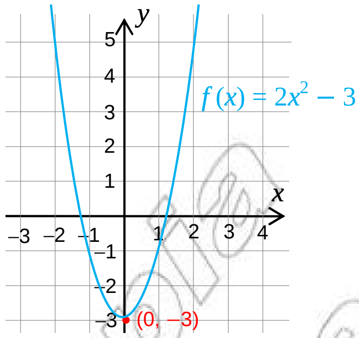

Figure 4.21

EXAMPLE 7 SKETCH THE GRAPH OF $f(x) = 2x^2 + 3$.

x	–2	–1	0	1	2
$f(x)$	11	5	3	5	11

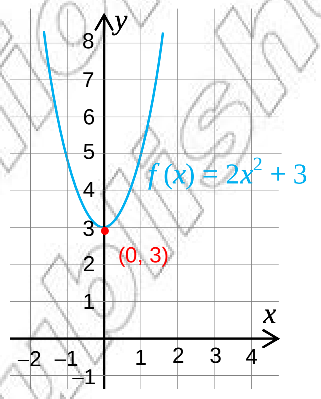

Figure 4.22

OBSERVE THAT THE GRAPHS ARE ALL PARABOLAS AND THEY ALL OPEN UPWARD BUT T IN DIFFERENT PLACES. ALSO NOTE THAT THE CORRESPONDING VALUES OF $f(x) = 2x^2 + 3$ ARE 3 UNITS MORE THAN THE VALUES OF $f(x) = 2x^2$ AND THE CORRESPONDING VALUES OF $f(x) = 2x^2 - 3$ ARE 3 UNITS LESS THAN THE VALUES OF $f(x) = 2x^2$. USING THIS, THE GRAPHS OF THE FUNCTIONS OF

$f(x) = 2x^2 - 3$ AND $f(x) = 2x^2 + 3$ CAN BE OBTAINED FROM THE GRAPH OF $f(x) = 2x^2$.

THIS LEADS US TO ANOTHER WAY OF SKETCHING GRAPHS OF QUADRATIC FUNCTIONS.

FROM GRAPHS OF QUADRATIC FUNCTIONS OF THE FORM

$f(x) = ax^2$ AND $f(x) = ax^2 + c$, $a \neq 0$, $c \in \mathbb{R}$, WE CAN SUMMARIZE:

Case 1: IF $a > 0$,

1. THE GRAPH OPENS UPWARD.
2. THE VERTEX IS (0, 0) FOR $f(x) = ax^2$ AND (0, c) FOR $f(x) = ax^2 + c$.
3. THE DOMAIN IS ALL REAL NUMBERS.
4. THE RANGE IS $\{y: y \geq 0\}$ FOR $f(x) = ax^2$ AND $\{y: y \geq c\}$ FOR $f(x) = ax^2 + c$.
5. THE VERTICAL LINE THAT PASSES THROUGH THE VERTEX IS THE AXIS OF THE PARABOLA (OR THE AXIS OF SYMMETRY).

Case 2: IF $a < 0$,

1. THE GRAPH OPENS DOWNWARD.
2. THE VERTEX IS (0, 0) FOR $f(x) = ax^2$ AND $(0, c)$ FOR $f(x) = ax^2 + c$.
3. THE DOMAIN IS ALL REAL NUMBERS.
4. THE RANGE IS $\{y: y \le 0\}$ FOR $f(x) = ax^2$ AND $\{y: y \le c\}$ FOR $f(x) = ax^2 + c$.
5. THE VERTICAL LINE THAT PASSES THROUGH THE VERTEX IS THE AXIS OF THE PARABOLA (OR THE AXIS OF SYMMETRY).

Sketching graphs of quadratic functions, using the shifting rule

SO FAR WE HAVE USED TABLES OF VALUES TO SKETCH GRAPHS OF QUADRATIC FUNCTIONS. NOW WE SHALL SEE HOW TO USE THE SHIFTING RULE TO SKETCH THE GRAPHS OF QUADRATIC FUNCTIONS. AS YOU HAVE SEEN IN EXAMPLES 5, 6 AND 7, YOU CAN SKETCH THE GRAPH OF $f(x) = 2x^2 + 3$ BY SHIFTING THE GRAPH OF $f(x) = 2x^2$ BY 3 UNITS UPWARD, AND THE GRAPH OF $f(x) = 2x^2 - 3$ CAN BE OBTAINED BY SHIFTING THE GRAPH OF $f(x) = 2x^2$ BY 3 UNITS DOWNWARD.

EXAMPLE 8 SKETCH THE GRAPH OF $f(x) = x^2 - 1$ AND $f(x) = x^2 + 1$ BY SHIFTING $f(x) = x^2$ AND DETERMINE THE VERTEX OF EACH GRAPH.

SOLUTION: THE GRAPH OF $f(x) = x^2$ IS AS SHOWN IN FIGURE 4.23A.

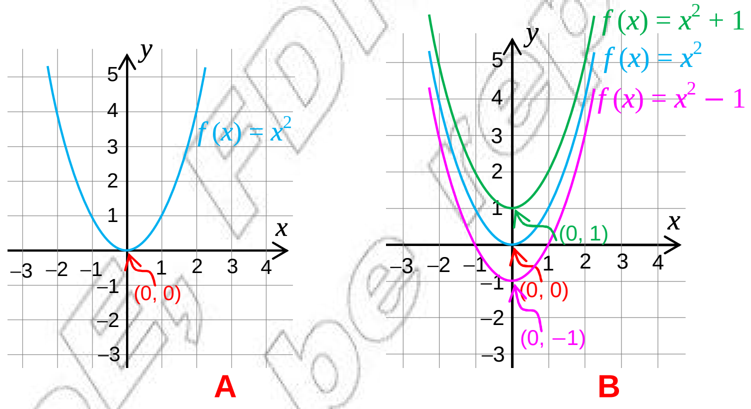

Figure 4.23

THE GRAPH OF $f(x) = x^2 - 1$ IS OBTAINED BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY 1 UNIT DOWNWARD GIVING A VERTEX AT $(0, -1)$. THE GRAPH OF $f(x) = x^2 + 1$ IS OBTAINED BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY 1 UNIT UPWARD, TO A VERTEX AT (0, 1). *See* FIGURE 4.23B.

EXAMPLE 9 SKETCH THE GRAPH OF

$f(x) = (x - 3)^2$ AND CONTRAST IT WITH THE GRAPH OF $f(x) = x^2$.

SOLUTION: BY CONSTRUCTING A TABLE OF VALUES, YOU CAN DRAW THE GRAPH OF $f(x) = (x - 3)^2$ AND SEE THAT IT IS A SHIFTING OF THE GRAPH OF $f(x) = x^2$ BY 3 UNITS TO THE RIGHT. THE VERTEX OF THE GRAPH IS (3, 0). (*See* FIGURE 4.24).

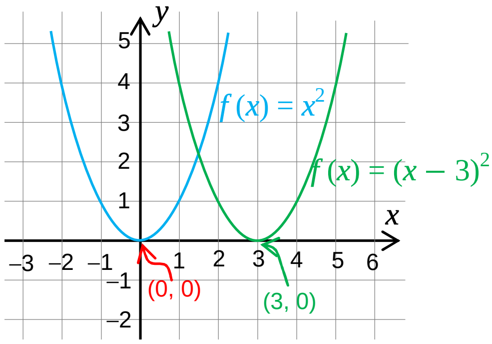

Figure 4.24

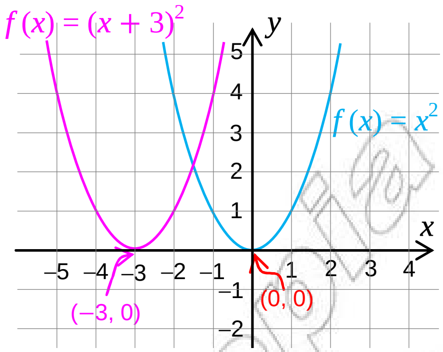

Figure 4.25

EXAMPLE 10 SKETCH THE GRAPH OF

$f(x) = (x + 3)^2$ AND CONTRAST IT WITH THE GRAPH OF $f(x) = x^2$.

SOLUTION: USING A TABLE OF VALUES, YOU GET THE GRAPH OF $f(x) = (x + 3)^2$ AND SEE THAT IT IS A SHIFTING OF THE GRAPH OF $f(x) = x^2$ BY 3 UNITS TO THE LEFT, GIVING A VERTEX AT (–3, 0) (*See* FIGURE 4.25).

LET $k > 0$, THEN THE GRAPH OF $f(x) = (x - k)^2$ IS OBTAINED BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY k UNITS TO THE RIGHT AND THE GRAPH OF $f(x) = (x + k)^2$ IS OBTAINED BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY k UNITS TO THE LEFT.

BY SHIFTING THE GRAPH OF $f(x) = x^2$ IN THE x AND y DIRECTIONS YOU CAN SKETCH GRAPHS OF QUADRATIC FUNCTIONS SUCH AS

A $f(x) = (x + 3)^2 + 2$ **B** $f(x) = (x - 3)^2 - 2$ **C** $f(x) = x^2 + 4x + 2$

EXAMPLE 11 SKETCH THE GRAPH OF $f(x) = (x + 3)^2 + 2$

SOLUTION: FIRST SKETCH THE GRAPH OF $f(x) = x^2$. TO OBTAIN THE GRAPH OF $f(x) = (x + 3)^2$ SHIFT THE GRAPH OF $f(x) = x^2$ TO THE LEFT BY 3 UNITS.

AFTER THIS, TO OBTAIN THE GRAPH OF $f(x) = (x + 3)^2 + 2$ SHIFT THE GRAPH OF $f(x) = (x + 3)^2$ BY 2 UNITS UPWARD.

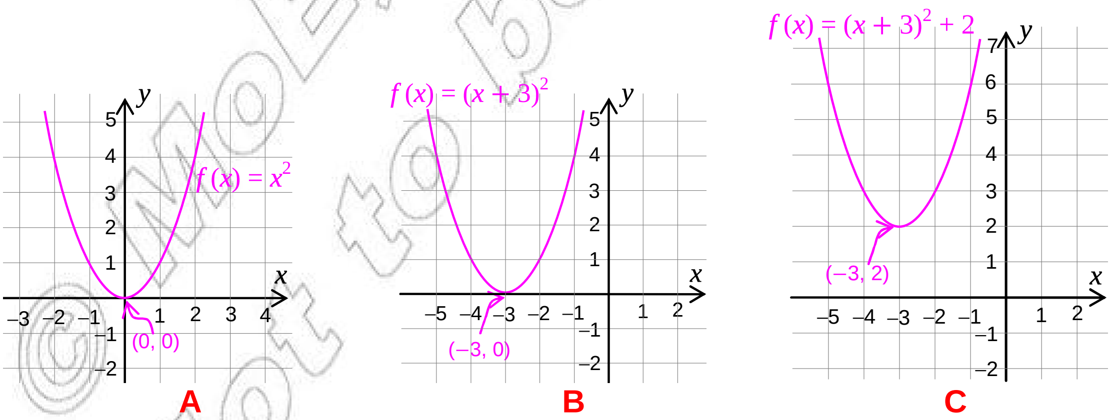

Figure 4.26

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EXAMPLE 12 SKETCH THE GRAPH OF $f(x) = (x - 3)^2 - 2$.

SOLUTION: FIRST SKETCH THE GRAPH OF $f(x) = x^2$.

TO OBTAIN THE GRAPH OF $f(x) = (x - 3)^2$ SHIFT THE GRAPH OF $f(x) = x^2$ TO THE RIGHT BY 3 UNITS SO THAT THE VERTEX IS AT (3, 0). AFTER THIS, TO OBTAIN THE GRAPH OF $f(x) = (x - 3)^2 - 2$, SHIFT THE GRAPH OF $f(x) = (x - 3)^2$ BY 2 UNITS DOWNWARD SO THAT THE VERTEX IS AT (3, –2).

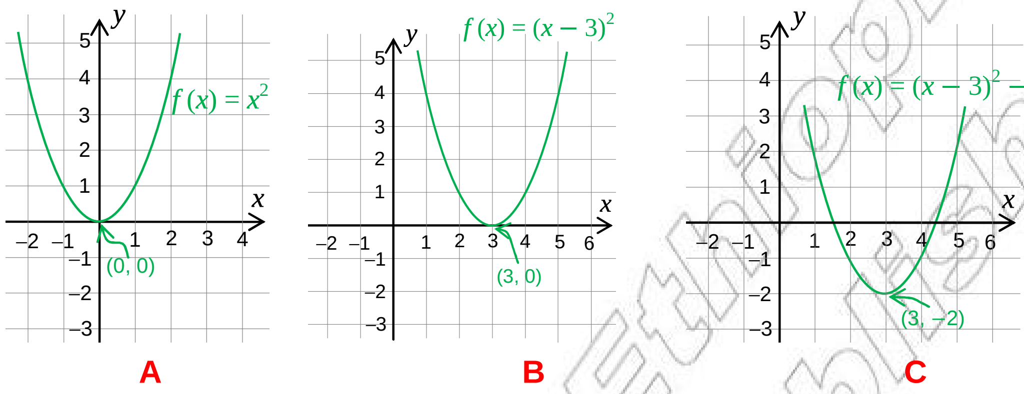

Figure 4.27

EXAMPLE 13 SKETCH THE GRAPH OF $f(x) = x^2 + 4x + 2$.

SOLUTION: IN ORDER TO SKETCH THE GRAPH OF $f(x) = x^2 + 4x + 2$, FIRST WE NEED TO TRANSFORM THIS FUNCTION INTO THE FORM OF $f(x) = (x + k)^2 + c$ BY COMPLETING THE SQUARE.

THEREFORE $f(x) = x^2 + 4x + 2$ CAN BE EXPRESSED AS

$$f(x) = (x + 2)^2 - 2$$

NOW YOU CAN SKETCH THE GRAPH OF $f(x) = (x + 2)^2 - 2$ AS ABOVE BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY 2 UNITS TO THE LEFT AND THEN BY 2 UNITS DOWNWARD.

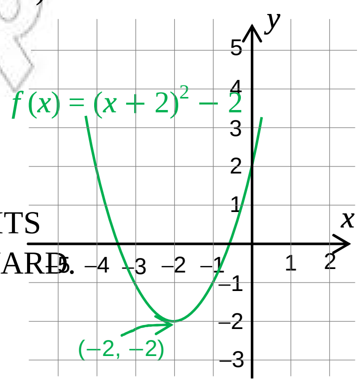

Figure 4.28

Note:

1 THE GRAPH OF $f(x) = (x + k)^2 + c$ OPENS UPWARD.

2 THE VERTEX OF THE GRAPH OF $f(x) = (x + k)^2 + c$ IS $(-k, c)$ AND THE VERTEX OF THE GRAPH OF $f(x) = (x - k)^2 - c$ IS $(k, -c)$. SIMILARLY THE VERTEX OF THE GRAPH OF $f(x) = (x + k)^2 - c$ IS $(-k, -c)$ AND THE VERTEX OF THE GRAPH OF $f(x) = (x - k)^2 + c$ IS (k, c).

Minimum and maximum values of quadratic functions

SUPPOSE YOU THROW A STONE UPWARD. THE STONE TURNS DOWN AFTER IT REACHES ITS MAXIMUM HEIGHT. SIMILARLY, A PARABOLA TURNS AFTER IT REACHES A MAXIMUM OR A MINIMUM VALUE.

Group Work 4.7

1 LET $f(x)$ BE A QUADRATIC FUNCTION. DISCUSS HOW TO DETERMINE THE MAXIMUM OR MINIMUM VALUE OF $f(x)$.

2 JUSTIFY YOUR CONCLUSION BY CONSIDERING SOME PARABOLAS.

RECALL THAT IF THE LEADING COEFFICIENT OF THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ IS POSITIVE ($a > 0$), THEN THE GRAPH OF THE FUNCTION OPENS UPWARD (IF $a < 0$, THEN THE GRAPH OPENS DOWNWARD). WHEN THE GRAPH OF A QUADRATIC FUNCTION OPENS U FUNCTION HAS A MINIMUM VALUE, WHEREAS IF THE GRAPH OPENS DOWNWARD, MAXIMUM VALUE. THE MINIMUM OR THE MAXIMUM VALUE OF A QUADRATIC FUNC OBTAINED AT THE VERTEX OF ITS GRAPH.

EXAMPLE 14 THE MINIMUM VALUE OF A QUADRATIC FUNCTION EXPRESSED AS

$$f(x) = (x + k)^2 + c \text{ IS } c$$

SIMILARLY, THE MAXIMUM VALUE OF $f(x) = -(x + k)^2 + c$ IS c.

EXAMPLE 15 SKETCH THE GRAPH OF $f(x) = x^2 + 6x - 5$ AND DETERMINE THE MINIMUM VALUE OF $f(x)$.

SOLUTION: $f(x) = x^2 + 6x + 9 - 9 - 5 = (x + 3)^2 - 14$.

HENCE THE GRAPH CAN BE SKETCHED BY SHIFTING THE GRAPH OF $f(x) = x^2$ BY 3 UNITS TO THE LEFT SIDE AND THEN DOWNWARD BY 14 UNITS.

HENCE, THE MINIMUM VALUE OF f IS -14.

IN THIS CASE, THE RANGE OF THE FUNCTION IS

$\{y: y \geq -14\} = [-14, \infty)$.

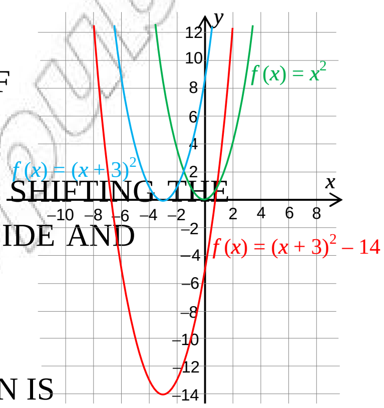

Figure 4.29

EXAMPLE 16 FIND THE MAXIMUM VALUE OF THE FUNCTION $f(x) = -x^2 + 6x - 8$, AND SKETCH ITS GRAPH.

SOLUTION: $f(x) = -x^2 + 6x - 9 + 9 - 8$

$= -(x^2 - 6x + 9) + 1;$

$f(x) = -(x - 3)^2 + 1.$

THE GRAPH OF $f(x) = -(x - 3)^2 + 1$ HAS VERTEX $(3, 1)$ AND HENCE THE MAXIMUM VALUE OF f IS 1.

IN THIS CASE, THE RANGE OF THE FUNCTION IS

$\{y: y \leq 1\} = (-\infty, 1]$

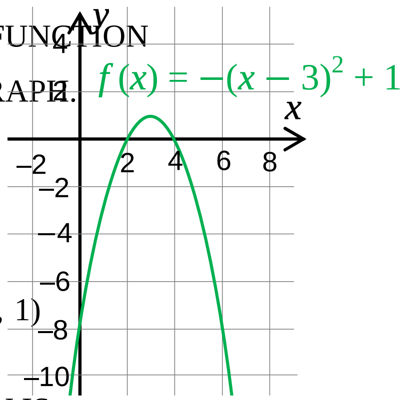

Figure 4.30

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Exercise 4.7

1 FOR EACH OF THE FOLLOWING QUADRATIC FUNCTIONS, a, b AND c:

A $f(x) = 2 + 3x - 2x^2$ **B** $f(x) = 3x^2 - 4x + 1$ **C** $f(x) = (x - 3)(2 - x)$

2 FOR EACH OF THE FOLLOWING QUADRATIC PREPARE A TABLE OF VALU INTERVAL $-3 \le x \le 3$.

A $f(x) = -4x^2$ **B** $f(x) = 3x^2 + 2$ **C** $f(x) = 2x^2 - 3x + 2$

3 SKETCH THE GRAPH FOR EACH OF THE FOLLOWING QUADRATIC FUNCTIONS BY S OF VALUES:

A $f(x) = -3x^2$ **B** $f(x) = 7x^2 - 3$ **C** $f(x) = 2x^2 + 6x + 1$

4 FIND THE DOMAIN AND REACH OF THE FOLLOWING FU

A $f(x) = 3 + 4x - x^2$ **B** $f(x) = x^2 + 2x + 1$ **C** $f(x) = (x - 3)(x - 2)$

D $f(x) = -3x^2 - 2$ **E** $f(x) = 3x^2 + 2$

5 SKETCH THE GRAPH EACH OF THE FOLLOWING QUADRATIC FUSING THE SHIFTING RULE:

A $f(x) = 9x^2 + 1$ **B** $f(x) = x^2 - 3$ **C** $f(x) = (x - 5)^2$

D $f(x) = (x - 2)^2 + 13$ **E** $f(x) = (x + 1)^2 - 7$ **F** $f(x) = 4x^2 + 7x + 3$

6 FIND THE VERTEX AND THE AXIS OF SYMMETRY OF THE F:

A $f(x) = x^2 - 5x + 8$ **B** $f(x) = (x - 4)^2 - 3$ **C** $f(x) = x^2 - 8x + 3$

7 DETERMINE THE MINIMUM OR THE VALUE OF EACH OF THE FOLLOWING FU AND DRAW THE G

A $f(x) = x^2 + 7x - 10$ **B** $f(x) = x^2 + 4x + 1$ **C** $f(x) = 2x^2 - 4x + 3$

D $f(x) = 4x^2 + 2x + 4$ **E** $f(x) = -x^2 - 4x$ **F** $f(x) = -6 - x^2 - 4x$

Key Terms

axis of symmetry	leading coefficient	turning point
combination of functions	linear functions	vertex
constant function	quadratic function	x-intercept
coordinate system	relation	y-intercept
domain	range	
function	slope	

Summary

1 IN A RELATION, TWO THINGS ARE RELATED TO EACH OTHER BY A RELATING PH

2 MATHEMATICALLY, A RELATION IS A SET OF ORDERED PAIRS. IF A AND B ARE NON-EMPTY SETS, THEN THE RELATION FROM A TO B IS A SUBSET OF A × B THAT SATISFIES THE RELATING PHRASE.

3 IF A AND B ARE ANY SETS, R ⊆ (A × B), WE CALL R A BINARY RELATION FROM A TO B OR A BINARY RELATION BETWEEN A AND B. A RELATION R ⊆ (A × A) IS CALLED A RELA OR ON A.

4 THE SET {x: (x, y) ∈ R FOR SOME y} IS CALLED THE DOMAIN OF THE REL.

THE SET {y: (x, y) ∈ R FOR SOME x} IS CALLED THE RANGE OF THE

5 A FUNCTION IS A SPECIAL TYPE OF RELATION IN WHICH EACH x-COORDINATE IS PAIRED WITH EXACTLY ONE UNIQUE y-COORDINATE.

6 A FUNCTION FROM A TO B CAN SOMETIMES BE WRITTEN f: A→B, WHERE THE DO OF f IS A AND THE RANGE OF f IS A SUBSET OF B, IN WHICH CASE B CONTAINS T THE ELEMENTS OF A BY THE f.

7 LET f AND g BE FUNCTIONS. WE DEFINE THE SUM $f + g$, THE DIFFERENCE $f - g$, THE PRODUCT fg, AND THE QUOTIENT $\frac{f}{g}$ AS:

$$f + g : (f + g)(x) = f(x) + g(x) \qquad fg : (fg)(x) = f(x)\, g(x)$$

$$f - g : (f - g)(x) = f(x) - g(x) \qquad \frac{f}{g} : \frac{f}{g}(x) = \frac{f(x)}{g(x)};\ g(x) \neq 0$$

8 IF a AND b ARE FIXED REAL NUMBERS, $a \neq 0$, THEN $f(x) = ax + b$ FOR $x \in \mathbb{R}$ IS CALLED A LINEAR FUNCTION. IF $a = 0$ THEN $f(x) = b$ IS CALLED A CONSTANT FUNCTION. LINEAR FUNCTIONS ARE $y = ax + b$.

9 IN $f(x) = ax + b$ FOR $a \neq 0$, $x \in \mathbb{R}$, a REPRESENTS THE SLOPE, (0, b) REPRESENTS THE y-INTERCEPT AND $\left(\frac{-b}{a}, 0\right)$ REPRESENTS THE x-INTERCEPT.

10 A FUNCTION DEFINED BY $f(x) = ax^2 + bx + c$ (a, b, $c \in \mathbb{R}$ AND $a \neq 0$) IS CALLED QUA FUNCTION. a IS CALLED THE LEADING C

11 WE CAN SKETCH THE GRAPH OF A LINEAR FUNCTION EITHER BY USING A TABLE OF VALUES OR THE x- AND y-INTERCEPTS.

12 WE CAN SKETCH THE GRAPH OF A QUADRATIC FUNCTION EITHER BY USING A TABLE OF VALUES OR THE SHIFTING RULE.

13 THE GRAPH OF $f(x) = ax^2 + bx + c$ OPENS UPWARD IF $a > 0$ AND DOWNWARD IF $a < 0$.

14 THE VERTEX IS THE POINT ON A COORDINATE SYSTEM AT WHICH THE GRAPH OF A QUADRATIC FUNCTION TURNS EITHER UPWARD OR DOWNWARD.

15 THE AXIS OF A PARABOLA (OR AXIS OF SYMMETRY) IS THE VERTICAL LINE THAT PASSES THROUGH THE VERTEX OF THE PARABOLA.

16 THE DOMAIN AND RANGE OF LINEAR FUNCTIONS IS THE SET OF ALL REAL NUMBERS.

17 THE DOMAIN OF A QUADRATIC FUNCTION IS THE SET OF ALL REAL NUMBERS, WHEREAS THE RANGE IS;

$\{y: y \geq k\}$ IF THE LEADING COEFFICIENT IS POSITIVE AND k IS THE VALUE OF y OF THE VERTEX.

$\{y: y \leq k\}$ IF THE LEADING COEFFICIENT IS NEGATIVE AND k IS THE VALUE OF y OF THE VERTEX.

18 THE MAXIMUM OR MINIMUM POINT (DEPENDING ON THE SIGN OF a) OF THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ IS $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Review Exercises on Unit 4

1 FOR THE RELATION {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)} FIND THE DOMAIN AND THE RANGE.

2 IF THE DOMAIN OF THE RELATION $R = \{(x, y): y = x + 3\}$ IS A = {1, 2, 3, 4} THEN LIST ALL THE ORDERED PAIRS THAT ARE MEMBERS OF THE RELATION AND FIND THE RANGE.

3 LET A = {1, 2, 3, 4, 5} AND B = $\{a, b\}$.

A FIND A × B.

B DETERMINE RELATIONS AS SUBSETS OF A × B SUCH THAT:

I $R_1 = \{(x, y): x$ IS ODD$\}$ **II** $R_2 = \{(x, y): 1 \leq x \leq 3\}$

4 LET A = {1, 2, 3, 4} AND B = {2, 4, 5}

A IF R IS A RELATION FROM A TO B THEN, IS IT TRUE THAT R IS ALSO A RELATION FROM B TO A? EXPLAIN YOUR ANSWER.

B IF $R \subseteq (A \times B)$ SUCH THAT R = {(2, 4), (2, 2), (4, 4), (4, 2)}, THEN IS R ALSO A RELATION FROM B TO A?

C WHAT CAN WE CONCLUDE FROM B?

5 LET R = {(x, y): x IS TALLER THAN y}

A DOES (x, x) BELONG TO THE RELATION? EXPLAIN.

B IS IT TRUE THAT IF (x, y) BELONGS TO R, THEN (y, x) ALSO BELONGS TO R?

C IF (x, y) AND (y, z) BELONG TO R, THEN IS IT TRUE THAT (x, z) BELONGS TO R?

6 LET R = {(x, y): y = x}. SHOW THAT EACH OF THE STATEMENTS IN QUESTION 5 IS TRUE.

7 FIND THE DOMAIN AND THE RANGE OF EACH OF THE FOLLOWING RELATIONS:

A $R = \{(x, y): y = 2x\}$ **B** $R = \{(x, y): y = |x|\}$

C $R = \{(x, y): x, y \in \{1, 2, 3, 4, 5\} \text{ AND } y = 2x - 1\}$

D $R = \{(x, y): y = \sqrt{x^2 - 4}\}$

8 SKETCH THE GRAPH OF EACH OF THE FOLLOWING RELATIONS AND DETERMINE THE DOMAIN AND THE RANGE:

A $R = \{(x, y): y \geq -2x + 3\}$ **B** $R = \{(x, y): y = 2x + 1\}$

C $R = \{(x, y): y < -x + 3\}$ **D** $R = \{(x, y): y \geq |x|\}$

E $R = \{(x, y): y \leq x \text{ AND } y \geq 1 - x\}$ **F** $R = \{(x, y): y \leq |x| \text{ AND } y \geq 0\}$

G $R = \{(x, y): y = x + 1 \text{ AND } y = 1 - x\}$

H $R = \{(x, y): y \leq x + 1, y \geq 1 - x \text{ AND } x \geq 0\}$

I $R = \{(x, y): y > x - 2, y \geq -x - 2 \text{ AND } y \leq 4\}$

9 FOR THE FOLLOWING GRAPH, SPECIFY THE RELATION AND WRITE DOWN THE DOMAIN AND RANGE:

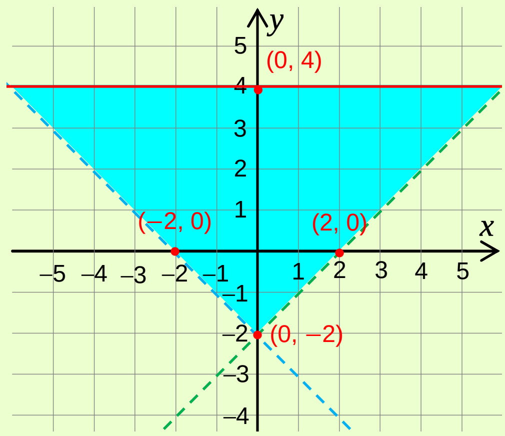


Figure 4.31

10 DETERMINE WHETHER EACH OF THE FOLLOWING RELATIONS IS A FUNCTION. IF IT IS NOT, GIVE A REASON.

A R = {(a, 1), (b, 2), (c, 3)}

B R = {(1,3), (2, 3), (3, 3), (4, 3), (5, 3)}

C R = {(1, 4), (1, 5), (1, 6), (5, 4), (5, 5)}

11 IF A = {2, 5, 7} AND B ={2, 3, 4, 6}, THEN IS A × B A FUNCTION? EXPLAIN YOUR ANSWER.

12 LET f = {(1, 2), (2, 3), (5, 6), (7, 8)}

A FIND THE DOMAIN AND RANGE OF f

B EVALUATE $f(2)$ AND $f(5)$

13 LET $f(x) = 2x + 1$ AND $g(x) = -3x - 4$

I DETERMINE:

A $f+g$ **B** $f-g$ **C** fg **D** $\frac{f}{g}$

II EVALUATE:

A $(2f+3g)(1)$ **B** $(3fg)(3)$ **C** $\frac{3f}{2g}(4)$

III FIND THE DOMAIN OF $\frac{f}{g}$:

14 LET $f(x) = \frac{x+4}{2x}$ AND $g(x) = \frac{2x+4}{x+1}$.

I DETERMINE:

A fg **B** $\frac{g}{f}$ **C** $2f - \frac{f}{g}$

II FIND THE DOMAINS OF

A fg **B** $\frac{g}{f}$ **C** $2f - \frac{f}{g}$

III EVALUATE

A $(f-g)(1)$ **B** $\frac{g}{f}(2)$ **C** $(2f - \frac{f}{g})(3)$

15 CONSTRUCT TABLES OF VALUES AND SKETCH THE GRAPH OF EACH OF THE FOLLOW

A $f(x) = 3x + 2$ **B** $x - 2y = 1$

C $f(x) = 2 - 7x$ **D** $f(x) = -3x^2 - 1$

E $f(x) = 3 - 2x + x^2$

16 SKETCH THE GRAPH OF EACH OF THE FOLLOWING USING x- AND y- INTERCEPTS:

A $f(x) = 7 + 2x$ **B** $f(x) = 3x - 5$

C $3x - y = 4$

17 BY USING SHIFTING RULE, SKETCH THE GRAPH OF EACH OF THE FOLLOWING:

A $f(x) = 4x^2 - 2x$ **B** $f(x) = x^2 - 8x + 7$

C $f(x) = 4x + 6 - 3x^2$

18 FOR THE FUNCTION $f(x) = 3x^2 - 5x + 7$, DETERMINE:

A WHETHER IT TURNS UPWARD OR DOWNWARD

B THE VERTEX

C THE AXIS OF SYMMETRY

19 DETERMINE THE MINIMUM (OR THE MAXIMUM) VALUE OF THE FOLLOWING FUNCTIO

A $f(x) = (x - 4)^2 - 5$ **B** $f(x) = 2x^2 - 6x + 7$

C $f(x) = 3x^2 - 5x + 8$ **D** $f(x) = -x^2 + 6x - 5$

E $f(x) = -2 + 4x - 2x^2$

20 DETERMINE THE RANGE OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = (x + 5)^2 + 3$ **B** $f(x) = x^2 - 9x + 10$

C $f(x) = -8 - x^2 - 6x$ **D** $f(x) = -x^2 + 2x + 4$

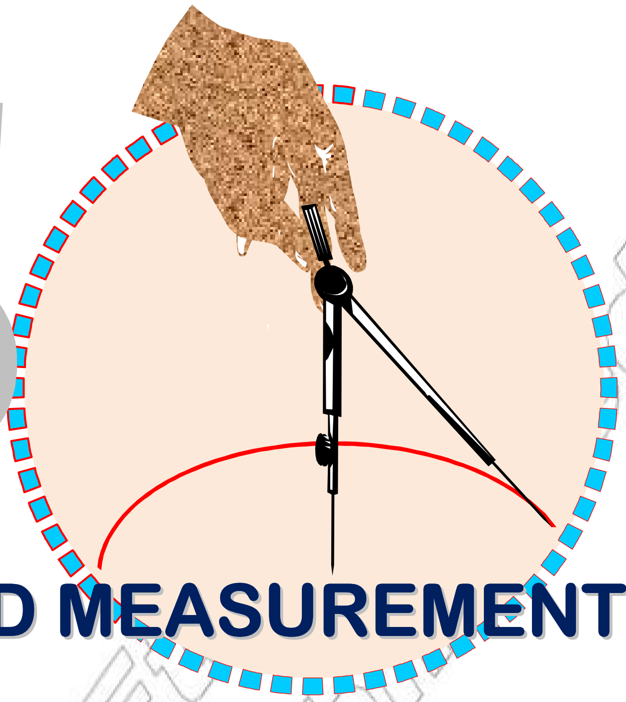
21 A MOBILE PHONE TECHNICIAN USES THE LINEAR FUNCTION $c(t) = 2t + 15$ TO DETERMINE THE COST OF REPAIR, WHERE t IS THE TIME IN HOURS AND $c(t)$ THE COST IN BIRR. HOW MUCH WILL YOU PAY IF IT TAKES HIM 3 HOURS TO REPAIR YOUR MOBILE?

22 A REAL ESTATE SELLS HOUSES FOR BIRR 200,000 PLUS BIRR 400 PER ONE SQUARE M

A FIND THE FUNCTION THAT REPRESENTS THE COST OF THE HOUSE THAT HAS AN M^2.

B CALCULATE THE COST OF THE HOUSE THAT HAS AN AREA OF 80 M^2.

Unit 5



GEOMETRY AND MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about regular polygons.*
- *apply postulates and theorems in order to prove congruence and similarity of triangles.*
- *construct similar figures.*
- *apply the concept of trigonometric ratios to solve problems in practical situations;*
- *know specific facts about circles.*
- *solve problems on areas of triangles and parallelograms.*

Main Contents

INTRODUCTION

YOU HAVE LEARNT SEVERAL CONCEPTS, PRINCIPLES AND THEOREMS OF (MEASUREMENT IN YOUR LOWER GRADES. IN THE P, YOU WILL LEARN MORE A GEOMETRY AND MEASUREMENT. REGULAR POLYGONS AND THEIR PROPERTIES, SIMILARITY OF TRIANGLES, RADIAN MEASURE OF AN ANGLE, TRIGONOM CIRCLES, PERIMETER AND AREA OF A SEGMENT AND A SECTOR OF A CIR(AND VOLUMES OF SOLIDS ARE THE MAJOR TOPICS COVERED IN THIS UNIT.

5.1 REGULAR POLYGONS

A Revision on polygons

THE FOLLOWING ACTIVITY MIGHT HELP YOU RECALL IMPORTANT FACTS ABOUT THAT YOU STUDIED IN PREVIOUS GRADES.

ACTIVITY 5.1

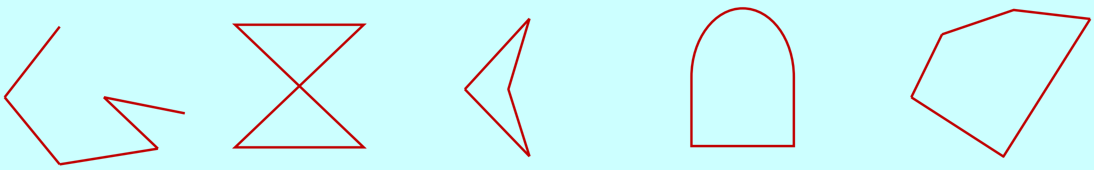
1 WHAT IS A POLYG

2 DISCUSS THE DIFFERENCE A CONVEX POLYGON AND A POLYGON.

3 FIND THE SUM OF THE MEASURES OF THE IN:

A A TRIANGLE. B A QUADRILATERAL. C A PENTA(

4 WHICH OF THE FOLLOWIN ARE POLYGONS?



A B C D E

Figure 5.1

Definition 5.1

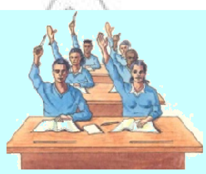
A polygon is a simple closed curve, formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

IN OTHER WORDS, A polygon IS A SIMPLE CLOSED PLANE SHAPE CONSISTING STRAIGHT-LINE SEGMENTS SUCH THAT NO TWO CONSECUTIVE LINE SEGMENTS ARE COLLINEAR.

B Interior and exterior angles of a polygon

WHEN REFERENCE IS MADE TO THE ANGLES OF A POLYGON, WE USUALLY M
THE NAME INDICATES, AN interior angle IS AN ANGLE IN THE INTERIOR OF A POLY

ACTIVITY 5.2



1 DRAW A DIAGRAM TO HAT IS MEANT BY AN INTERIOR A
POLYGON.

2 A HOW MANY INTERIOR ANGLE n-SIDED POLYGON HA
B HOW MANY DIAGONALS FROM A VE n-SIDED POLYGON H.
C INTO HOW MANY TRIANGL n-SIDED POLYGON BE PARTITIONED
DIAGONALS FROM ONE

3 WHAT RELATIONS ARE TH BETWEEN THE NUMBER OF SIDES, THE NUMB
AND THE NUMBER OF INTERIOR ANG n-SIDED POLYGON

NOTE THAT THE NUMBER OF *vertices*, *angles* AND *sides* OF A POLYGON ARE TH

Number of sides	Number of interior angles	Name of polygon
3	3	TRIANGLE
4	4	QUADRILATERAL
5	5	PENTAGON
6	6	HEXAGON
7	7	HEPTAGON
8	8	OCTAGON
9	9	NONAGON
10	10	DECAGON

Definition 5.2

An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called an exterior angle. It is formed between one side of the polygon and the extended adjacent side.

EXAMPLE 1 IN THE POLYGON ABCD IN FIGURE 5.2, ∠DCB IS AN INTERIOR ANGLE, ∠BCE AND ∠DCF ARE EXTERIOR ANGLES OF THE POLYGON AT THE VERTEX C.

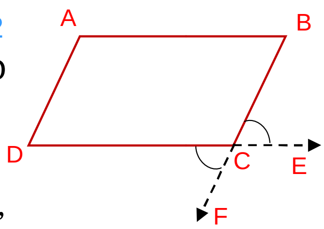


Figure 5.2

(THERE ARE TWO POSSIBLE EXTERIOR ANGLES AT ANY VERTEX, WHICH ARE EQUAL.)

C The sum of the measures of the interior angles of a polygon

LET USFIRST CONSIDER THE SUM OF THE THEINTERIOR ANGLES OF A

ACTIVITY 5.3

- DRAW A FAIRLY LARGE TRIANGLE ON A SHEET

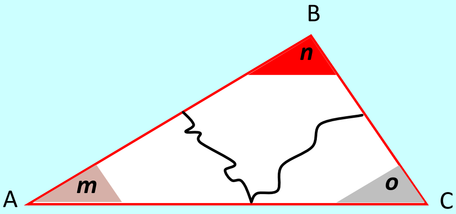


Figure 5.3

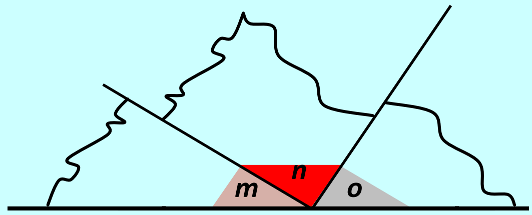


Figure 5.4

- NOW TEAR THE TRIANGLE INTO THREE PIECES, MAKING SURE E CORNER (AN
- FIT THESE THREE PIECES ALONG A STRAIGHT LINE IN FIGURE 5.4.

1 OBSERVE THAT THE SUM OF THE THREE ANGLANGLE.

2 WHAT IS THE SUM OF THE MEASUINTERIOR ANGLES OF ABC?

3 GIVEN BELOW ARE THE MEA $\angle$A, $\angle$B AND $\angle$C. CAN A TRIANGLE ABC BI WITH THE GIVEN ANGEXPLAIN.

A M ($\angle$A) = 36^{O}; M ($\angle$B) = 78^{O}; M ($\angle$C) = 66^{O}.
B M ($\angle$A) = 124^{C} ; M ($\angle$B) = 56^{O}; M ($\angle$C) = 20^{O}.
C M ($\angle$A) = 90^{O}; M ($\angle$B) = 74^{O}; M ($\angle$C) = 18^{O}.

BASED ON OBSERVATIONS FROM ACTIVITY, WE STATE THE FOLLOWIN

Theorem 5.1 Angle sum theorem

The sum of the measures of the three interior angles of any triangle is 180^{O}.

ACTIVITY 5.4

PARTITIONING A POLYGON INTO TRIANIN FIGURE 5.6 CAN HE YOU TO DETERMINE THE SUM OF THE INTERIOR ANG

COMPLETE THE FOLLOW

Number of sides of the polygon	Number of triangles	Sum of interior angles
3	1	$1 \times 180^{\circ}$
4	2	$2 \times 180^{\circ}$
5	3	$3 \times 180^{\circ}$
6		__ $\times 180^{\circ}$
7		
8		
n		__ $\times 180^{\circ}$

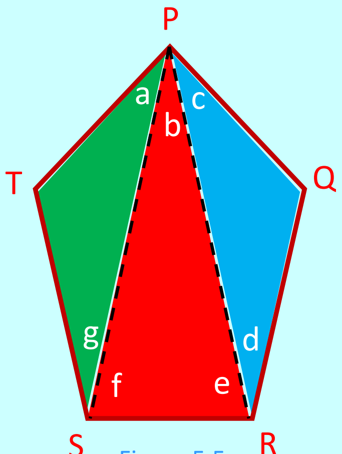


Figure 5.5

FROM THE ABOVE ACTIVITY, YO CAN GENERALIZE THE SUM OF INTERIOR ANGLES OF A FOLLOWS:

Theorem 5.2

If the number of sides of a polygon is n, then the sum of the measures of all its interior angles is equal to $(n - 2) \times 180^{\circ}$.

FROM ACTIVITY 5.4 AND THEOREM 5.2, YOU CAN ALSO OBSERVE THAT AN n-SIDED POLYGON C DIVIDED INTO $(n - 2)$ TRIANG SINCE THE SUM OF INTERIOR ANGLES OF A TRIANG^{o}, THE SUM OF THE ANGLES OF $(n - 2)$ TRIANGLES IS GIVEN BY:

$$S = (n - 2) \times 180^{\circ}.$$

ACTIVITY 5.5

1 USING FIGURE 5.6, VERIFY THE FO IS $S = (n - 2) \times 180^{o}$ GIVEN ABOVE.

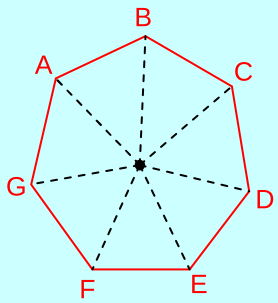


Figure 5.6

Hint: ANGLES AT A POINT ADD UP $^{\circ}$.

2 BY DIVIDING EACH OF THE FOLLOWING FIGURES INTO TRIANGLES, SHOW THAT THE FORMULA $S = (n - 2) \times 180^o$ FOR THE SUM OF THE MEASURES OF ALL INTERIOR ANGLES OF AN n-SIDED POLYGON IS VALID FOR EACH OF THE FOLLOWING POLYGONS:

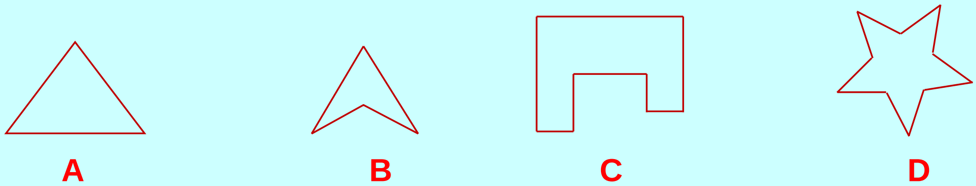


Figure 5.7

3 IN A QUADRILATERAL ABCD, IF m($\angle$A) = 80^o, m($\angle$B) = 100^o AND m($\angle$D) = 110^o, FIND m($\angle$C).

4 IF THE MEASURES OF THE INTERIOR ANGLES OF A HEXAGON ARE x^O, $2x^O$, 60^O, $(x + 30)^O$, $(x - 10)^O$ AND $(x + 40)^O$, FIND THE VALUE OF x.

5 **A** LET i_1, i_2, i_3 BE THE MEASURES OF THE INTERIOR ANGLES OF THE GIVEN TRIAN LET e_1, e_2 AND e_3 BE THE MEASURES OF THE EXTERIOR ANGLES, AS INDICA IN FIGURE 5.8.

EXPLAIN EACH STEP IN THE FOLLOWING:

$$i_1 + e_1 = 180^O$$
$$i_2 + e_2 = 180^O$$
$$i_3 + e_3 = 180^O$$

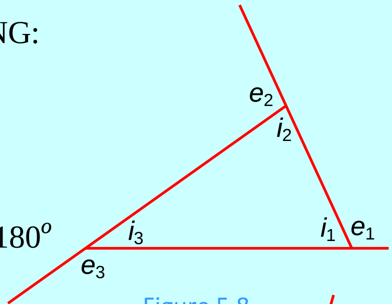


Figure 5.8

$$(i_1 + e_1) + (i_2 + e_2) + (i_3 + e_3) = 180^o + 180^o + 180^o$$

$$(i_1 + i_2 + i_3) + (e_1 + e_2 + e_3) = 3 \times 180^o$$

$$180^o + e_1 + e_2 + e_3 = 3 \times 180^o$$

$$e_1 + e_2 + e_3 = 3 \times 180^o - 180^o = 2 \times 180^o$$

$$e_1 + e_2 + e_3 = 360^o$$

THAT IS, *the sum of the measures of the exterior angles of a triangle, taking one angle at each vertex, is* 360^o.

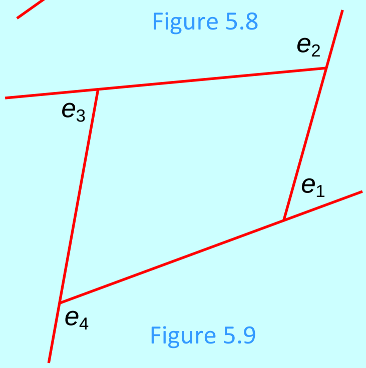


Figure 5.9

B REPEAT THIS FOR THE QUADRILATERAL GIVEN IN FIGURE 5.9. FIND THE SUM OF THE MEASURES OF THE EXTERIOR ANGLES OF THE QUADRILATERAL. I.E., FIND $e_1 + e_2 + e_3 + e_4$.

C IF $e_1, e_2, e_3 \ldots e_n$ ARE THE MEASURES OF THE EXTERIOR ANGLES OF AN n-SIDED POLYGON, THEN $e_1 + e_2 + e_3 + \ldots + e_n$ = ______.

6 SHOW THAT THE MEASURE OF AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO T MEASURES OF THE TWO OPPOSITE INTERIOR ANGLES.

5.1.1 Measures of Angles of a Regular Polygon

SUPPOSE WE CONSIDER A CIRCLE WITH CENTRE O AND RADIUS r AND DIVIDE THE CIRCLE INTO n EQUAL ARCS. (THE FIGURE GIVEN ON THE RIGHT SHOWS THIS WHEN $n = 8$.)

FOR EACH LITTLE ARC, WE DRAW THE CORRESPONDING CHORD. THIS GIVES A POLYGON WITH VERTICES $P_1, P_2, \ldots, P_n$. SINCE THE ARCS HAVE EQUAL LENGTHS, THE CHORDS (WHICH ARE THE SIDES OF THE POLYGON) ARE EQUAL. IF WE DRAW SEGMENTS FROM O TO EACH VERTEX OF THE POLYGON, WE GET n ISOSCELES TRIANGLES. IN EACH TRIANGLE, THE MEASURE OF THE CENTRAL ANGLE O IS GIVEN BY:

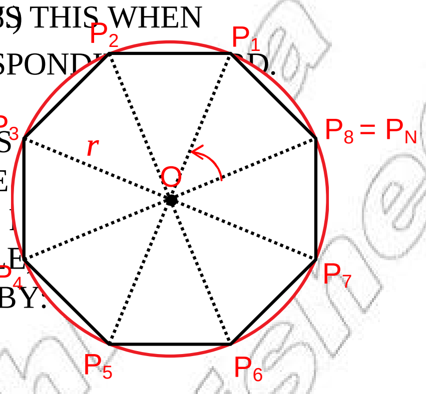


Figure 5.10

$$M(\angle O) = \frac{360^o}{n}.$$

SINCE THE VERTEX ANGLES AT O OF EACH ISOSCELES TRIANGLE HAVE EQUAL MEAS $\frac{360^o}{n}$, IT FOLLOWS THAT ALL THE BASE ANGLES OF ALL THE ISOSCELES TRIANGLES ARE

FROM THIS, IT FOLLOWS THAT THE MEASURES OF ALL THE ANGLES OF THE POLYGON MEASURE OF AN ANGLE OF THE POLYGON IS TWICE THE MEASURE OF ANY BASE ANGLE THE ISOSCELES TRIANGLES. SO, THE POLYGON HAS ALL OF ITS SIDES EQUAL AND AL EQUAL. A POLYGON OF THIS TYPE IS CALLED A regular polygon.

Definition 5.3

A regular polygon is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angles are equal.

NOTE THAT THE MEASURE OF AN INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{S}{n}$, WHERE $S = (n-2) \times 180^{O}$ IS THE SUM OF THE MEASURES OF ALL OF ITS INTERIOR ANGLES. HENCE, WE FOLLOWING:

THE MEASURE OF EACH INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{(n-2)180^o}{n}$.

A POLYGON IS SAID TO BE INSCRIBED IN A CIRCLE IF ALL OF ITS VERTICES LIE ON THE CIRC

FOR EXAMPLE, THE QUADRILATERAL SHOWN IN FIGURE 5.11 IS INSCRIBED IN THE CIRCLE.

ANY REGULAR POLYGON CAN BE INSCRIBED IN A CIRCLE. BECAUSE OF THIS, THE CENTRE AND THE RADIUS OF A CIRCLE CAN BE TAKEN AS THE CENTRE AND RADIUS OF AN INSCRIBED REGULAR POLYGON.

