MATHEMATICS

TEACHER'S GUIDE

GRADE 9

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA

MINISTRY OF EDUCATION

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INTRODUCTION

The study of mathematics at this cycle, grades 9 -10, prepares our students for the future, both practically and philosophically. Studying mathematics provides them not only with specific skills in mathematics, but also with tools and attitudes for constructing the future of our society. As well as learning to think efficiently and effectively, our students come to understand how mathematics underlies daily life and, on a higher level, the dynamics of national and international activity. The students automatically begin to apply high-level reasoning and values to daily life and also to their understanding of the social, economic, political and cultural realities of the country. In turn, this will help them to actively and effectively participate in the ongoing process of developing the nation.

At this cycle, our students gain a solid knowledge of the fundamental mathematical theories, theorems, rules and procedures. They also develop reliable skills for using this knowledge to solve problems independently. To this end, the objectives of mathematics learning at this cycle are to enable students to

- gain a solid knowledge of mathematics.
- appreciate the power, elegance and structure of mathematics.
- use mathematics in daily life.
- understand the essential contributions of mathematics to the fields of engineering, science, economics and so on.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The rote-learning paradigm has been replaced by the student-centered model. A student-centered classroom stimulates student inquiry, and the teacher serves as a mentor who guides students as they construct their own knowledge base and skills. A primary goal when you teach a concept is for the students to discover the concept for themselves, particularly as they recognize threads and patterns in the data and theories that they encounter under your guidance.

One of our teaching goals is particularly fostered by the student-oriented approach. We want our students to develop personal qualities that will help them in real life.

For example, student-oriented teachers encourage students' self confidence and their confidence in their knowledge, skills and general abilities. We motivate our students to express their ideas and observations with courage and confidence. Because we want them to feel comfortable addressing individuals and groups and to present themselves and their ideas well, we give them safe opportunities to stand before the class and present their work. Similarly, we help them learn to learn to answer questions posed directly to them by other members of the class.

Teamwork is also emphasized in a student-centered classroom. For example, the teacher creates favorable conditions for students to come together in groups and exchange ideas about what they have learned and about material they have read. In this process, the students are given many opportunities to openly discuss the knowledge they have acquired and to talk about issues raised in the course of the discussion.

This teacher's guide will help you teach well. For example, it is very helpful for budgeting your teaching time as you plan how to approach a topic. The guide suggests tested teaching-time periods for each subject you will teach. Also, the guide contains answers to the review questions at the end of each topic.

Each section of your teacher's guide includes student-assessment guidelines. Use them to evaluate your students' work. Based on your conclusions, you will give special attention to students who are working either above or below the standard level of achievement. Check each student's performance against the learning competencies presented by the guide. Be sure to consider both the standard competencies and the minimum competencies. Note that the *minimum requirement level* is not the *standard level of achievement*. To achieve the standard level, your students must fulfill all of their grade-level's competencies successfully.

When you identify students who are working either below the standard level or below the minimum level, give them extra help. For example, give them supplementary presentations and reviews of the material in the section, give them extra time to study, and develop extra activities to offer them. You can also encourage high-level students in this way. You can develop high-level activities and extra exercises for them and can offer high-level individual and group discussions. Be sure to show the high-level students that you appreciate their good performance, and encourage them to work hard. Also, be sure to discourage any tendencies toward complacency that you might observe.

Some helpful reference materials are listed at the end of this teacher's guide. For example, the internet is a rich resource for teachers, and searching for new web sites is well worth your time as you investigate your subject matter. Use one of the many search engines that exist – for example, Yahoo and Google.

Do not forget that, although this guide provides many ideas and guidelines, you are encouraged to be innovative and creative in the ways you put them into practice in your classroom. Use your own knowledge and insights in the same way as you encourage your students to use theirs.

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ACTIVE LEARNING AND CONTINUOUS ASSESSMENT REQUIRED!