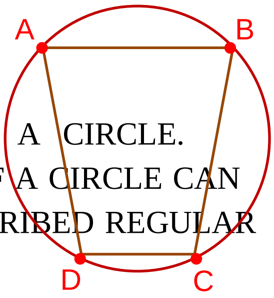


Figure 5.11

EXAMPLE 1

I FIND THE MEASURE OF EACH INTERIOR ANGLE AND EACH CENTRAL ANGLE OF A REGULAR POLYGON WITH:

A 3 SIDES **B** 5 SIDES

II FIND THE MEASURE OF EACH EXTERIOR ANGLE OF A REGULAR n-SIDED POLYGON

SOLUTION:

I **A** SINCE THE SUM OF INTERIOR ANGLES OF A TRIANGLE IS 180^o, EACH INTERIOR ANGLE IS $\frac{180^o}{3} = 60^o$.

RECALL THAT A 3-SIDED REGULAR POLYGON IS AN equilateral triangle.

TO FIND THE MEASURE OF A CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON, RECALL THAT the sum of the measures of angles at a point is 360°. HENCE, THE SUM OF THE MEASURES OF THE CENTRAL ANGLES IS 360° (FIGURE 5.11 ILLUSTRATES THIS FOR $n = 8$). SO, THE MEASURE OF EACH CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$. FROM THIS, WE CONCLUDE THAT THE MEASURE OF EACH CENTRAL ANGLE OF AN EQUIL IS $\frac{360^o}{3} = 120^o$.

B RECALL THAT THE SUM OF ALL INTERIOR ANGLES OF A 5-SIDED POLYGON IS $(5 - 2)180^o = 3 \times 180^o = 540^o$. SO, THE MEASURE OF EACH interior angle OF A regular pentagon IS $540^o \div 5 = 108^o$.

ALSO, THE MEASURE OF EACH central angle OF A regular pentagon IS $\frac{360^o}{5} = 72^O$.

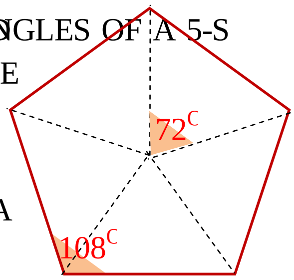


Figure 5.12

II TO FIND THE MEASURE OF EACH EXTERIOR ANGLE IN A REGULAR n-SIDED POLYGON, NOTICE THAT AT EACH VERTEX, THE SUM OF AN INTERIOR ANGLE AND AN EXTERIOR ANGLE IS 180°.

(*See* FIGURE 5.13).

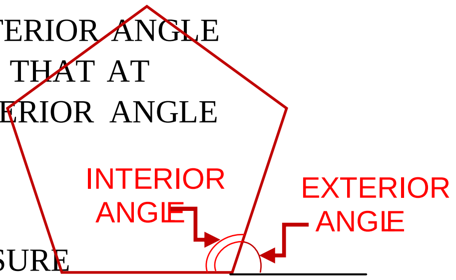


Figure 5.13

HENCE EACH EXTERIOR ANGLE WILL MEASURE

$$180^o - \left[\frac{(n-2)}{n} 180^o\right] = \frac{n180^o - (n-2)180^o}{n} = \frac{360^o}{n},$$

WHICH IS THE SAME AS THE MEASURE OF A CENTRAL ANGLE.

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WE CAN SUMMARIZE OUR RESULTS ABOUT ANGLE MEASURES IN REGULAR POLYGONS A

FOR ANY REGULAR n-SIDED POLYGON:

I MEASURE OF EACH INTERIOR ANGLE $= \dfrac{(n-2)180^o}{n}$

II MEASURE OF EACH CENTRAL ANGLE $= \dfrac{360^o}{n}$

III MEASURE OF EACH EXTERIOR ANGLE $= \dfrac{360^o}{n}$

Exercise 5.1

1 IN FIGURE 5.14A, NO TWO LINE SEGMENTS THAT ARE IN SUCCESSION ARE COLLINEAR AND NO TWO SEGMENTS INTERSECT EXCEPT AT THEIR END POINTS. YET THE FIGURE IS NO… WHY NOT?

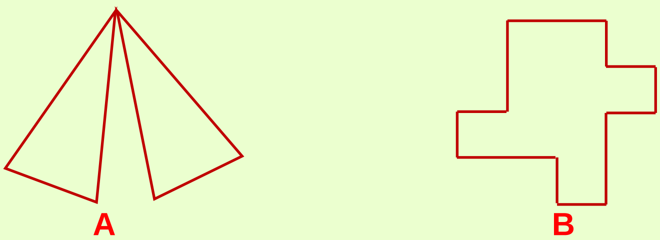


Figure 5.14

2 IS FIGURE 5.14B A POLYGON? HOW MANY SIDES DOES IT HAVE? HOW MANY VERTI DOES IT HAVE? WHAT IS THE SUM OF THE MEASURES OF ALL OF ITS INTERIOR ANG

3 ABCD IS A QUADRILATERAL SUCH THAT THE MEASURES OF THE INTERIOR ANGLES ARE GIVEN AS $m(\angle D) = 112^o$, $m(\angle C) = 75^o$ AND $m(\angle B) = 51^o$. FIND $m(\angle A)$.

4 FIND THE MEASURE OF AN INTERIOR ANGLE OF A REGULAR POLYGON WITH:

A 10 SIDES **B** 20 SIDES **C** 12 SIDES

5 FIND THE NUMBER OF SIDES OF A REGULAR POLYGON, IF THE MEASURE OF EACH OF ITS INTERIOR ANGLES IS:

A 150^o **B** 160^o **C** $147.\overline{27}^o$

6 IF THE MEASURE OF A CENTRAL ANGLE OF A REGULAR POLYGON IS 18^o, FIND THE MEASURE OF EACH OF ITS INTERIOR ANGLES.

7 **I** CAN A REGULAR POLYGON BE DRAWN SUCH THAT THE MEASURE OF EACH EXTERIOR ANGLE IS:

A 20^o? **B** 16^o? **C** 15^o?

IN EACH CASE, IF YOUR ANSWER IS NO, JUSTIFY IT; IF YES, FIND THE NUMBER OF SI

II CAN A REGULAR POLYGON BE DRAWN SUCH THAT THE MEASUR:

A 144^{o}? **B** 140^{o}? **C** 130^{o}?

IN EACH CASE IF YOUR ANS', FIND THE NUMBER OF SID JUSTIFY IT.

8 ABCDEFGH IS A REGULAR OCTAGON $\overline{AB}$ AND $\overline{DC}$ ARE PRODUCED TO MI FIND M(∠AND).

9 FIGURE 5.15 REPRESENTS PART OF A REGULAR POL $\overline{AB}$ ANI $\overline{BC}$ ARE SIDES, AND R IS THE CEOF THE CIRCLE IN WHICH THE POLYGON IS INSCI COMPLETE THE FOLLOW:

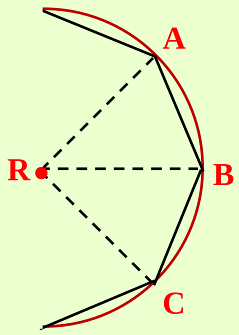


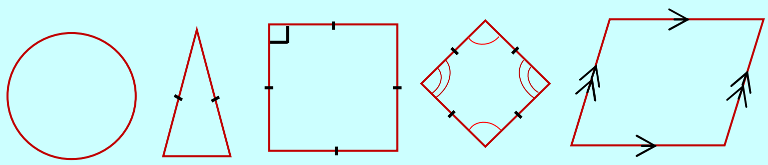
Figure 5.15

Number of sides	m(∠ARB) or m(∠BRC)	m(∠ABR) or m(∠CBR)	m (∠ABC)
3			
4			
5			
6			
	45^{C}		
9	40^{C}	70^{C}	140^{C}
			144^{C}
12			
15			
18			
20			

5.1.2 Properties of Regular Polygons

ACTIVITY 5.6

1 WHICH OF THE FOLLOWING PLANE FIGURES CAN BE D: TWO IDENTICAL PARTS BY DRAWING A LINI (IN OTHI WORDS, WHICH OF THE FOLLOWING PLANE FIGURES SYMMETRY?) DIS(



A **B** **C** **D** **E**

Figure 5.16

2 WHICH OF THE ABOVE FIGURES HAVE MORE THAN ONE

3 HOW MANY LINES OF SY! DOES A REGUL n-SIDED POLYGON

A FIGURE HAS A line of symmetry, IF IT CAN BE FOLDED SO THAT ONE HALF OF THE FIGURE COINCIDES WITH
THE ETHIOPIAN FLAG HAS A LINE OF SYMMETRY
BROKEN LINE SHOWN. THE RIGHT HALF IS A REFL
HALF, AND THE CENTRE LINE IS THE LINE OF REFL

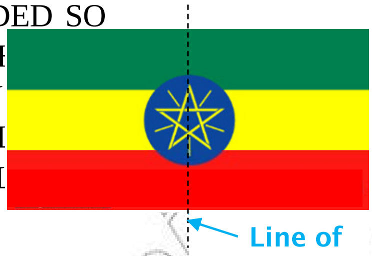


Figure 5.17

THE LINE OF REFLECTION IS ALSO CALLED THE line of symmetry.
A FIGURE THAT HAS AT LEAST ONE LINE OF SYMMETRY IS C symmetrical figure.

SOME FIGURES HAVE MORE THAN ONE LINE OF SYMMETRY. IN SUCH CASES, THE SYMMETRY ALWAYS INTERSECT AT ONE POINT.

NOTE THAT EQUILATERAL TRIANGLES, SQUARES AND REGULAR PENTAGONS HAVE A SYMMETRY AS THEIR SIDES.

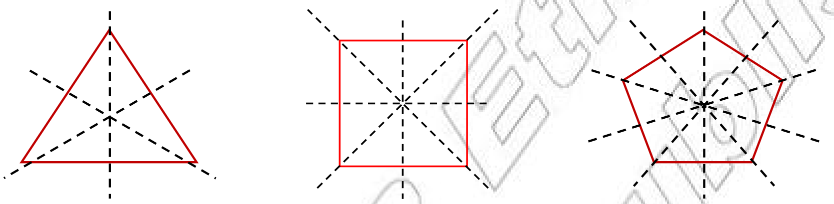


Figure 5.18

TO GENERALIZE, A REGULAR n-SIDED POLYGON ALWAYS HAS n LINES OF SYMMETRY.

Circumscribed regular polygons

A POLYGON WHOSE SIDES ARE TANGENT TO A CIRCLE IS SAID TO circumscribe the circle. FOR EXAMPLE, THE QUADRILATERAL KLMN *circumscribes* THE CIRCLE. THE CIRCLE IS *inscribed* IN THE QUADRILATERAL.

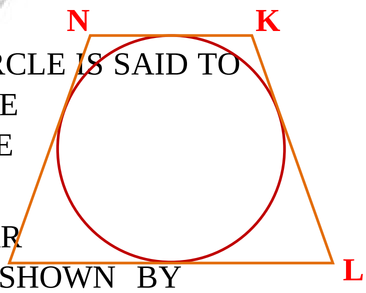


Figure 5.19

IT IS POSSIBLE TO CIRCUMSCRIBE ANY n-SIDED REGULAR POLYGON ABOUT A CIRCLE. THE METHOD IS SHOWN BY CIRCUMSCRIBING A 5-SIDED POLYGON, IN FIGURE 5.20.

THE IDEA IS THAT THE RADII TO THE POINTS OF TANGENCY MAKE 5 CONGRUENT ANGLES AT THE CENTRE WHOSE MEASURES ADD UP TO 360°. USING A PROTRACTOR, WE CONSTRUCT FIVE RADII, MAKING ADJACENT CENTRAL ANGLES OF $\frac{360^o}{5} = 72^o$. THE RADII END AT S, R, W, X, Y.

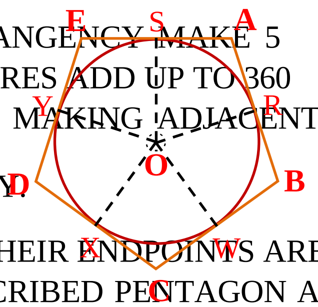


Figure 5.20

LINE SEGMENTS PERPENDICULAR TO THE RADII AT THEIR ENDPOINTS ARE TANGENT TO THE CIRCLE AND FORM THE CIRCUMSCRIBED PENTAGON AS DESIRED.

Can you see how the vertices A, B, C, D, E are determined?

REGULAR POLYGONS HAVE A SPECIAL RELATION TO CIRCLES. A REGULAR POLYGON INSCRIBED IN OR CIRCUMSCRIBED ABOUT A CIRCLE.

THIS LEADS US TO STATE THE FOLLOWING PROPERTY ABOUT REGULAR POLYGONS:

A CIRCLE CAN ALWAYS BE INSCRIBED IN OR CIRCUMSCRIBED ABOUT ANY GIVEN REGUL

IN FIGURE 5.20 ABOVE, THE RADIUS OX OF THE INSCRIBED CIRCLE IS THE DISTANCE F CENTRE TO THE SIDE (CD) OF THE REGULAR POLYGON. THIS DISTANCE FROM THE CENT OF THE POLYGON, DENOTED BY a, IS THE SAME. THIS DISTANCE IS CALLED THE apothem OF THE REGULAR POLYGON.

Definition 5.4

The distance a from the centre of a regular polygon to a side of the polygon is called the apothem of the polygon. That is, the apothem a of a regular polygon is the length of the line segment drawn from the centre of the polygon perpendicular to the side of the polygon.

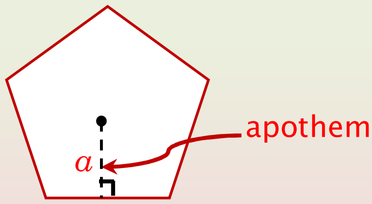


Figure 5.21

Regular Pentagon

THE FOLLOWING EXAMPLE ILLUSTRATES HOW TO FIND PERIMETER, AREA AND APOTHE

EXAMPLE 1 IN FIGURE 5.22, THE REGULAR PENTAGON ABCDE IS INSCRIBED IN A CIRCLE WITH CENTRE O AND RADIUS r. WRITE FORMULAE FOR THE SIDE s, THE PERIMETER P, APOTHEM a AND AREA A OF THE REGULAR PENTAGON.

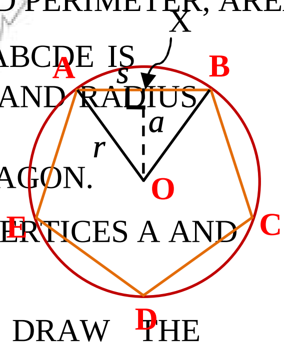


Figure 5.22

SOLUTION: TO SOLVE THE PROBLEM, JOIN O TO THE VERTICES A AND B AS SHOWN SO THAT ΔOAB IS FORMED.

ΔOAB IS AN ISOSCELES TRIANGLE (WHY?). DRAW THE PERPENDICULAR FROM O TO $\overline{AB}$. IT MEETS $\overline{AB}$ AT X. $\angle AOB$ IS A CENTRAL ANGLE OF THE REGULAR PENTAGON.

SO, $m(\angle AOB) = \dfrac{360^o}{5} = 72^o$.

NOW $\Delta AOX \cong \Delta BOX$ (*verify this*).

THEREFORE, $\angle AOX \cong \angle BOX$.

THEREFORE, $m(\angle AOX) = m(\angle BOX) = \dfrac{1}{2} m(\angle AOB) = \dfrac{1}{2}(72^o) = 36^o$.

LET s = AB, THE LENGTH OF THE SIDE OF THE REGULAR PENTAGON.

SINCE $\Delta AOX \cong \Delta BOX$, WE HAVE $\overline{AX} \cong \overline{XB}$. SO, $AX = \dfrac{1}{2} AB = \dfrac{1}{2} s$.

NOW IN THE RIGHT ANGLED TRIANGLE AOX YOU SEE THAT

$$\text{SIN}(\angle \text{AOX}) = \frac{\text{AX}}{\text{AO}}. \text{ I.E., SIN}\left(\frac{1}{2}(\angle AOB)\right) = \frac{\frac{1}{2}s}{r}$$

$$\text{SIN } 36^o = \frac{\frac{1}{2}s}{r}. \text{ SO } \frac{1}{2}s = r \text{ SIN } 36^o$$

THEREFORE, $s = 2r \sin 36^{\circ}$ . . . . (1)

PERIMETER P OF THE POLYGON IS

P = AB + BC + CD + DE + EA

BUT SINCE AB = BC = CD = DE = EA = s, WE HAVE $P = s + s + s + s + s = 5s$.

SINCE FROM (1) WE HAVE $s = 2r$ SIN 36^o, THE PERIMETER OF THE REGULAR PENTAGON IS

$P = 5 \times 2r \sin 36^{\circ}$

$\therefore$ $P = 10r \sin 36^{\circ}$ . . . . (2)

TO FIND A FORMULA FOR THE APOTHEM a, CONSIDER ΔAOX

$$\text{COS}(\angle \text{AOX}) = \frac{\text{XO}}{\text{AO}}$$

SINCE M($\angle$AOX) = 36^o, XO = a, AND AO = r.

$$\text{COS}(36^o) = \frac{a}{r}$$

SO, $a = r \cos 36^{\circ}$ . . . . (3)

TO FIND THE AREA OF THE REGULAR PENTAGON, FIRST FIND THE AREA OF ΔAOB. TAKING THE HEIGHT AND THE BASE OF ΔAOB AS OX AND AB, RESPECTIVELY, WE HAVE,

$$\text{AREA OF } \Delta \text{AOB} = \frac{1}{2} \text{AB} \times \text{OX} = \frac{1}{2} \times s \times a = \frac{1}{2} as$$

NOW THE AREA OF THE REGULAR PENTAGON ABCDE = AREA OF ΔAOB + AREA OF ΔBOC + AREA OF ΔCOD + AREA OF ΔDOE + AREA OF ΔEOA.

SINCE ALL THESE TRIANGLES ARE CONGRUENT, THE AREA OF EACH TRIANGLE IS $\frac{1}{2}as$.

SO, THE AREA OF THE REGULAR PENTAGON ABCDE $= 5 \times \left(\frac{1}{2}as\right) = \frac{1}{2}a(5s) = \frac{1}{2}aP$ . . . (4)

SINCE $36^o = \frac{180^o}{5}$, WHERE 5 IS THE NUMBER OF SIDES, WE CAN GENERALIZE THE A

FORMULAE FOR n-SIDED REGULAR POLYGON BY REPLACING 36^o BY $\frac{180^o}{n}$, AS FOLLOWS.

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Theorem 5.3

Formulae for the length of side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are

1. $s = 2r\sin\frac{180^o}{n}$
2. $a = r\cos\frac{180^o}{n}$
3. $P = 2nr\sin\frac{180^o}{n}$
4. $A = \frac{1}{2}aP$

EXAMPLE 2

A FIND THE LENGTH OF THE SIDE OF AN EQUILATERAL TRIANGLE IF ITS RADIUS IS $\sqrt{12}$ CM.

B FIND THE AREA OF A REGULAR HEXAGON WHOSE RADIUS IS 5 CM.

C FIND THE APOTHEM OF A SQUARE WHOSE RADIUS IS $\sqrt{8}$ CM.

SOLUTION:

A BY THE FORMULA, THE LENGTH OF THE SIDE IS $s = 2r\sin\frac{180^o}{n}$.

SO, REPLACING r BY $\sqrt{12}$ AND n BY 3, WE HAVE,

$$s = 2\times\sqrt{12}\times \text{SIN}\frac{180^o}{3} = 2\times\sqrt{12}\times\text{SIN }60^o$$

$$= 2\times\sqrt{12}\times\frac{\sqrt{3}}{2} = \sqrt{12\times 3} = \sqrt{36} = 6; \quad \left(\text{SIN }60^o = \frac{\sqrt{3}}{2}\right)$$

THEREFORE, THE LENGTH OF THE SIDE OF THE EQUILATERAL TRIANGLE IS 6 CM.

B TO FIND THE AREA OF THE REGULAR HEXAGON, WE USE THE FORMULA

$A = \frac{1}{2}aP$, WHERE a IS THE APOTHEM AND P IS THE PERIMETER OF THE REGULAR HEXAGON.

THEREFORE,

$$A = \frac{1}{2}aP = \frac{1}{2}\left(r\,\text{COS}\frac{180^o}{n}\right)\left(2nr\,\text{SIN}\frac{180^o}{n}\right) \quad (\textit{Substituting formulae for a and P})$$

$$= \frac{1}{2}\times\left(5\times\text{COS}\frac{180^o}{6}\right)\times\left(2\times 6\times 5\,\text{SIN}\frac{180^o}{6}\right)$$

$$= \frac{1}{2}\times 5\times\frac{\sqrt{3}}{2}\times 2\times 6\times 5\times\frac{1}{2}; \quad \left(\text{COS }30^o = \frac{\sqrt{3}}{2},\ \text{SIN }30^o = \frac{1}{2}\right)$$

$$= \frac{75\sqrt{3}}{2}\ \text{CM}^2$$

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c TO FIND THE APOTHEM OF THE SQUARE, WE USE THE FORMULA $a = r\cos\frac{180^o}{n}$.

REPLACING r BY $\sqrt{8}$ AND n BY 4, WE HAVE

$$a = \sqrt{8}\cos\frac{180^o}{4} = \sqrt{8}\cos 45^o \qquad \left(\cos 45^o = \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{8}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{16}}{2} = 2 \text{ CM.}$$

Exercise 5.2

1 WHICH OF THE CAPITAL LETTERS OF THE ENGLISH ALPHABET ARE SYMMETRICAL?

2 DRAW ALL THE LINES OF SYMMETRY ON A DIAGRAM OF A REGULAR:

a HEXAGON **b** HEPTAGON **c** OCTAGON

HOW MANY LINES OF SYMMETRY DOES EACH ONE HAVE?

3 IF A REGULAR POLYGON OF n SIDES HAS EVERY LINE OF SYMMETRY PASSING THROUGH A VERTEX, WHAT CAN YOU SAY ABOUT n?

4 STATE WHICH OF THE FOLLOWING STATEMENTS ARE TRUE AND WHICH ARE FALSE:

a A PARALLELOGRAM WHICH HAS A LINE OF SYMMETRY IS A RECTANGLE.

b A RHOMBUS WHICH HAS A LINE OF SYMMETRY MUST BE A SQUARE.

c AN ISOSCELES TRIANGLE WITH MORE THAN ONE LINE OF SYMMETRY IS AN EQUILATERAL TRIANGLE.

d A PENTAGON THAT HAS MORE THAN ONE LINE OF SYMMETRY MUST BE REGULAR.

5 SHOW THAT THE LENGTH OF EACH SIDE OF A REGULAR HEXAGON IS EQUAL TO THE LENGTH OF THE RADIUS OF THE HEXAGON.

6 SHOW THAT THE AREA A OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = 2r^2$.

7 DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE:

a THE AREA OF AN EQUILATERAL TRIANGLE WITH APOTHEM $\sqrt{3}$ CM AND SIDE 6 CM IS $9\sqrt{3}$ CM^2.

b THE AREA OF A SQUARE WITH APOTHEM $\sqrt{2}$ CM AND SIDE $2\sqrt{2}$ CM IS $8\sqrt{2}$ CM^2.

8 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR NINE-SIDED POLYGON WITH RADIUS 5 UNITS.

9 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR TWELVE-SIDED POLYGON WITH RADIUS 3 CM.

10 FIND THE RATIO OF THE PERIMETER OF A REGULAR HEXAGON TO ITS RADIUS AND SHOW THAT THE RATIO DOES NOT DEPEND ON THE RADIUS.

11 FIND THE RADIUS OF AN EQUILATERAL TRIANGLE WITH PERIMETER 24 UNITS.

12 FIND THE RADIUS OF A SQUARE WITH PERIMETER 32 UNITS

13 FIND THE RADIUS OF A REGULAR HEXAGON WITH PERIMETER 48 UNITS.

14 THE RADIUS OF A CIRCLE IS 12 UNITS. FIND THE PERIMETER OF A REGULAR INSCRIBED:

A TRIANGLE B HEPTAGON C DECAGON

5.2 FURTHER ON CONGRUENCY AND SIMILARITY

Congruency

TODAY, MODERN INDUSTRIES PRODUCE LARGE NUMBERS OF PRODUCTS. OFTEN MANY OF THESE ARE THE SAME SIZE AND/OR SHAPE. TO DETERMINE THESE SHAPES AND SIZES, THE IDEA OF CONGRUENCY IS VERY IMPORTANT.

TWO PLANE FIGURES ARE CONGRUENT IF THEY ARE EXACT COPIES OF EACH OTHER.

Group Work 5.1

1 LOOK CAREFULLY AT THE FIGURES GIVEN BELOW AND LIST THE PAIRS THAT APPEAR TO BE CONGRUENT.

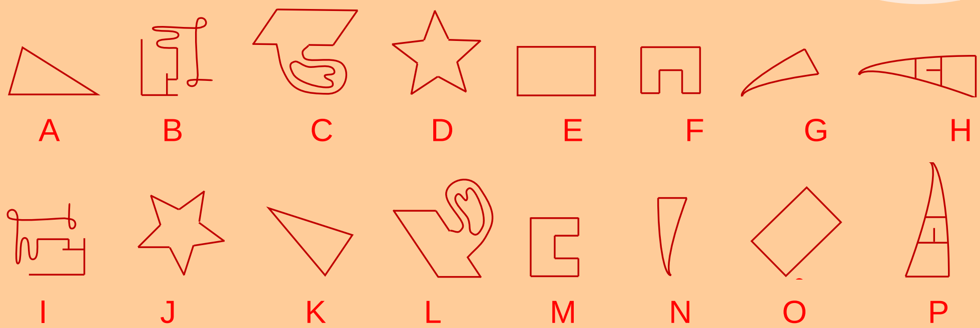

Figure 5.23

2 TEST WHETHER EACH PAIR IS, IN FACT, CONGRUENT BY TRACING ONE ON A THIN TRANSPARENT PAPER AND PLACING THE TRACING ON THE OTHER.

5.2.1 Congruency of Triangles

TRIANGLES THAT HAVE THE SAME SIZE AND SHAPE ARE CALLED congruent triangles. THAT IS, THE SIX PARTS OF THE TRIANGLES (THREE SIDES AND THREE ANGLES) ARE COR CONGRUENT. IF TWO TRIANGLES ABC AND DEF ARE CONGRUENT LIKE THOSE GIVEN BELOW, THE WE DENOTE THIS AS

$$\Delta ABC \cong \Delta DEF.$$

THE NOTATION "$\cong$" MEANS "is congruent to".

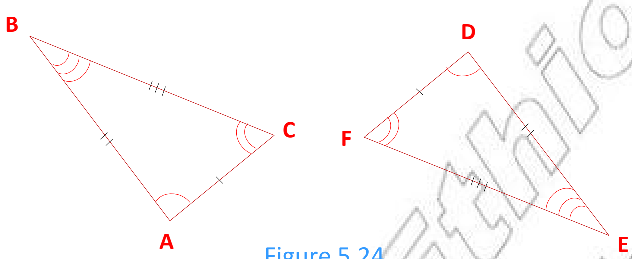

Figure 5.24

Congruent angles	Congruent sides
$\angle A \cong \angle D$; $\angle B \cong \angle E$; $\angle C \cong \angle F$	$\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$.

PARTS OF CONGRUENT TRIANGLES THAT "MATCH" ARE CALLED CORRESPONDING PART

THE TRIANGLES ABOVE, $\angle B$ CORRESPONDS TO $\angle E$ AND $\overline{AC}$ CORRESPONDS TO $\overline{DF}$.

TWO TRIANGLES ARE CONGRUENT WHEN ALL OF THE CORRESPONDING PARTS A HOWEVER, YOU DO NOT NEED TO KNOW ALL OF THE SIX CORRESPONDING PARTS TO THE TRIANGLES ARE CONGRUENT. EACH OF THE FOLLOWING THEOREMS STATE CORRESPONDING PARTS DETERMINE THE CONGRUENCE OF TWO TRIANGLES.

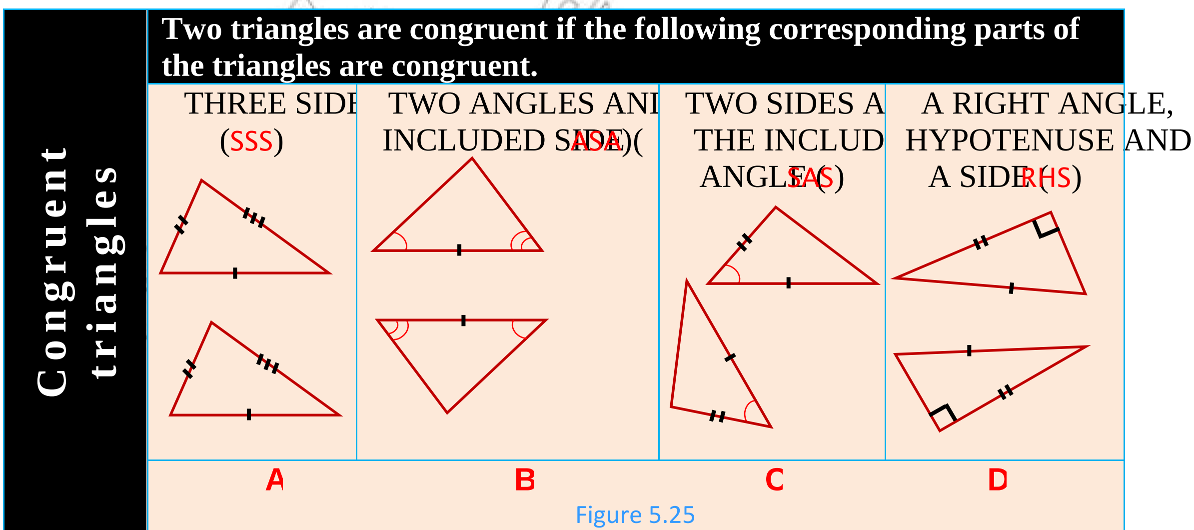

Figure 5.25

EXAMPLE 1 DETERMINE WHETHER EACH PAIR OF TRIANGLES IS CONGRUENT. IF SO, WRITE A CONGRUENCE STATEMENT AND STATE WHY THE TRIANGLES ARE CONGRUENT.

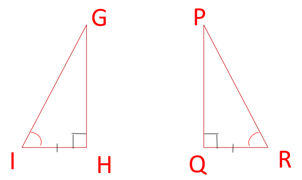

Figure 5.26

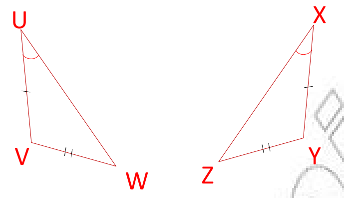

Figure 5.27

SOLUTION:

For the first two triangles: (FIGURE 5.26)

SINCE M($\angle$H) = 90^o AND

M($\angle$Q) = 90^o, $\angle H \cong \angle Q$

ALSO $\overline{IH} \cong \overline{RQ}$ (GIVEN)

$\angle I \cong \angle R$ (GIVEN)

$\therefore \Delta GHI \cong \Delta PQR$ (BY ASA)

For the last two triangles: (FIGURE 5.27)

$\overline{UV} \cong \overline{XY}$ (GIVEN)

$\angle VUW \cong \angle YXZ$ (GIVEN)

$\overline{VW} \cong \overline{YZ}$ (GIVEN)

SO TWO SIDES AND AN ANGLE ARE CONGRUENT. BUT THE ANGLE IS NOT INCLUDED BETWEEN THE SIDES. SO WE CANNOT CONCLUDE THAT THE TRIANGLES ARE CONGRUENT.

EXAMPLE 2 IN FIGURE 5.28, PQRS IS A SQUARE. A AND B ARE POINTS ON $\overline{QR}$ AND $\overline{SR}$ SUCH THAT $\overline{QA} \cong \overline{SB}$.

PROVE THAT $\angle PAQ \cong \angle PBS$

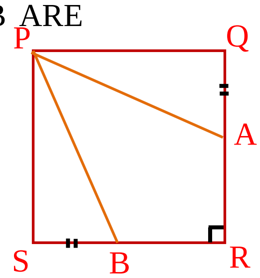

Figure 5.28

SOLUTION: $\overline{PQ} \cong \overline{PS}$ (*sides of a square*)

$\overline{QA} \cong \overline{SB}$ (*given*)

$\angle Q \cong \angle S$ (*right angles*)

THEREFORE, $\Delta PQA \cong \Delta PSB$ (BY SAS).

THEREFORE, $\angle PAQ \cong \angle PBS$ (CORRESPONDING ANGLES OF CONGRUENT TRIANGLES).

EXAMPLE 3 GIVEN $\Delta ABC \cong \Delta RST$.

FIND THE VALUE OF y, IF M($\angle$A) = 40^O AND M($\angle$R) = $(2y + 10)^O$.

SOLUTION: SINCE $\Delta ABC \cong \Delta RST$, THE CORRESPONDING ANGLES ARE CONGRUENT.

SO, $\angle A \cong \angle R$.

THEREFORE, M($\angle$A) = M($\angle$R).

I.E., $40^o = (2y + 10)^o$. SO, $y = 15^o$.

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Exercise 5.3

1 CHECK WHETHER THE FOLLOWING FOUR TRIANGLES ARE CONGRUENT:

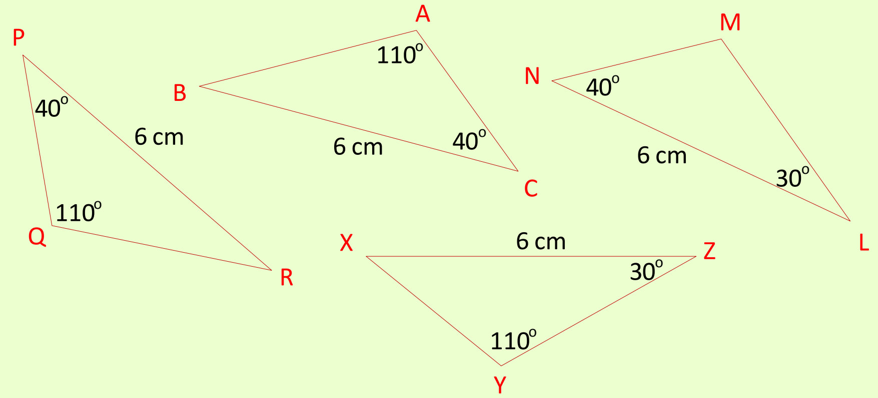


Figure 5.29

2 WHICH OF THE TRIANGLES ARE CONGRUENT TO THE BLUE-SHADED TRIANGLE? GIVE REASONS FOR YOUR ANSWER.

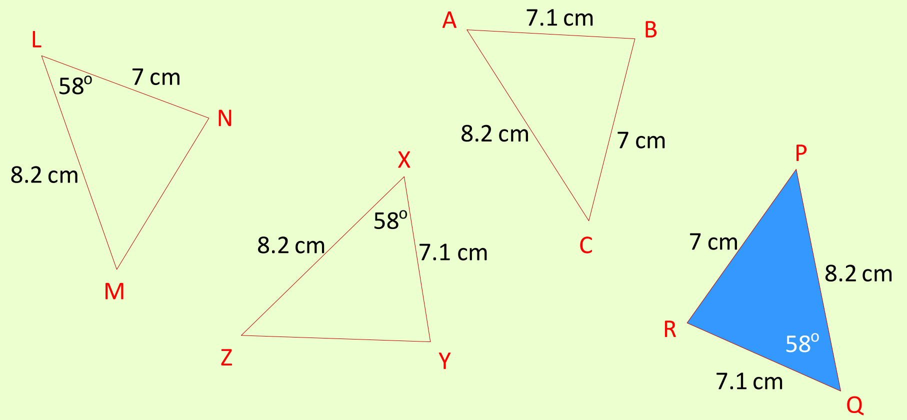


Figure 5.30

3 WHICH OF THE FOLLOWING PAIRS OF TRIANGLES ARE CONGRUENT? FOR THOSE THAT ARE CONGRUENT, STATE WHETHER THE REASON IS SSS, ASA, SAS OR RHS.

A

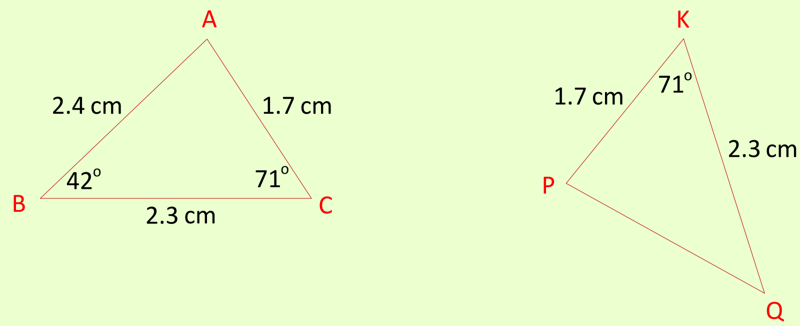


Figure 5.31

B

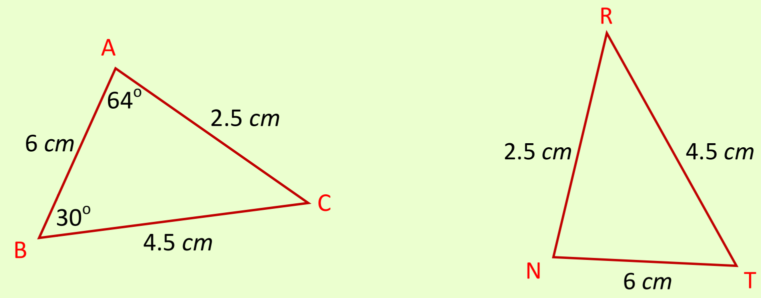


Figure 5.32

C

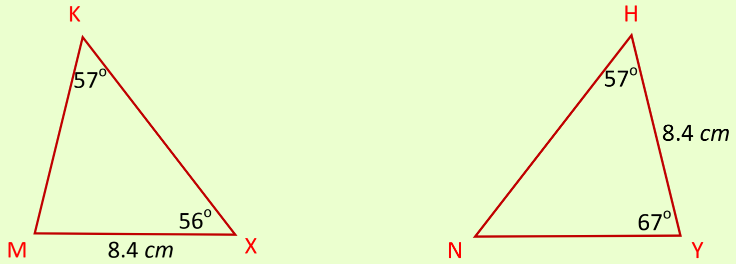


Figure 5.33

4 **A** ABC IS AN ISOSCELES TRIAN $\overline{AB} \cong \overline{AC}$, AND M IS THE MIDPO $\overline{BC}$. PROVE THAT $\angle ABC \cong \angle ACB$.

B IN FIGURE 5.34 BELOW, PROVE ΔBDF IS EQUILATERAL

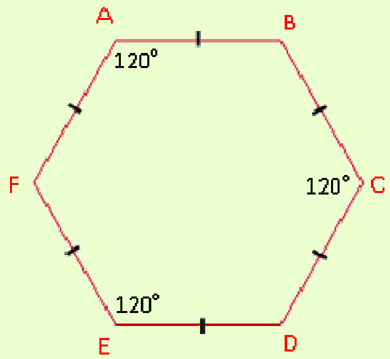


Figure 5.34

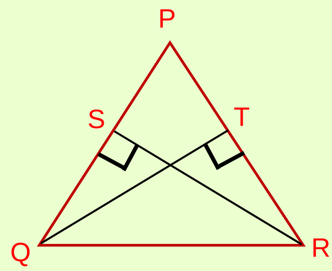


Figure 5.35

C IN FIGURE 5.35, PROVE IF $\overline{RS} \cong \overline{QT}$ THEN $\overline{PQ} \cong \overline{PR}$.

D ABC IS AN ISOSCELES TRIAN $\overline{AB} \cong \overline{AC}$. $\overline{AX}$ IS THE BISECT $\angle BAC$ MEETING $\overline{BC}$ AT X. PROVE THAT X IS THE MIDPOINT OF

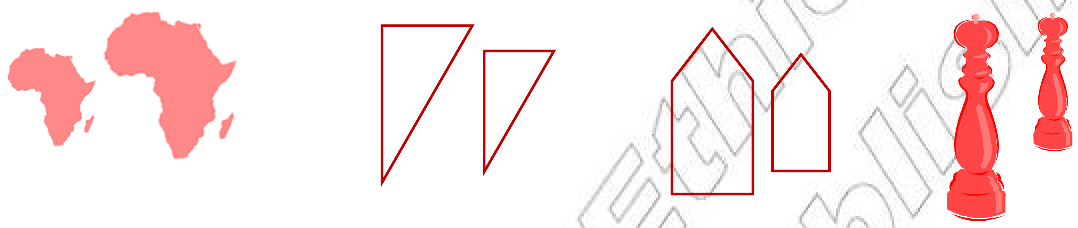
E ABCD IS A PARALLELOGRAM. P $\angle ABC \cong \angle ADC$.

Hint: FIRST JOIN AC AND USE ALTERNATE

5.2.2 Definition of Similar Figures

AFTER AN ARCHITECT FINISHES THE PLAN OF A BUILDING, IT IS USUAL TO PREPARE A BUILDING. IN DIFFERENT AREAS OF ENGINEERING, IT IS USUAL TO PRODUCE MODELS PRODUCTS BEFORE MOVING TO THE ACTUAL PRODUCTION. WHAT RELATIONSHIPS DO Y THE MODEL AND THE ACTUAL PRODUCT?

FIGURES THAT HAVE THE SAME SHAPE BUT THAT MIGHT HAVE DIFFERENT SIZES ARE C **similar.** EACH OF THE FOLLOWING PAIRS OF FIGURES ARE SIMILAR, WITH ONE SHAPE BEING AN OF THE OTHER.



A B C D

Figure 5.36

FROM YOUR GRADE 8 MATHEMATICS, RECALL THAT:

AN ENLARGEMENT IS A TRANSFORMATION OF A PLANE FIGURE IN WHICH EACH OF THE A, B, C IS MAPPED ONTO A', B', C' BY THE SAME SCALE FACTOR, k, FROM A FIXED POINT O. THE DISTANCES OF A', B', C' FROM THE POINT O ARE FOUND BY MULTIPLYING EACH OF THE D A, B, C FROM O BY THE SCALE FACTOR k.

$OA' = k \times OA$ $\qquad$ $OB' = k \times OB$ $\qquad$ $OC' = k \times OC$

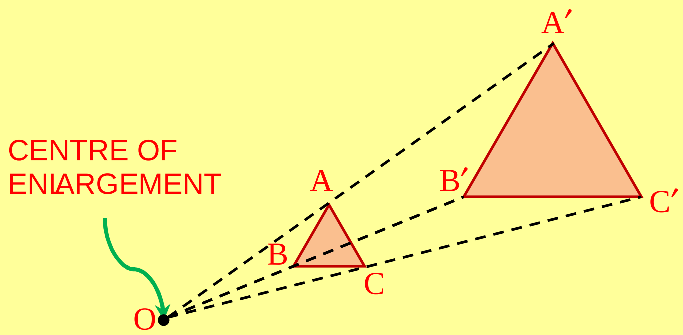


Figure 5.37

Group Work 5.2

1 ANSWER THE FOLLOWING QUESTIONS BASED ON FIGURE 5.38

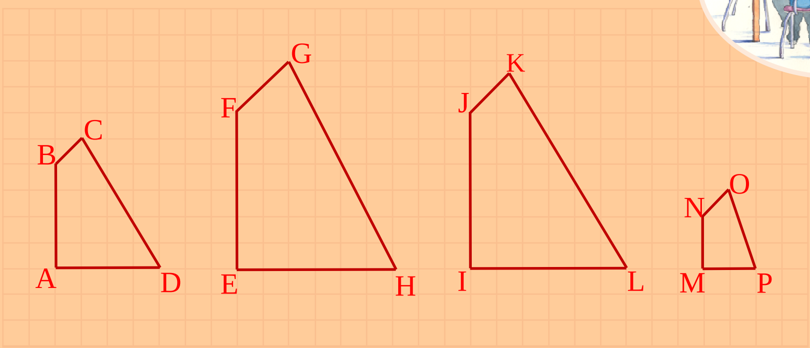


Figure 5.38

I IF A FIGURE IS ENLARGED, DO YOU ALWAYS GET A SIMILAR FIGURE?

II WHICH OF THE FIGURES ABOVE ARE SIMILAR FIGURES? DISCUSS.

III WHAT CAN YOU SAY ABOUT THE ANGLES B AND J?

IV WHICH OF THE ANGLES ARE EQUAL TO ANGLE A? WHICH OF THE ANGLES ARE EQUAL TO ANGLE D?

V WHAT OTHER EQUAL ANGLES CAN YOU FIND? DISCUSS.

VI WHAT CAN YOU SAY ABOUT THE ANGLES OF TWO SIMILAR POLYGONS? DISCUSS.

2 FIGURES ABCDE AND FGHIJ ARE SIMILAR AND

$$\frac{BC}{GH} = \frac{8}{4} = 2.$$

FIND THE RATIO OF OTHER CORRESPONDING SIDES OF ABCDE AND FGHIJ.

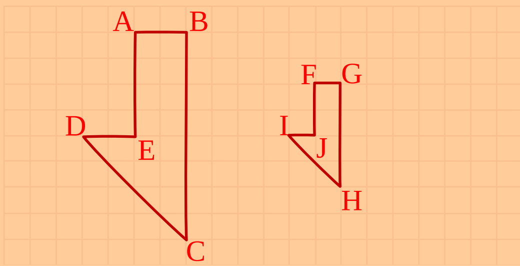


Figure 5.39

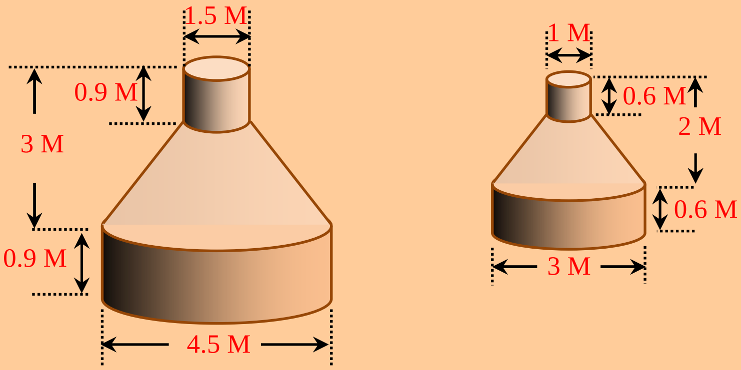


Figure 5.40

3 TWO SOLID FIGURES HAVE THE DIMENSIONS AS SHOWN IN FIGURE 5.40 ABOVE. ARE THE FIGURES SIMILAR? HOW CAN YOU MAKE SURE? DISCUSS.

4 **A** IS A RECTANGLE OF LENGTH 6 CM AND WIDTH 4 CM SIMILAR TO A RECTANGLE OF LENGTH 12 CM AND WIDTH 18 CM?

B IS A TRIANGLE, TWO OF WHOSE ANGLES ARE 75° AND 85°, SIMILAR TO A TRIANGLE TWO OF WHOSE ANGLES ARE 20° AND 85°?

HOW COULD YOU HAVE ANSWERED PARTS A AND B OF THIS QUESTION WITHOUT DRAWING? DISCUSS.

FROM THE ABOVE GROUP WORK, WE MAY CONCLUDE THE FOLLOWING.

IN SIMILAR FIGURES:

- **I** ONE IS AN ENLARGEMENT OF THE OTHER.
- **II** ANGLES IN CORRESPONDING POSITIONS ARE CONGRUENT.
- **III** CORRESPONDING SIDES HAVE THE SAME RATIO.

IN THE CASE OF A POLYGON, THE ABOVE FACTS CAN BE STAT

Similar polygons	TWO POLYGONS OF THE SAME NUMBER OF SIDES ARE SIMILAR, IF TH CORRESPONDING ANGLES ARE CONGRUENT AND THEIR CORRESPONDIN HAVE THE SAME RATIO.

EXAMPLE 1

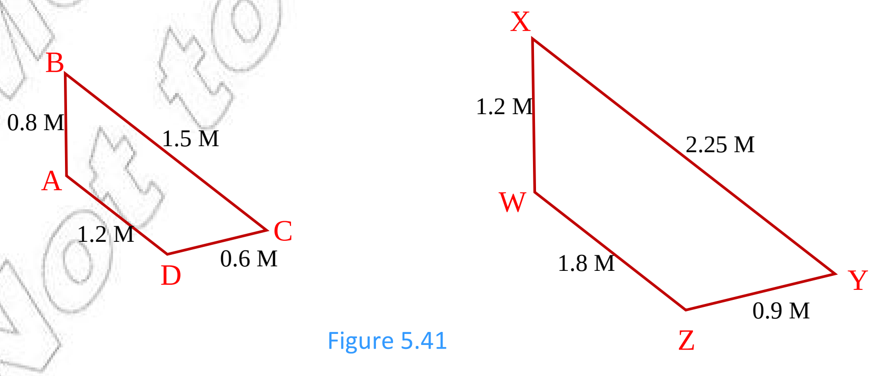


Figure 5.41

IF QUADRILATERAL ABCD IS SIMILAR TO QUADRILATERAL WXYZ, WE WRITE ABCD ~ WXYZ (THE SYMBOL ~ MEANS "is similar to").

CORRESPONDING ANGLES OF SIMILAR POLYGONS ARE CONGRUENT. YOU CAN USE A PROTRACTOR TO MAKE SURE THE ANGLES HAVE THE SAME MEASURE.

USING CORRESPONDING ANGLES AS A GUIDE, YOU CAN EASILY IDENTIFY THE CORRESPONDING SIDES.

$\overline{AB} \leftrightarrow \overline{WX}$, $\overline{BC} \leftrightarrow \overline{XY}$

$\overline{CD} \leftrightarrow \overline{YZ}$, $\overline{DA} \leftrightarrow \overline{ZW}$

(THE SYMBOL $\leftrightarrow$ MEANS "CORRESPONDS TO ").

$\angle A \cong \angle W$ $\qquad \angle B \cong \angle X$

$\angle C \cong \angle Y$ $\qquad \angle D \cong \angle Z$

A SPECIAL RELATIONSHIP ALSO EXISTS BETWEEN THE CORRESPONDING SIDES OF SIMILAR POLYGONS. COMPARE THE RATIOS OF LENGTHS OF THE CORRESPONDING SIDES:

$$\frac{AB}{WX} = \frac{0.8}{1.2} = \frac{2}{3},\ \frac{BC}{XY} = \frac{1.5}{2.25} = \frac{2}{3},\ \frac{CD}{YZ} = \frac{0.6}{0.9} = \frac{2}{3},\ \frac{DA}{ZW} = \frac{1.2}{1.8} = \frac{2}{3}.$$

YOU CAN SEE THE RATIOS OF THE LENGTHS OF THE CORRESPONDING SIDES ARE ALL EQUAL TO $\frac{2}{3}$.

EXAMPLE 2 REFERRING TO FIGURE 5.42, IF ABCDE ~ HIJKL, THEN FIND THE LENGTHS OF:

A $\overline{IJ}$ **B** $\overline{CD}$ **C** $\overline{HL}$

SOLUTION: SINCE ABCDE ~ HIJKL, WE HAVE,

$$\frac{AB}{HI} = \frac{BC}{IJ} = \frac{CD}{JK} = \frac{DE}{KL} = \frac{AE}{HL}$$

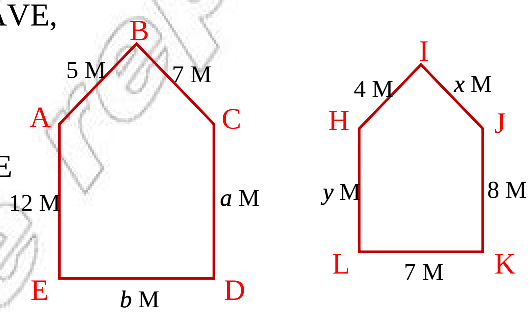


Figure 5.42

A TO FIND THE LENGTH OF $\overline{IJ}$, WE USE

$$\frac{AB}{HI} = \frac{BC}{IJ}$$

$$\frac{5}{4} = \frac{7}{x} \quad (AB = 5,\ HI = 4,\ BC = 7,\ IJ = x)$$

SO, $x = \frac{4 \times 7}{5} = 5.6$

THEREFORE, THE LENGTH OF $\overline{IJ}$ IS 5.6 M.