Dear mathematics teacher! For generations the technique of teaching mathematics at any level was dominated by what is commonly called the **direct instruction.** That is, students are given the exact tools and formulas they need to solve a certain mathematical problem, sometimes without a clear explanation as to why, and they are told to do certain steps in a certain order and in turn are expected to do them as such at all times. This leaves little room for solving varying types of problems. It can also lead to misconceptions and students may not gain the full understanding of the concepts that are being taught.

You just sit back for a while and try to think the most common activities that you, as a mathematics teacher, are doing in the class.

> *Either you explain (lecture) the new topic to them, and expect your students to remember and use the contents of this new topic or you demonstrate with examples how a particular kind of problem is solved and students routinely imitate these steps and procedures to find answers to a great number of similar mathematical problems.*

But this method of teaching revealed little or nothing of the meaning behind the mathematical process the students were imitating.

We may think that teaching is telling students something, and learning occurs if students remember it. But research reveals that teaching is not "pouring" information into students' brain and expecting them to process it and apply it correctly later.

Most educationalists agree that learning is an active meaning-making process and students will learn best by trying to make sense of something on their own with the teacher as a guide to help them along the way. This is the central idea of the concept Active Learning.

> **Active learning**, *as the name suggests, is a process whereby learners are actively engaged (involved) in the learning process, rather than "passively" absorbing lectures. Students are rather encouraged to think, solve problems, do activities carefully selected by the teacher, answer questions, formulate questions of their own, discuss, explain, debate, or brainstorm, explore and discover, work cooperatively in groups to solve problems and workout projects.*

The design of the course materials (student textbooks and teachers guides) for mathematics envisages active learning to be dominantly used. With this strategy, we feel that you should be in a position to help students understand the concepts through relevant, meaningful and concrete activities. The activities should be carried out by students to explore the world of mathematics, to learn, to discover and to develop interest in the subject. Though it is your role to exploit the opportunity of using active learning at an optimal level, for the sake of helping you get an insight, we recommend that you do the following as frequently as possible during your teaching:

- Engage your students in more relevant and meaningful activities than just listening.
- Include learning materials having examples that relate to students life, so that they can make sense of the information.
- Let students be involved in dialog, debate, writing, and problem solving, as well as higher-order thinking, e.g., analysis, synthesis, evaluation.
- Encourage students' critical thinking and inquiry by asking them thoughtful, open-ended questions, and encourage them to ask questions to each other.
- Have the habit of asking learners to apply the information in a practical situation. This facilitates personal interpretation and relevance.
- Guide them to arrive at an understanding of a new mathematical concept, formula, theorem, rule or any generalization, by themselves. You may realize this by giving them an activity in which students sequentially uncover layers of mathematical information one step at a time and discover new mathematics.
- Select assignments and projects that should allow learners to choose meaningful activities to help them apply and personalize the information. These need to help students undertake initiatives, discover mathematical results and even design new experiments to verify results.
- Let them frequently work in peers or groups. Working with other learners gives learners real-life experience of working in a group, and allows them to use their metacognitive skills. Learners will also be able to use the strengths of other learners, and to learn from others. When assigning learners for group work membership, it is advisable if it is based on the expertise level and learning style of individual group members, so that individual team members can benefit from one another's strengths.

In general, if mathematics is to develop creative and imaginative mathematical minds, you must overhaul your traditional methods of presentation to the more active and participatory strategies and provide learning opportunities that allow your students to be actively involved in the learning process.

While students are engaged with activities, group discussions, projects, presentations and many others they need to be continuously assessed.

CONTINUOUS ASSESSMENT

You know that continuous assessment is an integral part of the teaching learning process. Continuous assessment is the periodic and systematic method of assessing and evaluating a person's attributes and performance. Information collected from continuous behavioral change of students will help teachers to better understand their strengths and weaknesses in addition to providing a comprehensive picture of each student over a period of time. Continuous assessment will afford student to readily see his/her development pattern through the data. It will also help to strengthen the parent teacher relationship and collaboration. It is an ongoing process more than giving a test or exam frequently and recording the marks.