B IN THE SAME WAY,

$$\frac{AB}{HI} = \frac{CD}{JK}$$

$$\frac{5}{4} = \frac{a}{8}.\ \text{SO, } a = 10.$$

THEREFORE, CD = 10 M.

C

$$\frac{AB}{HI} = \frac{AE}{HL}.\ \text{I.E., } \frac{5}{4} = \frac{12}{y}$$

SO, $y = 9.6$.

THEREFORE, HL = 9.6 M.

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Exercise 5.4

1 **A** ALL OF THE FOLLOWING POLYGONS ARE REGULAR. IDENTIFY THE SIMILAR ONES.

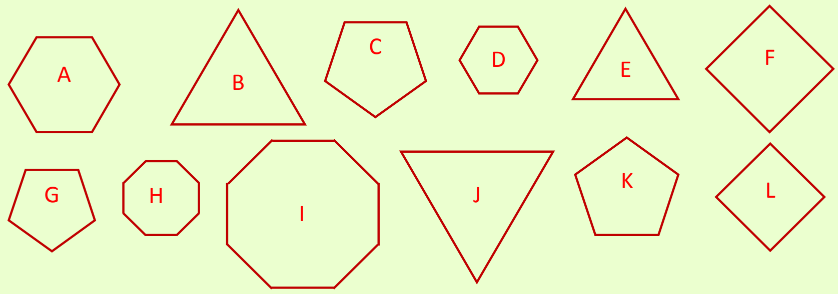

Figure 5.43

B EXPLAIN WHY REGULAR POLYGONS WITH THE SAME NUMBER OF SIDES ARE ALWAYS SIMILAR.

2 EXPLAIN WHY ALL CIRCLES ARE SIMILAR.

3 DECIDE WHETHER OR NOT EACH PAIR OF POLYGONS IS SIMILAR. EXPLAIN YOUR REASONING.

A

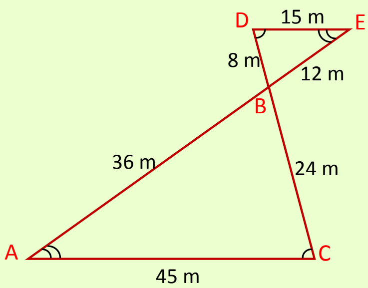

Figure 5.44

B

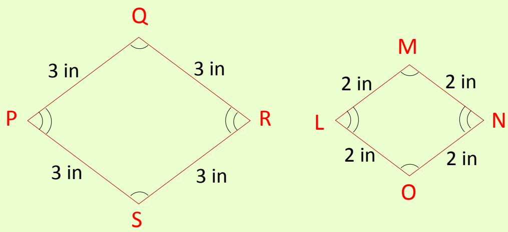

Figure 5.45

C

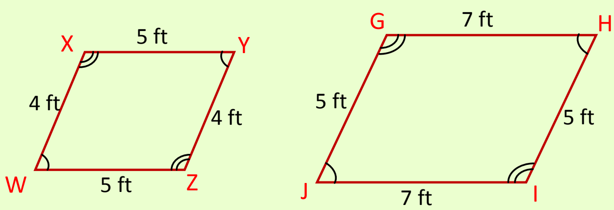

Figure 5.46

5.2.3 Theorems on Similarity of Triangles

YOU MAY START THIS SECTION BY RECALLING THE FOLLOWING FACTS ABOUT SIMILAR

Definition 5.5

Two triangles are said to be similar, if

1 their corresponding sides are proportional (have equal ratio), and

2 their corresponding angles are congruent.

That is, $\Delta ABC \sim \Delta DEF$ if and only if

$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ and

$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$

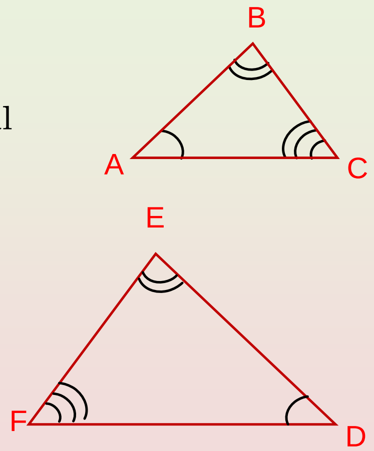

Figure 5.47

THE FOLLOWING THEOREMS ON SIMILARITY OF TRIANGLES WILL SERVE AS TESTS TO C NOT TWO TRIANGLES ARE SIMILAR.

Theorem 5.4 SSS similarity theorem

If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.

Restatement:

GIVEN ΔABC AND ΔDEF. IF $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$,

THEN $\Delta ABC \sim \Delta DEF$.

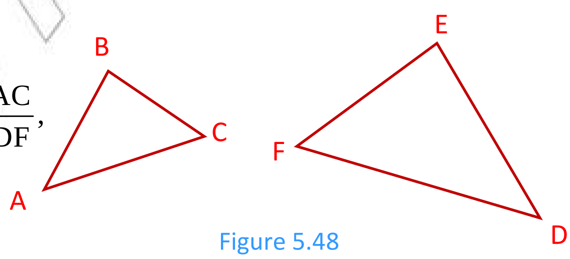

Figure 5.48

EXAMPLE 1 ARE THE TWO TRIANGLES IN FIGURE 5.49 SIMILAR?

SOLUTION: SINCE $\frac{PQ}{ST} = \frac{QR}{TU} = \frac{PR}{SU} = \frac{1}{2}$,

$\Delta PQR \sim \Delta STU$ (BY SSS SIMILARITY THEOREM).

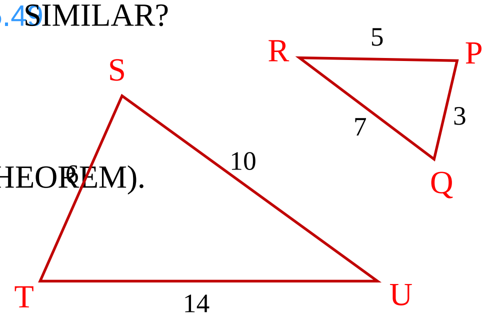

Figure 5.49

Group Work 5.3

1 INVESTIGATE IF THEOREM 5.3 WORKS FOR TWO POLYGON NUMBER OF SIDES IS GREATER THAN 3.

2 CONSIDER A SQUARE ABCD AND A RHOMBUS PQRS.

ARE THE RATIOS $\frac{AB}{PQ}$: $\frac{BC}{QR}$ AND $\frac{CD}{RS}$ EQU

WE NOW STATE THE SECOND THEOREM ON SIMILARITY OF TRIANGLES, WHICH IS side-angle-side (SAS) SIMILARITY THEOREM.

Theorem 5.5 SAS similarity theorem

Two triangles are similar, if two pairs of corresponding sides of the two triangles are proportional and if the included angles between these sides are congruent.

Restatement:

GIVEN TWO TRIANGLES ΔABC AND ΔPQR, IF

$\frac{AB}{PQ} = \frac{AC}{PR}$ AND $\angle A \cong \angle P$, THEN,

$\Delta ABC \sim \Delta PQR$.

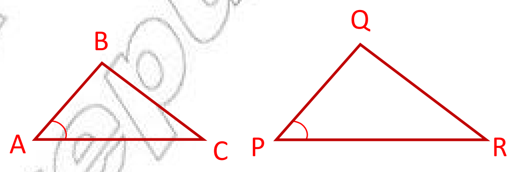

Figure 5.50

EXAMPLE 2 USE THE SAS SIMILARITY THEOREM TO CHECK WHETHER THE GIVEN TWO TRIANGLES ARE SIMILAR.

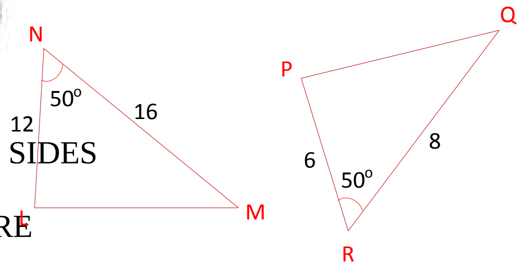

Figure 5.51

SOLUTION: SINCE $\frac{LN}{PR} = \frac{12}{6} = 2$, AND ALSO

$\frac{MN}{QR} = \frac{16}{8} = 2$, THE CORRESPONDING SIDES HAVE EQUAL RATIOS (I.E., THEY ARE PROPORTIONAL).

ALSO, SINCE $m(\angle N) = m(\angle R)$, IT FOLLOWS THAT THE INCLUDED ANGLES OF THE PROPORTIONAL SIDES ARE CONGRUENT.

THEREFORE, $\Delta LMN \sim \Delta PQR$ BY THE SAS SIMILARITY THEOREM.

FINALLY, WE STATE THE THIRD THEOREM ON SIMILARITY OF TRIANGLES, WHICH Angle-Angle (AA) SIMILARITY THEOREM.

Theorem 5.6 AA similarity theorem

If two angles of one triangle are congruent to two corresponding angles of another triangle, then the two triangles are similar.

Restatement:

GIVEN TWO TRIANGLES, NAMELY ΔABC AND ΔDEF. IF $\angle A \cong \angle D$ AND $\angle C \cong \angle F$, THEN $\Delta ABC \sim \Delta DEF$.

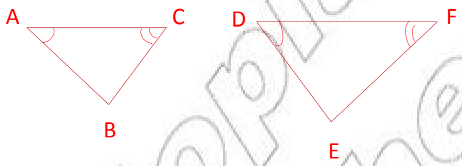

Figure 5.52

EXAMPLE 3 IN FIGURE 5.53, DETERMINE WHETHER THE TWO GIVEN TRIANGLES ARE SIMILAR.

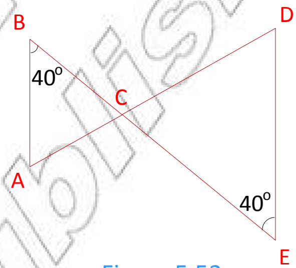

Figure 5.53

SOLUTION: IN ΔABC AND ΔDEC, $m(\angle B) = m(\angle E) = 40^o$.

SO, **I** $\angle B \cong \angle E$.

II $\angle ACB \cong \angle DCE$ (SINCE THEY ARE VERTICALLY OPPOSITE ANGLES).

THEREFORE, $\Delta ABC \sim \Delta DEC$ BY THE AA SIMILARITY THEOREM.

Exercise 5.5

1 STATE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE.

A IF TWO TRIANGLES ARE SIMILAR, THEN THEY ARE CONGRUENT.

B IF TWO TRIANGLES ARE CONGRUENT, THEN THEY ARE SIMILAR.

C ALL EQUILATERAL TRIANGLES ARE CONGRUENT.

D ALL EQUILATERAL TRIANGLES ARE SIMILAR.

2 WHICH OF THE FOLLOWING PAIRS OF TRIANGLES ARE SIMILAR? IF THEY ARE NOT SIMILAR, EXPLAIN WHY.

A

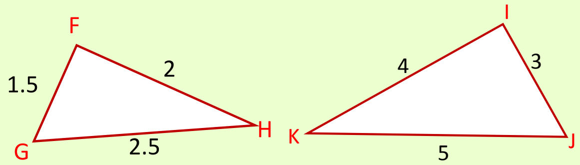

Figure 5.54

B

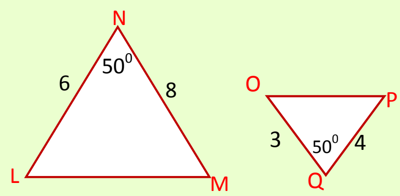

Figure 5.55

3 THE PAIRS OF TRI GIVEN BELOW ARE SIMILAR. FIND THE MEASURES BLANK SIDES.

A

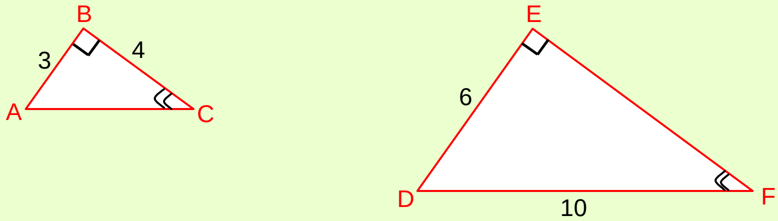

Figure 5.56

B

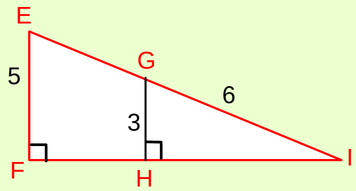

Figure 5.57

4 IN FIGURE 5.58, PROVE T:

A Δ ADC ~ Δ BEC

B Δ AFE ~ Δ BFD

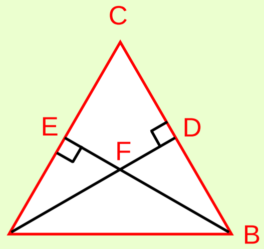

Figure 5.58

5 IN FIGURE 5.59, QUADRILA DEFG IS A SQUARE AND, ∠C IS A RIGHT ANGLE. PR:

A $\frac{AD}{EF} = \frac{DG}{EB}$

B $\frac{AD}{CG} = \frac{DG}{CF}$

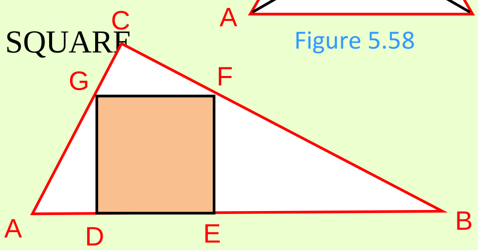

Figure 5.59

5.2.4 Theorems on Similar Plane Figures

Ratio of perimeters and ratio of areas of similar plane figures

ACTIVITY 5.7

CONSIDER FIGURE 5.60 GIVEN BELOW ANSWER THE FOLLOWING (

A SHOW THAT THE TWO RECTANGL

B WHAT IS THE RATIO OF THE CORRESP

C FIND THE PERIMET AND THE AREA OF EACH RECTANG

D WHAT IS THE RATIO OF PERIMETERS?

E WHAT IS THE RATIO OF THE

F WHAT IS THE RELATIONSHIP BETWEEN THE RATIO OF THE CORRESPONDING SIDES AND THE RATIO OF THE PERIMETERS?

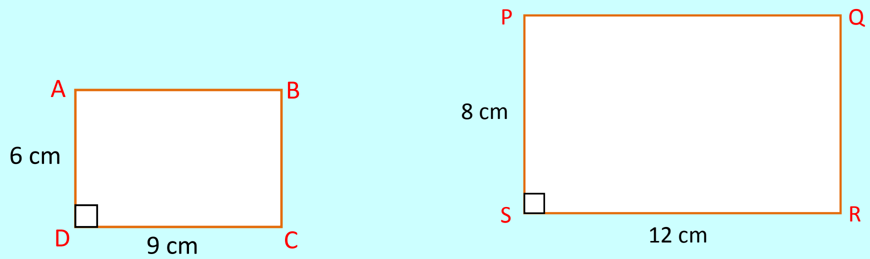

Figure 5.60

G WHAT IS THE RELATIONSHIP BETWEEN THE RATIO OF THE CORRESPONDING SIDES AND RATIO OF THE AREAS?

THE RESULTS OF THE ACTIVITY LEAD YOU TO THE FOLLOWING THEOREM.

Theorem 5.7

If the ratio of the lengths of the corresponding sides of two similar triangles is k, then

I the ratio of their perimeters is k

II the ratio of their areas is k^2.

Proof:-

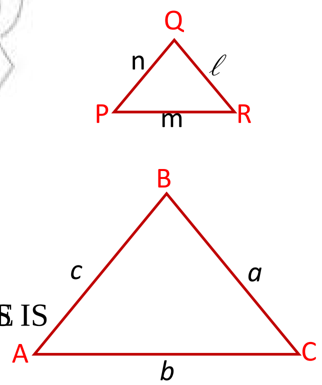

Figure 5.61

I GIVEN $\Delta ABC \sim \Delta PQR$.

THEN, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

I.E., $\frac{c}{n} = \frac{a}{l} = \frac{b}{m}$.

SINCE THE COMMON VALUE OF THESE RATIOS IS k, WE HAVE

$$\frac{c}{n} = \frac{a}{l} = \frac{b}{m} = k.$$

SO, $c = kn$, $a = kl$, $b = km$.

NOW THE PERIMETER OF ΔABC = AB + BC + CA = $c + a + b = kn + kl + km$.

FROM THIS, WE OBTAIN $c + a + b = kn + kl + km = k(n + l + m)$

THEREFORE, $\frac{c+a+b}{n+l+m} = k$.

THAT IS, $\frac{\text{PERIMETER OF } \Delta ABC}{\text{PERIMETER OF } \Delta PQR} = k$.

THIS SHOWS THAT THE RATIO OF THEIR PERIMETERS = THE RATIO OF THEIR CORRESPON

II TO PROVE THAT THE RATIO OF THEIR AREAS IS THE SQUARE OF THE RATIO OF ANY TWO CORRESPONDING SIDES:

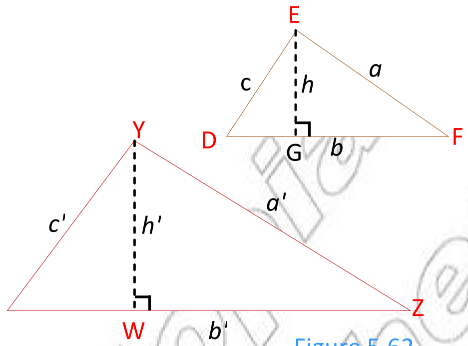

Figure 5.62

LET $\Delta DEF \sim \Delta XYZ$. THEN,

$$\frac{DE}{XY} = \frac{EF}{YZ} = \frac{DF}{XZ} = k.$$

THAT IS, $\frac{c}{c'} = \frac{a}{a'} = \frac{b}{b'} = k$.

LET $\overline{EG}$ BE THE ALTITUDE FROM E TO $\overline{DF}$ AND $\overline{YW}$ BE THE ALTITUDE FROM Y TO XZ.

SINCE ΔDEG AND ΔXYW ARE RIGHT TRIANGLES AND $\angle D \cong \angle X$, WE HAVE

$\Delta DEG \sim \Delta XYW$ (AA SIMILARITY)

THEREFORE, $\frac{h}{h'} = \frac{c}{c'} = k$.

NOW, AREA OF $\Delta DEF = \frac{1}{2}bh$, AND AREA OF $\Delta XYZ = \frac{1}{2}b'h'$.

THEREFORE, $$\frac{\text{AREA OF } \Delta DEF}{\text{AREA OF } \Delta XYZ} = \frac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \frac{b}{b'} \times \frac{h}{h'} = k \times k = k^2.$$

SO, THE RATIO OF THEIR AREAS IS THE SQUARE OF THE RATIO OF THEIR CORRESPONDIN

NOW WE STATE THE SAME FACT FOR ANY TWO POLYGONS.

Theorem 5.8

If the ratio of the lengths of any two corresponding sides of two similar polygons is k, then

I the ratio of their perimeters is k.

II the ratio of their areas is k^2.

Exercise 5.6

1 LET ABCD AND EFGH BE TWO QUADRILATERALS SUCH THAT ABCD ~ EFGH.

IF AB = 15 CM, EF = 18 CM AND THE PERIMETER OF ABCD IS 40 CM, FIND THE PERIMETER OF EFGH.

2 TWO TRIANGLES ARE SIMILAR. A SIDE OF ONE IS 2 UNITS LONG. THE CORRESPONDING SIDE OF THE OTHER IS 5 UNITS LONG. WHAT IS THE RATIO OF:

A THEIR PERIMETERS? **B** THEIR AREAS?

3 TWO TRIANGLES ARE SIMILAR. THE SIDES OF ONE ARE THREE TIMES AS LONG AS THE SIDES OF THE OTHER. WHAT IS THE RATIO OF THE AREAS OF THE SMALLER TO THE LARGER?

4 THE AREAS OF TWO SIMILAR TRIANGLES ARE 144 UNIT^2 AND 81 UNIT^2.

  A WHAT IS THE RATIO OF THEIR PERIMETERS?

  B IF A SIDE OF THE FIRST IS 6 UNITS LONG, WHAT IS THE LENGTH OF THE CORRESPONDING SIDE OF THE SECOND?

5 THE SIDES OF A POLYGON HAVE LENGTHS 5, 7, 8, 11 AND 15 UNITS. THE PERIMETER OF A SIMILAR POLYGON IS 75 UNITS. FIND THE LENGTHS OF THE SIDES OF THE LARGER P

5.2.5 Construction of Similar Figures

Enlargement

Group Work 5.4

Work with a partner

1 DRAW A TRIANGLE ABC ON SQUARED PAPER AS SHOWN.

2 TAKE A POINT O AND DRAW RAYS $\overrightarrow{OA}$, $\overrightarrow{OB}$ AND $\overrightarrow{OC}$; ON THESE RAYS MARK POINTS A′, B′ AND C′ SUCH THAT OA′ = 2OA; OB′ = 2 OB; OC′ = 2 OC

3 WHAT CAN YOU SAY ABOUT $\triangle ABC$ AND $\triangle A'B'C'$?

4 IS $\frac{OA'}{OA} = \frac{A'B'}{AB}$?

5 WHAT PROPERTIES HAVE NOT CHANGED?

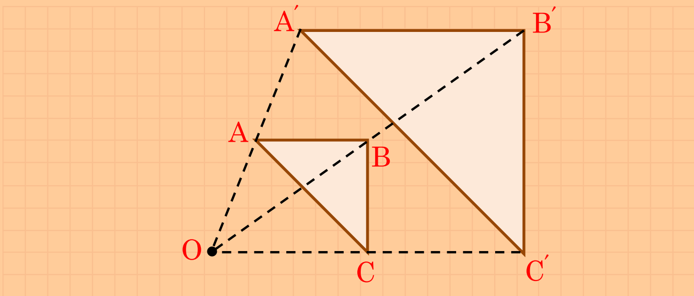

Figure 5.63

FIGURE 5.63 SHOWS TRIANGLE ABC AND ITS IMAGE TRIANGLE A′B′C′ UNDER THE TRANSFORMATION ENLARGEMENT. IN THE EQUATION OA′ = 2OA, THE FACTOR 2 IS CALLED THE scale factor AND THE POINT O IS CALLED the centre of enlargement.

IN GENERAL,

AN ENLARGEMENT WITH CENTRE O AND SCALE FACTOR k (WHERE k IS A REAL NUMBER) IS THE TRANSFORMATION THAT MAPS EACH POINT P TO A POINT P′ SUCH THAT

  I P′ IS ON THE RAY $\overrightarrow{OP}$ AND **II** $OP' = k\,OP$

IF AN OBJECT IS ENLARGED, THE RESULT IS AN IMAGE THAT IS MATHEMATICALLY SIMIL BUT OF DIFFERENT SIZE. THE IMAGE CAN BE EITHER LARGER, IF $k > 1$, OR SMALLER, IF $k < 1$.

EXAMPLE 1 IN FIGURE 5.64A BELOW, ΔABC IS ENLARGED TO FORM ΔA′B′C′. FIND THE CENTRE OF ENLARGEMENT.

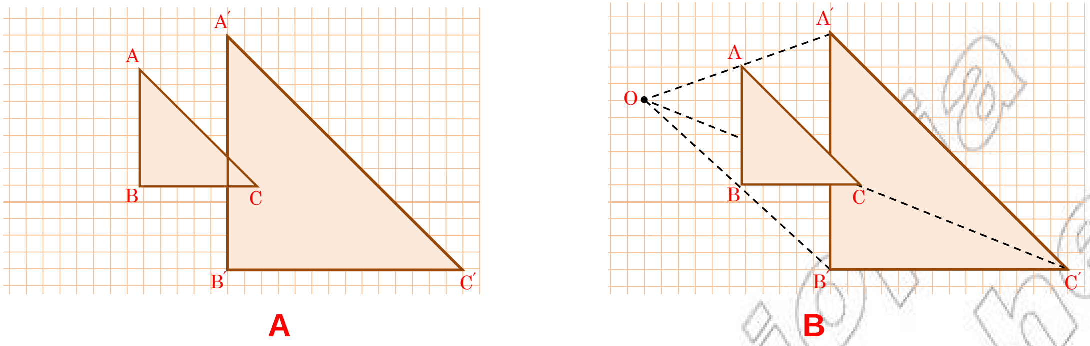

Figure 5.64

SOLUTION: THE CENTRE OF ENLARGEMENT IS FOUND BY JOINING CORRESPONDING POINTS ON THE OBJECT AND IMAGE WITH STRAIGHT LINES. THESE LINES ARE THEN EXTENDED TO MEET. THE POINT AT WHICH THEY MEET IS THE centre of enlargement O (*See* FIGURE 5.64B ABOVE).

EXAMPLE 2 IN FIGURE 5.65 BELOW, THE RECTANGLE ABCD UNDERGOES A TRANSFORMATION TO FORM RECTANGLE A′B′C′D′.

I FIND THE CENTRE OF ENLARGEMENT.

II CALCULATE THE SCALE FACTOR OF ENLARGEMENT.

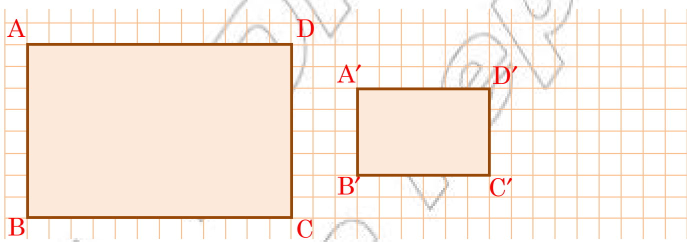

Figure 5.65

SOLUTION:

I BY JOINING CORRESPONDING POINTS ON BOTH THE OBJECT AND THE IMAGE, THE CENTRE OF ENLARGEMENT IS FOUND AT O, AS SHOWN IN FIGURE 5.66 BELOW.

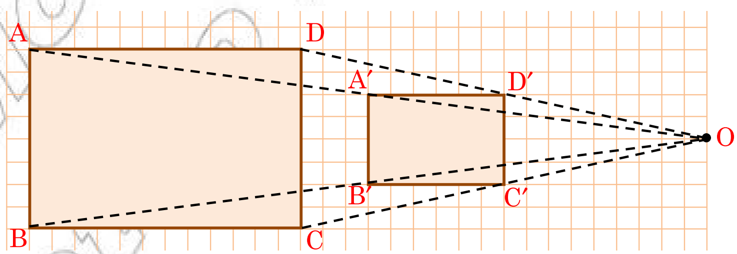

Figure 5.66

II THE SCALE FACTOR OF ENLARGEMENT $= \frac{A'B'}{AB} = \frac{4}{8} = \frac{1}{2}$

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IF THE SCALE FACTOR OF ENLARGEMENT IS GREATER THAN 1, THEN THE IMAGE IS L
OBJECT. IF THE SCALE FACTOR LIES BETWEEN 0 AND 1 THEN THE RESULTING IMAGE IS
THE OBJECT. IN THESE LATTER CASES, ALTHOUGH THE IMAGE IS SMALLER THAN T
TRANSFORMATION IS STILL KNOWN AS AN ENLARGEMENT.

Exercise 5.7

1 COPY THE FOLLOWING FIGURES AND FIND:

I THE CENTRE OF ENLARGEMENT. **II** THE SCALE FACTOR OF THE ENLARGEMENT.

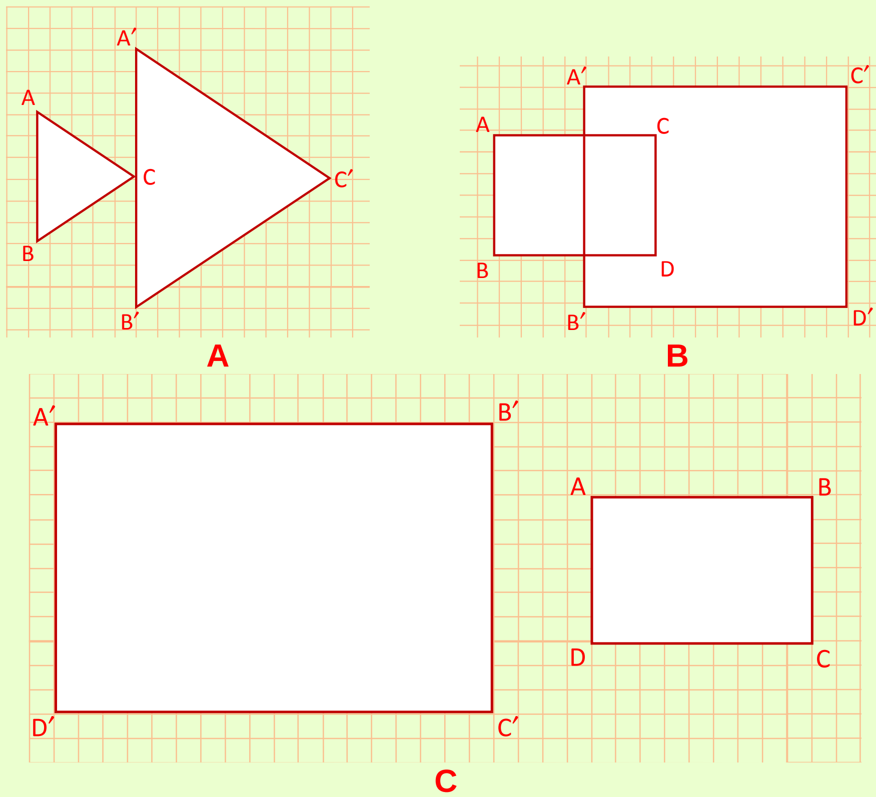


Figure 5.67

2 COPY AND ENLARGE EACH OF THE FOLLOWING FIGURES BY A SCALE FACTOR OF:

I 3 **II** $\frac{1}{2}$

(*O is the centre of enlargement*).

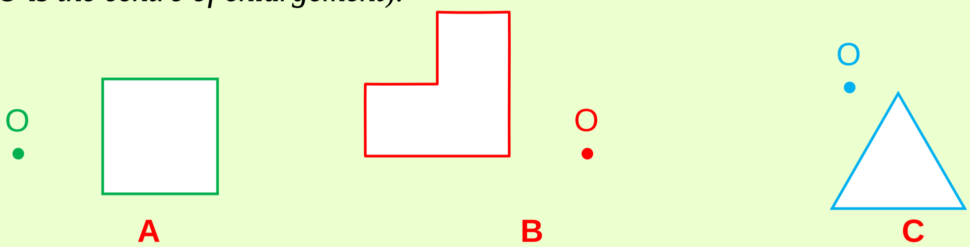


Figure 5.68

5.2.6 Real-Life Problems Using Congruency and Similarity

THE PROPERTIES OF CONGRUENCY AND SIMILARITY CAN BE APPLIED TO SOLVE SOME REAL-LIFE PROBLEMS AND ALSO TO PROVE CERTAIN GEOMETRIC PROPERTIES. FOR EX FOLLOWING EXAMPLES.

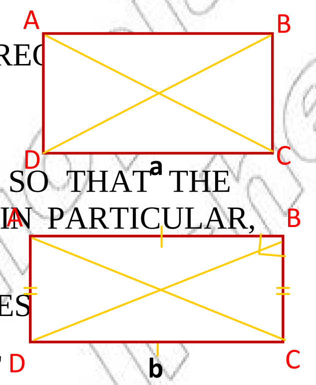


Figure 5.69

EXAMPLE 1 SHOW THAT THE DIAGONALS OF A REC CONGRUENT (FIGURE 5.69).

SOLUTION: SUPPOSE ABCD IS A RECTANGLE (FIGURE 5.69B).

THEN, ABCD IS A PARALLELOGRAM (WHY?) SO THAT THE OPPOSITE SIDES OF ABCD ARE CONGRUENT. IN PARTICULAR, $\overline{AB} \cong \overline{DC}$. CONSIDER ΔABC AND ΔDCB.

CLEARLY, $\angle ABC \cong \angle DCB$ (BOTH ARE RIGHT ANGLES

HENCE, $\Delta ABC \cong \Delta DCB$ BY THE SAS CONGRUENCE PROPERTY. CONSEQUENTLY, $\overline{AC} \cong \overline{DB}$, AS DESIRED.

CARPENTERS USE THE RESULT OF EXAMPLE 1 WHEN FRAMING RECTANGULAR SHAPES. THAT IS, T DETERMINE WHETHER A QUADRILATERAL IS A RECTANGLE, A CARPENTER CAN MEASU TO SEE IF THEY ARE CONGRUENT (IF SO, THE SHAPE IS A PARALLELOGRAM). THEN THE MEASURE THE DIAGONALS TO SEE IF THEY ARE CONGRUENT (IF SO, THE SHAPE IS A REC

EXAMPLE 2 WHEN ALI PLANTED A TREE 5 M AWAY FROM POINT A, IT JUST BLOCKED THE VIEW OF A BUILDING 50 M AWAY. IF THE BUILDING WAS 20 M TALL, HOW TA THE TREE?

SOLUTION: LABEL THE FIGURE AS SHOWN. LET x BE THE HEIGHT OF THE TREE.

WE HAVE $\frac{BE}{CD} = \frac{AE}{AD}$

$$\frac{x}{20} = \frac{5}{50}$$

$$x = 20 \times \frac{5}{50} = 2 \text{ M}$$

$\therefore$ THE HEIGHT OF THE TREE WAS 2 M.

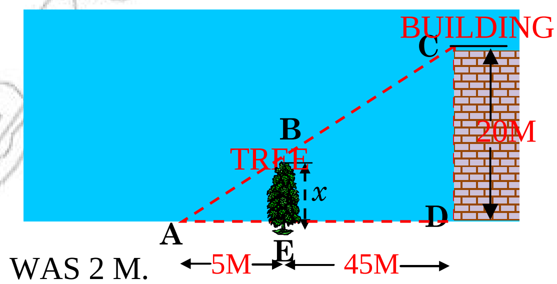


Figure 5.70

Exercise 5.8

1. AWEKE TOOK 1 HOUR TO CUT THE GRASS IN A SQUARE FIELD OF SIDE 30 M. HOW LONG WILL IT TAKE HIM TO CUT THE GRASS IN A SQUARE FIELD OF SIDE 120 M?
2. A LINE FROM THE TOP OF A CLIFF TO THE GROUND PASSES JUST OVER THE TOP OF A POLE 20 M HIGH. THE LINE MEETS THE GROUND AT A POINT 15 M FROM THE BASE OF THE PO 120 M AWAY FROM THIS POINT TO THE BASE OF THE CLIFF, HOW HIGH IS THE CLIFF
3. A TREE CASTS A SHADOW OF 30 M. AT THE SAME TIME, A 1 M POLE CASTS A SHADOW OF 12 M. FIND THE HEIGHT OF THE TREE.

5.3 FURTHER ON TRIGONOMETRY

5.3.1 Radian Measure of an Angle

AN ANGLE IS THE UNION OF TWO RAYS WITH A COMMON END POINT.

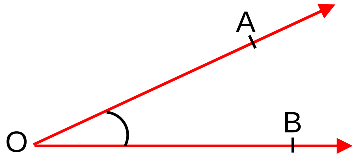


Figure 5.71

IN GENERAL, WE ASSOCIATE EACH ANGLE WITH A REAL NUMBER CALLED THE measure of the angle. THE TWO MEASURES THAT ARE MOST FREQUENTLY USED ARE DEGREE AND RADIAN.

I Measuring angles in degrees

WE KNOW THAT A RIGHT ANGLE CONTAINS 90^o, AND THAT A COMPLETE ROTATION CAN BE THOUGHT OF AS AN ANGLE OF 360^o. IN VIEW OF THIS LATTER FACT, WE CAN DEFINE A DEGREE AS FOLLOWS

Definition 5.6

A degree, denoted by (°), is defined as the measure of the central angle subtended by an arc of a circle equal in length to $\frac{1}{360}$ of the circumference of the circle.

✓ A minute WHICH IS DENOTED BY (′), IS $\frac{1}{60}$ OF A DEGREE.

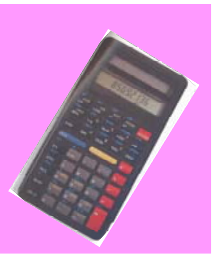
✓ A second WHICH IS DENOTED BY (″), IS $\frac{1}{60}$ OF A MINUTE.

SO, WE HAVE THE FOLLOWING RELATIONSHIP.

$1' = \left(\frac{1}{60}\right)^{o}$, $1'' = \left(\frac{1}{60}\right)'$ THAT IS $1'' = \left(\frac{1}{3600}\right)^{o}$ OR $1^o = 60'$ AND $1' = 60''$

Calculator Tip

Use your calculator to convert 20° 41'16'', which is read as 20 degrees, 41 minutes and 16 seconds, into degrees, (as a decimal).



II Measuring angles in radians

ANOTHER UNIT USED TO MEASURE ANGLES IS THE RADIAN. TO UNDERSTAND WHAT IS MEANT BY A RADIAN, WE AGAIN START WITH A CIRCLE. WE MEASURE A LENGTH r OF THE CIRCLE EQUAL TO THE R CIRCLE ALONG THE CIRCUMFERENCE OF THE CIRCLE, SO THAT $\widehat{AB}$ IS EQUAL TO THE RADIUS r. $\angle AOB$ IS THEN AN ANGLE OF 1 RADIAN. WE DEFINE THIS AS FOLLOWS.

Definition 5.7

A radian (rad) is defined as the measure of the central angle subtended by an arc of a circle equal in length to the radius of the circle.

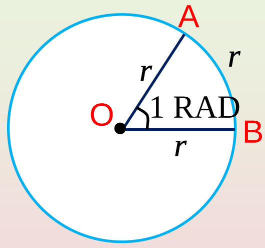


Figure 5.72

YOU KNOW THAT THE CIRCUMFERENCE OF A CIRCLE IS EQUAL TO 2 r. SINCE AN ARC OF LENGTH r ALONG THE CIRCLE GIVES 1 RAD, A COMPLETE ROTATION OF LENGTH 2 r GENERATES AN ANGLE OF 2 RADIANS. ON THE OTHER HAND, WE KNOW THAT A COMPLETE REVOLUTION REPRESEN 360^o. THIS GIVES US THE FOLLOWING RELATIONSHIP:

1 revolution = 360^o = 2 radians

I.E., 180^o = RADIANS, FROM WHICH WE OBTAIN,

$1 \text{ RADIAN} = \left(\frac{180}{}\right)^o \approx 57.3^o$. $\quad 1^O = \frac{}{180}$ RADIAN ≈ 0.0175 RADIAN.

THEREFORE, WE HAVE THE FOLLOWING CONVERSION RULES FOR DEGREES AND RADIANS.

TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^O}{}$

TO CONVERT DEGREES TO RADIANS MULTIPLY BY $\frac{}{180^O}$

EXAMPLE 1

I CONVERT EACH OF THE FOLLOWING TO RADIANS:

A 30^o **B** 90^o

II CONVERT EACH OF THE FOLLOWING TO DEGREES:

A $\frac{}{4}$ RA **B** $\frac{}{3}$ RAD

SOLUTION:

I **A** $30^o = 30^O \times \frac{}{180^O} = \frac{}{6}$ RAD. **B** $90^o = 90^O \times \frac{}{180^O} = \frac{}{2}$ RAD.

II **A** $\frac{}{4}$ RAD $= \frac{}{4} \times \frac{180^O}{} = 45^O$. **B** $\frac{}{3}$ RAD $= \frac{}{3} \times \frac{180^O}{} = 60^O$.

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5.3.2 Trigonometrical Ratios to Solve Right-angled Triangles

ACTIVITY 5.8

1 WHAT IS THE MEANA TRIGONOMETRIC RATIO?

2 GIVEN RIGHT-ANTRIANG, ΔABC ANDΔA'B'C',

IF M(∠A) = M(∠A'), WHAT CAN YOU SAY ABOUT THE T

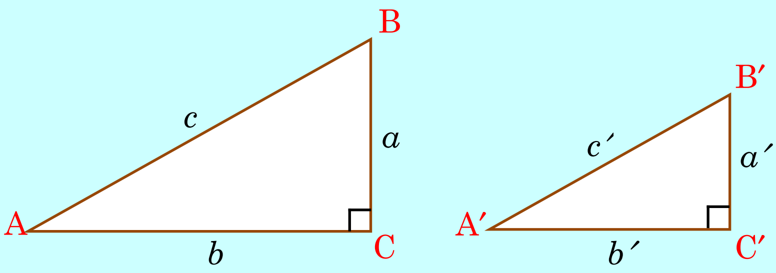

Figure 5.73

THE ANSWERS TO THESE QUESTIONS SHOULD HAVE LEAD YOU TO THE RELATIONSHIPS BETWEEN AN ANGLE AND THE SIDES OF A RIGHT-ANGLED TRIANGLE, WHICH HELP YOU TO SOLVE PROBLEMS THAT INVOLVE RIGHT-ANGLED TRIANGLES.

CONSIDER THE TWO TRIANGLES IN FIGURE 5.73 ABOVE.

GIVEN M(∠A) = M (∠A')

I ∠A ≅ ∠A'

II ∠C ≅ ∠C'

THEREFORE, ΔABC ~ ΔA'B'C' (BY AA SIMILARITY)

THIS MEANS $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$

FROM THIS WE GET,

1 $\frac{BC}{AB} = \frac{B'C'}{A'B'}$ **2** $\frac{AC}{AB} = \frac{A'C'}{A'B'}$ **3** $\frac{BC}{AC} = \frac{B'C'}{A'C'}$

OR $\frac{a}{c} = \frac{a'}{c'}$, $\frac{b}{c} = \frac{b'}{c'}$ AND $\frac{a}{b} = \frac{a'}{b'}$

THE FRACTIONS OR RATIOS IN EACH OF THESE PROPORTIONS ARE CALLED trigonometric ratios.

Sine:- THE FRACTIONS IN PROPORTION 1 ABOVE ARE FORMED BY DIVIDING THE SIDE OPPOSITE OF ∠A (OR ∠A') BY THE HYPOTENUSE OF EACH TRIANGLE. THIS RATIO IS CALLED THE SINE OF ∠A. IT IS ABBREVIATED TO SIN A.

Cosine:- THE FRACTIONS IN PRO 2 ARE FORMED BY DIVIDING THE adjacent side TO ∠A (OR ∠A') BY THE hypotenuse OF EACH TRIANGLE. THIS RATIO IS CALLED cosine OF ∠A. IT IS ABBREVIATED TO COS A.

Tangent:- THE FRACTION PROPORTION 3 ARE FORMED BY DIVIDING THE opposite side OF ∠A (OR ∠A') BY THE adjacent side. THIS RATIO IS CALLED THE ∠ A. IT IS ABBREVITO TAN A.

THE FOLLOWING ABBREVIATIONS ARE CO

ADJ = ADJACENT S   HYP = HYPOTENUSE   **OPP** = OPPOSI SIDE.

THE ABOVE DISCUSSION CAN BE SUMMARIZED AND EXP

SIN A $= \frac{OPP}{HYP} = \frac{BC}{AB}$;   COS A $= \frac{ADJ}{HYP} = \frac{AC}{AB}$

TAN A $= \frac{OPP}{ADJ} = \frac{BC}{AC}$

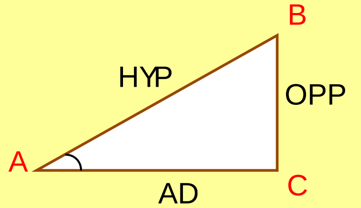

Figure 5.74

EXAMPLE 1 IN THE FOLLOWING RIGH, FIND THE VALUES OF SINE, COSINE AND TANGENT OF THE

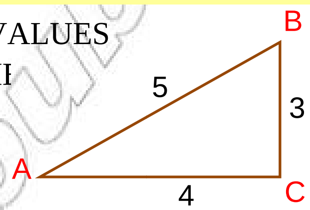

Figure 5.75

SOLUTION:

SIN A $= \frac{OPP}{HYP} = \frac{BC}{AB} = \frac{3}{5}$;   COS A $= \frac{ADJ}{HYP} = \frac{AC}{AB} = \frac{4}{5}$;   TAN A $= \frac{OPP}{ADJ} = \frac{BC}{AC} = \frac{3}{4}$

SIMILARLY, SIN B $= \frac{OPP}{HYP} = \frac{AC}{AB} = \frac{4}{5}$;   COS B $= \frac{ADJ}{HYP} = \frac{BC}{AB} = \frac{3}{5}$

TAN B $= \frac{OPP}{ADJ} = \frac{AC}{BC} = \frac{4}{3}$

ACTIVITY 5.9

1 USING RULER AND CES, DRAW AN EQUILATERAL TRIAN WHICH EACH SIDE CM LON DRAW THE ALTITUDE $\overline{AD}$ PERPENDICULAR $\overline{BC}$.

A WHAT IS M(∠ABD)? M(∠BAD) ? GIVE REASONS.

B FIND THE LENGTHS I AD (WRITE THE ANSWERS IN SIMPLIFIED RADICAL FORM).

C USE THESE TO FIND^o, TAN 30^o, COS 30^o, SIN 60^o, TAN 60^o, COS 60^o. WHAT DO YOU NOTICE

2 DRAW AN ISOSCELES TRIANGLE ABC IN WHICH ∠C IS A RIGHT ANGLE AND AC = 2 CM.

- **A** WHAT IS m(∠A)?
- **B** CALCULATE THE LENGTHS AB AND BC (LEAVE YOUR ANSWER IN RADICAL FORM).
- **C** CALCULATE SIN 45°, COS 45°, TAN 45°

FROM THE ABOVE ACTIVITY, YOU HAVE PROBABLY DISCOVERED THAT THE VALUES OF SI AND TANGENT OF THE ANGLES 30°, 45° AND 60° ARE AS SUMMARIZED IN THE FOLLOWING TABLE.

∠A	30°	45°	60°
SIN A	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
COS A	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
TAN A	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

THE ANGLES 30°, 45° AND 60° ARE CALLED **special angles**, BECAUSE THEY HAVE THESE EXACT TRIGONOMETRIC RATIOS.

EXAMPLE 2 A LADDER 6 M LONG LEANS AGAINST A WALL MAKING AN ANGLE OF 60° WITH THE GROUND. FIND THE HEIGHT OF THE WALL. HOW FAR FROM THE WALL I OF THE LADDER?

SOLUTION: CONSIDER △ABC IN THE FIGURE.

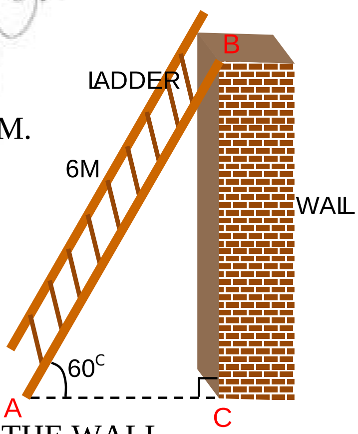

Figure 5.76

M(∠A) = 60°, M(∠C) = 90°, M(∠B) = 30° AND AB = 6 M.

WE WANT TO FIND BC AND AC.

TO FIND BC, WE USE SIN 60° = $\frac{BC}{AB}$. BUT, SIN 60° = $\frac{\sqrt{3}}{2}$

SO, $\frac{\sqrt{3}}{2} = \frac{BC}{6}$

THEREFORE, BC = $3\sqrt{3}$ M, WHICH IS THE HEIGHT OF THE WALL.

TO FIND THE DISTANCE BETWEEN THE FOOT OF THE LADDER AND THE WALL, WE U

COS 60° = $\frac{AC}{AB}$. COS 60° = $\frac{1}{2}$ AND AB = 6.

SO, $\frac{1}{2} = \frac{AC}{6}$ WHICH IMPLIES AC = 3 M.

In the above example, if the angle that the ladder made with the ground were 50°, how would you solve the problem?

TO SOLVE THIS PROBLEM, YOU WOULD NEED **trigonometric tables**, WHICH GIVE YOU THE VALUES OF SIN 50° AND COS 50°

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5.3.3 Trigonometrical Values of Angles from Tables

(*sin* , *cos* *and tan* , *for* $0^o \leq$ $< 180^o$)

IN THE PREVIOUS SECTION, WE CREATED A TABLE OF TRIGONOMETRIC RATIOS FOR THE SPECIAL (NAMELY 30^o, 45^o AND 60^o). THEORETICALLY, BY FOLLOWING THE SAME, A TABLE OF TRIGONOMETRIC RATIOS CAN BE CONSTRUCTED. THERE ARE TABLES OF APPROXIMATE VALUES OF TRIGONOMETRIC RATIOS OF ANGLES HAVE ALREADY CONSTRUCTED BY ADVANCED ARITHMETICAL PROCESSES. ONE IS INCLUDED AT THE END OF THIS BOOK.

ACTIVITY 5.10

USING THE TRIGONOMETRIC, FIND THE VALUE OF EACH OF THE

A COS 50^o **B** SIN 2^o **C** TAN 10^o **D** SIN 80^o

IF YOU KNOW THE VALUE OF ONE OF THE RATIOS OF AN ANGLE, YOU CAN USE OF TRIGONOMETRIC RATIOS TO FIND THE ANGLE. THE PROCEDURE IS ILLUSTR EXAMPLE.

EXAMPLE 1 FIND THE MEASURE OF THE ACUTE, CORRECT TO THE NEAR, IF SIN A ^{o} = 0.521.

SOLUTION: REFERRING TO THE "SINE" COLUMN TABLE, WE FIND THAT 0.521 DO APPEAR THERE. THE TWO VALUES IN THE TABLE CLOSEST TO ONE LARGER) ARE 0.515 AND 0.530. THESE VALUES TO 31^o AND 32^o, RESPECTIVELY.

NOTE THAT 0.521 IS CLOSER, WHOSE VALUE CORRESPONDS TO 31^o

THEREFORE, m(∠A) = 31^o (*to the nearest degree*)

ACTIVITY 5.11

1 USE YOUR TRIGONOMETRIC FIND THE VALUE OF THE ACUTE CORRECT TO THE NEAR.

A	SIN (A)= 0.92	**D**	SIN (A) = 0.981
B	COS (A) = 0.984	**E**	COS (A) = 0.422
C	TAN (A) = 0.38(	**F**	TAN (A) = 2.410

2 USE YOUR CALCULATOR TO FIN(check your calculator is in degrees mode)

USING TRIGONOMETRIC RATIOS, YOU CAN N-ANGLED TRIANGLES AN PROBLEMS. TO SOLVE-ARIGHT-ANGLED TRIANGLE MEANS TO FIND THE MISSING PARTS OF THE WHEN SOME PARTS ARE GIVEN. FOR EXAMPLE, IF YOU ARE GIVEN THE LE MEASURE OF AN ANGLE (OTHER THAN THE , YOU CAN USE THE APPI TRIGONOMETRIC RATIOS TO FIND THE REQUIRE

IN SHORT, IN SOLVING A RIGHT-ANGLED TRIANGLE, WE NEED TO USE

A THE TRIGONOMETRIC RATIOS OF

B Pythagoras theorem WHICH IS $a^2 + b^2 = c^2$, WHERE a IS THE LENGTH OF T OPPOSITE TO ∠A, b IS THE LENGTH OF THE SIDE OPPOSITE TO ∠B AND c IS THE LENGTH OF THE hypotenuse.

EXAMPLE 2 FIND THE LENGTHS OF THE SIDES INDICATED BY

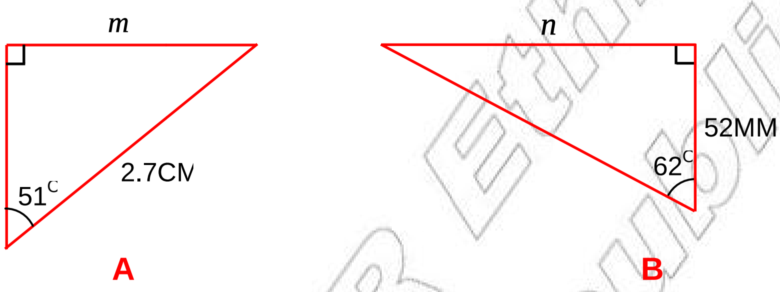


Figure 5.77

SOLUTION:

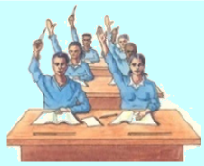
A $\sin 51^\circ = \dfrac{m}{2.7}$.