Continuous assessment enables you to assess a wide range of learning competencies and behaviors using a variety of instruments some of which are:

- Tests/ quizzes (written, oral or practical)
- Class room discussions, exercises, assignments or group works.
- Projects
- Observations
- Interview
- group discussions
- questionnaires

Different competencies may require different assessment techniques and instruments. For example, oral questions and interviews may serve to assess listening and speaking abilities. They also help to assess whether or not students are paying attention, and whether they can correctly express ideas. You can use oral questions and interviews to ask students to restate a definition, note or theorem, etc. Questionnaires, observations and discussions can help to assess the interest, participation and attitudes of a student. Written tests/exams can also help to assess student's ability to read, to do and correctly write answers for questions.

When to Assess

Continuous assessment and instruction are integrated in three different time frames namely, Pre-instruction, During-instruction and Post-instruction. To highlight each briefly

1. Pre-instruction assessment

This is to assess what students luck to start a lesson. Hence you should start a lesson by using opportunities to fill any observed gap. If students do well in the pre-instruction assessment, then you can begin instructing the lesson. Otherwise, you may need to revise important concepts.

The following are some suggestions to perform or make use of pre-instruction assessment.

i. assess whether or not students have the prerequisite knowledge and skill to be successful, through different approaches.
ii. make your teaching strategies motivating.
iii. plan how you form groups and how to give marks.
iv. create interest on students to learn the lesson.

2. Assessment During Instruction:

This is an assessment during the course of instruction rather than before it is started or after it is completed. The following are some of the strategies you may use to assess during instruction.

i. observe and monitor students' learning.

ii. check that students are understanding the lesson. You may use varying approaches such as oral questions, asking students to do their work on the board, stimulate discussion, etc.

iii. identify which students need extra help and which students should be left alone.

iv. ask a balanced type of exercise problems according to the students ability, help weaker students and give additional exercise for fast students.

v. monitor how class works and group discussions are conducted

3. Post Instruction Assessment: !!!

This is an assessment after instruction is completed. It is conducted usually for the purpose of documenting the marks and checking whether competencies are achieved. Based on the results students scored, you can decide whether or not there is anything the class didn't understand because of which you may revise some of the lessons or there is something you need to adjust on the approach of teaching. This also help you analyze whether or not the results really reflect what students know and what they can do, and decide how to treat the next lesson.

Forming and managing groups

You can form groups through various approaches: mixed ability, similar ability, gender or other social factors such as socioeconomic factors. When you form groups, however, care need to be taken in that you should monitor their effort. For example, if students are grouped by mixed ability the following problems may happen.

1. Mixed ability grouping may hold back high-ability students. Here, you should give enrichment activities for high ability students.
2. High ability students and low ability students might form a teacher – student relationship and exclude the medium ability students from group discussion. In this case you should group medium ability students together.

When you assign group work, the work might be divided among the group members, who work individually. Then the members get together to integrate, summarize and present their finding as a group project. Your role is to facilitate investigation and maintain cooperative effort.

Highlights about assessing students

You may use different instruments to assess different competencies. For example, consider each of the following competencies and the corresponding assessment instruments.

Competency 1. Define quadratic equation.

Instrument: Oral question.

Question: What is a quadratic equation?

Competency 2 - Students will solve quadratic equations.

Instrument: class work/homework/ quiz /test

Question: a. Solve $x^2 + 7x - 8 = 0$.

b. Solve $2x^2 + 6x - 8 = 0$.

Competency 3 – Apply quadratic equations in daily life problems.

Instrument: Exam/Assignment/project.

Question: A farmer has a rectangular plot of land whose area is $1120m^2$. If the perimeter is 136m, find the dimensions of the plot of land.

How often to assess !

Here are some suggestions which may help you how often to assess.