SO, $m = 2.7 \sin 51^\circ = 2.7 \times 0.777 \approx 2.1$ CM (*1 decimal place*)

B $\tan 62^\circ = \dfrac{n}{52}$.

SO, $n = 52 \tan 62^\circ = 52 \times 1.881 \approx 98$ MM (*to the nearest mm*)

ACTIVITY 5.12



USING FIGURE 5.78, WRITE EACH OF THE FOLLOWING IN TERMS OF THE LENGTHS a, b AND c.

1 **A** $\sin(\angle A)$ **B** $\cos(\angle A)$

C $\tan(\angle A)$ **D** $\dfrac{\sin(\angle A)}{\cos(\angle A)}$

E $\sin(\angle B)$ **F** $\cos(\angle B)$

G $\tan(\angle B)$ **H** $\dfrac{\sin(\angle B)}{\cos(\angle B)}$

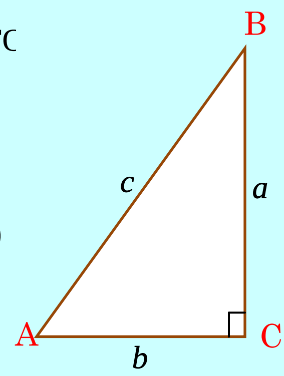


Figure 5.78

2 A $(\sin(\angle A))^2$
B $(\cos(\angle A))^2$
C WRITE THE VALUE OF $\sin^2(\angle A) + \cos^2(\angle A)$.

Notation: WE ABBREVIATE $(\sin(\angle A))^2$ AS $\sin^2(\angle A)$. SIMILARLY, WE WRITE $\cos^2(\angle A)$ AND $\tan^2(\angle A)$ INSTEAD OF $(\cos(\angle A))^2$ AND $(\tan(\angle A))^2$, RESPECTIVELY.

DO YOU NOTICE ANY INTERESTING RESULTS FROM THE ABOVE ACTIVITY? STATE THEM.

YOU MIGHT HAVE DISCOVERED THAT

1 IF $m(\angle A) + m(\angle B) = 90^o$, I.E., A AND B ARE COMPLEMENTARY ANGLES, THEN

I $\sin(\angle A) = \cos(\angle B)$ II $\cos(\angle A) = \sin(\angle B)$

2 $\tan(\angle A) = \dfrac{\sin(\angle A)}{\cos(\angle A)}$

3 $\sin^2(\angle A) + \cos^2(\angle A) = 1$

How can you use the trigonometric table to find the sine, cosine and tangent of obtuse angles such as 95^o, 129^o, and 175^o?

SUCH ANGLES ARE NOT LISTED IN THE TABLE.

BEFORE WE CONSIDER HOW TO FIND THE TRIGONOMETRIC RATIO OF OBTUSE ANG REDEFINE THE TRIGONOMETRIC RATIOS BY USING DIRECTED DISTANCE. TO DO THIS, W RIGHT ANGLE TRIANGLE POA AS SHOWN IN FIGURE 5.79. ANGLE POA IS THE ANTICLOCKWISE ANGLE FROM THE POSITIVE x-AXIS.

NOTE THAT THE LENGTHS OF THE SIDES CAN BE EXPRESSED IN TERMS OF THE COORDINA

I.E., OA = x, AP = y, AND USING PYTHAGORAS THEOREM, WE HAVE,

$$OP = \sqrt{x^2 + y^2}$$

AS A RESULT, THE TRIGONOMETRIC RATIOS OF $\angle POA$ CAN BE EXPRESSED IN TERMS OF x, y AND $\sqrt{x^2 + y^2}$, AS FOLLOWS:

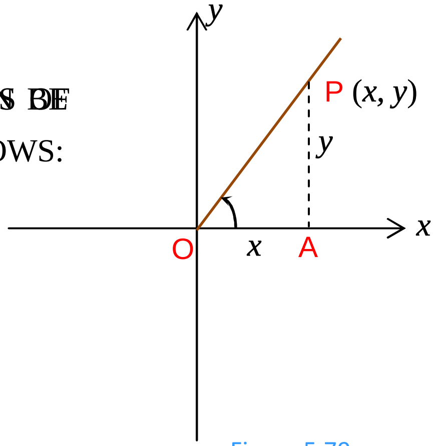


Figure 5.79

$$\sin(\angle POA) = \frac{\text{OPP}}{\text{HYP}} = \frac{AP}{OP} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos(\angle POA) = \frac{\text{ADJ}}{\text{HYP}} = \frac{OA}{OP} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\tan(\angle POA) = \frac{\text{OPP}}{\text{ADJ}} = \frac{AP}{OA} = \frac{y}{x}$$

I.E., $\sin(\angle POA) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $\cos(\angle POA) = \dfrac{x}{\sqrt{x^2 + y^2}}$; $\tan(\angle POA) = \dfrac{y}{x}$

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FROM THE ABOVE DISCUSSION, IT IS POSSIBLE TO COMPUTE THE VALUES OF TRIGONO USING ANY POINT ON THE TERMINAL SIDE OF THE ANGLE.

LET US NOW FIND THE SINE AND COSINE OF 129° USING THE TABLE. TO DO THIS, WE FIRST PUT 129° ON THE xy-PLANE, SO THAT ITS VERTEX IS AT THE ORIGIN AND ITS INITIAL SIDE ON THE POSITIVE x-AXIS.

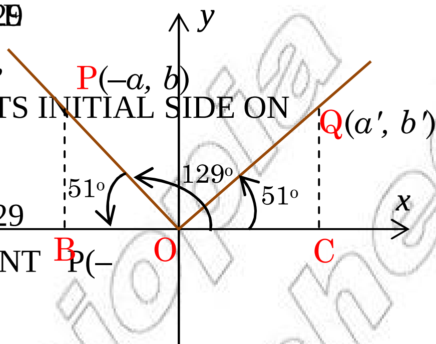

Figure 5.80

I TO FIND SIN 129°, WE FIRST EXPRESS SIN 129° IN TERMS OF THE COORDINATES OF THE POINT P(–a, b). SO, WE HAVE,

$$\sin 129^o = \frac{b}{\sqrt{a^2+b^2}}.$$

WHAT ACUTE ANGLE IN THE xy-PLANE HAS THE SAME y-VALUE (THAT IS b)?

IF WE DRAW THE ∠51° COQ SO THAT OP = OQ, THEN WE SEE THAT

ΔBOP ≅ ΔCOQ. SO WE HAVE

BP = CQ AND OB = OC

IT FOLLOWS THAT SIN 129° = SIN 51°. FROM THE TABLE SIN 51° = 0.777.

HENCE, SIN 129° = 0.777

NOTICE THAT SIN 129° = SIN (180° – 129°)

THIS CAN BE GENERALIZED AS FOLLOWS.

IF IS AN OBTUSE ANGLE, I.E., 90° < < 180°, THEN

SIN = SIN (180 –) **I**

II TO FIND COS 129°

HERE ALSO WE FIRST EXPRESS COS 129° IN TERMS OF THE COORDINATES OF P (–a, b). SO,

$$\cos 129^o = \frac{-a}{\sqrt{a^2+b^2}}.$$

BY TAKING 180° – 129°, WE FIND THE ACUTE ANGLE 51°.

SINCE ΔBOP ≅ ΔCOQ, WE SEE THAT OC = OB, BUT IN THE OPPOSITE DIRECTION. SO, TH x VALUE OF P IS THE OPPOSITE OF THE x VALUE OF Q. THAT IS $-a = a'$

THEREFORE, $\cos 129^o = \dfrac{a'}{\sqrt{a^2+b^2}} = -\cos 51^o$

FROM THE TRIGONOMETRIC TABLE, YOU HAVE COS 51° = 0.629

THEREFORE, COS 129° = – 0.629.

THIS DISCUSSION LEADS YOU TO THE FOLLOWING GENERALIZATION.

IF   IS AN OBTUSE ANGLE, THEN

COS = – COS (180° –)   II

EXAMPLE 3 WITH THE HELP OF THE TRIGONOMETRIC TABLE, FIND THE APPROXIMATE VALUES OF:

A COS 100°   **B** SIN 163°   **C** TAN 160°

SOLUTION: A USING THE RULE COS = – COS (180° –), WE OBTAIN;

COS 100° = – COS (180° – 100°) = – COS 80°

FROM THE TRIGONOMETRIC TABLE, WE HAVE COS 80° = 0.174.

THEREFORE, COS 100° = – 0.174.

B FROM THE RELATION SIN = SIN (180 –), WE HAVE

SIN 163° = SIN (180° – 163°) = SIN 17°

FROM THE TABLE SIN 17° = 0.292

THEREFORE, SIN 163° = 0.292

C TO FIND TAN 160°,

$$\tan 160^o = \frac{\sin 160^o}{\cos 160^o} = \frac{\sin 20^o}{-\cos 20^o} = -\left(\frac{\sin 20^o}{\cos 20^o}\right) = -\tan 20^o$$

FROM THE TABLE, WE HAVE TAN 20° = 0.364.

THEREFORE, TAN 160° = –0.364.

TO SUMMARIZE, FOR A POSITIVE OBTUSE ANGLE ,

SIN = SIN (180° –)

COS = – COS (180° –)

TAN = – TAN (180° –)

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Exercise 5.9

1 **I** EXPRESS EACH OF THE FOLLOWING RADIAN MEASURES IN DEGREES:

A $\frac{\ }{6}$ **B** **C** $\frac{\ }{3}$, **D** 2 **E** $\frac{3}{4}$ **F** 5

II EXPRESS EACH OF THE FOLLOWING IN RADIAN MEASURE:

A 270^o **B** 150^o **C** 225^o **D** 15^o

2 WITHOUT USING A TABLE, FIND THE VALUE OF EACH OF THE FOLLOWING. (ANSWERS MAY BE LEFT IN RADICAL FORM.)

A SIN $\frac{\ }{6}$ **B** TAN $\frac{3}{4}$ **C** COS 150^o **D** TAN $\frac{2}{3}$

3 IN ΔABC, IF M ($\angle$A) = 53^o, AC = 8.3 CM AND M ($\angle$C) = 90^o, FIND BC, CORRECT TO THE NEAREST WHOLE NUMBER.

4 A LADDER 20 FT LONG LEANS AGAINST A BUILDING, MAKING AN ANGLE OF 65^o WITH THE GROUND. DETERMINE, CORRECT TO THE NEAREST FT, HOW FAR UP THE BUILDIN REACHES.

5 EXPRESS EACH OF THE FOLLOWING IN TERMS OF COSINE OR SINE OF AN ACUTE ANGLE:

A COS 165^o **B** SIN 126^o **C** COS $\frac{3}{5}$ **D** SIN 139^o

6 IN EACH OF THE FOLLOWING, FIND THE LENGTH OF THE HYPOTENUSE (*a*).

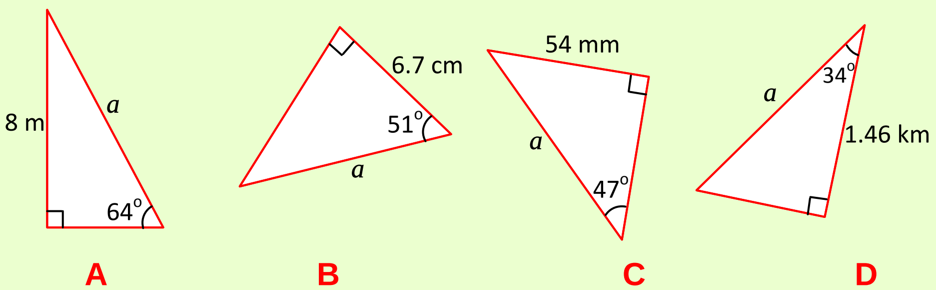

Figure 5.81

7 FIND THE SINE, COSINE AND TANGENT OF EACH OF THE FOLLOWING ANGLES FROM THE TABLE.

A 25^o **B** 63^o **C** 89^o

D 135^o **E** 142^o **F** 173^o

8 USE THE TRIGONOMETRIC TABLE INCLUDED AT THE END OF THE BOOK TO FIND THE DEGREE MEASURE OF $\angle$P IF:

A SIN P = 0.83 **B** COS P = 0.462 **C** TAN P = 0.945

D SIN P = $\frac{1}{4}$ **E** COS P = 0.824

5.4 CIRCLES

5.4.1 Symmetrical Properties of Circles

ACTIVITY 5.13

1 WHAT IS A CIRCLE

2 WHAT IS A LINE OF SYM

3 WHICH OF THE FOLLOWING FIGURES HAVE A LINE

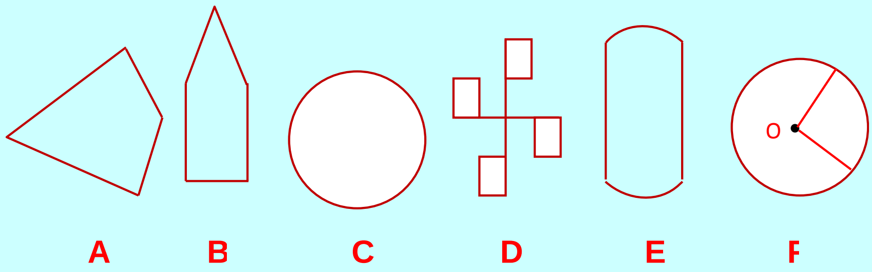

Figure 5.82

RECALL THAT A CIRCLE IS DEFINED AS THE SET OF POINTS IN A GIVEN PLANE, EACH OF WHICH IS THE SAME DISTANCE FROM A FIXED POINT OF THE PLANE. THE FIXED POINT IS CALLED THE centre, AND THE DISTANCE IS THE radius OF THE CIRCLE.

A LINE SEGMENT THROUGH THE CENTRE OF A CIRCLE WITH END POINTS ON THE CIRCLE IS CALLED A diameter. A chord OF A CIRCLE IS A LINE SEGMENT WHOSE END POINTS LIE ON THE CIRCLE.

IN SECTION 5.1.2, YOU LEARNED THAT IF ONE PART OF A FIGURE CAN BE MADE TO FIT THE REST OF THE FIGURE BY FOLDING IT ABOUT $\overleftrightarrow{AB}$, THE FIGURE IS SAID TO BE SYMMETRICAL ABOUT $\overleftrightarrow{AB}$, AND THE STRAIGHT LINE $\overleftrightarrow{AB}$ IS CALLED THE line of symmetry. FOR EXAMPLE, EACH OF THE FOLLOWING FIGURES IS SYMMETRICAL ABOUT THE LINE $\overleftrightarrow{AB}$.

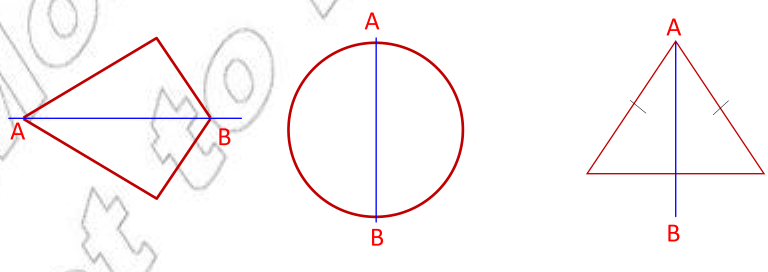

Figure 5.83

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OBSERVE THAT IN A SYMMETRICAL FIGURE THE LENGTH OF ANY LINE SEGMENT OR T… ANGLE IN ONE HALF OF THE FIGURE IS EQUAL TO THE LENGTH OF THE CORRESPONDING… THE SIZE OF THE CORRESPONDING ANGLE IN THE OTHER HALF OF THE FIGURE.

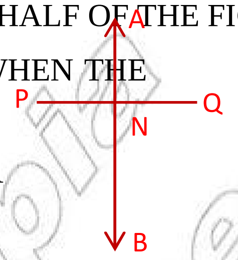

Figure 5.84

IF IN THE FIGURE ON THE RIGHT, P COINCIDES WITH Q WHEN THE FIGURE IS FOLDED ABOUT $\overleftrightarrow{AB}$ AND IF $\overline{PQ}$ INTERSECTS $\overleftrightarrow{AB}$ AT N

THEN $\angle PNA$ COINCIDES WITH $\angle QNA$ AND THEREFORE EACH IS A RIGHT ANGLE AND PN = QN.

THEREFORE,

IF P AND Q ARE CORRESPONDING POINTS FOR A LINE OF SYMMETRY $\overleftrightarrow{AB}$, THEN $\overleftrightarrow{AB}$ IS THE PERPENDICULAR BISECTOR OF $\overline{PQ}$. CONVERSELY, IF $\overleftrightarrow{AB}$ IS THE PERPENDICULAR BISECTOR OF $\overline{PQ}$, THEN P AND Q ARE CORRESPONDING POINTS FOR THE LINE OF SYMMETRY $\overleftrightarrow{AB}$ AND WE SAY THAT Q IS THE IMAGE OF P IN $\overleftrightarrow{AB}$ AND P IS THE IMAGE OF Q IN $\overleftrightarrow{AB}$.

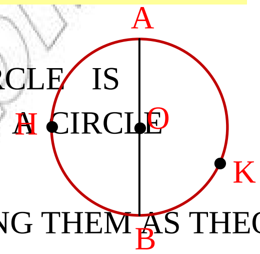

Figure 5.85

IN THE ADJACENT FIGURE, O IS THE CENTRE AND $\overline{AB}$ IS A DIAMETER OF THE CIRCLE. NOTE THAT A CIRCLE IS SYMMETRICAL ABOUT ITS DIAMETER. THEREFORE, A CIRCLE HAS AN INFINITE NUMBER OF LINES OF SYMMETRY.

WE NOW DISCUSS SOME PROPERTIES OF A CIRCLE, STATING THEM AS THEOREMS.

Theorem 5.9

The line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

Proof:-

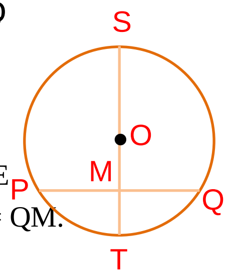

Figure 5.86

GIVEN: A CIRCLE WITH CENTRE O AND A CHORD $\overline{PQ}$ WHOSE MIDPOINT IS M.

WE WANT TO PROVE THAT $\angle OMP$ IS A RIGHT ANGLE.

CONSTRUCTION: DRAW THE DIAMETER ST THROUGH M. THEN THE CIRCLE IS SYMMETRICAL ABOUT THE LINE ST. BUT PM = QM.

SO, ST IS THE PERPENDICULAR BISECTOR OF PQ.

THIS COMPLETES THE PROOF.

Theorem 5.10

The line segment drawn from the centre of a circle perpendicular to a chord bisects the chord.

Proof:-

GIVEN: A CIRCLE WCENTRE O, AND THE LINE SEGMENT ON DRAWN FROM O PERPENDICULAR TAB AS SHOWN IN THE ADJACEN

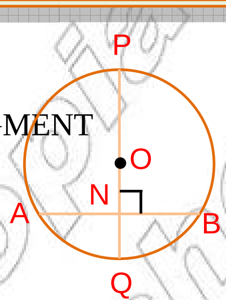

Figure 5.87

WE WANT TO PROVE THAT AN

CONSTRUCTION: DRAW TDIAMETER PQ THROUGH N.

THEN THE CIRCLE IS SYMMETRIC PQ. BUT PQ ⊥ AB AND A AND B ARE ON THE

THEREFORE, PQ IS THE PERPENDICULAR B

ACTIVITY 5.14

1 PROVE THEOREM 5.1 AND 5.11 USING CONGRUENCY OF TRIA

2 A CHORD OF LEN CM IS AT A DISTANCE OF 12 CM FR CENTRE OF A CIR FIND THE RA OF THE CIRCLE.

3 A CHORD OF A CIRCLE OF CM IS 8 CM LONG. FIND THE DISTANCE OF TI FROM THE CENTR

4 $\overline{AB}$ AND $\overline{CD}$ ARE EQUAL CHORDS IN A CIRCL CM. IF EACH CHOR CM, FIND THEIR DISTANCE FROM THE CENT

5 DEFINE WHAT YOU MI '*a line tangent to a circle*'.

6 HOW MANY TANGENTS ARE THERE FROM AN EXTERNAL POINT TO COMPARE THEIR G.

SOME OTHER PROPERTIES OF CIRCLE CAN ALSO BE PROVED BY USING THE FACT THAT A C SYMMETRICAL ABOUT ANY DIAMETER.

Theorem 5.11

i If two chords of a circle are equal, then they are equidistant from the centre.

ii If two chords of a circle are equidistant from the centre, then their lengths are equal.

Theorem 5.12

If two tangent segments are drawn to a circle from an external point, then,

i the tangents are equal in length, and

ii the line segment joining the centre to the external point bisects the angle between the tangents.

Restatement: IF TP IS A TANGENT TO A CIRCLE AT P WHOSE CENTRE IS O AND TQ IS ANOTHER TANGENT TO THIS CIRCLE AT Q, THEN,

I TP = TQ   II M(∠OTP) = M(∠OTQ)

Proof:-

I ΔOTP AND ΔOTQ ARE RIGHT ANGLED TRIANGLES WITH RIGHT ANGLES AT P AND

(A radius is perpendicular to a tangent at the point of tangency).

II OBVIOUSLY OT = OT

AND OP = OQ (*why?*)

III ∴ ΔOTP ≅ ΔOTQ (*why?*)

SO, TP = TQ AND M(∠OTP) = M(∠OTQ), AS REQUIRED.

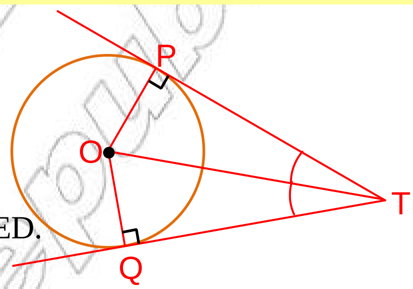

Figure 5.88

5.4.2 Angle Properties of Circles

WE START THIS SUBSECTION BY A REVIEW AND DISCUSSION OF SOME IMPORTANT TER
TO THE DIAGRAMS IN FIGURE 5.89 WILL HELP YOU TO UNDERSTAND SOME OF THESE TERMINOL
(IN EACH CIRCLE, O IS THE CENTRE.)

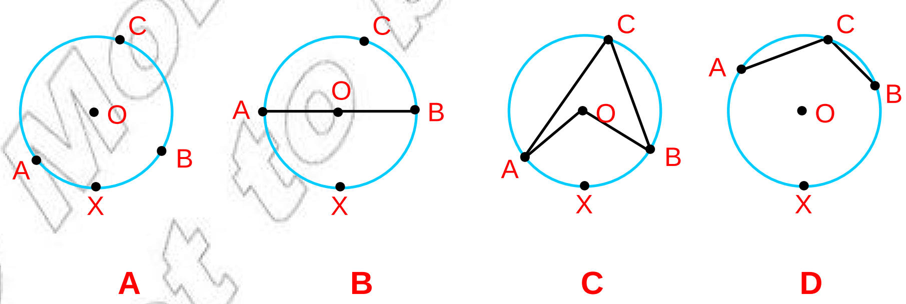

Figure 5.89

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- ✓ A PART OF A CIRCLE (PART OF ITS CIRCUMFERENCE) BETWEEN ANY TWO POINTS ON THE CIRCLE, SAY BETWEEN A AND B, IS CALLED AN arc AND IS DENOTED BY $\widehat{AB}$. HOWEVER, THIS NOTATION CAN BE AMBIGUOUS SINCE THERE ARE TWO ARCS OF THE CIRCLE WIT END POINTS. THEREFORE, WE EITHER USE THE TERMS MINOR ARC AND MAJOR ARC ANOTHER POINT, SAY X, ON THE DESIRED ARC AND THEN USE THE NOTATION $\widehat{AXB}$. FOR EXAMPLE, IN FIGURE 5.89A, $\widehat{AXB}$ IS THE PART OF THE CIRCLE WITH A AND B AS ITS EN POINTS AND CONTAINING THE POINT X. THE REMAINING PART OF THE CIRCLE, I.E WHOSE END POINTS ARE A AND B BUT CONTAINING C IS THE ARC $\widehat{ACB}$.

- ✓ IF AB IS A DIAMETER OF A CIRCLE (SEE FIGURE 5.89B), THEN THE ARC $\widehat{ACB}$ $\left(\text{OR } \widehat{AXB}\right)$ IS CALLED A semicircle. NOTICE THAT A SEMICIRCLE IS HALF OF THE CIRCUMFERENCE OF THE CIRCLE. AN ARC IS SAID TO BE A minor arc, IF IT IS LESS THAN A SEMICIRCLE AND A major arc, IF IT IS GREATER THAN A SEMICIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, $\widehat{AXB}$ IS MINOR ARC WHILE $\widehat{ACB}$ IS A MAJOR ARC.

A central angle OF A CIRCLE IS AN ANGLE WHOSE VERTEX IS AT THE CENTRE OF THE CIRCLE AND WHOSE SIDES ARE RADII OF THE CIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, THE ANGLE ∠AOB IS A CENTRAL ANGLE. IN THIS CASE, WE SAY THAT ∠AOB IS subtended BY THE ARC $\widehat{AXB}$ (OR BY THE CHORD AB). HERE, WE MAY ALSO SAY THAT THE ANGLE ∠AOB intercepts THE ARC $\widehat{AXB}$.

RECALL THAT THE MEASURE OF A CENTRAL ANGLE EQUALS THE ANGLE MEASURE OF TH THUS, IN FIGURE 5.89C,

$$m(\angle AOB) = m\left(\widehat{AXB}\right).$$

AN inscribed angle IN A CIRCLE IS AN ANGLE WHOSE VERTEX IS ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE. FOR EXAMPLE, IN FIGURE 5.89C, ∠ACB IS AN INSCRIBED ANGLE. HERE ALSO, THE INSCRIBED ANGLE ∠ACB IS SAID TO BE subtended BY THE ARC $\widehat{AXB}$ (OR BY THE CHORD $\overline{AB}$).

- ✓ OBSERVE THAT THE VERTEX OF AN INSCRIBED ANGLE ∠ACB IS ON THE ARC $\widehat{ACB}$. THIS ARC, $\widehat{ACB}$, CAN BE A SEMICIRCLE, A MAJOR ARC OR A MINOR ARC. IN SUCH CASES, WE THAT THE ANGLE ∠ACB IS INSCRIBED IN A SEMICIRCLE, MAJOR ARC OR MINOR A RESPECTIVELY. FOR EXAMPLE, IN FIGURE 5.89B, ∠ACB IS INSCRIBED IN A SEMICIRCLE, IN FIGURE 5.89C, ∠ACB IS INSCRIBED IN THE MAJOR ARC AND IN FIGURE 5.89D, ∠ACB IS INSCRIBED IN THE MINOR ARC.

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Theorem 5.13

The measure of a central angle subtended by an arc is twice the measure of an inscribed angle in the circle subtended by the same arc.

BOTH DRAWINGS IN FIGURE 5.90 ILLUSTRATE THIS THEOREM.

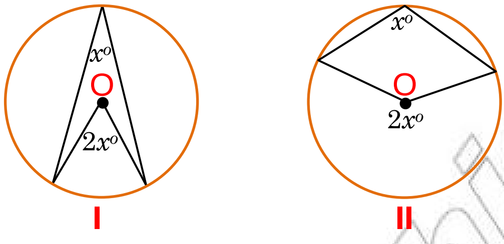


Figure 5.90

Exercise 5.10

IN EACH OF THE FOLLOWING FIGURES, O IS THE CENTRE OF THE CIRCLE. CALCULATE THE ANGLES MARKED x.

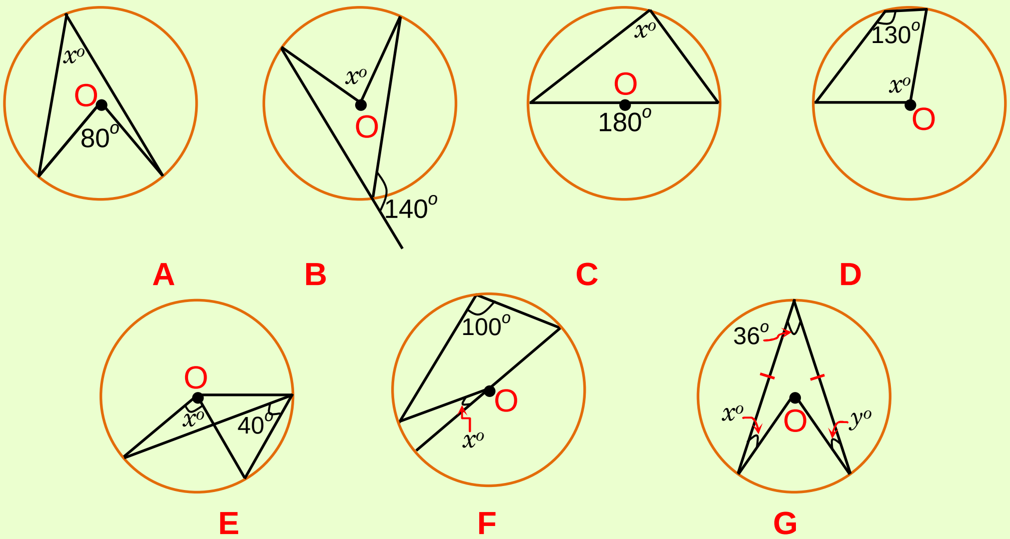


Figure 5.91

Corollary 5.13.1

Angles inscribed in the same arc of a circle (*i.e., subtended by the same arc*) are equal.

Proof:-

BY THE ABOVE THEOREM, EACH OF THE ANGLES INSCRIBED IN THE CIRCLE SUBTENDED BY THE ARC IS EQUAL TO HALF OF THE CENTRAL ANGLE SUBTENDED BY THE ARC. HENCE, THEY ARE EQUAL.

Corollary 5.13.2 Angle in a semicircle

The angle inscribed in a semi-circle is a right angle.

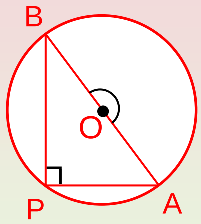


Figure 5.92

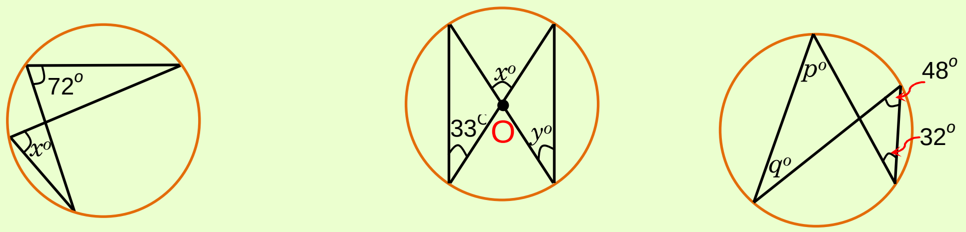
Proof:-

The given angle, ∠APB, is subtended by a semicircle. The corresponding central angle subtended by $\overline{AB}$ is straight angle. i.e., the central angle is 180^o. Hence, by Theorem 5.14 $m(\angle APB) = \frac{1}{2}m(\angle AOB) = \frac{1}{2}\times 180^O = 90^O$.

This completes the proof.

Exercise 5.11

1 Calculate the marked angles in each of the following figures.



A B C

Figure 5.93

2 In each of the following figures, O is the centre and $\overline{AB}$ is the diameter of the circle. Calculate the value of x in each case.

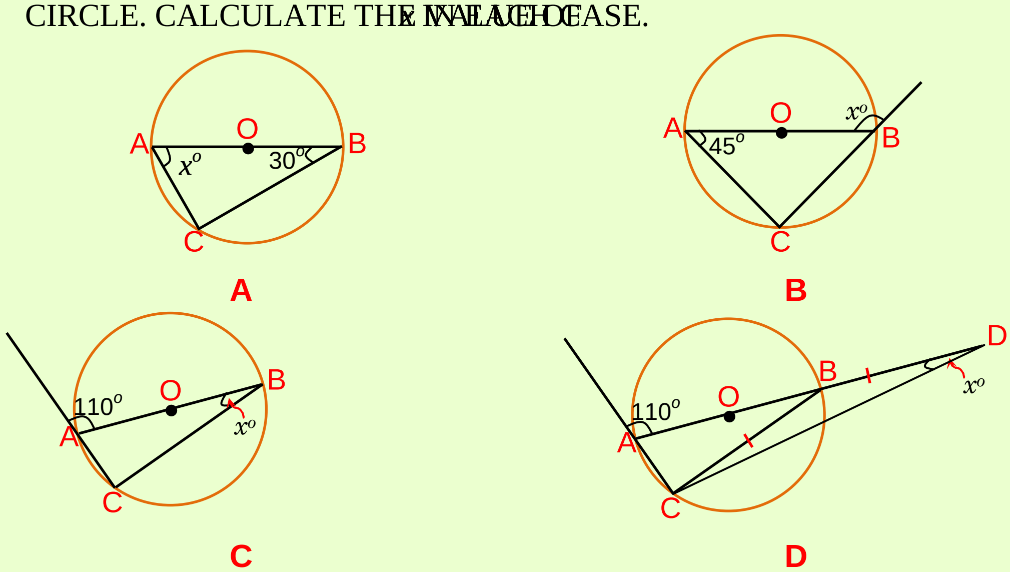


Figure 5.94

Corollary 5.13.3

Points P, Q, R and S all lie on a circle. They are called concyclic points.

Joining the points P, Q, R and S produces a cyclic quadrilateral.

The opposite angles of a cyclic quadrilateral are supplementary. i.e.,

$$m(\angle P) + m(\angle R) = 180^\circ \text{ and } m(\angle S) + m(\angle Q) = 180^\circ.$$

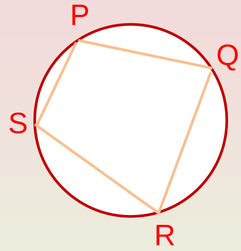


Figure 5.95

ACTIVITY 5.15

CALCULATE THE LAENTGLES IN EACH OF THE]:

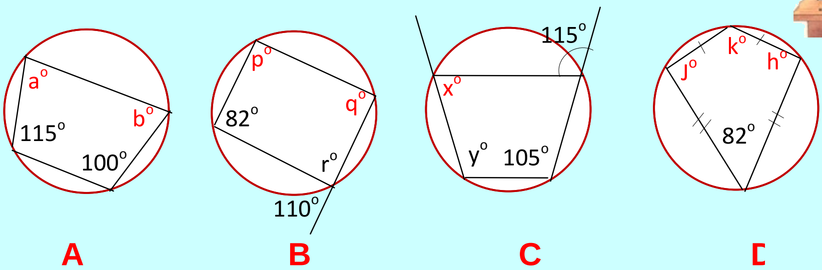


Figure 5.96

5.4.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

Remember that:

- ✓ CIRCUMFERENCE OF A C r OR d.
- ✓ AREA OF A CIRCLE^2.
- ✓ PART OF THE CIRCUMFERENCE OF A CIRarc.
- ✓ A segment OF A CIRCLE IS A REGION BOUNDED BY A C.
- ✓ A sector OF A CIRCLE IS BOUNDED BY TWO RA.

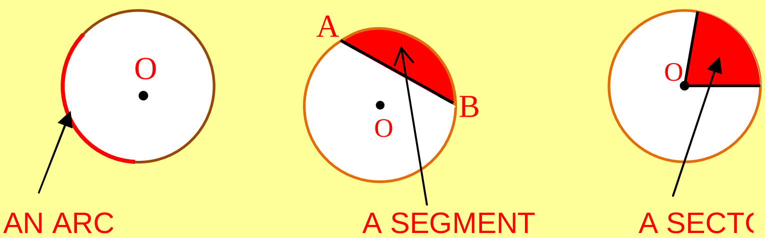


Figure 5.97

Group Work 5.5

- WHAT FRACTION OF A COMPLETE CIRCLE IS THE SHADED REGION IN FIGURE 5.98?
- WHAT IS THE AREA OF THE SHADED REGION IN THIS FIGURE?

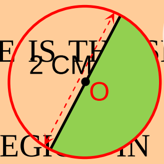


Figure 5.98

- WHAT IS THE AREA IF THE SHADED REGION IS A QUADRANT (FIGURE 5.99)?

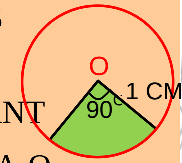


Figure 5.99

- WHAT IS THE AREA IF THE SHADED REGION IS PART OF A QUADRANT BOUNDED BY A CHORD AND AN ARC AS SHOWN IN FIGURE 5.100?

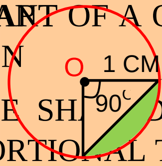


Figure 5.100

DISCUSS HOW TO FIND THE AREA OF EACH OF THE SHADED SECTORS SHOWN BELOW. IS THE AREA OF EACH SECTOR PROPORTIONAL TO THE ANGLE BETWEEN THE RADII BOUNDING THE SECTOR?

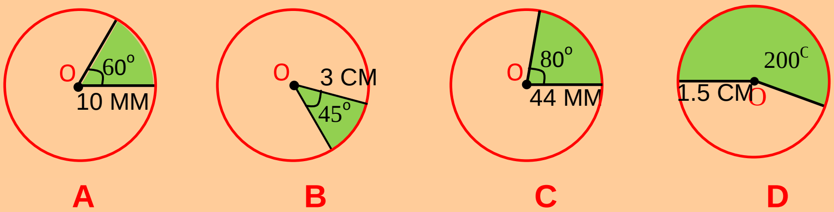


Figure 5.101

DISCUSS HOW TO FIND THE AREA OF EACH OF THE SHADED SEGMENTS SHOWN BELOW:

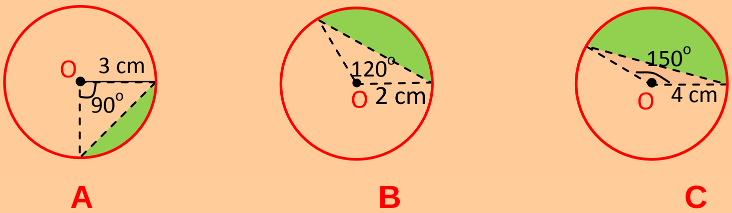


Figure 5.102

✓ Arc length

THE LENGTH ℓ OF AN ARC OF A CIRCLE OF RADIUS r THAT SUBTENDS AN ANGLE θ AT THE CENTRE IS GIVEN BY

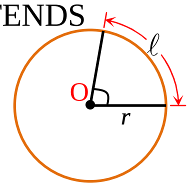


Figure 5.103

$$\ell = \frac{}{360^o} \times 2\ r = \frac{r}{180^o}$$

✓ The area and perimeter of a sector

THE AREA A OF A SECTOR OF RADIUS r AND CENTRAL ANGLE θ IS GIVEN BY

$$A = \frac{}{360^o} \times \ r^2 = \frac{r^2}{360^o}$$

THE PERIMETER *P* OF THE SECTOR IS THE SUM OF THE RADII AND THE ARC THAT BOUND IT.

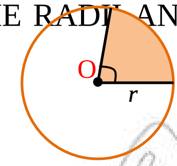


Figure 5.104

$$P = 2r + \frac{r}{180^o}$$

✓ The area and perimeter of a segment

THE AREA *A* AND PERIMETER *P* OF A SEGMENT OF A CIRCLE OF RADIUS *r* CUT OFF BY A CHORD SUBTENDING AN ANGLE AT THE CENTRE OF A CIRCLE ARE GIVEN BY

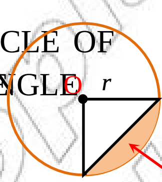


Figure 5.105

$$A = \frac{r^2}{360^o} - \frac{1}{2}r^2 \text{ SIN} \quad (\textit{sector area} - \textit{triangle area})$$

Note: THE AREA FORMULA FOR A TRIANGLE:

$A = \frac{1}{2}ab\text{ SIN}$ WHERE *a* AND *b* ARE THE LENGTHS OF ANY TWO SIDES OF THE TRIAN

AND θ IS THE MEASURE OF THE ANGLE INCLUDED BETWEEN THE GIVEN SIDES IS DIS THE NEXT SECTION (SECTION 5.5 OF THIS TEXTBOOK).

$$P = 2r\text{ SIN}\frac{}{2} + \frac{r}{180^o} \qquad (\textit{chord length} + \textit{arc length})$$

EXAMPLE 1 A SEGMENT OF A CIRCLE OF RADIUS 12 CM IS CUT OFF BY A CHORD SUBTENDING AN ANGLE 60^o AT THE CENTRE OF THE CIRCLE. FIND:

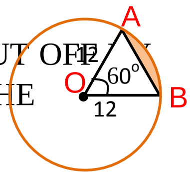


Figure 5.106

A THE AREA OF THE SEGMENT. **B** THE PERIMETER OF THE SECTOR.

SOLUTION:

A FROM THE FIGURE, AREA OF THE SEGMENT (THE SHADED PART)

= AREA OF THE SECTOR OAB – AREA OF TRIANGLE OAB;

AREA OF THE SECTOR OAB = $\frac{\times 12^2 \times 60}{360}$ = 24 CM^2.

AREA OF TRIANGLE OAB = $\frac{1}{2} \times 12^2 \times \text{SIN } 60^o = 36\sqrt{3}$ CM^2.

THEREFORE, SEGMENT AREA = $(24 - 36\sqrt{3})$ CM^2.

B PERIMETER OF THE SECTOR = 2 × RADIUS + LENGTH OF ARC AB

$$= 2 \times 12 + \frac{\times 12 \times 60}{180} = (24 + 4\ )\text{ CM}$$

Exercise 5.12

1 CALCULATE THE PERIMETER AND AREA OF EACH OF THE FOLLOWING FIGURES. ALL CURVES ARE SEMICIRCLES OR QUADRANTS.

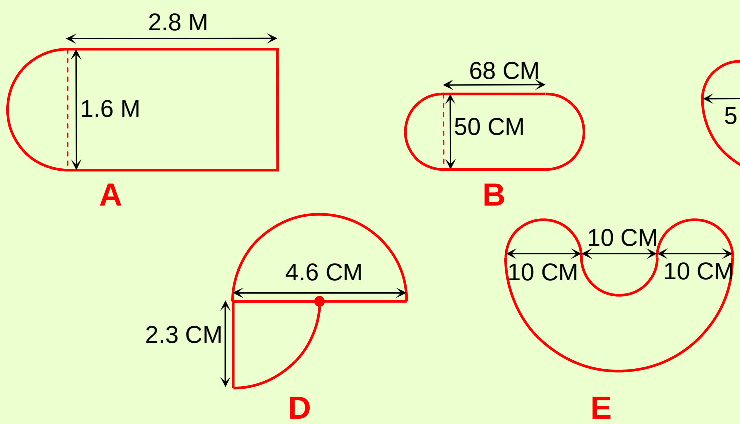


Figure 5.107

2 IN EACH OF THE FOLLOWING SECTORS OPQ FIND:

I THE LENGTH OF ARC PQ.

II AREA OF THE SECTOR OPQ.

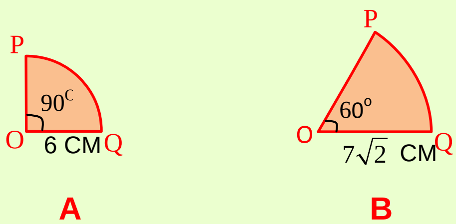


Figure 5.108

3 IN FIGURE 5.109, O IS THE CENTRE OF THE CIRCLE. IF THE RADIUS OF THE CIRCLE IS 4 CM AND $m(\angle AKB) = 30^o$, FIND THE AREA OF THE SEGMENT BOUNDED BY THE CHORD AB AND THE ARC AKB.

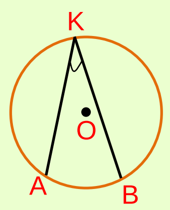
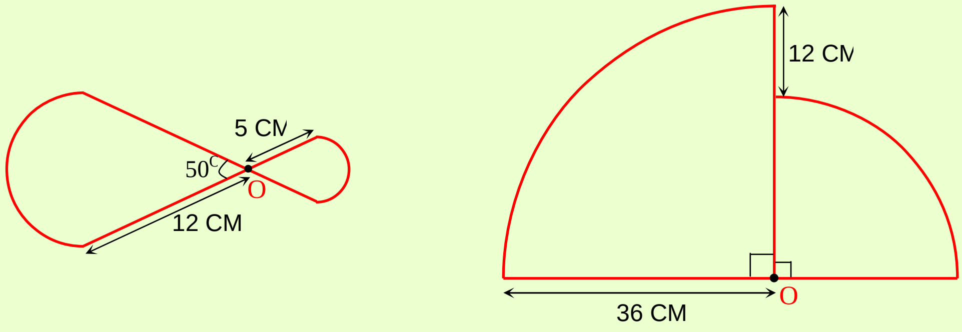


Figure 5.109

4 A SQUARE ABCD IS INSCRIBED IN A CIRCLE OF RADIUS 4 CM. FIND THE AREA OF THE MINOR SEGMENT CUT OFF BY THE CHORD $\overline{AB}$.

5 CALCULATE THE PERIMETER AND AREA OF EACH OF THE FIGURES, WHERE THE CURVES ARE ARCS OF A CIRCLE WITH COMMON CENTRE AT O.



A **B**

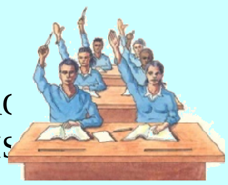
Figure 5.110

5.5 MEASUREMENT

5.5.1 Areas of Triangles and Parallelograms

A Areas of triangles

ACTIVITY 5.16



GIVEN THE RIGHT TRIANGLE SHOWN BELOW, VERIFY THAT EACH OF THE FOLLOWING EXPRESSIONS GIVES THE AREA OF $\triangle ABC$. IN EACH CASE, DISCUSS AND STATE THE FORMULA.

I AREA OF $\triangle ABC = \frac{1}{2}ac$

II AREA OF $\triangle ABC = \frac{1}{2}bh$

III AREA OF $\triangle ABC = \frac{1}{2}bc\,\text{SIN}\,(\measuredangle A)$

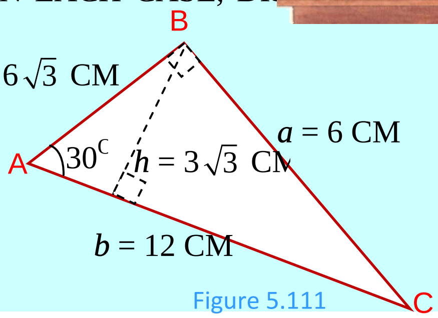


Figure 5.111

THE ABOVE ACTIVITY SHOULD HAVE REMINDED YOU WHAT YOU STUDIED IN LOWER GRADES EXCEPT FOR CASE III, WHICH YOU HAVE USED IN THE PRECEDING SECTION AND ARE GOING TO ABOUT NOW.

CASE I USES THE FOLLOWING:

THE AREA OF A RIGHT-ANGLED TRIANGLE WITH PERPENDICULAR SIDES OF LENGTH a AND b IS GIVEN BY

$$A = \frac{1}{2}ab$$

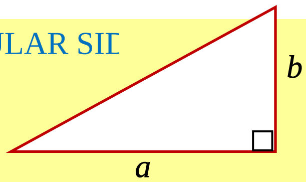


Figure 5.112

CASE II USES THE FOLLOWING FORMULA.

THE AREA A OF ANY TRIANGLE WITH BASE b AND THE CORRESPONDING HEIGHT h IS GIVEN BY

$$A = \frac{1}{2}bh$$

THE BASE AND CORRESPONDING HEIGHT OF A TRIANGLE MAY APPEAR IN ANY ONE OF THE FOLLOWING FORMS.

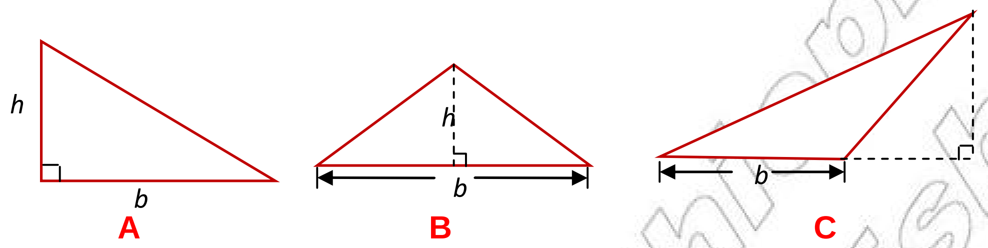


Figure 5.113

FROM THE VERIFICATION OF CASE II, WE COME TO THE FOLLOWING FORMULA.

THE AREA A OF ANY TRIANGLE WITH SIDES a AND b UNITS LONG AND ANGLE C ($\angle C$) INCLUDED BETWEEN THESE SIDES IS

$$A = \frac{1}{2}ab \text{ SIN } (\angle C)$$

Proof:-

LET Δ ABC BE GIVEN SUCH THAT BC = a AND AC = b.

Case i LET $\angle$C BE AN ACUTE ANGLE.

CONSIDER THE HEIGHT h DRAWN FROM B TO AC. IT MEETS AC AT D (SEE FIGURE 5.114).

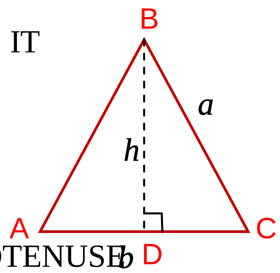


Figure 5.114

NOW, AREA OF ΔABC = $\frac{1}{2}bh$ ... (1)

SINCE Δ BCD IS RIGHT-ANGLED WITH HYPOTENUSE a,

$$\text{SIN } (\angle C) = \frac{h}{a}$$

$\therefore \; h = a \text{ SIN } (\angle C)$

REPLACING h BY a SIN ($\angle$C) IN 1 WE OBTAIN

AREA OF ΔABC = $\frac{1}{2}ab \text{ SIN}(\angle C)$ AS REQUIRED.

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Case ii LET ∠C BE AN OBTUSE ANGLE.

DRAW THE HEIGHT FROM B TO THE EXTENDED BASE AC. IT MEETS THE EXTENDED AT D. NOW,

AREA OF ∆ABC = AREA OF ∆ABD – AREA OF ∆BDC

$$= \frac{1}{2}\text{AD}.h - \frac{1}{2}\text{CD}.h = \frac{1}{2}h(\text{AD} - \text{CD})$$

$$= \frac{1}{2}h.\text{AC} = \frac{1}{2}hb \qquad (2)$$

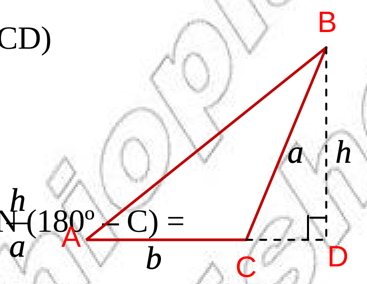

Figure 5.115

IN THE RIGHT-ANGLED TRIANGLE BCD, SIN (180° – C) = $\frac{h}{a}$

$\therefore \quad h = a$ SIN (180° – C)

SINCE SIN (180° – C) = SIN C, WE HAVE $h = a$ SIN (∠C)

For any two angles A and B if m (∠A) + m (∠B) = 180°, then, sin A = sin B.

∴ REPLACING h BY a SIN (∠C) IN 2 WE OBTAIN;

AREA OF ∆ABC = $\frac{1}{2}ab$ SIN (∠C) AS REQUIRED.

Case iii LET ∠C BE A RIGHT ANGLE.

$$A = \frac{1}{2}ab = \frac{1}{2}ab(\text{SIN } 90^{o}) \qquad (\text{SIN } 90^{o} = 1)$$

$$= \frac{1}{2}ab\,\text{SIN}(\angle C) \qquad (\textit{as required})$$

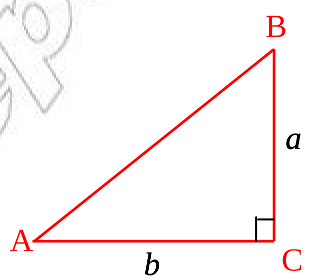

Figure 5.116

THIS COMPLETES THE PROOF.