- Class activities / class works: Every day (when convenient).
- Homework/Group work: as required.
- Quizzes: at the end of every one (or two) sub topics.
- Tests: at the end of every unit.
- Exams: once or twice in every semester.

How to Mark

The following are some suggestions which may help you get well prepared before you start marking:

- use computers to reduce the burden for record keeping.
- although low marks may diminish the students motivation to learn, don't give inflated marks for inflated marks can also cause reluctance.

The following are some suggestions on how to mark a semester's achievement.

1. One final semester exam 30%.
2. Tests 25%
3. Quizzes 10%
4. Homework 10%
5. Class activities, class work, presentation demonstration skills 15%
6. Project work, in groups or individually 10%.

Moreover

In a group work allow students to evaluate themselves as follows using format of the following type.

	A	***B***	***C***	***D***	
The ability to communicate					
The ability to express written works					
Motivation					
Responsibility					
Leadership quality					
Concern for others					
Participation					
Over all					

You can shift the leadership position or regroup the students according to the result of the self evaluation. You can also consider your observation.

Reporting students' progress and marks to parents

Parents should be informed about their children's progress and performance in the class room. This can be done through different methods.

1. The report card: two to four times per year.
2. Written progress report: Per week/two weeks/per month/two months.
3. Parent – teacher conferences (as scheduled by the school).

The report should be about the student performance say, on tests, quizzes, projects, oral reports, etc that need to be reported. You can also include motivation or cooperation behavior. When presenting to parents your report can help them appraise fast learner, pay additional concern and care for low achieving student, and keep track of their child's education. In addition, this provides an opportunity for giving parents helpful information about how they can be partners with you in helping the student learn more effectively.

The following are some suggested strategies that may help you to communicate with parents concerning marks, assessment and student learning.

1. Review the student's performance before you meet with parents.
2. Discuss with parents the students good and poor performances.
3. Do not give false hopes. If a student has low ability, it should be clearly informed to his/her parents.
4. Give more opportunities for parents to contribute to the conversation.
5. Do not talk about other students. Don't compare the student with another student.
6. Focus on solutions

NB. ***All you need to do is thus plan what type of assessment and how many of each you are going to use beforehand (preferably during the beginning of the year/semester).***

UNIT 1 THE NUMBER SYSTEM

INTRODUCTION

The main task of this unit is to survey systematically the numbers we have been dealing with so far and to calculate with them. The intention is to make the students familiar with the notion of real numbers and calculating with them. As an introduction, you can revise the number systems like integers and rational numbers and their essential properties that were covered in their previous grade levels.

The classification of the real numbers as rational and irrational numbers should be clear to the students. To let them know irrational numbers, locating a point on the number line and trying to represent it as a rational number will be considered. Locating a point on the number line was also dealt with previously. In Grade 8, students have already learned that there are points on the number line to which no rational number can be assigned. In this unit, students will learn about irrational numbers and real numbers. The notion of irrational number as infinite non periodic decimal will also be discussed in this unit. The correspondence between number and point on a number line should be stressed.

In addition to these, students should study how the concept of the square root of a number such as $\sqrt{2}$ leads to the definition of an irrational number. In relation to this, the concept of radicals, the notion of rationalization and its use in simplifying expressions involving radicals should be covered. After dealing with the number systems, some related concepts such as approximation, accuracy and scientific notation will be discussed.

Unit Outcomes

After completing this unit, students will be able to;

- *know basic concepts and important facts about real numbers.*
- *justify methods and procedures in computation with real numbers.*
- *solve mathematical problems involving real numbers.*

Suggested Teaching Aids in Unit1

Although teaching aids may not be excessively exploited for this unit, you can present different charts that manifest squares of whole numbers and multiplication tables. However, in the teaching of irrational numbers, you need a pair of compass and ruler to locate irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, etc. on the number line. You also need scientific calculators.

1.1 REVISION ON THE SET OF RATIONAL NUMBERS

Periods allotted: 3 periods

Competencies

At the end of this subunit, the students will be able to:

- *identify natural numbers, whole numbers and integers.*
- *define prime numbers and composite numbers.*
- *determine common factors and common multiples of pairs of numbers.*

Vocabulary: Factor, Multiple, Prime number, Composite number, Prime factorization

Introduction

This sub-unit deals with revising the set of rational numbers together with their important properties. However, this is done first by discussing the set of natural numbers, prime numbers, composite numbers and integers. Related to these numbers, the concepts of factors, multiples, prime factorization, common factors, common multiples, greatest common factor and least common multiple are discussed. For each concept, an activity and group work are provided to refresh the memory of the students or to guide them to the concepts.

Teaching Notes

So as to begin, it is better to motivate the students by giving an insight of the course and the units. You can also highlight the subtopics of this unit. Following these discussions, you can continue to discuss the subsections.

1.1.1 Natural Numbers, Integers, Prime Numbers and Composite Numbers

You can start the lesson by revising the natural numbers, prime numbers, composite numbers and integers. To do this, you may use Activity 1.1 for the purpose of revising student's prior knowledge on the various number systems. Group your students and let them discuss Activity 1.1. After they discuss in group, let some of the groups present their discussion to the whole class. You can then facilitate their discussion. For more facts about the sets of natural numbers, prime numbers, composite numbers and integers, you can proceed to Activity 1.2 given on page 3 of the student text. Group the students and ask them to do the activity. Let some of the groups present their work to the class. Then start discussing the answer to each question with the students. This will again lead you to discuss the definitions about the vocabularies (terms) indicated above. Make sure that students understand the definitions and concepts given in the lesson; in particular, make sure that they can distinguish between the set of natural numbers, and integers, prime numbers and composite numbers. Some students confuse prime numbers with odd numbers. Here, you are expected to make sure that students are able to distinguish between a prime number and an odd number.

Answers to Activity 1.1

1. a. $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, and $\mathbb{Q}$ b. $\mathbb{Z}$, $\mathbb{Q}$ c. $\mathbb{Q}$
   d. $\mathbb{Q}$ e. $\mathbb{Q}$
2. i. a. The set of natural numbers denoted by $\mathbb{N}$, is described by $\mathbb{N} = \{1, 2, 3, \ldots\}$.
   b. The set of whole numbers denoted by $\mathbb{W}$ is described by $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.
   c. The set of integers denoted by $\mathbb{Z}$ is described by $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
   d. The set of rational numbers denoted by $\mathbb{Q}$ is described by
   $$\mathbb{Q} = \left\{\frac{a}{b} : a \text{ and } b \text{ are integers and } b \neq 0\right\}.$$
   ii. a. $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

Answers to Activity 1.2

1. a. True
   b. False (because 0 is not in the set.)
   c. True
   d. True
   e. False (because 1 is neither prime nor composite)
   f. True
   g. True
   h. False (because units digit of 72 is neither 0 nor 5)
   i. True
   j. False (because 87 is not even. Thus 2 cannot be a common factor.)
   k. False (because there are infinite multiples such as, 36, 48, . ..)
   l. True
2. a. If there is a natural number c such that $b = a \times c$
   b. If there is a natural number d such that $a = b \times d$
   c. If there is a natural number d such that $a = b \times d$

In order for students to be able to distinguish between a prime number and a composite number, it is necessary that students should know how to find factors of a number. To this end, students need to revise the Divisibility Test. After introducing divisibility test given on page 4 of the student textbook as an illustration you have to discuss example 1 which helps students how to determine the divisibility of a certain number. Further to get into these concepts, you can let your students do Group work 1.1. The purpose of Group work 1.1 is to enable the students realize how to use divisibility test to find factors of a given number. So you are expected to discuss with the students how to find factors of a given number using the divisibility test. While applying the divisibility test; if a number does not have factors other than 1 and itself, then the number is prime otherwise it will be composite. That is, a number that has at least one factor other than 1 and itself is a composite number.