Group Work 5.6

1 USING THIS FORMULA, SHOW THAT THE AREA A OF A REGULAR n-SIDED POLYGON WITH RADIUS r IS GIVEN BY

$$A = \frac{1}{2}nr^2 \text{ SIN} \frac{360^{o}}{n}$$

2 SHOW THAT THE AREA A OF AN EQUILATERAL TRIANGLE INSCRIBED IN A CIRCLE OF RADIUS r IS

$$A = \frac{3\sqrt{3}}{4}r^2$$

NOW WE STATE ANOTHER FORMULA CALLED **Heron's formula**, WHICH IS OFTEN USED TO FIND THE AREA OF A TRIANGLE WHEN ITS THREE SIDES ARE GIVEN.

Theorem 5.14 Heron's formula

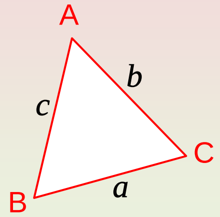

Figure 5.117

THE AREA A OF A TRIANGLE WITH SIDES a, b AND c UNITS LONG AND SEMI- PERIMETER $s = \frac{1}{2}(a+b+c)$ IS GIVEN BY

$$A = \sqrt{s(s-a)(s-b)(s-c)}\ .$$

EXAMPLE 1 GIVEN $\triangle$ABC. IF AB = 15 UNITS, BC = 14 UNITS AND AC = 13 UNITS, FIND

A THE AREA OF $\triangle$ABC.

B THE LENGTH OF THE ALTITUDE FROM THE VERTEX A.

C THE MEASURE OF $\angle$B.

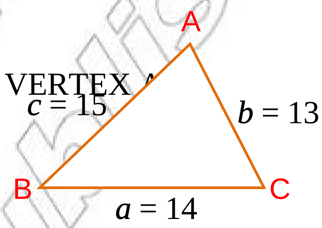

Figure 5.118

SOLUTION:

A $a = 14$ $\quad s - a = 7$

$b = 13$ $\quad s - b = 8$

$c = 15$ $\quad s - c = 6$

$a + b + c = 42$ $\quad (s - a) + (s - b) + (s - c) = 21$

$\therefore\ s = \dfrac{a+b+c}{2} = \dfrac{42}{2} = 21$

$\therefore$ AREA OF $\triangle$ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(7)(8)(6)} = 84$ UNIT^2

Why is the sum of s – a, s – b, s – c equal to s? This provides a useful check.

B LET THE ALTITUDE FROM THE VERTEX A TO THE CORRESPONDING BASE BC BE h, DRAWN MEETING BC AT D AS SHOWN.

THEN, AREA OF $\triangle$ABC $= \dfrac{1}{2}$ BC $\times h$

$\therefore\ 84 = \frac{1}{2}\times 14 \times h = 7h$

$\therefore\quad h = \dfrac{84}{7} = 12$ UNITS.

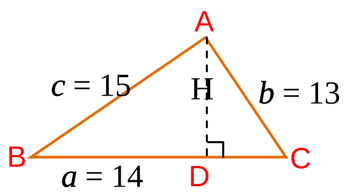

Figure 5.119

THEREFORE, THE ALTITUDE FROM THE VERTEX A IS 12 UNITS LONG.

C IN THE RIGHT ANGLED TRIANGLE ABD SHOWN ABOVE IN FIGURE 5.91, WE SEE THAT

$$\text{SIN}(\angle B) = \frac{AD}{AB} = \frac{h}{c} = \frac{12}{15} = 0.8$$

THEN, FROM TRIGONOMETRIC TABLES, WE FIND THAT THE CORRESPONDING ANGLE IS 53°.
I.E., M($\angle$B) = 53°.

B Area of parallelograms

ACTIVITY 5.17

1 WHAT IS A PARALLEL

2 SHOW THAT A DIAGONAL OF A PARALLELOGRAM DIVIDES THE PARALLELOGRAM INTO TWO CONGRUENT

Theorem 5.15

The area A of a parallelogram with base b and perpendicular height h is

$$A = bh$$

Proof:-

LET ABCD BE A PARALLELOGRAM WITH BASE BC = b. DRAW DIAGONAL AC. WE KNOW THAT AC DIVIDES THE PARALLELOGRAM INTO TWO CONGRUENT TRIANGLES. MOREOVER, NOTE THAT ANY TWO CONGRUENT TRIANGLES HAVE EQUAL AREAS. NOW, THE AREA OF ΔABC = $\frac{1}{2}$ bh.

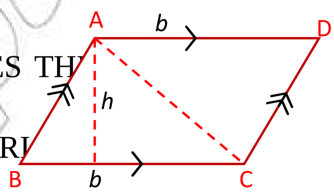

Figure 5.120

THEREFORE, THE AREA OF PARALLELOGRAM = $2\left(\frac{1}{2}bh\right) = bh.$

EXAMPLE 2 IF ONE PAIR OF OPPOSITE SIDES OF A PARALLELOGRAM HAVE LENGTH 40 CM AND THE DISTANCE BETWEEN THEM IS 15 CM, FIND THE AREA OF THE PARALLELOGRAM.

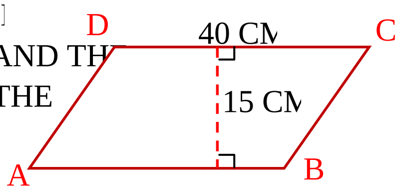

Figure 5.121

SOLUTION: AREA = 40 CM × 15 CM = 600 CM^2

Exercise 5.13

1 IN ΔABC, $\overline{BE}$ AND $\overline{CF}$ ARE ALTITUDES OF THE TRIANGLE. IF AB = 6 UNITS, AC = 5 UNITS AND CF = 4 UNITS FIND THE LENGTH OF BE.

2 IN ΔDEF, IF DE = 20 UNITS, EF = 21 UNITS AND DF = 13 UNITS, FIND:

A THE AREA OF ΔDEF

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b THE LENGTH OF ALTITUDE FROM THE VERTEX D

c SIN ($\angle$D)

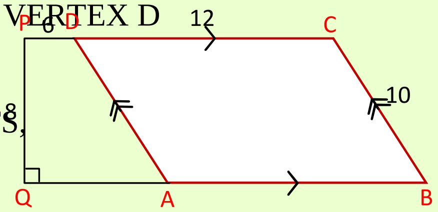

Figure 5.122

3 IN THE GIVEN FIG PD = 6 UNITS DC = 12 UNITS, PQ = 8 UNITS AND BC = 10 UNITS:

a THE AREA OF THE PARALLELO

b THE HEIGHT OF THE PARALLELOGRAM THAT COR AD.

4 PQRS IS A PARALLELOGRAM (CM^2. IF PQ = 5 CM AND QR = 4 CM, CALCULATE THE LENGTHS OF THE CORRESPON

5 IN ΔMNO IF MN = 5 CM, NO = 6 CM AND MO = 7 CM, FIND:

a THE AREA OF MNO. **b** THE LENGTH OF THE SHORT

(*leave your answers in radical form.*)

6 IN THE PARALLELOGRAM (SHOWN IN FIGURE 5.123 BELOW), AB = 2 CM, AD = 3 CM AND M($\angle$B) = 60^o. FIND THE LENGTH OF THE ALTITU $\overline{DC}$.

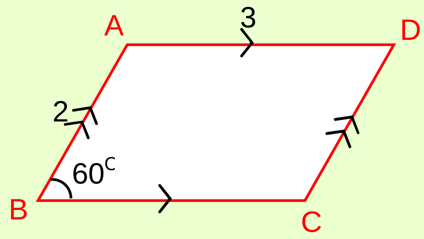

Figure 5.123

7 THE LENGTHS OF THE THREE SIDES OF A x, 4x AND 3x INCHES AND PERIMETER OF THE TRIANGLE INCHES FIND:

a THE LENGTHS OF THE SIDES OF

b THE AREA OF THE

8 FIND THE AREA OF A RHOMBUS WHOSE DIAGONALS ARE 5 INCHE

5.5.2 Further on Surface Areas and Volumes of Cylinders and Prisms

ACTIVITY 5.18

1 WHAT IS A SOLID F

2 WHICH OF THE FOLLOWING SOLIDS ARE PRISMS AND W WHICH OF THEM NEITHER PRISMS NOR C

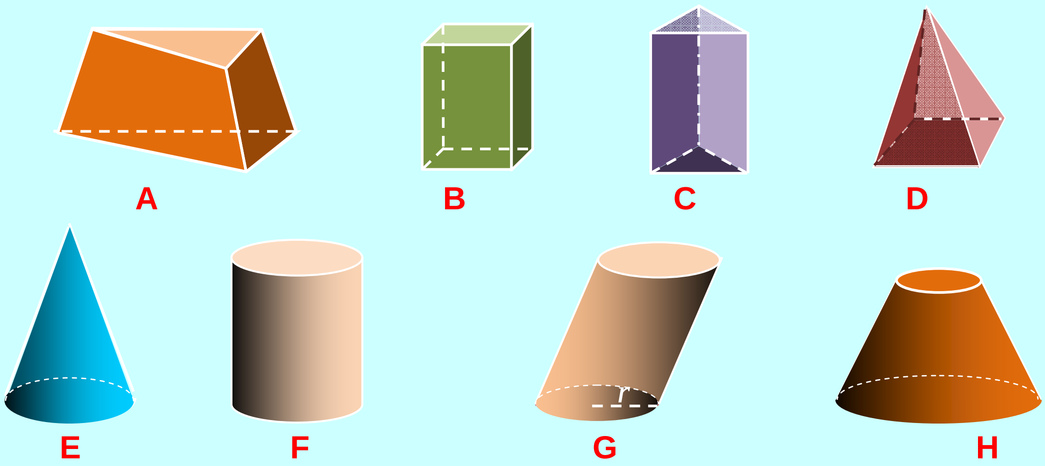

Figure 5.124

3 THE RADIUS OF THE BASE OF A RIGHT CIRCULAR CYLINDER IS 2 CM AND ITS ALTITUDE IS 3 CM. FIND ITS:

A CURVED SURFACE AREA **B** TOTAL SURFACE AREA **C** VOLUME

4 FIND A FORMULA FOR THE SURFACE AREA OF A RIGHT PRISM BY CONSTRUCTING A MODEL FROM SIMPLE MATERIALS.

5 ROLL A RECTANGULAR PIECE IN TO A CYLINDER. DISCUSS THE SURFACE AREA OF A RIGHT CIRCULAR CYLINDER.

A Prism

- A prism IS A SOLID FIGURE FORMED BY TWO CONGRUENT POLYGONAL REGIONS IN PARALLEL PLANES, ALONG WITH THREE OR MORE PARALLELOGRAMS, JOINING THE TWO PO POLYGONS IN PARALLEL PLANES ARE CALLED bases.
- A PRISM IS NAMED BY ITS BASE. THUS, A PRISM IS TRIANGULAR, RECTANGULAR, PENTAGONAL, ETC., IF ITS BASE IS A TRIANGLE, A RECTANGLE, A PENTAGON, ETC.,
- IN A PRISM,
  - ✓ THE LATERAL EDGES ARE EQUAL AND PARALLEL.
  - ✓ THE LATERAL FACES ARE PARALLELOGRAMS.
- A right prism IS A PRISM IN WHICH THE BASE IS PERPENDICULAR TO THE LATERAL EDGE. OTHERWISE IT IS AN oblique prism.
- IN A RIGHT PRISM
  - ✓ ALL THE LATERAL EDGES ARE PERPENDICULAR TO BOTH BASES.
  - ✓ THE LATERAL FACES ARE RECTANGLES.
  - ✓ THE ALTITUDE IS EQUAL TO THE LENGTH OF EACH LATERAL EDGE.
- A REGULAR PRISM IS A RIGHT PRISM WHOSE BASE IS A REGULAR POLYGON.

Surface area and volume of prisms

- THE LATERAL SURFACE AREA OF A PRISM IS THE SUM OF THE AREAS OF ITS LATERAL FACES.
- THE TOTAL SURFACE AREA OF A PRISM IS THE SUM OF ITS LATERAL AREA AND THE AREA OF THE BASES.
- THE VOLUME OF ANY PRISM IS EQUAL TO THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE.

✓ IF WE DENOTE THE LATERAL SURFACE AREA OF A PRISM BY A_L, THE TOTAL SURFACE AREA BY A_T, THE AREA OF THE BASE BY A_B AND ITS VOLUME BY V, THEN

I $A_L = Ph$

WHERE P IS THE PERIMETER OF THE BASE AND h IS THE ALTITUDE OR HEIGHT OF THE PRISM.

II $A_T = 2A_B + A_L$ **III** $V = A_B\,h.$

EXAMPLE 1 THE ALTITUDE OF A RECTANGULAR PRISM IS 4 UNITS AND THE WIDTH AND LENGTH OF ITS BASE ARE 3 AND 2 UNITS RESPECTIVELY. FIND:

A THE TOTAL SURFACE AREA OF THE PRISM. **B** THE VOLUME OF THE PRISM.

SOLUTION:

A TO FIND A_T, FIRST WE HAVE TO FIND THE BASE AREA AND THE LATERAL AREA

$$A_B = 2 \times 3 = 6 \text{ UNIT}^2$$

AND $A_L = Ph = (3 + 2 + 3 + 2) \times 4 = 40 \text{ UNIT}^2$

$$\therefore A_T = 2A_B + A_L = 2 \times 6 + 40 = 52 \text{ UNIT}^2$$

SO, THE TOTAL SURFACE AREA IS 52 UNIT²

B $V = A_B\,h = 6 \times 4 = 24 \text{ UNIT}^3$

EXAMPLE 2 THROUGH THE CENTRE OF A REGULAR HEXAGONAL PRISM WHOSE BASE EDGE IS 6 CM AND HEIGHT 8 CM, A HOLE WHOSE FORM IS A REGULAR TRIANGULAR PRISM WITH BASE EDGE 3 CM IS DRILLED AS SHOWN IN FIGURE 5.125. FIND:

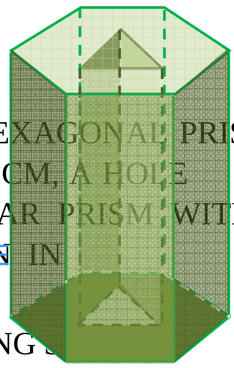

Figure 5.125

A THE TOTAL SURFACE AREA OF THE REMAINING SOLID.

B THE VOLUME OF THE REMAINING SOLID.

SOLUTION: RECALL THAT THE AREA OF A REGULAR n-SIDED POLYGON WITH RADIUS r IS

$$A = \frac{1}{2}nr^2 \text{ SIN} \frac{360^o}{n}.$$

ALSO, THE RADIUS AND THE LENGTH OF A SIDE OF A REGULAR HEXAGON ARE EQUAL.

SO, AREA OF THE GIVEN REGULAR HEXAGON $= \frac{1}{2} \times 6 \times 6^2 \times \text{SIN } 60^o = 54\sqrt{3} \text{ CM}^2.$

AREA OF THE EQUILATERAL TRIANGLE $= \frac{1}{2}ab \text{ SIN } C = \frac{1}{2} \times 3 \times 3 \times \text{SIN } 60^o = \frac{9\sqrt{3}}{4} \text{ CM}^2$

A **I** AREA OF THE BASES OF THE REMAINING SOLID = 2 × (AREA OF HEXAGON – AREA OF Δ)

$$= 2\times\left(54\sqrt{3}-\frac{9\sqrt{3}}{4}\right) = 108\sqrt{3}-\frac{9\sqrt{3}}{2} = \frac{207}{2}\sqrt{3}\ \text{CM}^2$$

II LATERAL SURFACE AREA OF THE REMAINING SOLID = LATERAL AREA OF HEXAGONAL PRISM + LATERAL AREA OF TRIANGULAR PRISM (INNER)

$= \text{PERIMETER OF HEXAGON} \times 8 + \text{PERIMETER OF TRIANGLE} \times 8$

$= 36 \times 8 + 9 \times 8 = 360\ \text{CM}^2.$

$\therefore$ TOTAL SURFACE AREA OF THE REMAINING SOLID = $\left(\frac{207}{2}\sqrt{3}+360\right)\text{CM}^2.$

B VOLUME OF THE REMAINING SOLID

= VOLUME OF HEXAGONAL PRISM – VOLUME OF TRIANGULAR PRISM

$$= \left(54\sqrt{3}\times 8-\frac{9\sqrt{3}}{4}\times 8\right)\text{CM}^3 = 414\sqrt{3}\ \text{CM}^3$$

B Cylinder

RECALL FROM YOUR LOWER GRADES THAT:

- A circular cylinder IS A SIMPLE CLOSED SURFACE BOUNDED ON TWO ENDS BY CIRCULAR BASES. (SEE FIGURE 5.126). A MORE GENERAL DEFINITION OF A CYLINDER REPLACES THE CIRCLE WITH ANY SIMPLE CLOSED CURVE. FOR EXAMPLE, THE CYLINDER SHOWN IN FIGURE 5.127 IS NOT A CIRCULAR CYLINDER.

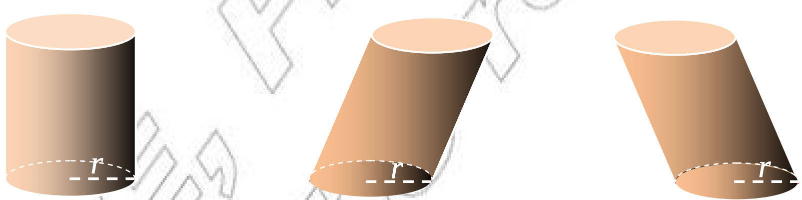

Figure 5.126 (circular cylinders)

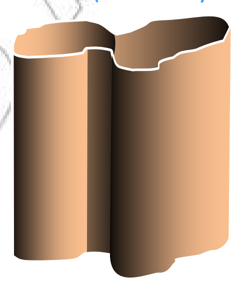

Figure 5.127

IN OUR PRESENT DISCUSSION, WE SHALL CONSIDER ONLY CYLINDERS WHOSE BASES ARE CIRCLES (I.E., CIRCULAR CYLINDERS).

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A CIRCULAR CYLINDER RESEMBLES A PRISM EXCEPT THAT ITS BASES ARE CIRCUL FIGURE 5.126A THE CYLINDER IS CALLED A RIGHT CIRCULAR CYLINDER IF THE LINE SEGMENT JOINING THE CENTRES OF THE BASES IS PERPENDICULAR TO THE CYLINDERS FIGURES 5.126B AND C ABOVE ARE NOT RIGHT CIRCULAR CYLINDERS; THEY A OBLIQUE CYLINDERS.

Surface area and volume of circular cylinders

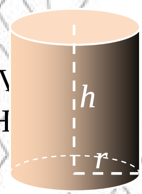

Figure 5.128

1 THE LATERAL SURFACE AREA (I.E., AREA OF THE CURVED SURFACE) OF A RIGHT CIRCULAR CYLINDER DENOTED BY A_L IS THE PRODUCT OF ITS HEIGHT h AND THE CIRCUMFERENCE C OF ITS BASE.

I.E. $A_L = hC$ OR $A_L = 2\ rh$

2 THE TOTAL SURFACE AREA (OR SIMPLY SURFACE AREA) OF A RIGHT CIRCULAR CYLINDER DENOTED BY A_T IS TWO TIMES THE AREA OF THE CIRCULAR BASE PLUS THE AREA OF THE CURVED SURFACE (LATERAL SURFACE AREA). SO, IF THE HEIGHT OF THE CYLINDER IS h AND THE RADIUS OF THE BASE CIRCLE IS r, WE HAVE

$$A_T = 2\ rh + 2\ r^2 = 2\ r\,(h + r)$$

3 THE VOLUME V OF THE RIGHT CIRCULAR CYLINDER IS EQUAL TO THE PRODUCT OF ITS BASE AREA AND HEIGHT.

SO, IF THE HEIGHT OF THE CYLINDER IS h AND ITS BASE RADIUS IS r THEN

$$V = \ r^2 h$$

EXAMPLE 3 IF THE HEIGHT OF A RIGHT CIRCULAR CYLINDER IS 8 CM AND THE RADIUS OF ITS BASE IS 5 CM FIND THE FOLLOWING GIVING YOUR ANSWERS IN TERMS OF

A ITS LATERAL SURFACE AREA **B** ITS TOTAL SURFACE AREA **C** ITS VOLUME

SOLUTION:

A THE LATERAL SURFACE AREA OF THE RIGHT CIRCULAR CYLINDER IS GIVEN BY

$A_L = 2\ rh$

$= 2\ \times 5 \times 8 = 80\ \ CM^2$

B $A_T = 2\ rh + 2\ r^2$

$= 2\ \times 5 \times 8 + 2\ \times 5^2 = 80\ + 50\ = 130\ \ CM^2$

C THE VOLUME OF THE CYLINDER IS

$V = \ r^2 h$

$= \ \times 5^2 \times 8 = 200\ \ CM^3$

EXAMPLE 4 A CIRCULAR HOLE OF RADIUS 2 UNITS IS DRILLED THROUGH THE CENTRE OF A RIGHT CIRCULAR CYLINDER WHOSE BASE HAS RADIUS 3 UNITS AND WHOSE ALTITUDE IS 4 UNITS. FIND THE TOTAL SURFACE AREA OF THE RESULTING FIGURE.

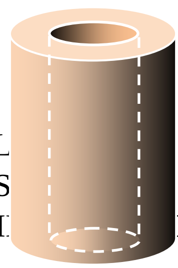
Figure 5.129

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SOLUTION: LET R BE THE RADIUS OF THE BIGGER CYLINDER AND r BE THE RADIUS OF THE SMALLER CYLINDER THEN

i AREA OF THE RESULTING BASE = 2 (R^2 – r^2)

$= 2 (\times 3^2 - \times 2^2)$ UNIT^2 = 10 UNIT^2

ii LATERAL SURFACE AREA OF THE RESULTING FIGURE

= LATERAL SURFACE AREA OF THE BIGGER CYLINDER

+ LATERAL SURFACE AREA OF INNER (SMALLER) CYLINDER

= (2 Rh + 2 rh) UNIT^2 = [2 (3) 4 + 2 (2) 4] UNIT^2

= 40 UNIT^2

THEREFORE, TOTAL SURFACE AREA OF THE RESULTING FIGURE = (10 + 40) UNIT^2 = 50 UNIT^2

Exercise 5.14

1 USING THE MEASUREMENTS INDICATED IN EACH OF THE FIGURES, FIND:

a THE TOTAL SURFACE AREA OF EACH FIGURE. **b** THE VOLUME OF EACH FIGURE.

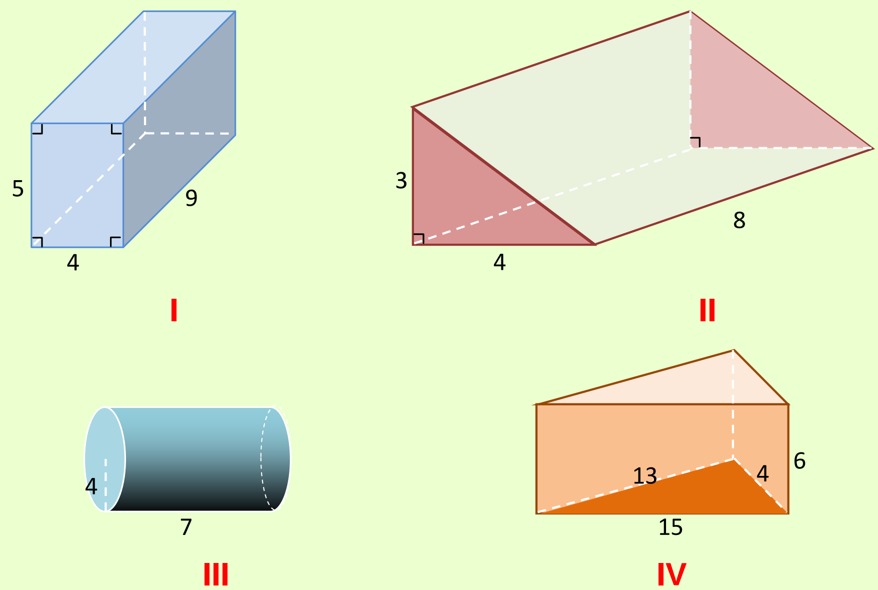


Figure 5.130

2 THE BASE OF A RIGHT PRISM IS AN ISOSCELES TRIANGLE WITH EQUAL SIDES 5 INCHES EACH, AND THIRD SIDE 4 INCHES. THE ALTITUDE OF THE PRISM IS 6 INCHES. FIND:

a THE TOTAL SURFACE AREA OF THE PRISM. **b** THE VOLUME OF THE PRISM.

3 FIND THE LATERAL SURFACE AREA AND TOTAL SURFACE AREA OF A RIGHT CIRCULAR CYLINDER IN WHICH:

a r = 4 FT, h = 12 FT **b** r = 6.5 CM, h = 10 CM

4 THROUGH A REGULAR HEXAGONAL PRISM WH 8 CM AND WHOSE HEIGHT IS CM, A HOLE IN THE SHAPE A RIGHT PRISM, WITH ITS END BEING A RHOM DIAGONALS CM AND CM IS DRILLED (SEE FIGURE 5.131). FIND:

A THE TOTAL SURFACE AREA OF THE R

B THE VOLUME OF THE REMAI

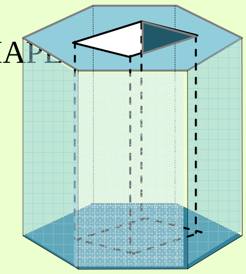


Figure 5.131

5 A MANUFACTURER MAKES A CLOSED RIGHT CYLINDRICAL CONTAIN 7 INCHES AND WHOSE HEIGHT MI INCHES. HE ALSO MANOTHER CYLINDRICAL CONTAINER WHOSE BASE HA INCHES AND WHOSE HEIGHT MECHES.

A WHICH CONTAIREQUIRES MORE METAL?

B HOW MUCH MORE METAL DOES ? GIVE YOUR ANSWER IN T .

Key Terms

apothem	lateral surface area	sector
arc	parallelogram	segment
area	polygon	similarity
circle	prism	total surface area
congruency	regular polygon	triangle
cylinder	rhombus	volume

Summary

1 A POLYGON IS A SIMPLE CLOSED CURVE FORMED BY THE UNION O SEGMENTS NO TWO OF WHICH IN SUCCESSION ARE COLLINEAR. T CALLED THE SIDES OF THE POLYGON AND THE END POINTS ITS vertices.

2 **A** A POLYGON IS SAID TO BE CONVEX IF EACH INTERIOR AN^o.

B A POLYGON IS SAID TO BE CNON CONVEX), IF AT LEAST ONE OF ITS INTERIOR ANGLES IS GREATER ^{o}.

3 A DIAGONAL OF A POLYGON IS A LINE SEGMENT THAT OF ITS NON-CONSECUTIVE VE

4 **A** THE SUM OF ALL THE INTERIOR ANGLES OF AN n-SIDED POLYGON IS GIVEN BY FORMULA

$$S = (n-2)\times 180^o$$

B THE SUM OF THE EXTERIOR ANGLES OF AN n-SIDED POLYGON IS GI

$$S = 360^o$$

5 A REGULAR POLYGON IS A CONVEX POLYGON WITH ALL SIDES EQUAL AND ALL ANGLES EQUAL.

6 **A** EACH INTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS

$$\frac{(n-2)\times 180^o}{n}$$

B EACH EXTERIOR ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$

C EACH CENTRAL ANGLE OF AN n-SIDED REGULAR POLYGON IS $\frac{360^o}{n}$

7 A FIGURE HAS A LINE OF SYMMETRY, IF IT CAN BE FOLDED SO THAT ONE HALF OF THE FIGURE COINCIDES WITH THE OTHER HALF.

A FIGURE THAT HAS AT LEAST ONE LINE OF SYMMETRY IS CALLED A **symmetrical figure.**

8 AN n-SIDED REGULAR POLYGON HAS n LINES OF SYMMETRY.

9 A CIRCLE CAN BE ALWAYS INSCRIBED IN OR CIRCUMSCRIBED ABOUT ANY GIVEN REGULAR POLYGON.

10 THE APOTHEM IS THE DISTANCE FROM THE CENTRE OF A REGULAR POLYGON TO A SIDE OF THE POLYGON.

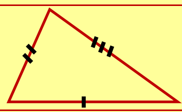
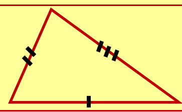
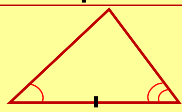
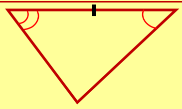
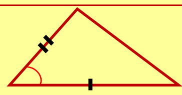
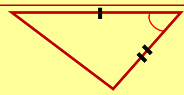
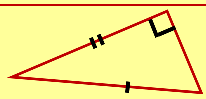
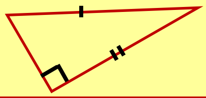
11 FORMULAE FOR THE LENGTH OF A SIDE s, APOTHEM a, PERIMETER P AND AREA A OF A REGULAR POLYGON WITH n SIDES AND RADIUS r ARE GIVEN BY

I $s = 2r \text{ SIN}\frac{180^o}{n}$ **II** $a = r \text{ COS}\frac{180^o}{n}$

III $P = 2nr \text{ SIN}\frac{180^o}{n}$ **IV** $A = \frac{1}{2}aP$ OR $A = \frac{1}{2}nr^2 \text{ SIN}\frac{360^o}{n}$

12 **Congruency**

TWO TRIANGLES ARE CONGRUENT, IF THE FOLLOWING CORRESPONDING PARTS OF ARE CONGRUENT.

I	THREE SIDES (SSS		
II	TWO ANGLES AN INCLUDED SIDE (ASA)		
III	TWO SIDES AND T INCLUDED ANGL (SAS)		
IV	A RIGHT ANGLE, HYPOTENUSE AN SIDE (RHS)		

13 Similarity

I TWO POLYGONS OF THE SAME NUMBER OF SIDES ARE SIMILAR, IF THEIR CORRESPONDING ANGLES ARE CONGRUENT AND THEIR CORRESPONDING SIDES HAVE THE SAME

II SIMILARITY OF TRIANGLES

A SSS-similarity theorem: IF THREE SIDES OF ONE TRIANGLE ARE PROPORTIONA TO THE THREE SIDES OF ANOTHER TRIANGLE, THEN THE TWO TRIANGLES

B SAS-similarity theorem: TWO TRIANGLES ARE SIMILAR, IF TWO PAIRS OF CORRESPONDING SIDES OF THE TRIANGLES ARE PROPORTIONAL AND TH ANGLES BETWEEN THE SIDES ARE CONGRUENT.

C AA-similarity theorem: IF TWO ANGLES OF ONE TRIANGLE ARE CORRESPONDINGLY CONGRUENT TO TWO ANGLES OF ANOTHER TRIANGLE TRIANGLES ARE SIMILAR.

14 IF THE RATIO OF THE LENGTHS OF ANY TWO CORRESPONDING SIDES OF TWO SIMILAR POLYGONS IS k THEN

I THE RATIO OF THEIR PERIMETERS IS k **II** THE RATIO OF THEIR AREAS IS k^2.

15 I Heron's formula

THE AREA A OF A TRIANGLE WITH SIDES a, b AND c UNITS LONG AND SEMI-PERIMETER

$s = \frac{1}{2}(a + b + c)$ IS GIVEN BY

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

II IF h IS THE HEIGHT OF THE TRIANGLE PERPENDICULAR TO THE BASE b, THEN THE AREA OF THE TRIANGLE IS $A = \frac{1}{2} bh$

III IF THE ANGLE BETWEEN THE SIDES a AND b IS THEN THE AREA A OF THE TRIANGLE IS

$$A = \frac{1}{2} ab \text{ SIN}$$

16 RADIANS MEASURE ANGLES IN TERMS OF THE LENGTHS OF THE ARC SWEPT OUT BY THE ANGLE. A RADIAN (RAD) IS DEFINED AS THE MEASURE OF THE CENTRAL ANGLE SUBTEND OF A CIRCLE EQUAL TO THE RADIUS OF THE CIRCLE.

$$1 \text{ RADIAN} = \left(\frac{180}{\ }\right)^{o} \approx 57.3^{o}$$

$$1^{o} = \frac{\ }{180} \text{ RADIAN} \approx 0.0175 \text{ RADIAN.}$$

✓ TO CONVERT RADIANS TO DEGREE, MULTIPLY BY $\frac{180^{o}}{\ }$

✓ TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\ }{180^{o}}$

17 **I** FOR ANY ACUTE ANGLE

SIN = COS (90^o –)

COS = SIN (90^o –)

II FOR ANY ANGLE BETWEEN 90^o AND 180^o

SIN = SIN (180^o –)

COS = –COS (180^o –)

TAN = –TAN (180^o –)

18 **A** A CIRCLE IS SYMMETRICAL ABOUT EVERY DIAMETER.

B A DIAMETER PERPENDICULAR TO A CHORD BISECTS THE CHORD.

C THE PERPENDICULAR BISECTOR OF A CHORD PASSES THROUGH THE CENTRE OF THE CIRCLE.

D IN THE SAME CIRCLE, EQUAL CHORDS ARE EQUIDISTANT FROM THE CENTRE.

E A TANGENT IS PERPENDICULAR TO THE RADIUS DRAWN AT THE POINT OF CONTACT.

F LINE SEGMENTS THAT ARE TANGENTS TO A CIRCLE FROM AN OUTSIDE POINT ARE EQUAL.

19 **Angle properties of a circle**

A THE MEASURE OF AN ANGLE AT THE CENTRE OF A CIRCLE IS TWICE THE MEASURE OF AN ANGLE AT THE CIRCUMFERENCE SUBTENDED BY THE SAME ARC.

B EVERY ANGLE AT THE CIRCUMFERENCE SUBTENDED BY A DIAMETER OF A CIRCLE IS A RIGHT ANGLE.

C INSCRIBED ANGLES IN THE SAME SEGMENT OF A CIRCLE ARE EQUAL.

20 **A** THE LENGTH ℓ OF AN ARC THAT SUBTENDS AN ANGLE AT THE CENTRE OF A CIRCLE WITH RADIUS r IS

$$\ell = \frac{R}{180^o}$$

B THE AREA A OF A SECTOR WITH CENTRAL ANGLE AND RADIUS r IS GIVEN BY

$$A = \frac{r^2}{360^o}$$

C THE AREA A OF A SEGMENT ASSOCIATED WITH A CENTRAL ANGLE AND RADIUS R IS GIVEN BY

$$A = \frac{r^2}{360^0} - \frac{1}{2} r^2 \sin\theta$$

21 IF A_L IS THE LATERAL SURFACE AREA, A_T IS THE TOTAL SURFACE AREA OF THE PRISM, A_B IS BASE AREA OF THE PRISM AND V IS THE VOLUME OF THE PRISM, THEN

I $A_L = Ph$, WHERE P IS THE PERIMETER OF THE BASE AND h IS THE ALTITUDE OR HEIGHT OF THE PRISM.

II $A_T = 2A_B + A_L$

III $V = A_B h$

Review Exercises on Unit 5

1 ABCDE IS A PENTAGON. IF M(∠A) = M(∠B) = M(∠C) = M(∠D) = 115^{O}, FIND M(∠E).

2 GIVEN A REGULAR CONVEX POLYGON WITH 20 SIDES, FIND THE MEASURE OF:

I EACH INTERIOR ANGLE. **II** EACH EXTERIOR ANGLE. **III** EACH CENTRAL ANGLE.

3 THE MEASURE OF EACH INTERIOR ANGLE OF A REGULAR CONVEX POLYGON IS 150°. HOW MANY SIDES DOES IT HAVE?

4 THE ANGLES OF A QUADRILATERAL, TAKEN IN ORDER, ARE y°, $3y^{\circ}$, $5y^{\circ}$, $7y^{\circ}$. VERIFY THAT TWO OF ITS SIDES ARE PARALLEL.

5 FIND THE AREA OF A REGULAR HEXAGON IF EACH SIDE IS 8 CM. (*Leave the answer in radical form*).

6 THE AREA OF A REGULAR HEXAGON IS GIVEN AS $384\sqrt{3}$ CM^2.

A HOW LONG IS EACH SIDE OF THE HEXAGON?

B FIND THE RADIUS OF THE HEXAGON. **C** FIND THE APOTHEM OF THE HEXAGON.

7 FIND THE VALUE OF x IN THE FOLLOWING PAIR OF CONGRUENT TRIANGLES:

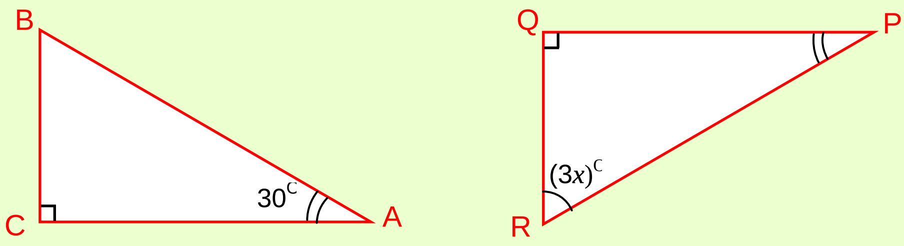


Figure 5.132

8 IN FIGURE 5.133 BELOW, BA = BC AND KA = KC. SHOW THAT M(∠BAK) = M(∠BCK).

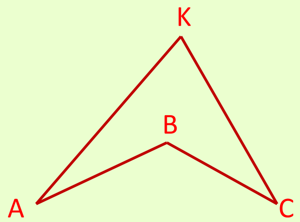


Figure 5.133

9 TWO TRIANGLES ARE SIMILAR. THE SIDES OF ONE ARE 4, 6 AND 7 CM. THE SHORTEST SIDE OF THE OTHER IS 10 CM. CALCULATE THE LENGTHS OF THE OTHER TWO SIDES OF THIS

10 IN THE FIGURE BELOW, ∠ABC AND ∠BDC ARE RIGHT ANGLES; IF AB = 5 CM, AD = 3 CM AND BD = 4 CM, FIND BC AND DC.

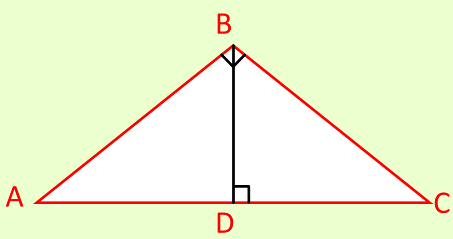


Figure 5.134

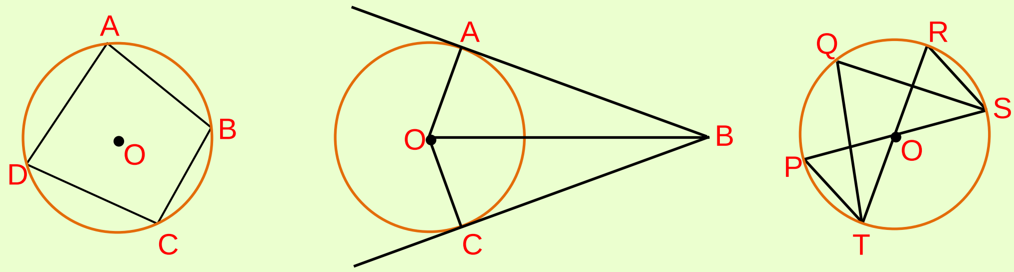
11 THE AREAS OF TWO SIMILAR TRIANGLES ARE 81 CM^2 AND 144 CM^2. IF ONE SIDE OF THE FIRST TRIANGLE IS 6 CM, WHAT IS THE LENGTH OF THE CORRESPONDING SIDE OF THE SEC

12 A CHORD OF A CIRCLE OF RADIUS 6 CM IS 8 CM LONG. FIND THE DISTANCE OF THE CHORD FROM THE CENTRE.

13 TWO CHORDS, AB AND CD, OF A CIRCLE INTERSECT AT RIGHT ANGLES AT A POINT INSIDE THE CIRCLE. IF m($\angle$BAC) = 35^o, FIND m($\angle$ABD).

14 IN EACH OF THE FOLLOWING FIGURES, O IS CENTRE OF THE CIRCLE. IN EACH FIGURE, IDENTIFY WHICH ANGLES ARE:

I SUPPLEMENTARY ANGLES. **II** RIGHT ANGLES. **III** CONGRUENT ANGLES.



A **B** **C**

Figure 5.135

15 FIND THE PERIMETER AND AREA OF A SEGMENT OF A CIRCLE OF RADIUS 8 CM, CUT OFF BY A CHORD THAT SUBTENDS A CENTRAL ANGLE OF:

A 120^o **B** $\frac{3}{4}$ RADIANS.

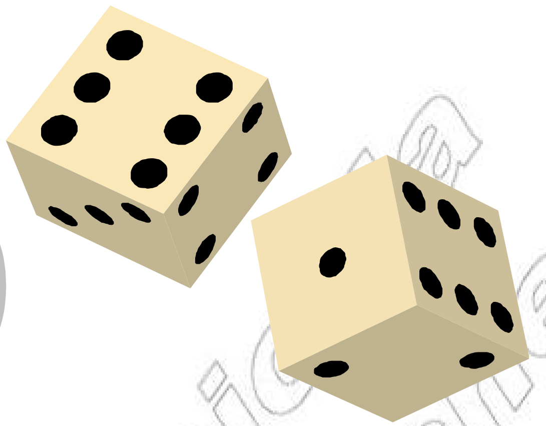
16 CALCULATE THE VOLUME AND TOTAL SURFACE AREA OF A RIGHT CIRCULAR CYLINDER OF HEIGHT 1 M AND RADIUS 70 CM.

17 A 40 M DEEP WELL WITH RADIUS $3\frac{1}{2}$ M IS DUG AND THE EARTH TAKEN OUT IS EVENLY SPREAD TO FORM A PLATFORM OF DIMENSIONS 28 M BY 22 M. FIND THE HEIGHT PLATFORM.

18 A GLASS CYLINDER WITH A RADIUS OF 7 CM HAS WATER UP TO A HEIGHT OF 9 CM. A METAL CUBE OF $5\frac{1}{2}$ CM EDGE IS IMMERSED IN IT COMPLETELY. CALCULATE THE HEIGHT BY THE WATER RISES IN THE CYLINDER.

19 AN AGRICULTURE FIELD IS RECTANGULAR, WITH DIMENSIONS 100 M BY 42 M. A 20 M DEEP WELL OF DIAMETER 14 M IS DUG IN A CORNER OF THE FIELD AND THE EARTH T SPREAD EVENLY OVER THE REMAINING PART OF THE FIELD. FIND THE INCREASE IN THE FIELD.

Unit 6



STATISTICS AND PROBABILITY

Unit Outcomes:

After completing this unit, you should be able to:

- *know methods and procedures in collecting and presenting simple statistical data.*
- *know basic concepts about statistical measures.*
- *understand facts and basic principles about probability.*
- *solve simple mathematical problems on statistics and probability.*

Main Contents

INTRODUCTION

YOU HAVE SOME KNOWLEDGE ABOUT STATISTICS AND ITS BASICS, SUCH AS COLLECTI PRESENTATION OF DATA, ETC., FROM YOUR PRIMARY GRADE MATHEMATICS. IN T FORMALLY DEFINE 'STATISTICS' AS A BRANCH OF APPLIED MATHEMATICS AND LEAI COLLECTION, PRESENTATION AND ANALYSIS OF NUMERICAL DATA. THE UNIT ALSO CONCEPT OF PROBABILITY, WHICH WAS INTRODUCED IN GRADE 8 AND TEACHES YOU MORE ABOUT EXPERIMENTAL AND THEORETICAL APPROACHES TO PROBABILITY AND HELPS YOU PROBLEMS BASED ON THESE APPROACHES.

6.1 STATISTICAL DATA

6.1.1 Collection and Tabulation of Statistical Data

Group Work 6.1

1 SPLIT THE CLASS INTO THREE GROUPS. LET THE CLA YEAR'S MATHEMATICS RESULT OF THE SCHOOL I THE SCHOOL OFFICE'S RECORDS.

LET GROUP B COLLECT INFORMATION ABOUT THE DISEASES TREATED IN YOUR N CENTRE, HOSPITAL, OR HEALTH POST. LET GROUP C MEASURE THE HEIGHT OF EAC YOUR CLASS AND CONSIDER ITS DISTRIBUTION BY AGE AND SEX.

ANSWER THE FOLLOWING QUESTIONS USING THE INFORMATION GATHERED BY EAC

A HOW MANY STUDENTS APPEARED FOR THE EXAM?

B HOW MANY STUDENTS SCORED A IN THE NATIONAL EXAM?

C WHAT WAS THE SCORE OBTAINED BY MOST OF THE STUDENTS?

D WHICH DISEASES ARE TREATED MOST FREQUENTLY?

E WHAT IS THE AVERAGE HEIGHT OF THE CLASS?

F ARE MALES OR FEMALES TALLER?

2 DISCUSS MORE ABOUT THE IMPORTANCE AND PURPOSE OF STATISTICS.

3 WHAT IS THE ANNUAL BIRTH RATE AND DEATH RATE IN ETHIOPIA? WHICH GOV AGENCY IS RESPONSIBLE FOR THE PREPARATION OF SUCH RECORDS?

4 WHY DOES THE GOVERNMENT OF ETHIOPIA CARRY OUT A CENSUS EVERY TEN YEA

THERE ARE MANY DEFINITIONS OF THE TERM statistics GIVEN BY DIFFERENT SCHOLARS. HOWEVER, FOR THE PURPOSE OF THIS UNIT, WE WILL CONFINE OURSELVES TO THE FOLLOWING:

Definition 6.1

Statistics is the science of *collecting*, *organizing*, *presenting*, *analyzing* and *interpreting data* (quantitative information) in order to draw conclusions.

BASICALLY, STATISTICS IS A PROCEDURAL PROCESS OF FIVE LOGICAL STEPS IN HANDLING NUMERICAL DATA. THESE ARE:

collection of data → organization of data → presentation of data → analysis of data → interpretation of data

1 Collection of data

THE FIRST STEP OF STATISTICS IS COLLECTION OF DATA. THIS IS THE PROCESS OF OBTAINING MEASUREMENTS OR COUNTS. FOR EXAMPLE, MEASURING THE HEIGHTS OF STUDENTS I OR COUNTING THE NUMBER OF PERSONS ADMITTED TO A CERTAIN HOSPITAL ARE EXA data collection.

2 Organization of data

THE SECOND STEP OF STATISTICS REFERS TO THE ORGANIZATION OF DATA. COLLECTE ORGANIZED IN A SUITABLE FORM TO UNDERSTAND THE INFORMATION GATHERED. THE MUST BE edited, classified AND tabulated.

3 Presentation of data

THE MAIN PURPOSE OF DATA PRESENTATION IS TO FACILITATE STATISTICAL ANALYS DONE BY ILLUSTRATING THE DATA USING GRAPHS AND DIAGRAMS LIKE BAR GRAPH, H CHARTS, PICTOGRAMS, FREQUENCY POLYGONS, ETC.

4 Analysis of data

IN ORDER TO MEET THE DESIRED PURPOSE OF INVESTIGATION, DATA HAS TO BE AN PURPOSE OF ANALYZING DATA IS TO HIGHLIGHT INFORMATION USEFUL FOR DECISION M

5 Interpretation of data

BASED ON ANALYZED DATA, CONCLUSIONS HAVE TO BE DRAWN. THIS STEP USUAL DECISION MAKING ABOUT A LARGE COLLECTION OF OBJECTS (THE POPULATION INFORMATION GATHERED FROM A SMALL COLLECTION OF SIMILAR OBJECTS (THE SAMP

THE DECISION MAKING PROCESSES USED BY THE MANAGERS OF MODERN BUSINESSES A IS GOVERNED BY STATISTICAL APPLICATION. STATISTICAL METHODS CAN BE APPLIED T WHERE NUMERICAL INFORMATION IS GATHERED WITH THE OBJECTIVE OF MAKING RAT IN THE FACE OF UNCERTAINTY.

THE FOLLOWING EXAMPLES SHOW US HOW STATISTICS PLAYS A MAJOR ROLE IN DECISIO
DIFFERENT SECTORS.

EXAMPLE 1 INFORMATION GATHERED ABOUT THE INCIDENCE OR PREVALENCE OF DIS COMMUNITY PROVIDES USEFUL INFORMATION ON CHANGING TRENDS I STATUS, MORTALITY, NUTRITIONAL STATUS OR ENVIRONMENTAL HAZARDS

EXAMPLE 2 STATISTICS IS USED TO STUDY EXISTING CONDITIONS AND THE PREVALEN HIV/AIDS IN ORDER TO DESIGN NEW PROGRAMS OR TO STUDY THE MERI DIFFERENT METHODS ADOPTED TO CONTROL HIV/AIDS. IT ASSISTS IN DETE THE EFFECTIVENESS OF NEW MEDICATION AND THE IMPORTANCE OF COUNS

EXAMPLE 3 DEMOGRAPHIC DATA ABOUT POPULATION SIZE, ITS DISTRIBUTION BY AGE AND THE RATE OF POPULATION GROWTH, ETC., ALL HELP POLICY MA DETERMINING FUTURE NEEDS SUCH AS FOOD, CLOTHING, HOUSING, ED HEALTH FACILITIES, WATER, ELECTRICITY AND TRANSPORTATION SYSTEMS

EXAMPLE 4 RECORDING ANNUAL TEMPERATURES IN A COUNTRY PROVIDES THE COMM TIMELY WARNING OF ENVIRONMENTAL HAZARDS.

EXAMPLE 5 STATISTICAL DATA COLLECTED ON CUSTOMER SERVICES PROVIDES FEEDB HELP TO REFORM POLICIES AND SYSTEMS.

IN THE ABSENCE OF ACCURATE AND TIMELY DATA, IT IS IMPOSSIBLE TO FORM SUITA STATISTICS ALSO PLAYS A VITAL ROLE IN MONITORING THE PROPER IMPLEMENTATIO AND POLICIES.

IN ITS ORDINARY USAGE, POPULATION REFERS TO THE NUMBER OF PEOPLE LIVING I COUNTRY. IN STATISTICS, HOWEVER, **population** REFERS TO THE COMPLETE COLLECTION OF INDIVIDUALS, OBJECTS OR MEASUREMENTS THAT HAVE A COMMON CHARACTERISTIC.

GAINING ACCESS TO AN ENTIRE GROUP (OR POPULATION) IS OFTEN DIFFICULT, EXP SOMETIMES DESTRUCTIVE. THEREFORE, INSTEAD OF EXAMINING THE ENTIRE GROUP, EXAMINES A SMALL PART OF THE GROUP, CALLED A **sample**

DATA CAN BE CLASSIFIED AS EITHER **qualitative** OR **quantitative**. HOWEVER, STATISTICS DEALS MAINLY WITH QUANTITATIVE DATA.

EXAMPLE 6 DATA COLLECTED FROM THE POPULATION OF STUDENTS IN ETHIOPIA COUL

I **Qualitative** IF THE DATA IS BASED ON SOME CHARACTERISTIC WHOSE VALUES A NUMBERS, SUCH AS THEIR EYE COLOUR, SEX, RELIGION OR NATIONALITY.

II **Quantitative** IF THE DATA IS NUMERICAL SUCH AS HEIGHT, WEIGHT, AGE OR SCO TESTS.

A RULE WHICH GIVES A CORRESPONDING VALUE TO EACH MEMBER OF A POPULATIO **population function**.

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EXAMPLE 7 HERE IS A TABLE SHOWING THE APPROXIMATE SIZES OF MAJOR LAKES IN ET

TABLE 6.1: SIZE OF MAJOR LAKES IN ETHIOPIA

Name of Lake	Length (km)	Width (km)	Area (km^2)
ABAYA	60	20	1160
ABAYATA	17	15	205
ASHENGE	5	4	20
HAWASSA	16	9	229
CHAMO	26	22	551
HAYK	7	5	35
KOKA	20	15	205
LANGANO	18	16	230
SHALLA	28	12	409
TANA	70	60	3600
ZIWAY	25	20	434

WE CAN THINK OF THE SET OF THE ELEVEN LAKES AS THE POPULATION AND THE FUNCT
L: LENGTH, W: WIDTH, A: AREA, ETC AS FUNCTIONS ON THIS POPULATION.

EXAMPLE 8 THE FOLLOWING TABLE SHOWS THE AGE OF 10 STUDENTS IN A CERTAIN CL

TABLE 6.2: AGE OF STUDENTS

Name of student x	Age (in years) A(x)
ABEBE	18
ABDU	17
BAYISSA	16
FATUMA	17
HIWOT	15
KIDANE	14
LEMLEM	18
MESERET	17
OMOD	15
ZEHARA	16

THE STUDENTS ARE MEMBERS OF THE POPULATION AND THEIR AGE, A IS THE POPULAT
STATISTICS CAN BE CLASSIFIED INTO TWO TYPES: DESCRIPTIVE STATISTICS AND INFERE

Definition 6.2

Descriptive statistics is a branch of statistics concerned with summarizing and describing a large amount of data without drawing any conclusion about a particular bit of data.

DESCRIPTIVE STATISTICS DESCRIBES INFORMATION COLLECTED THROUGH NUMERICA CHARTS, GRAPHS AND TABLES. THE MAIN PURPOSE OF DESCRIPTIVE STATISTICS IS T OVERVIEW OF THE INFORMATION COLLECTED.

Definition 6.3

Inferential statistics is a branch of statistics concerned with interpreting data and drawing conclusions.

WE CAN CLASSIFY DATA AS primary data AND secondary data.

1 Primary data

DATA IS SAID TO BE primary, IF IT IS OBTAINED FIRST HAND FOR THE PARTICULAR PURPO WHICH ONE IS CURRENTLY WORKING. PRIMARY DATA IS ORIGINAL DATA, OBTAINED PE PRIMARY SOURCES BY OBSERVATION, INTERVIEW OR DIRECT MEASUREMENT.

EXAMPLE 9 IF YOU MEASURE THE HEIGHTS OF STUDENTS IN YOUR CLASS, THIS IS PRIMARY DATA.

EXAMPLE 10 THE DATA GATHERED BY THE MINISTRY OF EDUCATION ABOUT THE NU STUDENTS ENROLLED IN DIFFERENT UNIVERSITIES OF ETHIOPIA IS PRIMAR THE MINISTRY ITSELF. (*If you were to use this data, it would be secondary data for you.*)

2 Secondary data

DATA WHICH HAS BEEN COLLECTED PREVIOUSLY (FOR SIMILAR OR DIFFERENT PURPOS secondary data. SECONDARY DATA REFERS TO THAT DATA WHICH IS NOT ORIGINATE RESEARCHER HIMSELF/HERSELF, BUT WHICH HE/SHE OBTAINS FROM SOMEONE ELSE'S F SOURCES OF SECONDARY DATA ARE OFFICIAL PUBLICATIONS, JOURNALS, NEWSPAI STUDIES, NATIONAL STATISTICAL ABSTRACTS, ETC.

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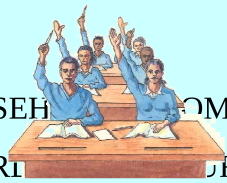
EXAMPLE 11 REPORTS ON THE NUMBER OF VICTIMS OF HIV/AIDS BY THE MINISTRY HEALTH IS SECONDARY DATA FOR ANYONE OTHER THAN THE MINISTRY.

EXAMPLE 12 THE 2007 CENSUS OF POPULATION SIZE OF REGIONS BY SEX REPORTED B CENTRAL STATISTICAL AGENCY (CSA) IS SECONDARY DATA FOR THE GOVE

INFORMATION EXPRESSED IN QUANTITATIVE FORM CAN RESULT IN SUCH A LARGE AMO UNLESS THESE FIGURES ARE PRESENTED IN SOME ORGANIZED MANNER, THEIR SIGNIFIC LOST. ONE OF THE BASIC METHODS OF PRESENTING STATISTICAL DATA IS PUTTING IT I tables. TO DO THIS, OFTEN THE DATA NEEDS TO BE CLASSIFIED.

Classification IS THE PROCESS OF ARRANGING THINGS INTO GROUPS OR CLASSES.

ACTIVITY 6.1



1. CLASSIFY THE EMPLOYEES IN YOUR SCHOOL BY HOUSEH OME.
2. GROUP THE NUMBER OF HIV/AIDS VICTIMS RECORD UR NEAREST HEALTH CENTRE ACCORDING TO THEIR AGE GROUP.
3. COLLECT DATA ON AGE, HEIGHT AND MATHEMATICS EXAM SCORE OF THE STUD IN YOUR CLASS. CLASSIFY OR TABULATE THE DATA COLLECTED.

DIFFERENT PEOPLE OR ORGANIZATIONS COLLECT DATA FOR DIFFERENT REASONS A CLASSIFICATION THEY USE IS ALSO DIFFERENT ACCORDINGLY. TO SEE THIS, CONSIDER EXAMPLES.

EXAMPLE 13 AN ECONOMIST IN THE LABOUR DEPARTMENT OF A REGIONAL SOCIAL A BUREAU MAY CLASSIFY THE HOUSEHOLDS IN A CERTAIN LOCALITY BY I INCOME AS SHOWN IN THE TABLE BELOW.

TABLE 6.3: MONTHLY INCOME OF 300 HOUSEHOLDS

Income (in Birr)	Number of households
UNDER 350	85
BETWEEN 350 AND 650	72
BETWEEN 651 AND 950	64
BETWEEN 951 AND 1250	48
BETWEEN 1251 AND 155	21
ABOVE 1550	10
TOTAL	300

EXAMPLE 14 ACCORDING TO THE 2007 ETHIOPIAN CENSUS, THE ETHIOPIAN CENTRAL STA AGENCY (CSA) HAS CLASSIFIED THE POPULATION BY SEXAS FOLLOWS.

TABLE 6.4: POPUIATION BY SEX (2007 ETHIOPIAN CENSUS)

Region	Male (in 1000)	Female (in 1000)	Both sexes (in 1000)
TIGRAY	2124.8	2189.6	4314.4
AFFAR	786.3	624.7	1411.0
AMHARA	8636.9	8577.2	17214.1
OROMIYA	13676.2	13482.3	27158.5
SOMALE	2468.8	1970.4	4439.2
BENSHANGU	340.4	330.5	670.9
SNNP	7482.0	7560.5	15042.5
GAMBELA	159.7	147.2	306.9
HARARI	92.3	91.1	183.4
ADDIS ABAB	1304.5	1433.7	2738.2
DIRE DAWA	171.9	170.9	342.8
Total	37243.8	36577.4	73821.2

A STATISTICAL TABLE IS A SYSTEMATIC PRESENTATION OR ORGANIZATION OF NUM COLUMNS AND ROWS. COLUMNS ARE VERTICAL ARRANGEMENTS AND ROWS ARE HO MAIN PURPOSE OF A STATISTICAL TABLE IS TO ALLOW THE READER TO QUICKLY A INFORMATION. A TITLE AND ROW AND COLUMN HEADERS ARE IMPORTANT.

Exercise 6.1

1. WHAT ARE THE STEPS USED IN DOING A STATISTICAL STUDY?
2. WHAT DO WE MEAN BY ORGANIZING OR PRESENTING DATA?
3. EXPLAIN EACH OF THE FOLLOWING STATISTICAL TERMS BY GIVING EXAMPLES.
   - A QUALITATIVE DATA B QUANTITATIVE DATA C POPULATION
   - D POPULATION FUNCTION E SAMPLE
4. MENTION FOUR USES OF STATISTICS.
5. WHAT IS DESCRIPTIVE STATISTICS?
6. DESCRIBE IN YOUR OWN WORDS THE DIFFERENCE BETWEEN A POPULATION AND A
7. DETERMINE WHETHER THE FOLLOWING DATA IS QUALITATIVE OR QUANTITATIVE.
   - A GENDER B TEMPERATURE C ZIP CODE
   - D NUMBER OF DAYS E RELIGIONS F OCCUPATIONS
   - G AGES H COLOURS I NATIONALITY

8 MENTION SOME ADVANTAGES OF TABULAR PRESENTATION OF DATA.

9 WHY IS IT NECESSARY TO ORGANIZE DATA IN A SYSTEMATIC MANNER AFTER IT HAS BEEN COLLECTED?

10 DRAFT A TABLE TO SHOW THE FOLLOWING DATA, COLLECTED FROM EMPLOYEES I

- **A** SEX
- **B** THREE RANKS: SUPERVISORS, ASSISTANTS AND CLERKS
- **C** YEARS: 2000 AND 2001
- **D** AGE GROUP: 18 YEARS AND UNDER, OVER 18 BUT LESS THAN 50 YEARS, OVER 5

6.1.2 Distributions and Histograms

INFORMATION (DATA) IS OBTAINED FROM A CENSUS, EXISTING DATA SOURCES, SURVEY EXPERIMENTS. AFTER DATA IS COLLECTED, IT MUST BE ORGANIZED INTO A MANAGEAB THAT IS NOT ORGANIZED IS REFERRED TO AS raw data.

Definition 6.3

A quantity that we measure from observation is called a variate or variable denoted by V. The distribution of a population function is the function which associates with each variate of the population function a corresponding frequency denoted by f.

METHODS FOR ORGANIZING RAW DATA INCLUDE THE DRAWING OF TABLES OR GRAPH QUICK OVERVIEW OF THE INFORMATION COLLECTED.

EXAMPLE 1 SUPPOSE THERE ARE 10 STUDENTS IN A GROUP WHOSE SCORES IN A MATH QUIZ WERE AS FOLLOWS:

13, 12, 14, 13, 12, 12, 13, 14, 15, 12

ORGANIZE THE DATA IN TABULAR FORM. WHAT ARE THE VARIATES? GIVE THE FREQU VARIATE.

SOLUTION: THE DATA GIVEN ABOVE IS RAW DATA.

WE MAY NOW TABULATE THE GIVEN DATA IN THE FORM GIVEN BELOW.

Score (V)	12	13	14	15
Number of students (f)	4	3	2	1

THE TABLE GIVEN ABOVE IS CALLED THE frequency distribution table. THE SCORES ARE THE VARIATE AND THE NUMBER OF STUDENTS GETTING A PARTICULAR SCORE IS THE frequency OF THE VARIATE.

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Definition 6.4

A frequency distribution is a tabular or graphical representation of a data showing the frequency associated with each data value.

EXAMPLE 2 ORGANIZE THE DATA BELOW INTO A FREQUENCY DISTRIBUTION TABLE.

8,	9,	8,	7,	10,	9,	6,	4,	9,	8,
7,	8,	10,	9,	8,	6,	9,	7,	8,	8

SOLUTION: (WRITE THE VALUES IN ASCENDING ORDER.)

Value(V)	4	5	6	7	8	9	10	TOTAL
Frequency(f)	1	0	2	3	7	5	2	20

QUANTITATIVE DATA CAN ALSO BE REPRESENTED GRAPHICALLY, THROUGH A histogram

Definition 6.5

A histogram is a graphical representation of a frequency distribution in which the variate (V) is plotted on the x-axis (horizontal axis) and the frequency (f) is plotted on the y-axis (vertical axis).

WHEN DRAWING A HISTOGRAM:

I CONSTRUCT A FREQUENCY DISTRIBUTION TABLE OF THE GIVEN DATA.

II THE x-AXIS

- **A** DETERMINE A SUITABLE SCALE FOR THE HORIZONTAL AXIS. DETERMINE THE NUMBER OF RECTANGLES NEEDED TO REPRESENT EACH VARIATE OR GROUP OF VARIATES AS
- **B** TRY NOT TO BREAK THE x-AXIS.

III THE y-AXIS

- **A** DISPLAY INFORMATION ABOUT FREQUENCY ON THE VERTICAL (y) AXIS.
- **B** DETERMINE THE LENGTH OF THE y-AXIS.

IV DRAW BARS FOR EACH VARIATE (V)

V LABEL THE HISTOGRAM WITH A TITLE, AND LABEL THE AXES.

Note: **I** THE HEIGHT OF EACH RECTANGLE IS THE FREQUENCY.

II THE WIDTH OF EACH RECTANGLE SHOULD BE THE SAME.

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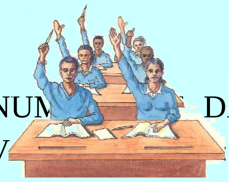
ACTIVITY 6.2



CONSIDER THE FOLLOWING DATA THAT SHOWS THE NUM DAYS 25 INDIVIDUALS PARTICIPATED IN SOIL AND WATER CONSERV

3	8	7	4	8
5	9	8	5	9
7	8	3	7	5
8	5	6	8	8
10	7	4	4	7

CONSTRUCT A FREQUENCY DISTRIBUTION TABLE AND A HISTOGRAM FOR THE ABOVE D

EXAMPLE 3 THE TEMPERATURE IN °C FOR THE FIRST 14 DAYS OF SEPTEMBER IN A CERTAI TOWN WERE RECORDED AS

22,	27,	19,	23,	19,	18,	27,
27,	25,	23,	26,	27,	28,	23

CONSTRUCT A FREQUENCY DISTRIBUTION TABLE AND A HISTOGRAM FOI DATA.

SOLUTION: NOW CONSTRUCT THE FREQUENCY DISTRIBUTION TABLE FROM THE RAW D

Temperature (in °C) (*V*)	18	19	20	21	22	23	24	25	26	27	28
frequency (*f*)	1	2	0	0	1	3	0	1	1	4	1

USING THE ABOVE TABLE, WE DRAW A HISTOGRAM AS SHOWN BELOW.

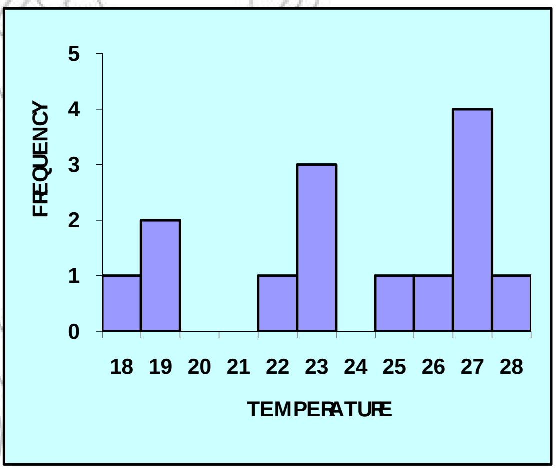


Figure 6.1

EXAMPLE 4 THE FOLLOWING HISTOGRAM SHOWS THE DAILY INCOME (IN BIRR) O EMPLOYEES IN A FACTORY.

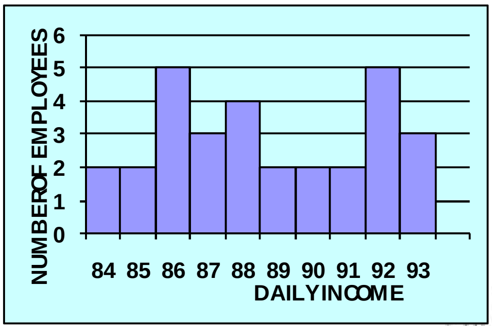


Figure 6.2

FROM THE HISTOGRAM, ANSWER THE FOLLOWING QUESTIONS:

A HOW MANY EMPLOYEES HAVE A DAILY INCOME OF BIRR 92?

B HOW MANY EMPLOYEES COLLECT A DAILY INCOME OF MORE THAN BIRR 90?

C WHAT IS THE HIGHEST FREQUENCY?

D WHAT PERCENT OF THE EMPLOYEES EARN A DAILY INCOME OF MORE THAN BI

SOLUTION:

A 5 EMPLOYEES HAVE A DAILY INCOME OF BIRR 92.

B 10 EMPLOYEES EARN A DAILY INCOME OF MORE THAN BIRR 90.

C THE HIGHEST FREQUENCY IS 5.

D PERCENTAGE

$$= \frac{\text{SUM OF THE FREQUENCIES OF EMPLOYEES EARNING MORE THAN 89}}{\text{TOTAL FREQUENCY}} \times 100\%$$

$$= \frac{2+2+5+3}{30} \times 100\% = \frac{12}{30} \times 100\% = 40\%$$

I.E., 40% OF THE EMPLOYEES EARN A DAILY INCOME OF MORE THAN BIRR 89.

Exercise 6.2

1 GIVE TWO REASONS WHY RAW DATA SHOULD BE SUMMARIZED INTO A FREQUENCY D

2 WHAT IS THE DIFFERENCE BETWEEN A FREQUENCY DISTRIBUTION TABLE AND A HIST

3 THE AGES (TO THE NEAREST YEAR) OF 40 CHILDREN IN A CERTAIN VILLAGE ARE AS FOLLOWS:

10	7	4	5	1	9	3	6	5	4
2	7	5	3	2	5	6	2	8	9
5	8	9	9	5	2	1	3	9	4
3	5	7	9	6	3	6	8	1	2

PREPARE A FREQUENCY DISTRIBUTION TABLE AND A HISTOGRAM FOR THE GIVEN D

4 COLLECT THE SCORE THE STUDENTS IN YOUR CLASS OBTAINED IN THEIR MID-SEMESTER MATHEMATICS EXAM AND

- **A** PREPARE A FREQUENCY DISTRIBUTION TABLE.
- **B** DRAW A HISTOGRAM.
- **C** WHAT SCORE IS MOST FREQUENT?
- **D** WHAT IS THE LEAST SCORE OBTAINED?

5 A SAMPLE OF 50 COUPLES MARRIED FOR 10 YEARS WERE ASKED HOW MANY CHILDREN THEY HAD. THE RESULT OF THE SURVEY IS AS FOLLOWS:

0	4	2	2	1	3	0	3	2	4
3	3	1	3	3	3	3	3	2	2
1	3	3	2	4	3	1	5	2	2
2	0	0	2	1	2	2	2	3	2
3	3	3	4	3	1	3	0	3	2

- **A** CONSTRUCT A FREQUENCY DISTRIBUTION.
- **B** CONSTRUCT A HISTOGRAM.
- **C** WHAT PERCENTAGE OF COUPLES HAVE TWO CHILDREN?
- **D** WHAT PERCENTAGE OF COUPLES HAVE AT LEAST TWO CHILDREN?

6 HERE ARE QUINTALS OF FERTILIZER DISTRIBUTED TO 50 FARMERS.

20	24	22	19	20	10	18	24	10	15
21	20	20	19	20	10	14	22	10	18
18	15	14	18	20	15	14	22	14	20
15	14	15	20	21	10	20	20	15	24
10	10	15	22	14	21	20	14	15	10

- **A** CONSTRUCT A FREQUENCY DISTRIBUTION.
- **B** CONSTRUCT A HISTOGRAM.

7 SUPPOSE THE FOLLOWING DATA REPRESENTS THE NUMBER OF PERSONS W
COUNSELLING ON HIV/AIDS ON 40 CONSECUTIVE DAYS:

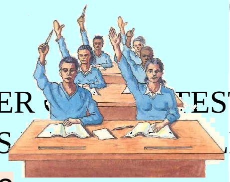
10	5	10	3	4	5	12	9	11	13
10	9	6	10	8	7	3	7	9	10
4	6	8	6	7	6	4	4	11	8
10	9	5	8	8	7	8	8	6	12

- **A** CONSTRUCT A FREQUENCY DISTRIBUTION TABLE FROM THE DATA.
- **B** CONSTRUCT A HISTOGRAM.
- **C** ON WHAT PERCENT OF DAYS DID MORE THAN 10 PEOPLE TAKE COUNSELLING?

6.1.3 Measures of Location (Mean, Median and Mode(s))

QUANTITATIVE VARIABLES CONTAINED IN RAW DATA OR IN FREQUENCY TABLES SUMMARIZED BY MEANS OF A FEW NUMERICAL VALUES. A KEY ELEMENT OF THIS SU CALLED THE measure of average OR measure of location. THE THREE COMMONLY USED MEASURES OF LOCATION ARE THE arithmetic mean (OR THE MEAN), the median AND THE mode(s).

ACTIVITY 6.3



1 AFTER COMPLETING A UNIT, A MATHEMATICS TEACHER TEST MARKED OUT OF 10, AND THE SCORES OF 22 STUDENTS LOWS:

6, 5, 8, 10, 6, 7, 3, 9, 3, 2, 9, 6, 7, 2, 6, 5, 4, 8, 6, 4,8, 3

- A DID THE GROUP DO WELL IN THE TEST?
- B PREPARE A FREQUENCY DISTRIBUTION TABLE FROM THE GIVEN DATA.
- C WHAT IS THE AVERAGE SCORE OF THE GROUP?
- D HOW MANY STUDENTS SCORE ABOVE AVERAGE?
- E FROM THE AVERAGE OBTAINED, CAN WE SAY SOMETHING ABOUT THE PERFOR THE GROUP?
- F WHAT RELATION CAN WE SEE BETWEEN THE SINGLE VALUE OBTAINED IN C AN MARKS OF THE STUDENTS? CAN THE SINGLE VALUE SUMMARIZE THE DATA?

2 RECORD THE HEIGHT AND AGE OF EACH STUDENT IN YOUR CLASS.

- A WHAT IS THE AVERAGE HEIGHT AND AGE OF THE STUDENTS?
- B WHAT IS THE MIDDLE VALUE OF HEIGHT AND AGE OF THE STUDENTS?
- C WHAT VALUE OF HEIGHT AND AGE IS MOST FREQUENT (OR HAS THE HIGHEST FREQUENCY)?

3 SUPPOSE A STUDENT SCORED THE FOLLOWING MARKS IN FIVE SUBJECTS:

85, 89, 78, 92, 91

- A WHAT IS THE AVERAGE SCORE OF THE STUDENT?
- B WHAT IS THE MIDDLE VALUE OF THE SCORE?

4 CONSIDERING THE FOLLOWING DATA

20, 21, 21, 22, 23, 23, 25, 27, 27, 27, 29, 98, 98

- A FIND THE MEAN, MEDIAN AND MODE.
- B WHICH MEASURE OF LOCATION DOES NOT GIVE A CLEAR IDEA OF THE CENTRE OF THE DISTRIBUTION?

5 COULD YOU FIND THE ARITHMETIC MEAN OF QUALITATIVE DATA? WHAT ABOUT MODE?

1 The arithmetic mean

WHEN USED IN EVERYDAY LANGUAGE THE WORD "AVERAGE" OFTEN STANDS FOR THE ARITHMETIC MEAN

Definition 6.6

The **arithmetic mean** (or the **mean**) of a variable is the sum of all the data values, divided by the total frequency (number of observations).

If $x_1, x_2, x_3, \ldots, x_n$ are the n observations of a variable, then the mean, $\bar{x}$, is given by

$$\text{Mean}: \ \bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\text{SUM OF VALUES}}{\text{TOTAL NUMBER OF VAL}}$$

EXAMPLE 1 FIND THE MEAN OF THE FOLLOWING DATA:

7, 21, 2, 17, 3, 13, 7, 4, 9, 7, 9

SOLUTION:

$$\bar{x} = \frac{7+21+2+17+3+13+7+4+9+7+9}{11} = \frac{99}{11} = 9$$

Note: THE MEAN OF A POPULATION FUNCTION CAN ALSO BE CALCULATED FROM ITS FREQUENCY DISTRIBUTION. SO, IF THE VALUES $x_1, x_2, x_3, \ldots, x_n$ OCCUR $f_1, f_2, f_3, \ldots, f_n$ TIMES, RESPECTIVELY, THEN THE MEAN ($\bar{x}$) IS GIVEN BY

$$\text{MEAN: } \bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + \ldots + f_n}$$

EXAMPLE 2 THE FOLLOWING TABLE SHOWS THE AGE OF 14 STUDENTS IN A CERTAIN CLASS:

Age in years (*V*)	12	13	16	18
Number of students (*f*)	3	4	2	5

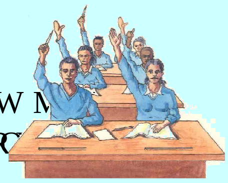
COMPUTE THE MEAN AGE OF THE STUDENTS.

SOLUTION:

$$\bar{x} = \frac{12\times3+13\times4+16\times2+18\times5}{3+4+2+5} = \frac{36+52+32+90}{14} = \frac{210}{14} = 15 \text{ YEARS}$$

Properties of the mean

ACTIVITY 6.4



THERE ARE FIVE STUDENTS IN A GROUP. LEMLEM WANTS TO KNOW HOW MUCH MONEY EACH STUDENT HAS AND ASKED ALL THE MEMBERS OF THE GROUP. SHE FOUND THE FOLLOWING AMOUNTS:

BIRR 6, BIRR 9, BIRR 8, BIRR 4 AND BIRR 3.

A WHAT IS THE MEAN OF THE AMOUNT OF MONEY WITHIN THE GROUP?

B IF LEMLEM GIVES BIRR 2 TO EACH MEMBER OF THE GROUP, WHAT WILL BE TH MEAN?

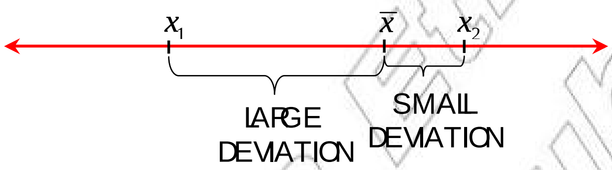
C IF THE AMOUNT OF MONEY IN THE POCKET OF EACH MEMBER IS MULTIPLIED I WHAT WILL BE THE NEW MEAN?

D IF YOU SUBTRACT THE MEAN OF THE DATA OBTAINED FROM EACH VALUE, WH THE SUM OF THE DIFFERENCES OBTAINED?

E DISCUSS, WHAT YOU OBSERVED FROM YOUR ANSWERS TO **A**, **B**, **C** AND **D**.

THE ABOVE ACTIVITY SHOULD HELP YOU TO OBSERVE DIFFERENT PROPERTIES OF THE MEAN.

THE DIFFERENCE BETWEEN A SINGLE DATA VALUE x AND THE MEAN IS CALLED THE DEVIATION FROM THE MEAN (OR SIMPLY THE DEVIATION) AND IS GIVEN BY $(x - \overline{x})$. A DATA POINT THAT IS CLOSE TO THE MEAN WILL HAVE A SMALL DEVIATION, WHEREAS DATA POINTS FAR FROM THE M LARGE DEVIATIONS AS SHOWN IN THE FIGURE BELOW.



1 THE SUM OF THE DEVIATIONS OF INDIVIDUAL OBSERVATIONS FROM THE MEAN ($\overline{x}$) IS ZERO. THAT IS, LET $x_1, x_2, x_3, \ldots, x_n$ BE n OBSERVATIONS WITH MEAN $\overline{x}$. THEN THE SUM OF THE DEVIATIONS OF THE OBSERVATIONS FROM THE MEAN IS GIVEN BY

$$(x_1 - \overline{x}) + (x_2 - \overline{x}) + (x_3 - \overline{x}) + \ldots + (x_n - \overline{x}) = 0$$

Proof:-

SINCE THE MEAN OF n OBSERVATIONS $x_1, x_2, x_3, \ldots, x_n$ IS GIVEN BY $\overline{x}$,

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} \text{ WHICH SHOWS } x_1 + x_2 + x_3 + \ldots + x_n = n\overline{x}$$

NOW, $(x_1 - \overline{x}) + (x_2 - \overline{x}) + (x_3 - \overline{x}) + \ldots + (x_n - \overline{x})$

$$= (x_1 + x_2 + x_3 + \ldots + x_n) - \underbrace{(\overline{x} + \overline{x} + \overline{x} \ldots + \overline{x})}_{n \text{ TIMES}}$$

$$= (x_1 + x_2 + x_3 + \ldots + x_n) - n\overline{x}$$

$$= n\overline{x} - n\overline{x} = 0 \text{ AS REQUIRED.}$$

EXAMPLE 3 LET THE AGES OF 5 CHILDREN BE 2, 3, 6, 9, 10. THEN, THE MEAN AGE

$$\overline{x} = \frac{2+3+6+9+10}{5} = \frac{30}{5} = 6$$

THE SUM OF THE DEVIATIONS FROM THE MEAN IS:

(2 – 6) + (3 – 6) + (6 – 6) + (9 – 6) + (10 – 6) = –4 – 3 + 0 + 3 + 4 = 0

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2 IF A CONSTANT K IS ADDED TO (OR SUBTRACTED FROM) EACH DATA VALUE, THEN THE NEW MEAN IS THE SUM (OR THE DIFFERENCE) OF THE OLD MEAN AND THE CONSTANT k

Proof:- LET $\bar{x}$ BE THE MEAN OF THE DATA VALUES AND k BE THE CONSTANT.

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \bar{x}$$

ADDING k TO EACH DATA VALUE, THE NEW MEAN IS THEN

$$\frac{(x_1+k)+(x_2+k)+(x_3+k)+\ldots+(x_n+k)}{n} =$$

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n + k + k + k + \ldots + k}{n}$$

$$= \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} + \frac{nk}{n}$$

$$= \bar{x} + k \text{ (THE OLD MEAN PLUS } k\text{).}$$

A SIMILAR PROOF CAN BE DONE FOR THE CASE WHEN k IS SUBTRACTED FROM EACH DATA VALUE.

3 THE MEAN OF THE SUM OR DIFFERENCE OF TWO POPULATION FUNCTIONS (OF EQUAL NUMBERS OF OBSERVATIONS) IS EQUAL TO THE SUM OR DIFFERENCE OF THE MEANS OF THE POPULATION FUNCTIONS.

Proof:-

LET $\frac{x_1+x_2+x_3+\ldots+x_n}{n} = \bar{x}$ AND $\frac{y_1+y_2+y_3+\ldots+y_n}{n} = \bar{y}$

THEN THE MEAN OF THEIR SUM,

$$\text{MEAN}(x+y) = \frac{(x_1+y_1)+(x_2+y_2)+\ldots+(x_n+y_n)}{n}$$

$$= \frac{(x_1+x_2+x_3+\ldots+x_n)+(y_1+y_2+y_3+\ldots+y_n)}{n}$$

$$= \frac{(x_1+x_2+x_3+\ldots+x_n)}{n} + \frac{(y_1+y_2+y_3+\ldots+y_n)}{n}$$

$$= \bar{x} + \bar{y} \text{ (THE SUM OF THE MEANS)}$$

EXAMPLE 4 THE MEAN OF 2, 4, 6, 8 IS 5 AND THE MEAN OF 5, 7, 9, 7 IS 7. THEN, THE MEAN OF THE SUM 7, 11, 15, 15 IS 5+7 = 12.

4 THE MEAN OF A CONSTANT TIMES A POPULATION FUNCTION IS EQUAL TO THE CONSTANT TIMES THE MEAN OF THE POPULATION FUNCTION. THAT IS,

IF $\bar{x}$ IS THE MEAN OF THE POPULATION FUNCTION $x_1, x_2, x_3, \ldots, x_n$ AND IF k IS A CONSTANT, THEN THE MEAN OF $kx_1, kx_2, kx_3, \ldots, kx_n$ IS EQUAL TO $k\bar{x}$

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Proof:-

$$\frac{kx_1 + kx_2 + kx_3 + \ldots + kx_n}{n} = \frac{k(x_1 + x_2 + x_3 + \ldots + x_n)}{n} = k\bar{x}$$

EXAMPLE 5 THE MEAN OF 8, 9, 6, 8, 4, IS 7. IF YOU MULTIPLY EACH OF VALUE BY 5, YC WILL OBTAIN 40, 45, 30, 40, 20. THEN THE NEW MEAN IS 5×7 = 35

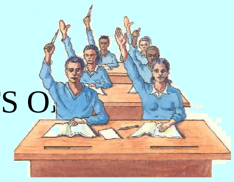
Note:

1. THE MEAN IS UNIQUE.
2. THE MEAN IS AFFECTED BY EXTREME VALUES.

2 The median

THE FOLLOWING ACTIVITY WILL HELP YOU TO REVISE WHAT YOU LEARNED IN PREVIOUS GRADE

ACTIVITY 6.5



1. FIND THE MEDIAN FOR EACH OF THE FOLLOWING SETS O
   - **A** 5, 2, 9, 7, 3
   - **B** 12 , 8, 10, 14, 13, 9
2. WHAT DID YOU OBSERVE ABOUT THE MIDDLE TERM WHEN THE NUMBER OF OBSERVATIONS IS ODD OR EVEN?

A SECOND MEASURE OF LOCATION OF QUANTITATIVE DATA IS THE median

Definition 6.7

The median is the value that lies in the middle of the data when it is arranged in ascending or descending order. So, half the data is below the median and half the data is above the median.

EXAMPLE 6 FIND THE MEDIAN OF EACH OF THE FOLLOWING:

A 6, 7, 9, 7, 11, 13, 15 **B** 27, 23, 36, 38, 27, 40, 45, 39

SOLUTION:

A FIRST ARRANGE THE DATA IN ASCENDING ORDER AS 6, 7, 7, 9, 11, 13, 15

THERE ARE SEVEN VALUES (AN ODD NUMBER OF VALUES) AND THE MIDDLE THE 4TH ELEMENT OF THE LIST WHICH IS 9.

THEREFORE 9 IS THE MEDIAN OF THE DATA.

B FIRST, ARRANGE THE DATA IN ASCENDING ORDER AS 23, 27, 27, 36, 38, 39, 40, 45

THERE ARE EIGHT VALUES (AN EVEN NUMBER). THE TWO MIDDLE VALUES AI 4TH AND 5TH ELEMENTS OF THE LIST WHICH ARE 36 AND 38. THE MEDIAN IS HALF TH OF 36 AND 38. SO, THE MEDIAN IS $\frac{36+38}{2} = 37$.

EXAMPLE 7 FIND THE MEDIAN OF THE FOLLOWING DISTRIBUTION

V	1	2	3	4	5
f	2	3	2	4	2

SOLUTION: THERE ARE 13 DATA VALUES. SO, THE MEDIAN IS THE 7TH PIECE OF DATA, WHICH IS 3.

NOTE THAT THE MEDIAN OF A SET OF DATA WITH VALUES ARRANGED IN ASCENDING ORDER IS:

- **I** THE MIDDLE VALUE OF THE LIST IF THERE IS AN ODD NUMBER OF VALUES.
- **II** HALF OF THE SUM OF THE TWO MIDDLE VALUES IF THERE IS AN EVEN NUMBER OF VALUES.

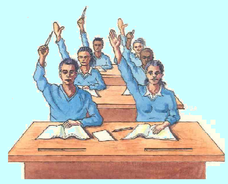
Properties of the median

1. THE MEDIAN CAN BE OBTAINED EVEN WHEN SOME OF THE DATA VALUES ARE NOT
2. IT IS NOT AFFECTED BY EXTREME VALUES.
3. IT IS UNIQUE FOR A GIVEN DATA SET.

3 The mode

THE FOLLOWING ACTIVITY SHOULD HELP YOU TO RECALL WHAT YOU HAVE LEARNT ABOUT MODE PREVIOUSLY.

ACTIVITY 6.6



1. FIND THE MODE(S) OF THE FOLLOWING DATA
   - **A** 5, 7, 8, 7, 9, 11
   - **B** M, F, M, F, F
2. CAN YOU FIND THE MEAN AND MEDIAN FOR THE ABOVE DATA?
3. DISCUSS YOUR OBSERVATION.

A THIRD MEASURE OF LOCATION IS THE mode. THE MODE CAN BE FOUND FOR BOTH QUANTITATIVE AND QUALITATIVE DATA.

Definition 6.8

The value of the variable which occurs most frequently in a data set is called the mode.

EXAMPLE 8 FIND THE MODE OF EACH OF THE FOLLOWING DATA SETS:

- **A** 4, 6, 12, 10, 7
- **B** 12, 10, 11, 13, 10, 14, 12, 18, 17
- **C** 9, 8, 7, 10, 6, 8

SOLUTION:

A IT HAS NO MODE BECAUSE EACH VALUE OCCURS ONLY ONCE.

B THE VALUES 10 AND 12 BOTH OCCUR TWICE, WHILE THE OTHERS OCCUR ONLY IT HAS TWO MODES AND THE DATA IS A BIMODAL.

C 8 IS THE MODE BECAUSE IT OCCURRED TWICE (MOST FREQUENTLY).

EXAMPLE 9 FIND THE MEAN, MEDIAN AND MODE OF THE FOLLOWING DISTRIBUTI TEMPERATURES IN A CERTAIN TOWN FOR ONE MONTH.

Temperature in $^oC(V)$	20	21	23	24	26	28
Number of days(f)	2	4	5	9	3	7

SOLUTION: MEAN $\bar{x} = \dfrac{(20\times 2)+(21\times 4)+(23\times 5)+(24\times 9)+(26\times 3)+(28\times 7)}{2+4+5+9+3+7}$

$$= \frac{40+84+115+216+78+196}{30} = \frac{729}{30} = 24.3$$

THEREFORE, THE MEAN IS 24.3ºC.

THE NUMBER OF OBSERVATIONS IS AN EVEN NUMBER WHICH IS 30. SO, THE MEDIA THE SUM OF THE 15^{TH} AND 16^{TH} VALUES.

I.E., MEDIAN $= \dfrac{15^{TH}\text{ VALUE} + 16^{TH}\text{ VALUE}}{2} = \dfrac{24+24}{2} = 24$

THEREFORE, THE MEDIAN IS 24ºC.

THE VALUE WITH HIGHEST FREQUENCY IS THE NUMBER 24. THEREFORE, THE MODE

NOTE THAT A SET OF DATA CAN HAVE NO MODE, ONE MODE (unimodal), TWO MODES (bimodal) OR MORE THAN TWO MODES (multimodal). IF THERE IS NO OBSERVATION THAT OCCURS WITH T HIGHEST FREQUENCY, WE SAY THE DATA HAS no mode

Properties of the Mode

1. THE MODE IS NOT ALWAYS UNIQUE.
2. IT IS NOT AFFECTED BY EXTREME VALUES.
3. THE MODE CAN ALSO BE USED FOR QUALITATIVE DATA.

Exercise 6.3

1 **A** FIND THE MEAN, MODE AND MEDIAN OF THE FOLLOWING DATA.

11, 9,14, 3, 11, 4, 10, 21, 8, 15, 350

B WHICH MEASURE OF LOCATION IS PREFERABLE FOR THIS DATA?

2 GIVEN BELOW IS A FREQUENCY DISTRIBUTION OF VALUES V.

A FIND THE MEAN, MODE AND MEDIAN OF THE FOLLOWING DISTRIBUTION.

B HOW MANY OF THE VALUES ARE NON-NEGATIVE?

V	–2	–3	0	1	2	3
f	3	2	3	6	5	1

3 GIVEN THE NUMBERS 5, 6, 7, 10, 12, WHICH NUMBER MUST BE REMOVED IN ORDER T MAKE THE MEAN OF THE RESULTING VALUES 7.5?

4 GIVEN THE NUMBERS 10, 12, 19, 15, 8, WHAT NUMBER COULD BE INCLUDED SO THAT THE MEDIAN IS 11? (EXPLAIN)

5 GIVEN 3, 4, X, 5, Y, 12. FIND THE VALUES OF x AND y IF THE MODE OF THE DATA IS 3 AND THE MEAN IS 6.

6 IF THE MEAN OF a, b, c, d IS k, THEN WHAT IS THE MEAN OF

A $a + b,\ 2b,\ c + b,\ d + b$? **B** $ab,\ b^2,\ cb,\ db$?

7 CALCULATE THE MEAN, MEDIAN AND MODE OF THE FOLLOWING DATA;

Value	10	15	20	25	30	35	40
Frequency	15	10	50	4	10	8	3

8 IN A SURVEY OF THE NUMBER OF OCCUPANTS OF CARS, THE FOLLOWING DATA RES

Number of occupants	1	2	3	4
Number of cars	7	11	7	x

A IF THE MEAN NUMBER OF OCCUPANTS IS $2\frac{1}{3}$, FIND x.

B IF THE MODE IS 2, FIND THE LARGEST POSSIBLE VALUE OF x.

C IF THE MEDIAN IS 2, FIND THE LARGEST POSSIBLE VALUE OF x

9 A RESEARCHER TABULATED THE NUMBER OF CASES HEARD BY 8 JUDGES ON A GIV COURT AND FOUND THE FOLLOWING DATA:

Judges	1	2	3	4	5	6	7	8
Count of cases	6	3	1	2	0	5	5	4

A FIND THE MEAN, MEDIAN AND MODE.

B THE RESEARCHER REPORTED THAT OVER HALF OF THE JUDGES HEARD ABOVE WHAT DOES THE RESEARCHER MEAN BY THE "AVERAGE"?

10 THE FOLLOWING RAW DATA REPRESENTS THE NUMBER OF HIV/AIDS PATIENTS WAI COUNSELLING AT 8:00 AM ON 40 CONSECUTIVE SATURDAYS AT A CERTAIN HOSPITA

11	6	5	8	11	6	3	7	4	6
5	4	13	14	9	11	13	8	10	9
10	9	6	5	10	7	8	7	8	3
8	7	8	9	6	10	11	8	8	4

A DRAW A FREQUENCY DISTRIBUTION TABLE.

B CALCULATE THE MEAN, MEDIAN AND MODAL NUMBER OF HIV/AIDS PATIENTS

C DRAW A HISTOGRAM.

11 IN A MATHEMATICS TEST THE SCORES FOR BOYS WERE 7, 8, 7, 5 AND 6 AND THE SCORES FOR GIRLS WERE 6, 3, 9, 8, 2, 2, 5, 7,3

A FIND THE MEAN SCORE FOR THE BOYS.

B FIND THE MEAN SCORE FOR THE GIRLS.

C FIND THE MEAN SCORE FOR BOTH THE BOYS AND GIRLS.

D WHAT DO YOU CONCLUDE?

12 THE MODE OF SOME DATA IS 20. IF EACH VALUE IN THE DATA IS INCREASED BY 2, W BE THE MODE OF THE NEW DATA?

13 FIND THE MEAN, MEDIAN AND MODE OF THE DATA REPRESENTED BY THE HISTOGR

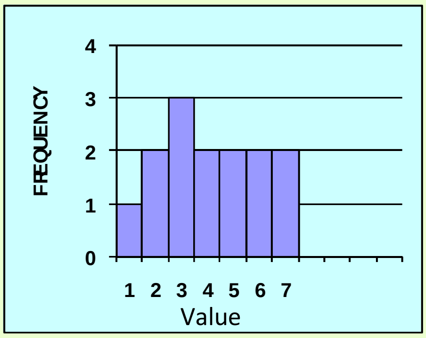


Figure 6.3

14 AN AGRICULTURAL DEVELOPMENT STATION SELLS SEEDLINGS OF PLANT THROUGH CLAIMS THAT THE AVERAGE HEIGHT OF THE PLANTS AFTER ONE YEAR'S GROWTH A SAMPLE OF 24 OF THE PLANTS WERE MEASURED AFTER ONE YEAR WITH THE FOL RESULTS (IN CM).

6	7	7	9	34	56	85	89	89	90	90	91
91	92	93	93	94	95	95	96	97	97	99	93

A FIND THE MEAN AND THE MEDIAN HEIGHT OF THE SAMPLE.

B IS THE STATION'S CLAIM ABOUT AVERAGE HEIGHT JUSTIFIED?

15 IN ORDER TO RECEIVE A GRADE OF A IN HER MATHEMATICS EXAM, ABEBA NEEDS A MEAN SCORE OF 90 AND ABOVE ON 4 TESTS. SO FAR ABEBA HAD SCORED 80, 91 AND 93 ON WHAT IS THE LOWEST SCORE THAT SHE MUST GET IN HER LAST TEST IN ORDER GRADE OF A?

6.1.4 Measures of Dispersion for Ungrouped Data

WHEN COMPARING SETS OF DATA, IT IS USEFUL TO HAVE A WAY OF MEASURING TH
SPREAD OF THE DATA.

Group Work 6.2

CONSIDER THE FOLLOWING THREE SETS OF DATA.

Group	Values							Total	Mean	Mode	Median
A	7	7	7	7	7	7	7				
B	4	5	6	7	7	9	1				
C	1	7	12	7	2	19	1				

A COMPLETE THE TABLE BY FINDING THE SUM OF EACH GROUP AND THE MEAN, AND MODE.

B ARE THE MEANS EQUAL? ARE THE MODES EQUAL? ARE THE MEDIANS THE SAM

C COMPARE THE VARIATION OF EACH GROUP?

- **I** WHICH GROUP SHOWS MOST VARIATION?
- **II** WHICH GROUP SHOWS NO VARIATION?
- **III** WHICH GROUP SHOWS SLIGHT VARIATION?

D COMPARE THE DIFFERENCE BETWEEN THE MEAN AND EACH OBSERVED VALU GROUP A, B AND C.

- **I** IN WHICH GROUP IS THE MEAN CLOSEST TO EACH VALUE?
- **II** IN WHICH GROUP IS THE DIFFERENCE BETWEEN THE MEAN AND EACH DA VALUE THE LARGEST?

E CALCULATE THE RANGE FOR EACH GROUP.

Dispersion OR Variation IS THE SCATTER (OR SPREAD) OF DATA VALUES FROM A MEASU CENTRAL TENDENCY.

THERE ARE SEVERAL MEASURES OF DISPERSION THAT CAN BE CALCULATED FOR A SET SECTION, WE WILL CONSIDER ONLY THREE OF THEM, NAMELY, range, variance, AND THE standard deviation.

1 Range

THE SIMPLEST AND THE MOST CRUDE MEASURE OF DISPERSION OF QUANTITATIVE DATA

Definition 6.9

The range R of a set of numerical data is the difference between the highest and the lowest values. i.e.,

Range = Highest value – Lowest value

EXAMPLE 1 THE AGES OF SIX STUDENTS ARE 24, 20, 18, 13, 16, 15 YEARS, RESPECTIVELY. WHAT IS THE RANGE?

SOLUTION: RANGE = HIGHEST VALUE – LOWEST VALUE = 24 – 13 = 11 YEARS.

EXAMPLE 2 FIND THE RANGE OF THE DISTRIBUTION GIVEN IN THE TABLE BELOW.

V	2	8	9	13	15	18
f	3	4	2	1	5	4

SOLUTION: THE MAXIMUM VALUE IS 18 AND THE MINIMUM VALUE IS 2.

RANGE = MAXIMUM VALUE – MINIMUM VALUE = 18 – 2 = 16

2 Variance (^{2})

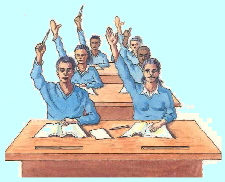
Definition 6.10

Variance, denoted by (^{2}), is defined as the mean of the squared deviations of each value from the arithmetic mean.

3 Standard deviation ( )

THE FOLLOWING ACTIVITY WILL HELP YOU TO LEARN THE STEPS USED TO FIND VARIANCE AND DEVIATION.

ACTIVITY 6.7



CONSIDER THE FOLLOWING DATA SET:

2, 3, 10, 6, 9

A FIND THE MEAN $\bar{x}$.

B FIND THE DEVIATION OF EACH DATA VALUE FROM THE MEAN ($x - \bar{x}$).

C SQUARE EACH OF THE DEVIATIONS ($(x - \bar{x})^2$)

D FIND THE MEAN OF THESE SQUARED DEVIATIONS AND ITS PRINCIPAL SQUARE ROO

THE standard deviation IS THE MOST VALUABLE AND WIDELY USED MEASURE OF DISPERSI

Definition 6.11

Standard deviation, denoted by , is defined as the positive square root of the mean of the squared deviations of each value from the arithmetic mean.

The actual method of calculating can be summarized in the following steps:

Step 1 FIND THE ARITHMETIC MEAN $\bar{x}$ OF THE DISTRIBUTION.

Step 2 FIND THE DEVIATION OF EACH DATA VALUE FROM THE MEAN $(x-\bar{x})$.

Step 3 SQUARE EACH OF THESE DEVIATIONS, $(x-\bar{x})^2$.

Step 4 FIND THE MEAN OF THESE SQUARED DEVIATIONS. THIS VALUE IS CALLED variance AND IS DENOTED BY ^{2}.

Step 5 TAKE THE PRINCIPAL SQUARE ROOT OF ^{2}, I.E.,

STANDARD DEVIATION = $\sqrt{\text{VARIANCE}}$

EXAMPLE 3 FIND THE VARIANCE ^{2} AND THE STANDARD DEVIATION FOR THE FOLLOWING DATA:

3, 5, 8, 11, 13

SOLUTION:

x	$(x-\bar{x})$	$(x-\bar{x})^2$
3	−5	25
5	−3	9
8	0	0
11	3	9
13	5	25
TOTAL 40		68

$$\text{VARIANCE } (^2) = \frac{68}{5} = 13.6$$

$$\text{Standard deviation } (\ ) = \sqrt{\ ^2} = \sqrt{13.6} \approx 3.7$$

EXAMPLE 4 FIND THE VARIANCE AND STANDARD DEVIATION OF THE POPULATION WHOSE DISTRIBUTION IS GIVEN IN THE FOLLOWING TABLE.

V	2	3	5	6	8
f	3	4	4	5	4

SOLUTION: FIRST, THE MEAN HAS TO BE CALCULATED.

$$\bar{x} = \frac{3\times2+4\times3+4\times5+5\times6+4\times8}{3+4+4+5+4} = \frac{100}{20} = 5$$

x	f	xf	$x-\overline{x}$	$(x-\overline{x})^2$	$f(x-\overline{x})^2$
2	3	6	-3	9	27
3	4	12	-2	4	16
5	4	20	0	0	0
6	5	30	1	1	5
8	4	32	3	9	36
Total	20	100	0		84

VARIANCE $(\sigma^2) = \frac{84}{20} = 4.2$

STANDARD DEVIATION $(\sigma) = \sqrt{} \approx$ 4

THEREFORE, THE POPULATION VARIANCE AND STANDARD DEVIATION ARE 4.2 AND 2.05

Properties of variance and standard deviation

Group Work 6.3

CONSIDER THE FOLLOWING DATA WHICH SHOWS THE SUGAR IN KILOGRAMS SOLD BY A SMALL SHOP FOR FIVE DAYS.

6, 4, 8, 9, 3

I FIND THE MEAN.

II FIND THE VARIANCE AND STANDARD DEVIATION.

III IN THE NEXT FIVE DAYS, IF THE DAILY SALES EXACTLY TWICE,

- **A** FIND THE MEAN OF SALES FOR THE NEXT FIVE DAYS.
- **B** FIND THE VARIANCE AND STANDARD DEVIATION FOR THE NEXT FIVE DAYS.
- **C** COMPARE YOUR ANSWERS ABOVE WITH THOSE OBTAINED IN A AND B ABOVE.
- **D** DISCUSS THE COMPARISON YOU DID ABOVE.

IV IF THE DAILY SALES GIVEN FOR THE FIRST FIVE DAYS ARE DOUBLED, I.E. IF THE DAILY SALES WERE 12, 8, 16, 18 AND 6,

- **A** FIND THE MEAN, VARIANCE AND STANDARD DEVIATION.
- **B** COMPARE THE ABOVE RESULT WITH THOSE OBTAINED IN I AND II AND DISCUSS THE RESULTS.

THE ABOVE GROUP WORK WILL HELP YOU TO OBSERVE THE FOLLOWING PROPERTIES.

1 IF A CONSTANT c IS ADDED TO EACH VALUE OF A POPULATION FUNCTION, THEN THE NEW VARIANCE IS THE SAME AS THE OLD VARIANCE. THE NEW STANDARD DEVIATION IS THE SAME AS THE OLD STANDARD DEVIATION.

Proof:-

LET $x_1, x_2, x_3, \ldots, x_n$ BE n OBSERVATIONS WITH MEAN $\bar{x}$ AND VARIANCE ^{2}.

ADDING c: $x_1+c,\ x_2+c,\ x_3+c,\ \ldots,\ x_n+c$. THEN THE NEW MEAN IS $\bar{x}+c$.

$$\text{NEW VARIANCE} = \frac{\left[(x_1+c)-(\bar{x}+c)\right]^2+\left[(x_2+c)-(\bar{x}+c)\right]^2+\ldots+\left[(x_n+c)-(\bar{x}+c)\right]^2}{n}$$

$$= \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{n} = {}^2.$$ (*The original variance*)

AND THE NEW STANDARD DEVIATION IS $\sqrt{\text{VARIANCE}} = \sqrt{\ ^2} =$ (*the original standard deviation*)

EXAMPLE 5 GIVEN 1, 2, 6, 3

A FIND THE VARIANCE **B** FIND THE STANDARD DEVIATION.