Answers to Group work 1.1

1. To list all the factors of 24, we check all those natural numbers that divide 24 starting from the smallest natural numbers.

Thus 1 divides 24 ∴ 1 is a factor of 24
2 divides 24 ∴ 2 is a factor of 24
3 divides 24 ∴ 3 is a factor of 24
4 divides 24 ∴ 4 is a factor of 24
5 does not divide 24 ∴ 5 is not a factor of 24
6 divides 24 ∴ 6 is a factor of 24.
7 does not divide 24 ∴ 7 is not a factor of 24
8 divides 24 ∴ 8 is a factor of 24
9, 10 , 11 do not divide 24 ∴ they are not factors of 24
12 divides 24 ∴ 12 is a factor of 24
24 divides 24 ∴ 24 is a factor of 24
∴ All factors of 24 are the numbers 1, 2, 3, 4, 6, 8, 12 and 24.

2. To find the possible length and width of the rectangle means to find all factors of 432. We find that all factors of 432 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 108, 144, 216, 432.

3. To find the prime factorization of 360, we first factorize 360 into 36 and 10 and then factorize the factors into prime factors. Thus $360 = 36 \times 10$

$$= 4 \times 9 \times 2 \times 5$$
$$= 2^2 \times 3^2 \times 2 \times 5$$
$$= 2^3 \times 3^2 \times 5$$

$\therefore 360 = 2^3 \times 3^2 \times 5$

From the group work, you may ask the students to envisage if there is any short cut way to determine the divisibility of a number so that obtaining its factors can be easier. Following their imagination, it could be possible to discuss divisibility tests. The divisibility tests for checking the divisibility of a number are listed on page 4 of the student text.

Application of divisibility test in finding factors

Example 1 Find all factors of 24.

Solution i. obviously 24 is divisible by 1 $\therefore 24 = 1 \times 24$

ii. by the divisibly test, 24 is divisible by 2 $\therefore 24 = 2\times 12$

iii. by the divisibly test, 24 is divisible by 3$\therefore 24 = 3\times 8$

iv. by the divisibly test, 24 is divisible by 4 $\therefore 24 = 4\times 6$

$\therefore$ All Factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

Divisibility tests are also useful to check whether or not a prime number divides a given number, and these divisibility tests can also be used to find all prime numbers that are less than or equal to a given number.

Example 2 List all prime numbers that are less than or equal to 50 using divisibility tests.

Solution: Make an array of the first 50 natural numbers as shown in the table below. Skip the number 1. Then cross out all the numbers greater than 2 that are divisible by 2 (every even number except 2).

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
1	~~12~~	1	~~14~~	~~15~~	~~16~~	1	~~18~~	1	~~20~~
~~21~~	~~22~~	2	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	2	~~30~~
3	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	3	~~38~~	~~39~~	~~40~~
4	~~42~~	4	~~44~~	~~45~~	~~46~~	4	~~48~~	~~49~~	~~50~~

Next, cross out all numbers divisible by 3 (even if it had been crossed out before) but don't cross out 3. Notice that 4, the next consecutive number, has already been crossed out. Cross out all numbers divisible by 5 except 5 itself. Continue this process for every natural number up to 50 that has not been crossed out. When this task is completed, circle the numbers not crossed out; the array will look like the table shown.

The numbers circled in this array are prime numbers less than or equal to 50 and the numbers crossed out are composite numbers less than or equal to 50. Notice again that 1 is neither prime nor composite. That is why 1 is excluded in the above example. You can give such a type of exercises for students to do.

The procedure used above to find the prime numbers is called the **sieve of Eratosthenes**.

After making sure that the students can determine factors and the prime numbers below some number, you can proceed to finding the prime factorization of numbers which is a very useful way of determining factors of a number.