C ADD 2 TO EACH VALUE AND FIND THE VARIANCE AND STANDARD DEVIATION OF THE RESULTING NUMBERS

SOLUTION: $\bar{x} = \dfrac{1+2+6+3}{4} = 3$

$x-\bar{x}: -2, -1, 3, 0$ AND $(x-\bar{x})^2: 4, 1, 9, 0$

A $^2 = \dfrac{4+1+9+0}{4} = \dfrac{14}{4} = 3.5$

B $= \sqrt{3.5} \approx 1.87$

C ADDING 2: 3, 4, 8, 5

NEW MEAN: $\bar{x} = \dfrac{3+4+8+5}{4} = \dfrac{20}{4} = 5 = 3+2$

$x-\bar{x}: -2, -1, 3, 0$ AND $(x-\bar{x})^2: 4, 1, 9, 0$

NEW: $^2 = \dfrac{4+1+9+0}{4} = \dfrac{14}{4} = 3.5$

NEW $= \sqrt{3.5} \approx 1.87$

THEREFORE, THE OLD VARIANCE = THE NEW VARIANCE

THE OLD STANDARD DEVIATION = THE NEW STANDARD DEVIATION.

2 IF EACH VALUE OF A POPULATION FUNCTION IS MULTIPLIED BY A CONSTANT k, THEN

I THE NEW VARIANCE IS k^2 TIMES THE OLD VARIANCE

II THE NEW STANDARD DEVIATION IS $|k|$ TIMES THE OLD STANDARD DEVIATION.

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Proof:-

CONSIDER $x_1, x_2, \ldots, x_n$ WHOSE MEAN IS $\bar{x}$ AND VARIANCE IS s^2.

MULTIPLYING EACH DATA VALUE BY c GIVES US A NEW MEAN OF $c\bar{x}$.

$$\text{THEN, NEW VARIANCE} = \frac{(cx_1 - c\bar{x})^2 + (cx_2 - c\bar{x})^2 + (cx_3 - c\bar{x})^2 + \ldots + (cx_n - c\bar{x})^2}{n}$$

$$= \frac{c^2[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2]}{n}$$

$$= c^2 \times \text{THE OLD VARIANCE} = c^2 s^2$$

THEREFORE, NEW STANDARD DEVIATION = $\sqrt{c^2 s^2} = |c|s$

Exercise 6.4

1 FIND THE RANGE, VARIANCE AND STANDARD DEVIATION OF THE FOLLOWING DATA.

4, 2, 3, 3, 2, 1, 4,3, 2, 6

2 FIND THE RANGE, VARIANCE AND STANDARD DEVIATION OF THE DISTRIBUTION IN THE TABLE BELOW.

V	–1	–2	0	1	2
f	2	1	3	3	1

3 FIND THE RANGE, VARIANCE AND STANDARD DEVIATION OF THE HISTOGRAM IN THE FIGURE BELOW.

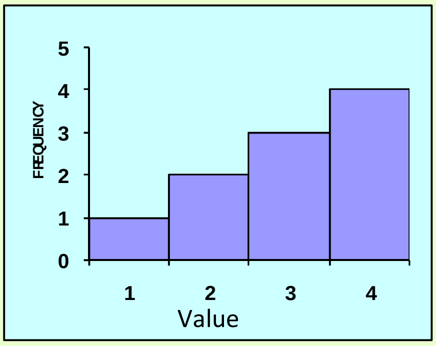


Figure 6.4

4 WHAT IS THE VALUE OF y, IF THE STANDARD DEVIATION OF 8, 8, 8, 8, y, 8 IS 0?

5 IF THE VARIANCE OF a, b, c, d IS k, THEN WHAT IS

A THE VARIANCE OF $a + c, b + c, 2c, d + c$?

B THE STANDARD DEVIATION OF $a + c, b + c, 2c, d + c$?

C THE VARIANCE OF ac, bc, c^2, dc?

D THE STANDARD DEVIATION OF ac, bc, c^2, dc?

6 IF A POPULATION FUNCTION x HAS MEAN $M(x) = 2$ AND $M(x^2) = 8$, FIND ITS STANDARD DEVIATION.

6.2 PROBABILITY

"The true logic of this world is the calculus of probabilities". James Clerk Maxwell

HISTORICAL NOTE:

The first inquiry into the science of Probability was made by Girolamo Cardano (1501-1576), an Italian physician and mathematician. Cardano predicted the date of his own death. Since he was healthy at the end of the day, he poisoned himself to make his prediction come true!

IN YOUR GRADE 8 LESSONS, YOU HAVE DISCUSSED THE WORD PROBABILITY. YOU OFTEN USE IT AS "*The probability of winning a game is low*", OR *there is a high probability that it will rain today*", ETC. IN THESE TWO SENTENCES, THE WORD PROBABILITY DESCRIBES ESTIMATES OF THE POSSIBILITIES.

PROBABILITY IS A NUMERICAL VALUE THAT DESCRIBES THE LIKELIHOOD OF THE OC EVENT IN AN EXPERIMENT.

THE FOLLOWING GROUP WORK WILL HELP YOU RECALL WHAT YOU HAVE LEARNED ON GRADE 8.

Group Work 6.4

ABEL THROWS A FAIR DIE ONCE. BASED ON THIS EXPERIMENT, DISCUSS THE FOLLOWING:

1 IS IT POSSIBLE TO PREDICT THE NUMBER THAT WILL SHOW UP ON THE UPPER FACE OF THE DIE? WHY?

2 LIST THE SET OF ALL POSSIBLE OUTCOMES.

3 WRITE AN EXAMPLE OF AN EVENT FROM THE EXPERIMENT.

4 WHAT CAN YOU SAY ABOUT THE FOLLOWING EVENTS?

- **I** THE NUMBER ON THE UPPER FACE OF THE DIE IS SEVEN
- **II** THE NUMBER ON THE UPPER FACE OF THE DIE IS AN INTEGER.
  - **A** WHICH OF THE ABOVE EVENTS IS CERTAIN?
  - **B** WHICH OF THE ABOVE EVENTS IS IMPOSSIBLE?

5 DETERMINE THE PROBABILITIES OF THE FOLLOWING EVENTS.

- **A** THE NUMBER ON THE UPPER FACE OF THE DIE IS 2.
- **B** THE NUMBER ON THE UPPER FACE OF THE DIE IS 7.
- **C** THE NUMBER ON THE UPPER FACE OF THE DIE IS LESS THAN 7.

6 DISCUSS THE FOLLOWING TERMS.

A EXPERIMENT **B** POSSIBILITY SET **C** EVENT
D IMPOSSIBLE EVENT **E** CERTAIN EVENT

Definition 6.12

An experiment is a trial by which an observation is obtained but whose outcome cannot be predicted in advance.

Experimental probability

PROBABILITY DETERMINED USING DATA COLLECTED FROM REPEATED EXPERIMENTS IS CALLED EXPERIMENTAL PROBABILITY.

EXAMPLE 1 THE NUMBERS 1 TO 20 ARE EACH WRITTEN ON 20 IDENTICAL CARDS. ONE CARD IS CHOSEN AT RANDOM.

A LIST THE SET OF ALL POSSIBLE OUTCOMES.

B LIST THE ELEMENTS OF THE FOLLOWING EVENTS:

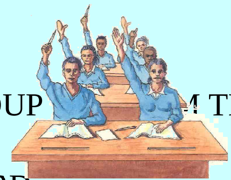
- **I** THE NUMBER IS LESS THAN 5.
- **II** THE NUMBER IS GREATER THAN 15.
- **III** THE NUMBER IS GREATER THAN 21.
- **IV** THE NUMBER IS DIVISIBLE BY 5.
- **V** THE NUMBER IS A PRIME.

SOLUTION:

A S = {1, 2, 3, …, 19, 20}

B
- **I** {1, 2, 3, 4}
- **II** {16, 17, 18, 19, 20}
- **III** { } OR ∅ SINCE NO CARD HAS A NUMBER GREATER THAN 20.
- **IV** {5, 10, 15, 20}
- **V** {2, 3, 5, 7, 11, 13, 17, 19}

ACTIVITY 6.8



ARRANGE YOURSELVES INTO GROUPS OF 5. LET EACH GROUP PERFORM THE FOLLOWING ACTIVITIES.

1 TAKE A COIN, TOSS IT 5 TIMES, 10 TIMES AND 15 TIMES AND RECORD YOUR OBSERVATIONS IN THE FOLLOWING TABLE.

	Number of tosses			Total
Number of times a coin is tossed	5	10	15	
Number of times the coin shows up Heads				
Number of times the coin shows Tails				

WHAT PROPORTION OF THE NUMBER OF TOSSES SHOWS HEADS? A TAILS? WHAT IS THE PROBABILITY THAT THE OUTCOME IS HEAD?

2 THROW A DIE 20 TIMES. RECORD THE OBSERVATION AND COMPLETE THE FOLLOWING TABLE.

Number on the upper face of the die	1	2	3	4	5	6
Number of times it shows up						

A FIND THE NUMBER OF TIMES 3 IS ON THE UPPER FACE OF THE DIE.

B FIND THE NUMBER OF TIMES 6 IS ON THE UPPER FACE OF THE DIE.

C FIND THE NUMBER OF TIMES 7 IS ON THE UPPER FACE OF THE DIE.

D WRITE THE PROPORTION OF EACH NUMBER.

E WHAT IS THE PROBABILITY THAT THE NUMBER THAT SHOWS ON THE UPPER FACE OF THE DIE IS 4?

SUPPOSE WE TOSS A COIN 100 TIMES AND GET A HEAD 45 TIMES, AND A TAIL 55 TIMES. T

WOULD SAY THAT IN A SINGLE TOSS OF A COIN, THE PROBABILITY OF GETTING A HEAD $\frac{45}{100} = \frac{9}{20}$

AGAIN SUPPOSE WE TOSS A COIN 500 TIMES AND GET A HEAD 260 TIMES, AND A TAIL 24
THEN WE SAY THAT IN A SINGLE TOSS OF A COIN, THE PROBABILITY OF GETTING
$\frac{260}{500} = \frac{13}{25}$. SO FROM VARIOUS EXPERIMENTS, WE MIGHT OBTAIN DIFFERENT PROBABILITIES FOR THE SAME EVENT. HOWEVER, IF AN EXPERIMENT IS REPEATED A SUFFICIENTLY LARGE NUM
THE RELATIVE FREQUENCY OF AN OUTCOME WILL TEND TO BE CLOSE TO THE THEORE
OF THAT OUTCOME.

Definition 6.13

The possibility set (or sample space) for an experiment is the set of all possible outcomes of the experiment.

EXAMPLE 2

A GIVE THE SAMPLE SPACE FOR TOSSING A COIN.

B WHAT IS THE SAMPLE SPACE FOR THROWING A DIE?

SOLUTION:

A WHEN WE TOSS A COIN THERE ARE ONLY TWO POSSIBLE OUTCOMES: HEADS (H) OR TAILS (T). SO S = {H,T}.

B WHEN WE THROW A DIE THE SCORE CAN BE ANY OF THE NUMBERS 1, 2, 3, 4, 5, 6, SO S = {1,2,3,4,5,6}.

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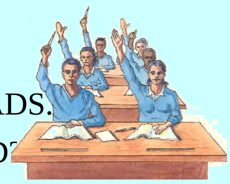
Definition 6.14

An event is a subset of the possibility set (sample space).

ACTIVITY 6.9



SUPPOSE WE TOSS A COIN 1000 TIMES AND OBTAIN 495 HEADS.

A HOW MANY TIMES WAS THE EXPERIMENT PERFORMED?

B IF OUR EVENT IS HEADS, HOW MANY TIMES DOES THIS EVENT OCCUR?

C WHAT IS THE PROBABILITY OF HEADS BASED ON THE RESULT OF THIS EXPERIMENT?

Definition 6.15

If an experiment has n equally likely outcomes and if m of these represent a particular event, then the probability of this event occurring is $\frac{m}{n}$.

EXAMPLE 3 IN AN EXPERIMENT OF SELECTING STUDENTS AT RANDOM, A TEACHER FOUND THE FOLLOWING RESULT AFTER 50 TRIALS.

Student	BOY	GIRL	TOTAL
Number	20	30	50

WHAT IS THE PROBABILITY THAT A RANDOMLY SELECTED STUDENT IS A GI

SOLUTION THE PROBABILITY THAT A RANDOMLY SELECTED STUDENT WILL BE A GIRL IS THE RATIO OF THE NUMBER OF GIRLS TO THE TOTAL NUMBER OF TRIALS.

$$P(\text{A GIRL WILL BE SELECTED}) = \frac{30}{50} = \frac{3}{5}$$

IN DECIMAL FORM THE PROBABILITY IS 0.6.

A TREE DIAGRAM IS ONE WAY OF SHOWING THE POSSIBLE OUTCOMES OF A I EXPERIMENT.

EXAMPLE 4 IN AN EXPERIMENT OF TOSSING TWO COINS,

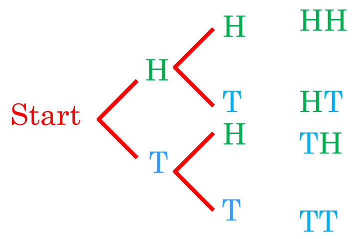
A WHAT ARE THE POSSIBLE OUTCOMES?

B HOW MANY DIFFERENT POSSIBLE OUTCOMES ARE THERE?

C WHAT IS THE PROBABILITY OF THE COINS LANDING WITH

I TWO HEADS? **II** TWO TAILS? **III** ONE HEAD?

SOLUTION: USING A TREE DIAGRAM, WE GET



A THE SET OF POSSIBLE OUTCOMES IS S = {HH, HT, TH, TT}.

B THERE ARE 4 POSSIBLE OUTCOMES.

C **I** THE EVENT TWO HEADS = {HH} HAS ONE MEMBER, SO

$$\text{P (TWO HEADS)} = \frac{1}{4}$$

II $\text{P (TWO TAILS)} = \frac{1}{4}$

III THE EVENT ONE HEAD = {HT, TH} HAS TWO MEMBERS, SO

$$\text{P (ONE HEAD)} = \frac{2}{4} = \frac{1}{2}$$

IN REAL SITUATIONS, IT MIGHT NOT ALWAYS BE POSSIBLE TO PERFORM AN EXPE CALCULATE PROBABILITY. IN SUCH SITUATIONS, WE NEED TO DEVELOP ANOTHER AP THE PROBABILITY OF AN EVENT.

IN THE NEXT SECTION, YOU WILL DISCUSS A THEORETICAL APPROACH OF FINDING PROB

Theoretical probability of an event

Definition 6.16

THE THEORETICAL PROBABILITY OF AN EVENT E, WRITTEN AS $P(E)$, IS DEFINED AS FOLLOWS:

$$P(E) = \frac{\text{NUMBER OF OUTCOMES FAVOURABLE TO THE EVENT } E}{\text{TOTAL NUMBER OF POSSIBLE OUTCOMES } S}$$

YOU CAN WRITE THE PROBABILITY OF AN EVENT AS A FRACTION, A DECIMAL, OR A PER

EXAMPLE 5 A FAIR COIN IS TOSSED ONCE. WHAT IS THE PROBABILITY OF GETTING A HEAD?

SOLUTION:

S = { H, T}

E = {H}

$$\text{P (HEAD)} = \frac{n(E)}{n(S)} = \frac{1}{2} = 0.5$$

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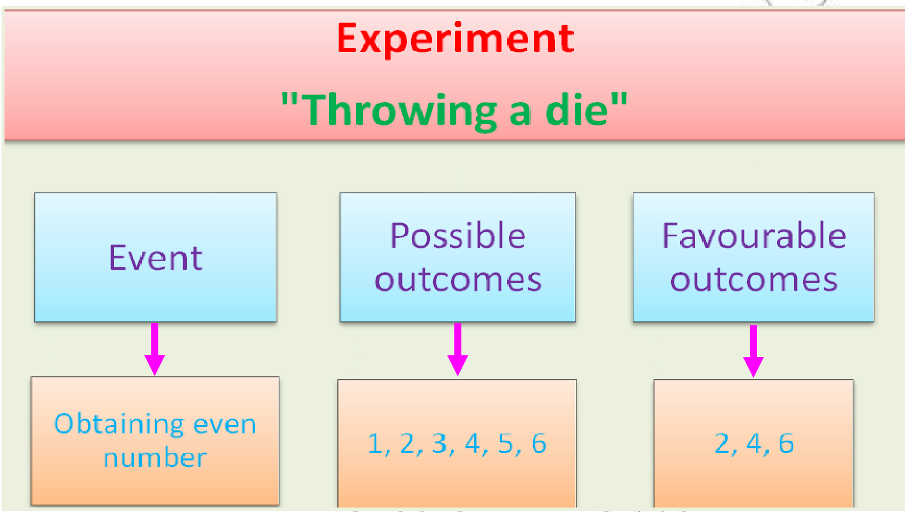
EXAMPLE 6 IF WE THROW A DIE ONCE, WHAT IS THE PROBABILITY THAT AN EVEN NUMBER WILL SHOW ON THE UPPER FACE OF THE DIE?

SOLUTION:

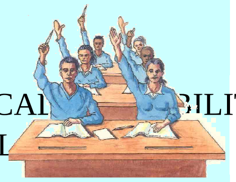
S = {1, 2, 3,4, 5, 6}

E = {2, 4, 6}

$$P(\text{EVEN}) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$



ACTIVITY 6.10



WE ARE GOING TO INVESTIGATE WHETHER THE THEORETICAL ...ILITY OF A COIN LANDING ON HEADS IS BACKED UP BY EXPERIMENTAL ...

A TOSS A COIN 10 TIMES, 20 TIMES, 30 TIMES . . . .

B KEEP A RECORD OF YOUR RESULTS,

Number of throws	Number of heads
10	
20	
30	

C FOR EACH ROW IN THE TABLE, WHAT PROPORTION OF THE THROWS LANDED AS HEADS?

HOW DO YOUR ANSWERS COMPARE WITH $P(\text{HEAD}) = \frac{1}{2}$ (THE THEORETICAL PROBABILITY)?

> **Definition 6.17**
>
> Let S be the possibility set of an experiment and each element of S be equally likely to occur. Then the probability of the event E occurring, denoted by $P(E)$, is defined as:
>
> $$P(E) = \frac{\text{NUMBER OF ELEMENTS IN } E}{\text{NUMBER OF ELEMENTS IN } S} = \frac{n(E)}{n(S)}$$

EXAMPLE 7 A DIE IS THROWN ONCE. WHAT IS THE PROBABILITY THAT THE NUMBER APPEARING WILL BE

A 3? **B** A NUMBER LESS THAN 5?

SOLUTION: THERE ARE SIX POSSIBLE OUTCOMES: {1, 2, 3, 4, 5, 6}. HENCE, $n(S)$ = 6.

A ONLY ONE OF THESE OUTCOMES IS 3. HENCE THE PROBABILITY THAT 3 WILL BE ON THE UPPER FACE OF THE DIE IS $\frac{1}{6}$.

B {1, 2, 3, 4} IS THE REQUIRED SET, WHICH HAS FOUR ELEMENTS. HENCE THE PROBABILITY IS $\frac{4}{6} = \frac{2}{3}$.

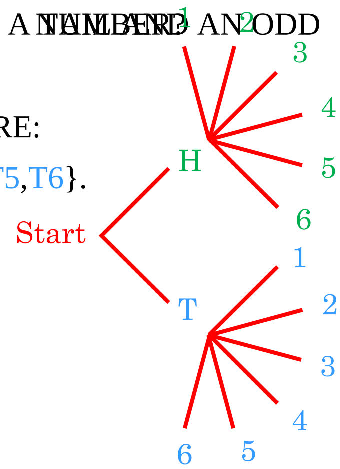
EXAMPLE 8 A DIE AND A COIN ARE TOSSED TOGETHER.

A SKETCH A TREE DIAGRAM SHOWING THE OUTCOMES OF THIS EXPERIMENT.

B WHAT IS THE PROBABILITY OF GETTING A HEAD AND AN EVEN NUMBER?

C WHAT IS THE PROBABILITY OF GETTING A TAIL AND AN ODD NUMBER?

SOLUTION:



A THE OUTCOMES OF THIS EXPERIMENT ARE:

S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

SO, $n(S)$ = 12

B E_1 = {H2, H4, H6}. HENCE $P(E_1) = \frac{3}{12} = \frac{1}{4}$.

C E_2 = {T1, T3, T5}. HENCE $P(E_2) = \frac{3}{12} = \frac{1}{4}$.

EXAMPLE 9 USE A TREE DIAGRAM TO LIST THE SAMPLE SPACE (POSSIBILITY SET) SHOWING THE POSSIBLE ARRANGEMENT OF BOYS AND GIRLS IN A FAMILY WITH EXACTLY THREE CHILDREN.

A WHAT IS THE PROBABILITY THAT ALL THREE CHILDREN ARE BOYS?

B WHAT IS THE PROBABILITY THAT TWO CHILDREN ARE BOYS AND ONE IS A GIRL?

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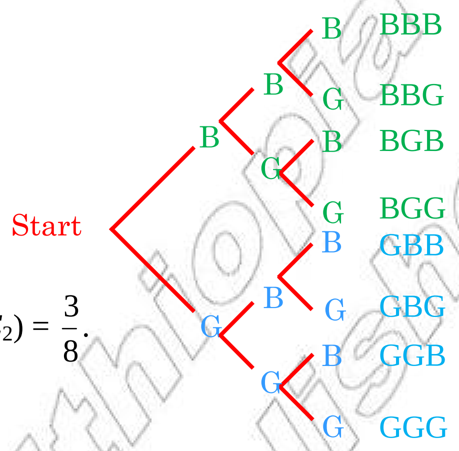
C WHAT IS THE PROBABILITY THAT NONE OF THE CHILDREN IS A BOY?

D WHAT IS THE PROBABILITY THAT AT LEAST ONE OF THE CHILDREN IS A GIRL?

E WHAT IS THE PROBABILITY THAT ALL THREE CHILDREN ARE OF THE SAME SEX?

SOLUTION:



S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

THUS, $n(S) = 8$.

A E_1 = {BBB}. HENCE $P(E_1) = \frac{1}{8}$.

B E_2 = {BBG, BGB, GBB}. HENCE $P(E_2) = \frac{3}{8}$.

C E_3 = {GGG}. HENCE $P(E_3) = \frac{1}{8}$.

D E_4 = {BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

HENCE $P(E_4) = \frac{7}{8}$.

(ALTERNATIVELY, HAVING AT LEAST ONE GIRL IS ALL OUTCOMES EXCEPT BBB

I.E., 8 – 1 = 7 OUTCOMES, GIVING THE SAME RESULT, $P(E_4) = \frac{7}{8}$).

E E_5 = {BBB, GGG}. HENCE, $P(E_5) = \frac{2}{8} = \frac{1}{4}$.

Note: FOR ANY EVENT E,

- ✓ $0 \leq P(E) \leq 1$.
- ✓ IF $P(E) = 1$ THEN THE EVENT IS certain
- ✓ IF $P(E) = 0$, THEN THE EVENT IS impossible

FOR EXAMPLE, IF A BALL IS TAKEN FROM A BOX CONTAINING ONLY RED BALLS, THEN

P (BALL IS RED) = 1 AND P (BALL IS BLACK) = 0.

Exercise 6.5

1 TWO DICE ARE SIMULTANEOUSLY THROWN ONCE. IN WHICH OF THE FOLLOWING EVENTS CAN OCCUR.

A A= THE SAME NUMBER IS SHOWN ON EACH DIE.

B B = THE SUM OF THE NUMBERS IS 13.

C C = THE PRODUCT OF THE TWO NUMBERS IS 1.

D D = THE QUOTIENT OF THE TWO NUMBERS IS 7.

2 THREE COINS ARE TOSSED AT THE SAME TIME. SKETCH A TREE DIAGRAM FOR THE OUTCOMES OF THIS EXPERIMENT. WHAT IS THE POSSIBILITY SET?

3 A BAG CONTAINS FOUR RED BALLS AND THREE BLACK BALLS. WHAT IS THE POSSIBILITY SET FOR COLOUR, IF 2 BALLS ARE CHOSEN AT RANDOM?

4 TOSS A COIN AND KEEP A RECORD OF WHETHER IT LANDS ON HEADS OR TAILS. DO THIS AT LEAST 20 TIMES FOR EACH EXPERIMENT AND PERFORM AT LEAST FIVE EXPERIME YOUR RESULTS IN A TABLE LIKE THE FOLLOWING.

Experiment	Number of coin tosses	Number of heads obtained
1		
2		
3		
4		
5		
TOTAL		

A DO YOU FEEL THAT THE TWO OUTCOMES “HEAD” AND “TAIL” ARE EQUALLY LIKELY?

B DO YOUR EXPERIMENTAL RESULTS SUPPORT THIS FEELING?

C WHAT IS THE RATIO OF THE NUMBER OF HEADS TO THE NUMBER OF TOSSES IN EACH EXPERIMENT?

D WHAT RATIO DO YOU HAVE FOR THE TOTAL NUMBER OF HEADS TO THE TOTAL NUMBER OF TOSSES?

5 A FAIR DIE IS ROLLED ONCE. CALCULATE THE PROBABILITY OF GETTING:

A AN ODD NUMBER **B** A SCORE OF 5

C A PRIME NUMBER **D** A SCORE OF 0

6 A NUMBER IS SELECTED AT RANDOM FROM THE WHOLE NUMBERS 1 TO 20, BOTH INCLUSIVE. FIND THE PROBABILITY THAT THE NUMBER SELECTED IS:

A EVEN **C** A MULTIPLE OF 3 **E** THE SQUARE OF 2

B A MULTIPLE OF 2 AND 3 **D** EVEN OR ODD **F** THE SQUARE OF 6

7 A BAG CONTAINS FIVE RED BALLS, THREE BLACK BALLS AND FOUR WHITE BALLS. A BALL IS DRAWN OUT OF THE BAG AT RANDOM. WHAT IS THE PROBABILITY THAT THE BALL

A WHITE? **B** RED? **C** BLACK?

8 A BAG CONTAINS 100 IDENTICAL CARDS ON WHICH THE NUMBERS 1 TO 100 ARE MARKED. A CARD IS DRAWN RANDOMLY. FIND THE PROBABILITY THAT THE NUMBER ON THE C

A AN EVEN NUMBER **B** AN ODD NUMBER **C** A MULTIPLE OF 7

D A MULTIPLE OF 5 **E** A MULTIPLE OF 3 **F** LESS THAN 76

G GREATER THAN 32 **H** A FACTOR OF 24

Key Terms

analysis	measure of central tendency	range
arithmetic mean	measure of dispersion	raw data
average	measure of location	sample
classification	median	sample space
collection	mode	secondary data
descriptive statistics	outcomes	standard deviation
equally likely	presentation	statistical data
event	population	statistics
frequency	population function	tabulation
frequency distribution	possibility set	variable (or variate)
histogram	primary data	variance
interpretation	probability	

Summary

1 STATISTICS IS THE SCIENCE OF COLLECTING, ORGANIZING, PRESENTING, ANALYSING AND INTERPRETING DATA IN ORDER TO DRAW CONCLUSIONS.

2 A POPULATION IS THE COMPLETE COLLECTION OF INDIVIDUALS, OBJECTS OR MEASUREMENTS THAT HAVE A CHARACTERISTIC IN COMMON.

3 A SMALL PART (OR A SUBSET) OF A POPULATION IS CALLED A SAMPLE.

4 IF THE CATEGORIES OF A CLASSIFICATION ARE BASED ON AN ATTRIBUTE OR CHARACTERISTICS WHOSE VALUES ARE NOT NUMBERS, THEN IT IS CALLED QUALITATIVE CLASSIFICA

5 IF THE CHARACTERISTIC OF INTEREST IS NUMERICAL, THEN IT IS CALLED QUANTITATIVE CLASSIFICATION.

6 DESCRIPTIVE STATISTICS IS A BRANCH OF STATISTICS CONCERNED WITH SUMMARIZING AND DESCRIBING A LARGE AMOUNT OF DATA.

7 DATA IS SAID TO BE primary, IF IT IS OBTAINED FIRST-HAND FOR THE PARTICULAR PURPOSE FOR WHICH ONE IS CURRENTLY WORKING.

8 DATA THAT HAS BEEN PREVIOUSLY COLLECTED FOR A SIMILAR PURPOSE IS CALLED SECONDARY DATA.

9 A STATISTICAL TABLE IS A SYSTEMATIC PRESENTATION OF DATA IN COLUMNS AND ROWS.

10 THE QUANTITY THAT WE MEASURE FROM OBSERVATION IS CALLED VARIATE (OR VARIABLE).

11 THE DISTRIBUTION OF A POPULATION FUNCTION IS THE FUNCTION THAT ASSOCIATES WITH EACH VARIATE OF THE POPULATION FUNCTION THE CORRESPONDING FREQUENCY.

12 A FREQUENCY DISTRIBUTION IS A DISTRIBUTION SHOWING THE NUMBER OF OBSERVATIONS ASSOCIATED WITH EACH DATA VALUE.

13 A HISTOGRAM IS A PICTORIAL REPRESENTATION OF A FREQUENCY DISTRIBUTION IN WHICH THE VARIABLE (x) VALUES ARE PLOTTED ON THE x-AXIS AND THE FREQUENCY OF OCCURRENCE IS PLOTTED ON THE y-AXIS.

14 IF $x_1, x_2, x_3, \ldots x_n$ ARE THE n OBSERVATIONS OF A VARIABLE x, THEN THE MEAN ($\bar{x}$) IS GIVEN BY

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

15 THE MEDIAN OF A VARIABLE IS THE VALUE THAT LIES IN THE MIDDLE OF THE DATA WHEN ARRANGED IN ASCENDING OR DESCENDING ORDER.

16 THE MODE OF A VARIABLE IS THE MOST FREQUENTLY OCCURRING VALUE OF THE VARIABLE THAT OCCURS IN THE DATA SET.

17 THE RANGE R OF A SET OF NUMERICAL DATA IS THE DIFFERENCE BETWEEN THE MAXIMUM AND MINIMUM VALUES.

RANGE = MAXIMUM VALUE − MINIMUM VALUE

18 STANDARD DEVIATION IS THE SQUARE ROOT OF THE MEAN OF THE SQUARED DEVIATION OF EACH VALUE FROM THE ARITHMETIC MEAN.

19 THE OUTCOMES OF AN EXPERIMENT ARE SAID TO BE EQUALLY LIKELY, WHEN THE EXPERIMENT IS REPEATED A LARGE NUMBER OF TIMES, EACH OUTCOME OCCURS EQUALLY OFTEN.

20 THE POSSIBILITY SET FOR AN EXPERIMENT IS THE SET OF ALL POSSIBLE OUTCOMES OF THE EXPERIMENT. IT IS ALSO KNOWN AS THE SAMPLE SPACE.

21 AN EVENT IS A SUBSET OF THE POSSIBILITY SET.

22 IF S IS THE POSSIBILITY SET OF AN EXPERIMENT AND EACH OUTCOME IS EQUALLY LIKELY, THEN THE PROBABILITY OF EVENT E OCCURRING, DENOTED BY $P(E)$, IS DEFINED AS:

$$P(E) = \frac{\text{NUMBER OF ELEMENTS IN } E}{\text{NUMBER OF ELEMENTS IN } S} = \frac{n(E)}{n(S)}$$

Review Exercises on Unit 6

1 WHAT IS MEANT BY SUMMARIZING AND DESCRIBING DATA?

2 THE MARKS OF 30 STUDENTS IN A MATHEMATICS TEST ARE GIVEN BELOW.

3	5	4	6	8	12	14	5	6	5
8	5	9	10	9	10	12	10	12	10
12	13	10	15	14	15	14	15	14	14

A CONSTRUCT A FREQUENCY DISTRIBUTION TABLE.

B DRAW A HISTOGRAM TO REPRESENT THE DATA.

C WHAT PERCENT OF THE STUDENTS HAVE SCORED LESS THAN 15?

3 REFER TO THE FOLLOWING HISTOGRAM TO ANSWER THE QUESTIONS THAT FOLLOW.

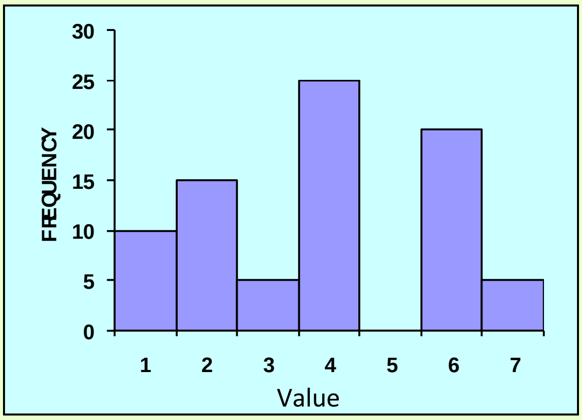


Figure 6.5

A PREPARE A FREQUENCY DISTRIBUTION TABLE.

B WHAT IS THE HIGHEST VARIABLE?

C WHAT IS THE HIGHEST FREQUENCY?

D HOW MANY VARIATES OCCUR 5 TIMES?

E WHICH VARIATES HAVE THE MINIMUM FREQUENCY?

4 FIND THE MEAN, MEDIAN, MODE, RANGE, VARIANCE AND STANDARD DEVIATION OF THE POPULATION FUNCTION WHOSE DISTRIBUTION IS GIVEN IN THE TABLE BELOW.

V	2	3	4	5	6
f	2	4	1	2	3

5 FIND THE MEAN, MEDIAN, MODE, RANGE AND VARIANCE OF THE HISTOGRAM GIVEN BELOW.

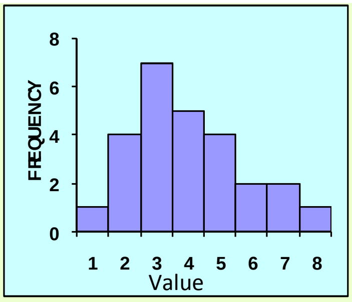


Figure 6.6

6 WHY CAN THE PROBABILITY OF AN EVENT NOT BE $2\sqrt{2}-$, $-\frac{}{2}$ OR $\frac{3}{2}$?

7 AN INTEGER n, $1 \leq n \leq 144$, IS PICKED AT RANDOM. WHAT IS THE PROBABILITY THAT I
THE SQUARE OF AN INTEGER?

8 GIVEN THE FOLLOWING VALUES OF A POPULATION FUNCTION:

5, 4, 7, 3, 6, 5, 3, 1, 5, 7, 5, 9.

FIND THE PROBABILITY THAT A RANDOMLY CHOSEN VALUE FROM THE DATA IS

A A MODAL VALUE;

B BELOW THE MEAN VALUE;

C ANY OF THE NUMBERS 1, 4, 6 OR 9;

D AN ODD NUMBER GREATER THAN THE MEAN VALUE.

9 TWO FAIR DICE ARE ROLLED ONCE. WHAT IS THE PROBABILITY THAT THE DIFFERENCE OF THE TWO NUMBERS SHOWN IS 1?

10 GIVEN BELOW IS THE FREQUENCY DISTRIBUTION OF A POPULATION V.

V	−10	−5	0	5	10
f	5	10	5	20	10

IF AN ELEMENT IS DRAWN RANDOMLY FROM THE POPULATION FIND THE PROBABIL

A NON-NEGATIVE; **B** NON-ZERO;

C LESS THAN OR EQUAL TO −5; **D** POSITIVE;

11 THE MEDIAN OF $x-4$, x, $2x$ AND $2x+12$ IS 9, WHERE x IS A POSITIVE INTEGER. FIND THE VALUE OF x.

12 THE TABLE BELOW SHOWS THE NUMBER OF STUDENTS WHO SCORED MARKS 3, 4 OR 5 IN A MATHS TEST.

Mark	3	4	5
Number of Students	3	x	4

IF THE MEAN MARK IS 4.1, HOW MANY STUDENTS SCORED 4?

13 IN A CLASS OF BOYS AND GIRLS, THE MEAN WEIGHT OF 8 BOYS IS 55 KG AND THE MEAN WEIGHT OF A GROUP OF GIRLS IS 48 KG. THE MEAN WEIGHT OF ALL THE CHILDREN HOW MANY GIRLS ARE THERE?

14 THERE ARE 24 RIGHT-HANDED STUDENTS IN A CLASS OF 30. WHAT IS THE PROBABILITY THAT A STUDENT CHOSEN AT RANDOM WILL BE LEFT-HANDED?

15 SUPPOSE YOU WRITE THE DAYS OF THE WEEK ON IDENTICAL PIECES OF PAPER. YOU MIX THEM IN A BOWL AND SELECT ONE AT A TIME. WHAT IS THE PROBABILITY THAT T SELECT WILL HAVE THE LETTER R IN IT?

16 A PAIR OF DICE ARE ROLLED. FIND THE PROBABILITY THAT THE SUM OF THE NUMBERS ON THE UPPER FACES IS:

A 9; **B** GREATER THAN 9; **C** EVEN; **D** NOT GREATER THAN 9;

E GREATER THAN 9 AND EVEN; **F** GREATER THAN 9 OR EVEN.

17 FROM THE MEMBERS OF A FARMERS' ASSOCIATION, ONLY 50 CULTIVATED WHEAT. AN AGRICULTURAL EXPERT WANTS TO STUDY THE FARMERS' YIELD IN TERMS OF HARVESTED PER HECTARE AND FOUND THE FOLLOWING

50	45	45	50	46	48	55	48	52	54
51	52	45	55	46	50	55	54	49	51
48	46	51	52	47	45	49	54	46	48
53	52	48	46	55	47	51	47	50	53
47	53	48	45	54	48	50	46	52	54

A PREPARE A FREQUENCY DISTRIBUTION TABLE THAT REPRESENTS THE DATA.

B DRAW A HISTOGRAM.

C FIND THE MODE OF THE DATA.

D IF THE WEREDA AGRICULTURE OFFICE WANTS TO PRAISE FARMERS WHO PRODUCED MORE THAN 52 QUINTALS PER HECTARE, HOW MANY FARMERS WILL THEY PRAISE?

18 WHICH OF THE FOLLOWING IS TRUE?

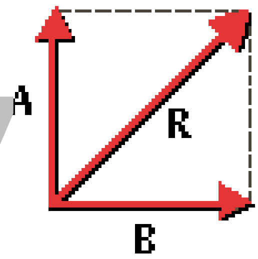
A THE MEAN, MODE AND MEDIAN OF A POPULATION FUNCTION CANNOT BE EQUAL.

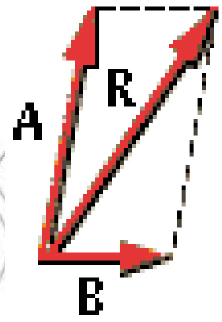
B THE RANGE AND THE STANDARD DEVIATION OF A POPULATION FUNCTION ARE INVERSELY RELATED.

C THE RANGE OF A POPULATION FUNCTION CAN BE A NON-POSITIVE NUMBER.

D THE SUM OF THE DEVIATIONS OF EACH VALUE OF A POPULATION FUNCTION FROM THE MEAN WILL ALWAYS BE ZERO.

Unit 7





VECTORS IN TWO DIMENSIONS

Unit Outcomes:

After completing this unit, you should be able to;

- *know basic concepts and specific facts about vectors.*
- *perform operations on vectors.*

Main Contents

7.1 Introduction to vectors and scalars

7.2 Representation of a vector

7.3 Addition and subtraction of vectors and multiplication of a vector by a scalar

7.4 Position vector of a point

Key Terms

Summary

Review Exercises

INTRODUCTION

FROM PREVIOUS GRADES, YOU KNOW ABOUT MEASUREMENTS OF DIFFERENT KINDS SU HEIGHT, WEIGHT, TEMPERATURE, DISTANCE, ANGLE MEASURE, AREA, ETC. SUCH QUANTITIES NUMBERS AS THEIR MEASURE (WITH SOME UNIT OF MEASUREMENT). FOR EXAMPLE, TI A ROOM IS 3 M, THE WEIGHT OF A QUINTAL IS 100 KG, THE DISTANCE BETWEEN TWO CLASSROOM IS 8 M, THE TEMPERATURE OF A NORMAL PERSON IS 36.5 °C, THE AREA OF A TRIANGLE ABC IS 6 CM^2, ETC. NOT ALL QUANTITIES, HOWEVER, ASSUME ONLY A SINGLE REAL N THEIR MEASURE. THERE ARE SOME QUANTITIES THAT ASSUME MEASURES INVOLVING D

EXAMPLE SUPPOSE WE ARE IN SCHOOL A, AND SOMEONE HAS TOLD US THAT HE STUD NEARBY SCHOOL B THAT IS d KM AWAY. DO WE HAVE ENOUGH INFORMATION TO FIND B? OF COURSE NOT, BECAUSE B COULD BE AT ANY POINT ON A CIR RADIUS d KM CENTRED AT A. IN ADDITION TO THE DISTANCE, WE NEED TO KNO DIRECTION IN ORDER TO FIND B.

THERE ARE MANY PHYSICAL QUANTITIES WHOSE MEASUREMENTS INVOLVE BOTH MA DIRECTION. THESE INCLUDE VELOCITY, FORCE, ACCELERATION, ELECTRIC OR MAGNI SUCH QUANTITIES ARE CALLED **vectors**. TODAY VECTORS HAVE MANY APPLICATIONS. ALL BRANC OFCLASSICAL AND MODERN PHYSICS ARE REPRESENTED BY USING THE LANGUAGE VECTORS ARE ALSO USED WITH INCREASING FREQUENCY IN THE SOCIAL AND BIOLOGI THIS UNIT, YOU WILL DEAL WITH VECTORS, IN PARTICULAR VECTORS IN TWO DIMENSIO

HISTORICAL NOTE:

Sir William Rowan Hamilton (1805-1865)

The study of vectors started with Hamilton's invention of quaternions. Quaternions were developed as mathematical tools for the exploration of physical space. As quaternions contained both scalar and vector parts, difficulties arose when these two parts were treated simultaneously.

Scientists soon learned that many problems could be dealt with by considering vector parts separately, and the study of vector analysis began.

7.1 INTRODUCTION TO VECTORS AND SCALARS

Group Work 7.1

1 DISCUSS SOME QUANTITIES THAT CAN BE EXPRESS USING A SINGLE MEASUREMENT (WITH UNITS) .

2 DISCUSS SOME QUANTITIES THAT REQUIRE BOTH SIZE AND DIRECTION.

IN GENERAL, THERE ARE TWO TYPES OF PHYSICAL ME: THOSE INVOLVING ONLY MAGNITUDE AND NO D, CALLED scalars AND OTHERS INVOLVING MTUDE AND A DEFINITE DIRECTION, CALLED vectors. IN MANY APPLICATIONS OF MATHEMATICS T AND BIOLOGICAL SCIENCES AND ENGINEERING, SCIENTISTS QUANTITIES THAT HAVE BOTH MAGNITUDECTION. AS MENTIONED ABOVE, EXAMPLES INCLU FORCE, VELOCITY, ACCELERATION, ANIT IS USEFUL TO BE ABLE EXPRESS THESE QUANTITIES (VECTORS) BOTH GEOMETRICALLY

ACTIVITY 7.1

CONSIDER THE FOLLOWING QUANTITIES AND EACH IS A SC. QUANTITY OR VECTOR QUAI

A	AMOUNT OF RAINFAL	**B**	AREA OF A PLANE
C	TEMPERATURE IN A	**D**	FORCE OF WATER HITTINC
E	GRAVITY	**F**	ACCELERATION MOTORE
G	VOLUME OF A SOLID	**H**	SPEED OF AN EAI

Scalar quantities

Definition 7.1

Scalar quantities are those quantities of measures that have only magnitude and no direction. (Simply represented by a real number and a specified unit).

EXAMPLE 1 THE LENGTH OF A SIDE OF A TI CM. SINCE 4 IS A REAL NUMB NO DIRECTION THE LENGTH REPRE

EXAMPLE 2 THE HEIGHT OF MT RAS DASHEN IS 4550 METRES. HERE THE HEIGHT IS REPRESENTED BY A SINGLE REA. HENCE IT REPRESENTS

EXAMPLE 3 THE DAYTIME TEMPERATURE OF MERCURY RISES TO 430°C. SINCE 30 IS A REAL NUMBER, TTEMPERATURE REPRESENTS A SCALAR.

Vector quantities

Definition 7.2

Vector quantities are those quantities of measure that have both magnitude (length) and direction.

EXAMPLE 4 THE VELOCITY OF A C KM/H IN THE DIRECTION OF THIS IS A VECTOR.

EXAMPLE 5 SUPPOSE HELEN M, FROM A, 10M TO THE EAST [E] AND M TO THE NORTH [N] TO REACH SHOW, AS A VECTOR, HELEN'S DISPLAC.

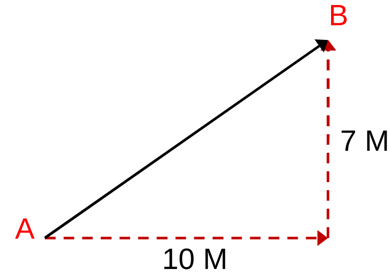


Figure 7.1

SOLUTION: TAKEN TOGETHER, THE DISTANCE AND DIRECTION OF THE CALLED THE displacement FROM A TO B, AND IS REPRESENTED BY FIGURE 7.1.

THE ARROWHEAD TELLS US THAT WE ABOUT THE DISPLACEMENT OF HELE B. THIS IS AN EXAMPLE OF A

7.2 REPRESENTATION OF A VECTOR

ACTIVITY 7.2

1 DISCUSS ALGEBRAIC AND GEOMETRIC RES OF VECTORS.

2 REPRESENT THE $\overrightarrow{OP}$ GEOMETRICALLY, WHERE O IS AND P = (2, 3) IN THE xy-COORDINATE SYSTEM.

3 DISCUSS THE MAGNITUDE AND DIRECTION O

4 FIND THE MAGNITUDE AND THE DIRECTION $\overrightarrow{OP}$.

5 WHEN ARE TWO VECTOI

A VECTOR CAN BE REPRESENTED EITHER OR GEOMETRICALLY. OFTEN, THE MOST CONVENIENT WAY OF REPRESENTING VECTORS IS GEOMETR VECTOR IS REPRESENTED BY AN ARROW SEGMENT.

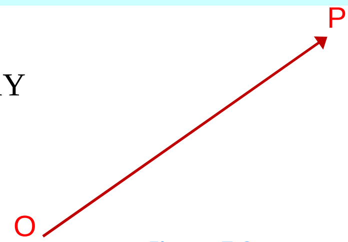


Figure 7.2

WHEN A VECTOR IS REPRESENTED B (*see* $\overrightarrow{OP}$ ABOVE), THE POINT O IS CALL initial point AND P IS CALLED terminal point. SOMETIMES, VECTORS ARE RE USING LETTERS OR A LETTER WITH A BA $\vec{u}$, $\vec{v}$, ETC.

EXAMPLE 1 WHAT DOES THE VECTOR IN THE FOLLOWING FIGURE REPRESENT?

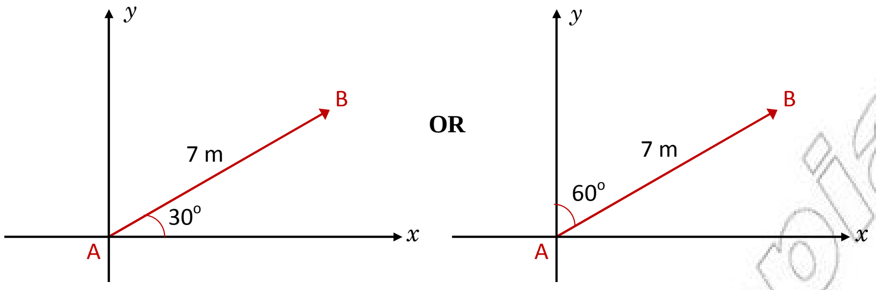


Figure 7.3

SOLUTION: THE VECTOR $\overrightarrow{AB}$ HAS A LENGTH OF 7 M AND DIRECTION OF NORTH OF EAST 30° [E30°N] (OR A DIRECTION OF NORTH 60° EAST [N60°E]). ITS INITIAL POINT IS A AND ITS TERMINAL POINT IS B.

WHAT DO YOU THINK IS THE MAGNITUDE (LENGTH) OF A VECTOR AND THE DIRECTION O

EXAMPLE 2 THE FOLLOWING ARE EXAMPLES OF VECTOR REPRESENTATION. CAN YOU DETERMINE THEIR LENGTHS AND DIRECTIONS?

Hint: USE RULER AND PROTRACTOR

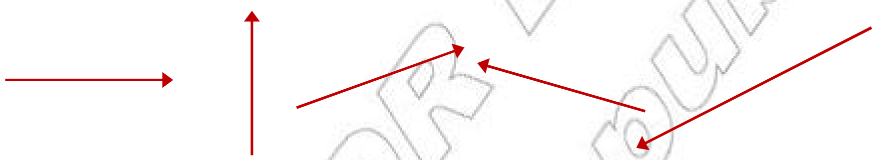


Figure 7.4

Magnitude (length) of vectors

THE MAGNITUDE (LENGTH) OF A VECTOR $\overrightarrow{OP}$ OR SIMPLY $\vec{u}$ IS THE LENGTH OF THE LINE SEGMENT FROM THE INITIAL POINT O TO THE TERMINAL POINT P, (THE LENGTH OF THE DIRECTED

Notation: MAGNITUDE OF VECTOR $\overrightarrow{OP}$ IS DENOTED AS $|\overrightarrow{OP}|$.

EXAMPLE 3 DETERMINE THE LENGTH OF THE VECTOR $\overrightarrow{OP}$.

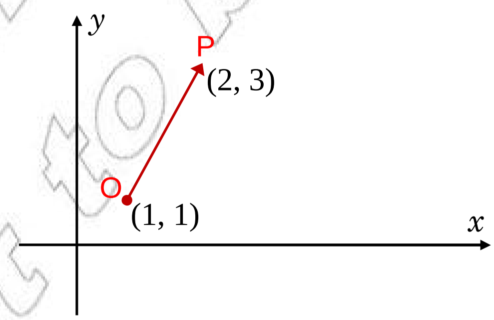


Figure 7.5

SOLUTION: THE MAGNITUDE OF THE VECTOR $\overrightarrow{OP}$ IS $|\overrightarrow{OP}| = \sqrt{5}$ (HOW?)

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EXAMPLE 4 DETERMINE THE LENGTH IN CENTIMETRES OF EACH OF THE VECTORS SHO FOLLOWING FIGURE 7.6.

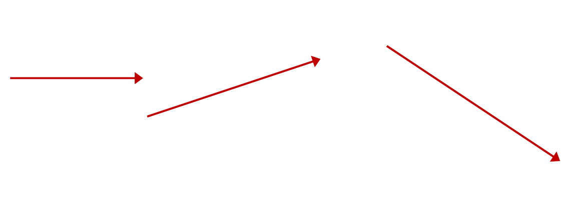

Figure 7.6

EXAMPLE 5 A FORCE OF 10 POUNDS IS EXERTED VERTICALLY DOWN TO THE SURFACE O EARTH AND A FORCE OF 20 POUNDS IS EXERTED PARALLEL TO THE SURFACE EARTH FROM LEFT TO RIGHT. THE GEOMETRIC REPRESENTATION IS

Figure 7.7

HERE, NOTICE THAT THE ARROWS (DIRECTED LINE SEGMENTS) ARE DRAWN WITH LENG TO THE MAGNITUDES. THE ARROW REPRESENTING 10 POUNDS IS HALF THE LENGTH REPRESENTING 20 POUNDS.

FROM THIS, WE REALIZE THAT THE MAGNITUDE OF A VECTOR IS REPRESENTED BY THE L ARROW THAT REPRESENTS THE VECTOR.

Direction of vectors

THE DIRECTION OF A VECTOR IS THE ANGLE THAT IT FORMS (THE ARROW THAT REPRESENTS THE VECTOR) WITH THE HORIZONTAL LINE AT ITS INITIAL POINT (OR WITH THE VERTICAL L COMPASS DIRECTIONS).

EXAMPLE 6 THE DIRECTION OF THE VECTOR $\vec{u}$ FROM THE HORIZONTAL LINE AT ITS INITIAL POIN AS REPRESENTED BELOW, IS 45^{o} (OR N45^{o}E)

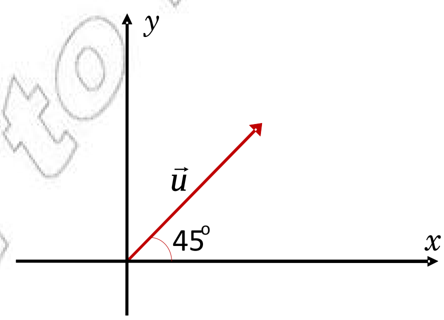

Figure 7.8

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CONSIDER THE FOLLOWING PAIRS OF VECTORS $\vec{u}$ AND $\vec{v}$.

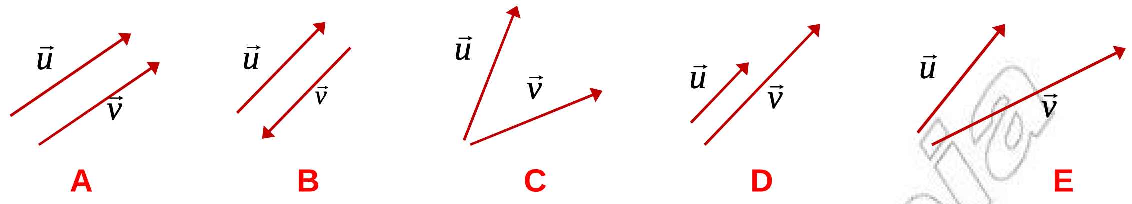

Figure 7.9

WHAT DO YOU OBSERVE? DO THEY HAVE THE SAME LENGTH? DO THEY HAVE THE SAME

THE TWO VECTORS IN A HAVE THE SAME LENGTH AND THEY HAVE THE SAME DIRECTION. THE TWO VECTORS IN B HAVE THE SAME LENGTH BUT THEY HAVE OPPOSITE DIRECTIONS. THE TWO VECTORS IN C HAVE THE SAME LENGTH AND DIFFERENT DIRECTIONS. THE VECTORS IN D HAVE DIFFERENT LENGTH, BUT THEY HAVE SAME DIRECTION. AND THE TWO VECTORS IN E DIFFER IN BOTH LENGTH AND DIRECTION.

Note: 1 IF TWO VECTORS HAVE OPPOSITE DIRECTIONS, THEY ARE CALLED OPPOSITE

2 VECTORS THAT HAVE EITHER THE SAME OR OPPOSITE DIRECTIONS AR PARALLEL VECTORS.

EXAMPLE 7 FROM THE VECTORS GIVEN IN FIGURE 7.9 ABOVE, A, B AND D ARE PARALLEL VECTORS.

WHEN WE REPRESENT VECTORS BY USING DIRECTED ARROWS AS GIVEN ABOVE, WE SIMILARITIES OR DIFFERENCES IN LENGTH OR DIRECTION. WHAT DO YOU OBSERV FOLLOWING VECTORS?

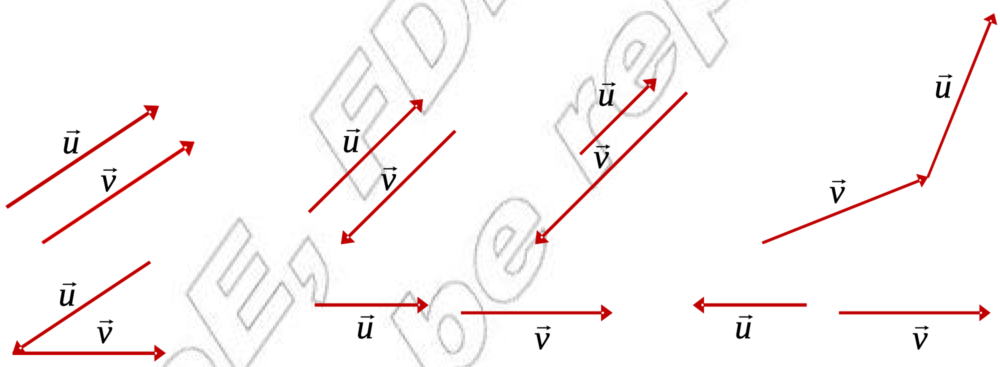

Figure 7.10

Equality of vectors

TWO VECTORS ARE SAID TO BE EQUAL, IF THEY HAVE THE SAME LENGTH AND THE SAM

EXAMPLE 8 THE FOLLOWING TWO VECTORS $\vec{u}$ AND $\vec{v}$, ARE EQUAL SINCE THEY HAVE THE SAME LENGTH AND THE SAME DIRECTION. THE ACTUAL LOCATION OF THESE VECTORS IS NOT SPECIFIED. WE CALL SUCH VECTORS free vectors.

Figure 7.11

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EXAMPLE 9 IN EACH OF THE DIAGRAMS BELOW, ALL THE VECTORS ARE EQUAL.

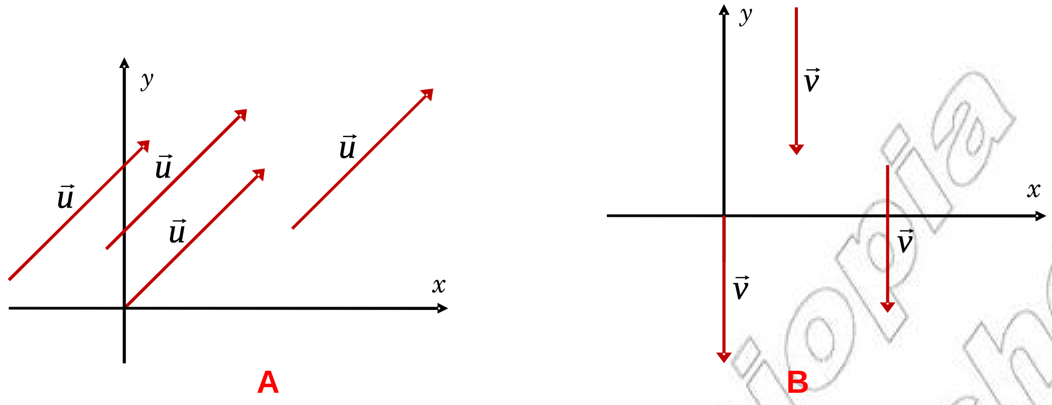

Figure 7.12

Group Work 7.2

1 SUPPOSE VECTORS $\vec{u}$ AND $\vec{v}$ ARE EQUAL,

- **A** CAN WE CONCLUDE THAT THEY HAVE THE SA ?
  WHY?
- **B** DO THEY HAVE THE SAME LENGTH? WHY?
- **C** DO THEY HAVE THE SAME DIRECTION? WHY?

2 SUPPOSE VECTORS $\vec{u}$ AND $\vec{v}$ ARE OPPOSITE,

- **A** CAN WE CONCLUDE THAT THEY MUST START FROM THE SAME INITIAL POINT?
- **B** DO THEY HAVE THE SAME LENGTH? WHY?
- **C** DO THEY HAVE THE SAME DIRECTION? WHY?

3 SUMMARIZE WHAT YOU HAVE CONCLUDED.

Exercise 7.1

1 DETERMINE THE MAGNITUDE AND DIRECTION OF EACH OF THE FOLLOWING VECTO

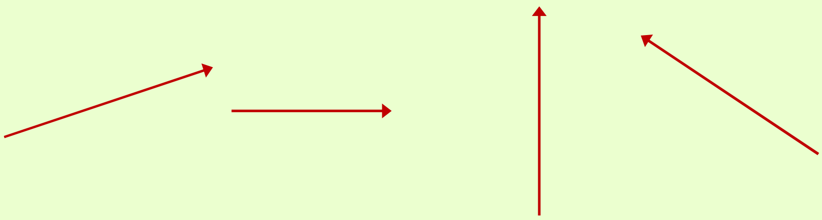

Figure 7.13

2 LOCATE EACH OF THE FOLLOWING VECTORS ON A COORDINATE SYSTEM.

- **A** $\overrightarrow{OP}$ WHOSE LENGTH IS 3 CM AND DIRECTION IS [N40°E]
- **B** $\overrightarrow{AB}$ WHOSE LENGTH IS 5 CM AND DIRECTION IS [S45°E]
- **C** $\overrightarrow{CD}$ WHOSE INITIAL POINT IS (1, 2), LENGTH IS 3 CM AND DIRECTION IS [N60°W]

3 FROM THE FOLLOWING, IDENTIFY THE PAIRED VECT, OR OPPOS.

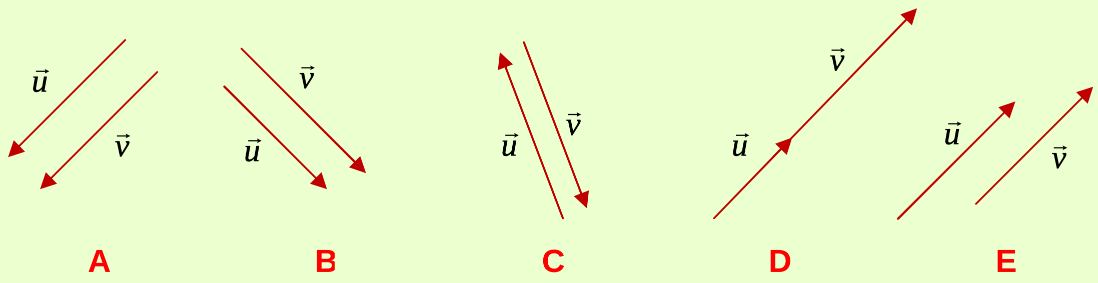

Figure 7.14

7.3 ADDITION AND SUBTRACTION OF VECTORS AND MULTIPLICATION OF A VECTOR BY A SCALAR

7.3.1 Addition of Vectors

ACTIVITY 7.3

GIVEN BELOW ARE PAIRS OF $\overrightarrow{AB}$ AND $\overrightarrow{CD}$. TRANSLATE $\overrightarrow{CD}$ SO THA ITS INITIAL POINT IS AT THE TERMINA $\overrightarrow{AB}$. THEN, HOW DO Y EXPRESS $\overrightarrow{AD}$ IN TERMS OF $\overrightarrow{AB}$ AND $\overrightarrow{CD}$?

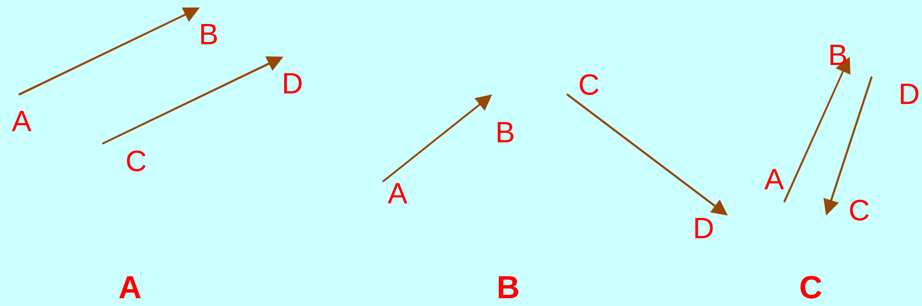

Figure 7.15

1 WHAT IS MEANT BY ADDITION OF VI

2 HOW WOULD YOU ADD VECT

3 IS THE LENGTH OF THE SUM OF TWO VECTORS ALWAYS EQUAL TO T EACH VECTOR? W

SUPPOSE YOU HAVE THREE CITIES A, B ASSUMING YOU KNOW THE DISTANC DIRECTION FROM A TO FROM B TO C AS SHOWN IN FIGURE 7.16

IF YOU WANT TO GO DIRECTLY FROM A TO C, WHAT WOULD BE THE DISTANCE AND DIRECTION?

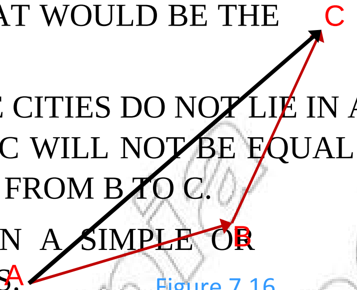

Figure 7.16

THE FIRST THING TO NOTICE IS THAT IF THE THREE CITIES DO NOT LIE IN A STRAIGHT LINE, THEN THE DISTANCE FROM A TO C WILL NOT BE EQUAL TO THE SUM OF THE DISTANCES FROM A TO B AND FROM B TO C.

ALSO, THE DIRECTION MAY NOT BE RELATED IN A SIMPLE OR OBVIOUS WAY TO THE TWO SEPARATE DIRECTIONS.

YOU WILL SEE, HOWEVER, THAT THE SOLUTION IS EASY IF WE WORK WITH THE COMPO DISPLACEMENT VECTORS. LET THE COMPONENTS OF THE VECTOR FROM A TO B IN T NORTH DIRECTIONS BE a, b AND FROM B TO C IN THE EAST AND NORTH DIRECTIONS BE a', b' RESPECTIVELY. THEN WE CAN SEE THAT THE COMPONENT OF THE DISPLACEMENT VEC C IN THE EAST DIRECTION IS $a + a'$ AND IN THE NORTH DIRECTION IS $b + b'$.

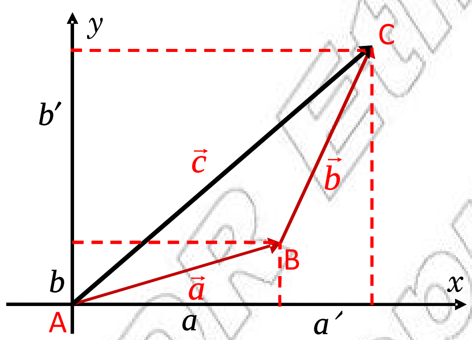

Figure 7.17

FROM THIS, WE CAN CONCLUDE THAT $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ OR $\vec{a}+\vec{b}=\vec{c}$.

WE SHALL DISCUSS ADDITION OF VECTORS USING TWO APPROACHES: THE triangle law AND THE parallelogram law OF ADDITION OF VECTORS.

Group Work 7.3

1 DISCUSS THE TRIANGLE LAW OF VECTOR ADDITION

2 DISCUSS THE PARALLELOGRAM LAW OF VECTOR A

3 WHAT RELATION AND DIFFERENCE DO BOTH LAWS HAV

Triangle law of addition of vectors

CONSIDER THE FOLLOWING (FIGURE 7.18).

OBSERVE THAT $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

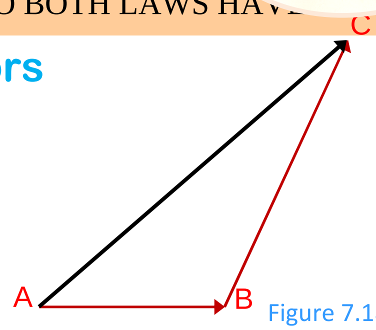

Figure 7.18

Definition 7.3 Triangle law of vector addition

Let $\vec{a}$ and $\vec{b}$ be two vectors in a coordinate system. If $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$, then their sum, $\vec{a}+\vec{b} = \overrightarrow{AB}+\overrightarrow{BC}$ is the vector represented by the directed line segment $\overrightarrow{AC}$. That is $\vec{a}+\vec{b} = \overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$.

ACTIVITY 7.4

1 CONSIDER THE FOLLOWING. DETERMINE THE SUM $\overrightarrow{AB}+\overrightarrow{BC}$ OF EACH PAIR OF VECTORS.

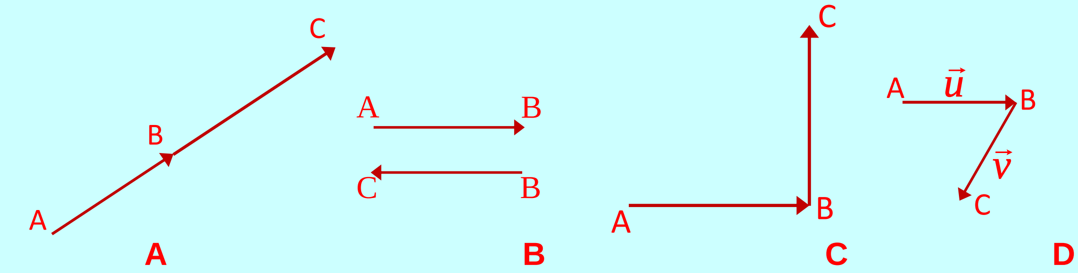

Figure 7.19

BY WRITING THE VECTOR ADDITION $\overrightarrow{AB}+\overrightarrow{BC}$, WE ARE LOOKING FOR THAT VECTOR WHOSE INITIAL POINT IS A AND WHOSE TERMINAL POINT IS C. $\overrightarrow{AC}$ IS SOMETIMES CALLED THE resultant displacement.

VECTOR ADDITION CAN BE DONE EITHER GRAPHICALLY OR BY SEPARATING COMPONENTS. WE SHALL DISCUSS THE ADDITION OF VECTOR COMPONENTS IN THIS UNIT.

EXAMPLE 1 A CAR TRAVELS 4 KM TO THE NORTH AND THEN $4\sqrt{3}$ KM TO THE EAST. WHAT IS THE DISPLACEMENT OF THE CAR FROM ITS FINAL POSITION?

SOLUTION:

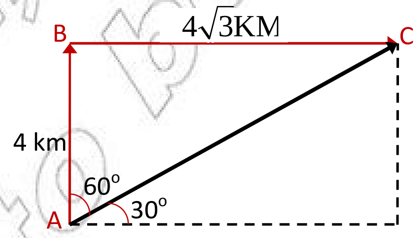

Figure 7.20

THE MAGNITUDE AC $= \sqrt{4^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16+48} = \sqrt{64} = 8$ AND

$\tan(\measuredangle BAC) = \dfrac{\text{OPPOSITE}}{\text{ADJACENT}} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$. THEREFORE, $\measuredangle BAC = 60^o$

SO THE DISPLACEMENT IS THAT OF $\overrightarrow{AC}$, WHICH IS 8 KM IN THE DIRECTION OF 30^O EAST.

EXAMPLE 2 A PERSON MOVED 10 M TO THE EAST FROM A TO B AND THEN 10 M TO THE W FROM B TO A. FIND THE RESULTANT DISPLACEMENT.

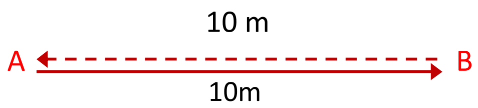

Figure 7.21

SOLUTION: HERE WE SEE THAT THE PERSON ENDS UP AT A, HENCE HIS DISPLACEMENT IS ZERO. FROM THIS WE SEE THAT IF WE ADD $\overrightarrow{AB}$ AND $\overrightarrow{BA}$, THEN THE SUM OF THESE VECTORS $\overrightarrow{AB}+\overrightarrow{BA}$ VANISHES IN THE SENSE THAT THE INITIAL POINT AND THE TERMINAL POINT COINCIDE. SUCH A VECTOR IS CALLED A null vector AND IS DENOTED BY $\vec{0}$ OR SIMPLY 0. I.E., $\overrightarrow{AB}+\overrightarrow{BA}=0$.

GIVEN $\overrightarrow{AC}$, IF $\vec{u}$ IS A VECTOR PARALLEL TO $\overrightarrow{AC}$ BUT IN OPPOSITE DIRECTION, THEN $\vec{u}$ IS SAID TO BE AN opposite vector TO $\overrightarrow{AC}$. $-\overrightarrow{AC}$ REPRESENTS THE VECTOR EQUAL IN MAGNITUDE BUT OPPOSITE IN DIRECTION TO $\overrightarrow{AC}$. THAT IS, $-\overrightarrow{AC}=\overrightarrow{CA}$. NOTICE THAT $\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AC}-\overrightarrow{AC}=0$

EXAMPLE 3 THE FOLLOWING ARE ALL OPPOSITE TO VECTOR $\overrightarrow{AC}$.

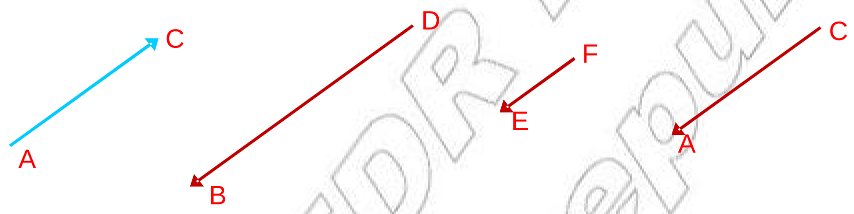

Figure 7.22

THAT IS, VECTORS $\overrightarrow{CA}$, $\overrightarrow{DB}$ AND $\overrightarrow{FE}$ ARE ALL OPPOSITE TO $\overrightarrow{AC}$ (BUT NOT EQUAL IN MAGNITUDE)

EXAMPLE 4 CONSIDER THE VECTORS $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{CB}$ AND $\overrightarrow{AD}$. DETERMINE THE FOLLOWING VECTORS.

A $\overrightarrow{AC}+\overrightarrow{CB}$ **B** $\overrightarrow{AC}+\overrightarrow{CA}+\overrightarrow{AD}$

C $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$ **D** $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}+\overrightarrow{DA}$

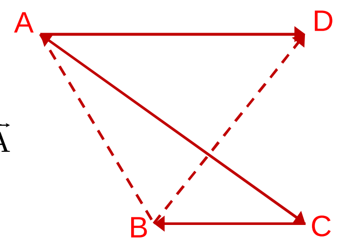

Figure 7.23

SOLUTION:

A $\overrightarrow{AC}+\overrightarrow{CB}=\overrightarrow{AB}$ **B** $\overrightarrow{AC}+\overrightarrow{CA}+\overrightarrow{AD}=\overrightarrow{AD}$

C $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}=\overrightarrow{AD}$ **D** $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}+\overrightarrow{DA}=\vec{0}$

Parallelogram law of addition of vectors

IN THE ABOVE, WE SAW HOW THE TRIANGLE LAW OF ADDITION IS APPLICABLE WHERE THE INITIAL POINT OF ONE VECTOR IS THE TERMINAL POINT OF THE OTHER. WE MAY SOM VECTORS WHOSE INITIAL POINT IS THE SAME, YET WE NEED TO FIND THEIR SUM.

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ACTIVITY 7.5

CONSIDER THE VE $\overrightarrow{AC}$ AND $\overrightarrow{AD}$ AS GIVEN BELOW.

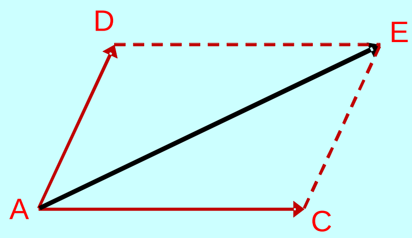


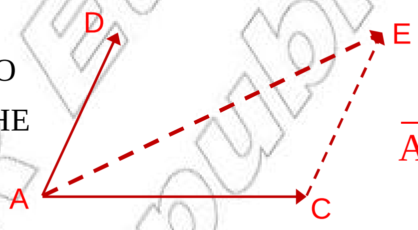
Figure 7.24

DISCUSS HOW TO DETERMINE $\overrightarrow{AC}+\overrightarrow{AD}$.

FROM PREVIOUS DISCU YOU NOTICE THAT IF TWO VECTORS HAVE THE S DIRECTION, THEN THEY A

IN FIGURE 7.25, $\overrightarrow{AD}$ AND $\overrightarrow{CE}$ ARE EQUAL. SO $\overrightarrow{AC}+\overrightarrow{AD}$ AND $\overrightarrow{AC}+\overrightarrow{CE}$ REPRESENT THE SAME VECTOR. BUT, FROM TRIANGLE

$\overrightarrow{AC}+\overrightarrow{CE}=\overrightarrow{AE}$



$\overrightarrow{AD}=\overrightarrow{CE}$

Figure 7.25

THEREFORE $\overrightarrow{AC}+\overrightarrow{AD}=\overrightarrow{AE}$.

NOW LET'S SEE HOW WE CAN CONSTRUCT A PARALLEL THE SUM OF TWO VE (WITH THE SAME INITIAL POINT) AS THE DIAGONAL OF THE CONSTRUCTED

EXAMPLE 5 GIVEN THE VE $\overrightarrow{AC}$, $\overrightarrow{AE}$, $\overrightarrow{AD}$, $\overrightarrow{AF}$, $\overrightarrow{AK}$ AND $\overrightarrow{AG}$ DESCRIBED IN FIGURE 7.26, DETERMINE THE FOLLOWING V.

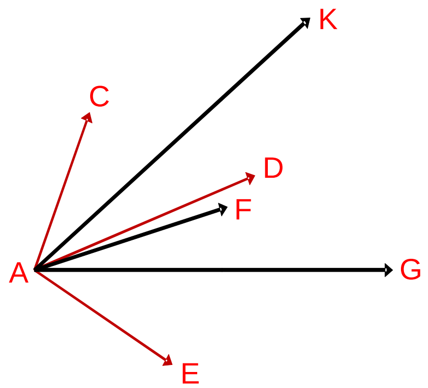


Figure 7.26

A $\overrightarrow{AC}+\overrightarrow{AE}$ **B** $\overrightarrow{AC}+\overrightarrow{AD}$

C $\overrightarrow{AE}+\overrightarrow{AD}$

SOLUTION: CONSTRUCT A PARALLELOGRA

A $\overrightarrow{AC}+\overrightarrow{AE}=\overrightarrow{AF}$

B $\overrightarrow{AC}+\overrightarrow{AD}=\overrightarrow{AK}$

C $\overrightarrow{AE}+\overrightarrow{AD}=\overrightarrow{AG}$

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Subtraction of vectors

Group Work 7.4

IF YOU HAVE VECTORS $\overrightarrow{AB}$, $\overrightarrow{BC}$ AND $\overrightarrow{AC}$ SUCH THAT $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

A HOW WOULD YOU RE $-\overrightarrow{AB}$ GEOMETRICALLY?

B CAN YOU SHOW GEOMETRIC $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$?

C DISCUSS VECTOR SUB, AND MULTIPLICATION OF A VECTO

D HOW DO YOU REPRESENT VECTOR SUBTRACTION AND SCALATOR GEOMETRICALLY

YOU HAVE DISCUSSED ADDITION OF VECTORS THAT ARE GEOMETRICALLY DES INITIAL POINT OF ONE VECTOR TO THE TERMINAL POINT OF THE OTHER WITHOUT CH MAGNITUDE AND DIRECTION. NOW YOU SHALL CONSIDER THE GEOMETRIC SUBTRACTION OF VECTORS.

ACTIVITY 7.6

CONSIDER THE FOLLOWING $\overrightarrow{AC}$ AND $\overrightarrow{AD}$ WHAT DO YOU THIN OTHER VECTORS IN DIAGRAMS A, B AND C DESCRIBE?

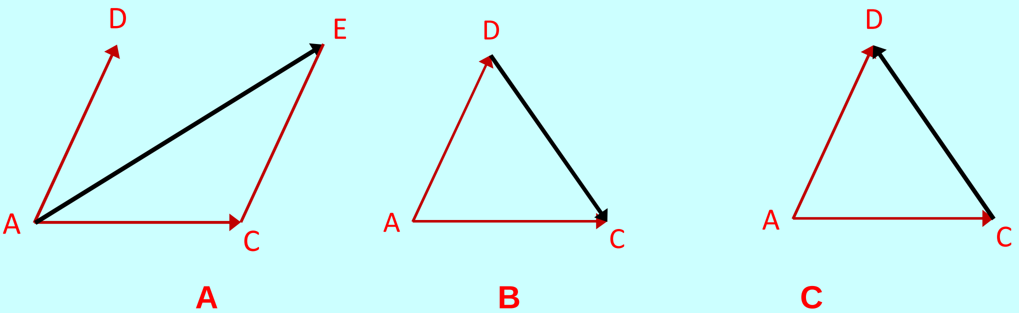

Figure 7.27

FROM ADDITION OF VECTORS, WE $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$,

FROM WHICH WE CAN SEE $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$.

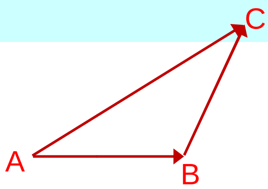

Figure 7.28

EXAMPLE 6 DESCRIBE THE VECTORS $\overrightarrow{BC}$ AND $\overrightarrow{BD}$ AND DETERMINE $\overrightarrow{CD}$ IN TERMS OF $\overrightarrow{BC}$ AND $\overrightarrow{BD}$

SOLUTION: $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$ FROM TRIANGLE LAW.

THEREFORE, $\overrightarrow{CD} = \overrightarrow{BD} - \overrightarrow{BC}$.

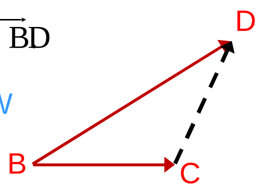

Figure 7.29

Multiplication of vectors by scalars

ACTIVITY 7.7

CONSIDER A VECTOR $\overrightarrow{AC}$ AND DETERMI

A $\overrightarrow{AC}+\overrightarrow{AC}$ **B** $\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}$ **C** $-\overrightarrow{AC}-\overrightarrow{AC}-\overrightarrow{AC}$

WHAT DO YOU OBSERVE? IT SEEMS VERY $\overrightarrow{AC}+\overrightarrow{AC} = 2\overrightarrow{AC}$. GEOMETRICALL MEANS WE ARE DOUBLING THE MAGNITUDE (LENG $\overrightarrow{AC}$ WITHOUT CHANGI DIRECTION.

IN THE SAME WAY IF WE HAVE $3\overrightarrow{AC}$, $\frac{1}{2}\overrightarrow{AC}$, $-\overrightarrow{AC}$ AND $-2\overrightarrow{AC}$, THEN IN $3\overrightarrow{AC}$ WE ARE TRIPLING THE MAGNITUDE OF $\overrightarrow{AC}$ AND WE ARE TAKING HALF OF THE $\overrightarrow{AC}$ IN $\frac{1}{2}\overrightarrow{AC}$.

WHAT DO YOU THINK $-\overrightarrow{AC}$ AND $-2\overrightarrow{AC}$ MEAN?

IF WE HAVE $k\overrightarrow{AC}$ WHERE k IS ANY REAL NUMBER, THEN DEPENDING ON k EITHER WE ARE ENLARGING THE VECTOR $\overrightarrow{AC}$ OR WE ARE SHORTENING THE $\overrightarrow{AC}$. WHEN $k > 0$, THE DIRECTION OF $\overrightarrow{AC}$ AND $k\overrightarrow{AC}$ ARE THE SAME, IF $k < 0$, THEN THE VECTORS $\overrightarrow{AC}$ AND $k\overrightarrow{AC}$ ARE IN OPPOSITE DIRECTIONS.

Scalar multiplication of a vector

Definition 7.4

Let $\overrightarrow{AC}$ be any given vector and k be any real number. The vector $k\overrightarrow{AC}$ is the vector whose magnitude is k times the magnitude of $\overrightarrow{AC}$ and,

A the direction of $k\overrightarrow{AC}$ is the same as the direction of $\overrightarrow{AC}$ if $k > 0$

B the direction of $k\overrightarrow{AC}$ is opposite to that of $\overrightarrow{AC}$ if $k < 0$.

EXAMPLE 7 FIGURE 7.30 SHOWS VECTOR $\overrightarrow{AB}$ AND THE RESULT OF MULTIPLYING I RESULT OF MULTIPLY -1.

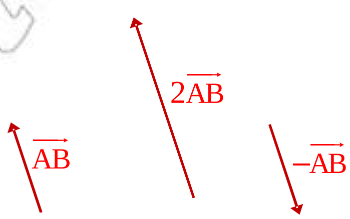

Figure 7.30

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EXAMPLE 8 CONSIDER THE FOLLOWING VECTORS.

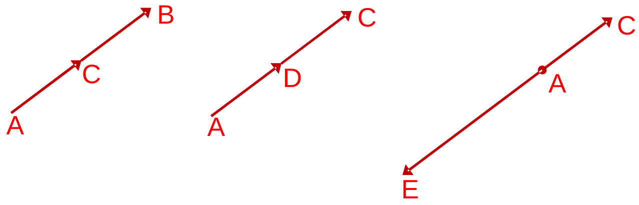

Figure 7.31

$\overrightarrow{AB} = 2\overrightarrow{AC}$, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$ AND $\overrightarrow{AE} = -2\overrightarrow{AC}$

Note: IF $k \neq 0$, THEN ANY VECTOR $\overrightarrow{AC}$ AND $k\overrightarrow{AC}$ ARE PARALLEL VECTORS.

Exercise 7.2

1. ALMAZ WALKS 3 M SOUTH AND THEN 4.35 M WEST. WHAT IS HER DISPLACEMENT FROM THE INITIAL POINT?
2. SIMON MOVES 2 KM WEST FROM A AND 5 KM NORTH TOWARDS B. WHAT IS THE DISPLACEMENT OF SIMON FROM A TO B?
3. BY DRAWING TIP TO TAIL ADD THE FOLLOWING THREE VECTORS:

   $\vec{u} = 25.0$ M NORTH, $\vec{v} = 35.0$ M AT 45 DEGREES EAST OF NORTH AND $\vec{w} = 12.0$ M EAST.
4. FROM FIGURE 7.32, GIVE A SINGLE VECTOR TO REPRESENT THE FOLLOWING

   A $\overrightarrow{AC} + \overrightarrow{CB}$ **B** $\overrightarrow{AD} - \overrightarrow{AC}$

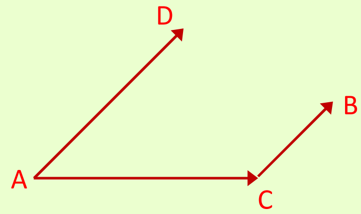

Figure 7.32

5. IF $\vec{u} = 20$ M DUE NORTH AND $\vec{v} = 10$ M AT 30 DEGREES E OF N, FIND $\vec{u} + \vec{v}$ AND $\vec{u} - \vec{v}$
6. FROM THE VECTOR $\overrightarrow{AC}$ GIVEN HERE, DRAW

   A $-3\overrightarrow{AC}$ **B** $\frac{4}{3}\overrightarrow{AC}$ **C** $4\overrightarrow{AC}$ **D** $-\overrightarrow{AC}$

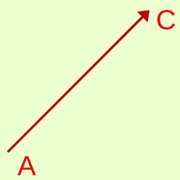

Figure 7.33

7.4 POSITION VECTOR OF A POINT

UP UNTIL NOW, YOU HAVE USED THE GEOMETRIC REPRESENTATION OF VECTORS. NE DISCUSS COMPONENTS OF VECTORS, AND VECTOR OPERATIONS THAT INCLUDE MAGNITUDE AND DIRECTION BY THE USE OF COMPONENTS OF A VECTOR. YOU WILL AL TO USE VECTORS TO DESCRIBE THE POSITION OF A POINT IN A COORDINATE SYSTEM.

Group Work 7.5

1 CONSIDER THE FOLLOWING VECTORS, WHICH *xy*-A... COORDINATE SYSTEM. MOVE EACH VECTOR SO T... IAL POINTS ARE AT THE ORIGIN.

A HOW DO YOU DIFFERENTIATE ONE VECTOR FROM ANOTHER?

B THE INITIAL POINT OF ANY OF THOSE VECTORS IS (0, 0). HOW DO YOU EXPRESS TERMINAL POINT?

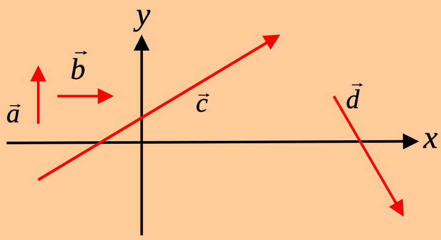

Figure 7.34

2 IF $\overrightarrow{AB}$ IS THE VECTOR WITH INITIAL POINT A= (1, 2) AND THE TERMINAL POINT (3, 4) W WILL ITS TERMINAL POINT BE IF ITS INITIAL POINT IS MOVED TO THE ORIGIN?

3 IF $\vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ REPRESENTS A VECTOR WITH INITIAL POINT AT THE ORIGIN, THEN HOW D EXPRESS $\vec{v}$ IN TERMS OF THE COORDINATES (2, 0) AND (0, 5)?

FROM PREVIOUS DISCUSSIONS, NOTICE THAT THE VECTORS REPRESENTED IN FIGURE 7.35 THAT HAVE DIFFERENT INITIAL PONTS ARE EQUAL. OF THESE VECTORS, THE ONE WHOSE INITIAL POINT IS THE ORIGIN IS CALLED THE standard form OF THE PRESENTATION OF THE VECTOR (OR SIMPLY, THE position vector).

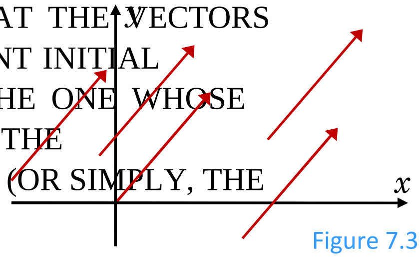

Figure 7.35

ANALYTICALLY, WE USUALLY EXPRESS VECTORS IN COMPONENT FORM. WE DO THIS B THE VECTOR WITH THE ORIGIN AS ITS INITIAL POINT AND WRITE THE COORDINATES POINT AS A "COLUMN VECTOR". FOR EXAMPLE, IN TWO DIMENSIONS, IF $\vec{v} = \overrightarrow{OP}$, WHERE O IS (0, 0) AND P IS THE POINT (x, y) THEN $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Note: SUCH COLUMN VECTORS ARE WRITTEN VERTICALLY, TO DISTINGUISH THEM FROM COORDINATES. ITS GEOMETRIC REPRESENTATION IS AS GIVEN BELOW.

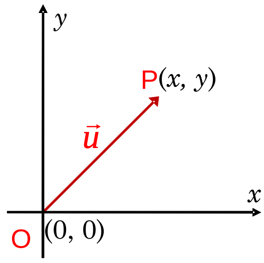

Figure 7.36

SINCE THE VECTOR $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ HAS O(0, 0) AS ITS INITIAL POI P(x, y) AS ITS TERMINAL P(

MAGNITUDE $|\vec{u}|$ IS $\sqrt{x^2 + y^2}$ WHICH IS THE LENGTH OF THE DIRECT (0, 0) TO P(x, y).

ACTIVITY 7.8

CONSIDER A VECTOR $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

1. REPRESENT IT GEOME.
2. APPLYING THE TRI.LAW OF VECTOR ADDITION DETERMINE THE COMPO $\vec{u}$.
3. FIND THE MAGNIT $\vec{u}$.
4. DETERMINE THE DIRECTION OF $\vec{u}$.

CONSIDER THE FOLLOWI

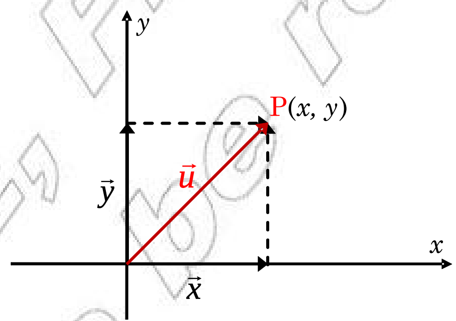

Figure 7.37

YOU CAN SEE THAT THE $\vec{u}$ CAN BE EXPRESSED AS SUM OF THE VE $\vec{x}$ AND $\vec{y}$ AS $\vec{u} = \vec{x} + \vec{y}$ FROM THE PARALLELOGRAM LA TRIANGLE LAW WHERE $\vec{x} = x\vec{i}$ AND $\vec{y} = y\vec{j}$ IN WHICH $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ AND $\vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. FROM THIS, WHEN $\vec{u} = x\vec{i} + y\vec{j}$, THE x AND y IN $\vec{u} = x\vec{i} + y\vec{j}$ ARE CALLED **components** OF $\vec{u}$.

THESE COMPONENTS ARE USEFUL IN DETERMINING THE DIF

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EXAMPLE 1 REPRESENT THE FOLLOWING VECTOR AS A POSITION VECTOR.

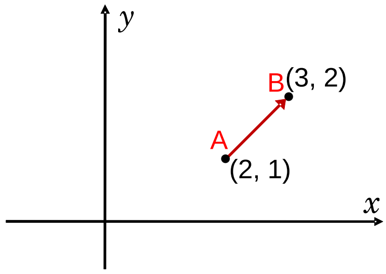

Figure 7.38

TO REPRESENT A POSITION VECTOR $\overrightarrow{AB}$, WE NEED TO CONSTRUCT A VECTOR WHICH HAS THE SA LENGTH AND SAME DIRECTION AS THE VECTOR $\overrightarrow{AB}$. THAT IS, WE NEED TO CONSTRUCT A VECTOR WHOSE ORIGIN IS O (0, 0) AND WHOSE TERMINAL IS $P(x_2 - x_1, y_2 - y_1)$ WHERE (x_1, y_1) IS THE INITIAL AND (x_2, y_2) IS THE TERMINAL POINT OF THE GIVEN VECTOR.

HENCE THE POSITION VECTOR OF THE VECTOR $\overrightarrow{AB}$ GIVEN ABOVE IS

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ OR } \vec{u} = \vec{i} + \vec{j} \text{ WHERE } \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ AND } \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

FROM THIS, WE CAN DETERMINE THE MAGNITUDE AND THE DIRECTION OF THE VECTOR

EXAMPLE 2 FOR THE VECTOR GIVEN BY $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, ITS GEOMETRIC REPRESENTATION IS GIVEN BELOW. FIND THE MAGNITUDE AND DIRECTION OF THE VECTOR.

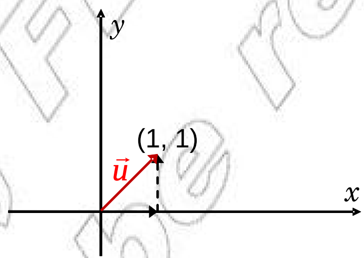

Figure 7.39

SOLUTION: FROM THIS GEOMETRIC REPRESENTATION AND FROM THE TRIGONOMETRI THAT YOU DISCUSSED IN CHAPTER FIVE, WE CAN DETERMINE THE DIRECT VECTOR.

$$\text{TAN} = \frac{\text{OPPOSITE}}{\text{ADJACENT}} \Rightarrow \text{TAN} = \frac{1}{1} =$$ THE ACUTE ANGLE WHOSE TANGENT VALUE IS 1 IS 45°

HENCE, THE DIRECTION OF THE VECTOR IS 45°

THE MAGNITUDE OF THE VECTOR IS ALSO $|\vec{u}| = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$

EXAMPLE 3 FIND THE POSITION VECTOR OF THE FOLLOWING VECTORS WHOSE INI TERMINAL POINTS ARE AS GIVEN BELOW.

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A INITIAL POINT (1, 2) AND TERMINAL POINT (2, 5)

B INITIAL POINT (−2, 3) AND TERMINAL POINT (1, 4)

SOLUTION:

A THE POSITION VECTOR $\vec{u}$ OF THE VECTOR WHOSE INITIAL POINT IS (1, 2) AND TERMINAL POINT IS (2, 5) IS (2 − 1, 5 − 2) = (1, 3).

THAT IS, $\vec{u} = \vec{i} + 3\vec{j}$ WHICH WILL BE REPRESENTED AS $\vec{u} = \begin{pmatrix}1\\3\end{pmatrix}$

B THE POSITION VECTOR $\vec{u}$ OF THE VECTOR WHOSE INITIAL POINT IS (−2, 3) AND TERMINAL POINT IS (1, 4) IS (1 − (−2), 4 − 3) = (3, 1). THAT IS, $\vec{u} = 3\vec{i} + \vec{j}$ WHICH WILL BE $\vec{u} = \begin{pmatrix}3\\1\end{pmatrix}$.

THE GEOMETRIC REPRESENTATION OF THESE VECTORS IS GIVEN BELOW.

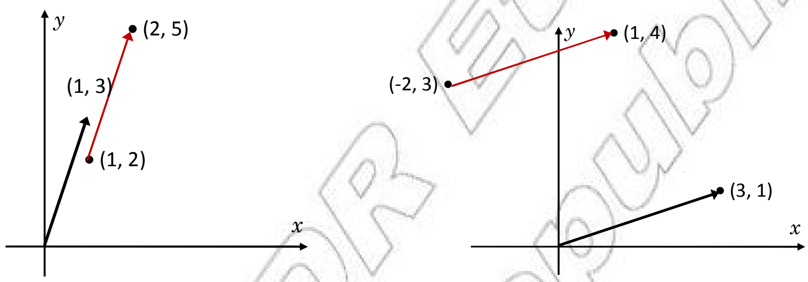

Figure 7.40

Exercise 7.3

1 WRITE EACH THE FOLLOWING POSITION VECTORS IN FIGURE 7.41 AS COLUMN VECTORS.

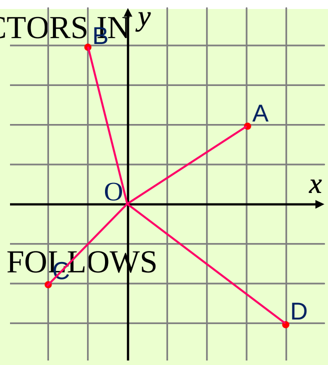

Figure 7.41

A $\overrightarrow{OA}$ **B** $\overrightarrow{OB}$

C $\overrightarrow{OC}$ **D** $\overrightarrow{OD}$

2 THE COORDINATES OF SOME POINTS ARE AS FOLLOWS

P (2, 5), Q (–1, –2), R (0, –4) AND S (3, –7)

WHAT IS THE POSITION VECTOR OF

A Q RELATIVE TO P? **B** R RELATIVE TO S? **C** S RELATIVE TO Q?

3 THE POSITION VECTOR OF X RELATIVE TO Y IS $\overrightarrow{YX} = \begin{pmatrix}2\\-5\end{pmatrix}$.

A WHAT IS THE POSITION VECTOR OF Y RELATIVE TO X?

B IF X HAS COORDINATES (−1, 3), WHAT ARE THE COORDINATES OF Y?

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c IF M IS THE MIDPOI $\overrightarrow{XY}$ WHAT IS $\overrightarrow{XM}$?

d WHAT IS THE POSITION V $\overrightarrow{OM}$?

4 REPRESENT THE VECTORS, WHOSI (I) AND TERMINAL POI ARE GIVEN BE GEOMETRICALLY ON A C SYSTEM.

a I(1, 4) AND T(3, : b I(–2, 2) AND T(1, (

5 DETERMINE THE POSITION VECTOR OF EACH OF TH QUESTION ABOVE.

6 DETERMINE THE MAGNITUDE AND THE DIRECTION OF EACH OF QUESTION ABOVE.

Key Terms

addition of vectors	position vector
direction of a vector	scalar quantities
equality of vectors	subtraction of vectors
magnitude(length of) a vector	triangle law of vector addition
parallelogram law of vector addition	vector quantities

Summary

1 A **scalar** IS A MEASURE THAT INVOLVES ONI AND NO DIRECTIOE A VECTOR INVOLVES BOTH MAGNITUDE

2 A **vector** IS DENOTED A DIRECTED ARROW. ITS LENGTH IS CALLED THE MAGN DIRECTION IT POINTS I THE DIRECTION OF THE VECTOR

3 VECTORS INCLUDE VELOCITY, FORCE, ACCELERATION, ELE, ETC.

4 A VECTOR IS REPRESENTED BY A $\overrightarrow{OP}$); THE POINT O IS CALLED THE IN AND P IS CALLED THE TERMINAL POINT. SOMETIMES, VECTORS AR LETTERS OR A LETTER WITH A BAR $\vec{u}$, $\vec{v}$, ETC.

5 THE MAGNITUDE VELOCITY IS THE SPEED; THE MAGNITUDE OF A DISTANCE. THUS, SPEED AND DISTANCE ARE S

6 A MAGNITUDE IS ALWAYS A POSITI

7 VECTORS CAN BE DESCRIBED GEOMETRICALLY: GEOMETRICALLY DIRECTED ARROV GEBRAICALL COLUMN VECTOR.

8 TWO VECTORS ARE SA **equal** IF THEY HAVE THE SAME MAGNITUDE A DIRECTION.

9 IF TWO VECTORS HAVE SAME OR OPPOSITE DIRECT **parallel**.

10 FOR ANY TWO VE $\overrightarrow{AB}$ AND $\overrightarrow{BC}$, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ (THE TRIANG AV)

11 A VECTOR THAT HAS NO MAGNITUDE AND DIRECTION IS CALLED A ZERO VECTOR O

12 THE DIAGONAL OF A PARALLELOGRAM IS THE SUM OF THE SIDE VECTORS. THIS IS CALLED THE Parallelogram Law.

13 SUBTRACTION OF VECTORS $\overrightarrow{AE}$ AND $\overrightarrow{AC}$, GIVEN AS $\overrightarrow{AE}-\overrightarrow{AC}=\overrightarrow{CE}$ IS THE SAME AS $\overrightarrow{AC}+\overrightarrow{CE}=\overrightarrow{AE}$

14 MULTIPLYING A VECTOR BY A SCALAR k ENLARGES OR SHORTENS THE VECTOR. IF $|k|>1$, IT ENLARGES THE VECTOR AND IF $|k|<1$ IT SHORTENS THE VECTOR. IF $k>0$, THE DIRECTION OF THE VECTOR IS UNCHANGED; MULTIPLYING A VECTOR BY $k<0$ CHANGES THE DIRECTION OF THE VECTOR INTO THE OPPOSITE DIRECTION.

15 IF THE INITIAL AND TERMINAL POINTS OF A VECTOR ARE (x_1, y_1) AND (x_2, y_2), THEN ITS POSITION VECTOR CAN BE CALCULATED AS $P = (x_2 - x_1, y_2 - y_1)$ AND IS DENOTED BY $P = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$.

Review Exercises on Unit 7

1 DESCRIBE WHAT IS MEANT BY SCALAR AND VECTOR QUANTITIES.

2 WRITE DOWN SOME EXAMPLES OF SCALAR AND VECTOR QUANTITIES.

3 HOW DO YOU REPRESENT A VECTOR GEOMETRICALLY?

4 SKETCH A VECTOR WHOSE MAGNITUDE IS 3 CM IN THE DIRECTION OF

A EAST **B** NORTH 30° EAST

5 SKETCH A VECTOR OF LENGTH 5 CM WHOSE DIRECTION IS

A NORTH 45° EAST **B** WEST **C** SOUTH 20° EAST

6 WHEN ARE TWO VECTORS PARALLEL?

7 SHOW, BY A DIAGRAM, THAT THE SUM OF TWO VECTORS $\overrightarrow{AB}$ AND $\overrightarrow{BC}$ IS $\overrightarrow{AC}$.

8 IF THE MAGNITUDE OF $\overrightarrow{AC}$ IS 4 CM, FIND THE MAGNITUDE OF

A $3\overrightarrow{AC}$ **B** $\frac{1}{4}\overrightarrow{AC}$ **C** $-\overrightarrow{AC}$

9 IF A IS THE POINT $(-2, 4)$ AND B IS THE POINT $(-3, -6)$, WHAT IS THE POSITION VECTOR OF B RELATIVE TO A?

10 THE POSITION VECTOR OF P (RELATIVE TO THE ORIGIN) IS $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$. IF THE MAGNITUDE OF $\overrightarrow{OP}$ IS 5 UNITS, FIND THE SET OF ALL POSSIBLE VALUES OF $\begin{pmatrix} x \\ y \end{pmatrix}$, WITH $x, y \in \mathbb{Z}$.

Table of Trigonometric Functions

	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	.....	1.000	.....	**90°**
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	**89°**
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	**88°**
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	**87°**
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	**86°**
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	**85°**
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	**84°**
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	**83°**
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	**82°**
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	**81°**
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	**80°**
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	**79°**
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	**78°**
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	**77°**
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	**76°**
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	**75°**
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	**74°**
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	**73°**
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	**72°**
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	**71°**
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	**70°**
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	**69°**
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	**68°**
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	**67°**
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	**66°**
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	**65°**
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	**64°**
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	**63°**
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	**62°**
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	**61°**
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	**60°**
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	**59°**
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	**58°**
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	**57°**
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	**56°**
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	**55°**
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	**54°**
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	**53°**
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	**52°**
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	**51°**
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	**50°**
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	**49°**
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	**48°**
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	**47°**
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	**46°**
45°	0.7071	0.7071	1 .0000	1.000	1.414	1.414	**45°**
	cos	**sin**	**cot**	**tan**	**csc**	**sec**	

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