Finding prime factorization

For the purpose of finding prime factorization of a number, note to the students that it will be essential to consider that the number under consideration is composite. This is so because, if the number is prime, there is no need to look for prime factorization. If the number is composite, however, there is prime factorization and let the students understand that there are two methods for determining prime factorization of a number. The first method consists of repeated division starting with the smallest prime that divides the number. That is, dividing by the smallest prime number as long as it divides the last quotient; then going to the next larger prime; and continuing until all prime factors have been obtained. Notice that this method is employed in Example 2 of page 6 in the student text.

For example, to find the prime factors of 72, we proceed as follows.

$72 \div 2 = 36$
$36 \div 2 = 18$
$18 \div 2 = 9$ here 9 cannot be divided by 2, and thus we need to consider the next larger prime number. That is, 3.
$9 \div 3 = 3$
$3 \div 3 = 1$
$\therefore 72 = 2\times2\times2\times3\times3 = 2^3\times3^2$.

The second method involves factoring the number into any two easily recognized factors and then factoring the factors.

$$72 = (12)\,(6) = (4\times3)(3\times2) = (2\times2\times3)\times(3\times2) = 2\times2\times2\times3\times3$$

To illustrate this method, **factor trees are** sometimes employed as discussed in the student text on page 5. Whichever method is employed to find the prime factors of a composite number you have to make students notice that the set of prime factors involved in the prime factorization is unique. For example, the prime factorization of 48 is $2^4\times3$. Thus, the set {2, 3} is the only set to express 48 in terms of prime factors. To help slow learners to understand this concept you may ask them if the expression 6×8 is the only way to express 48. That is, you ask if the set {6, 8} is the only set of composite factors of 48. After you discuss this you state the fundamental theorem of arithmetic which is a generalization of your discussion.

1.1.2 Common Factors, Common Multiples, Greatest Common Factor and Least Common Multiple

From the previous discussions, guide the students to revise the concepts of common factors and common multiples before treating each separately. To do so, ask the students to tell, for example

i. the set of common multiples of 3 and 4.
ii. the set of common factors of 12 and 18.

Let some of the students present their work to the class. Leaving them with open eyes to the concepts tell them that they are going to discuss each one by one. But, before they proceed to discuss each it is advisable to revise some of the ideas related to factors, multiples, common factors, common multiples, prime factorization, etc with active participation of the students.

A Common factors and the greatest common factor

Students are expected to recall the discussions they held and what they have learned from Questions 1 (h), 1 (i), 1 (j) and Question 2 of Activity 1.2 of last subsection 1.1.1. You may write these questions on the blackboard and you can ask students to answer these questions again. After you make sure that students have grasped the concept of what is meant by "a is a factor of b" when a and b are natural numbers and what is meant by a prime factorization of a natural number, you may start the lesson by making the student do Activity 1.3. The purpose of Activity 1.3 is to help students go further than simply finding common factors. So, let students do the activity in groups and based on their results, give the correct generalization of what is meant by greatest common factor.

As indicated, Example 1 given in page 6 of student textbook illustrates one of the methods used to find the GCF of two or more natural numbers. After you finish the discussion of Activity 1.3 you are expected also to discuss in the class the method used in the example. One way you do this is that you may let one of the fast learner students to do and explain the example on the black board. At this point it is necessary to assess or check whether the students have understood the concept. To this end, you may give Question 9 and 10 of Exercise 1.1 on page 11 or similar questions you may think as a class work or homework. In addition to this, you may assign to the fast learners Question 3 of review exercises on unit one on page 61 or similar questions.

Answers to Activity 1.3

1.	a.	{1, 3, 5, 15}	b.	15
2.	a.	{1, 2, 3, 6}	b.	6

Using the Divisibility Test, you may discuss how to list the factors of a natural number say 24. Another method to find factors is to use (apply) the prime factorization of the number.

Example 3 Find all the factors of 24. (you may discuss examples of such a type in tutorial periods)

Solution: $24 = 2^3 \times 3$.

Therefore, all factors of 24 are made up of product of at most 3 twos and 1 three (i.e., 2 occurring three times and 3 occurring only once). All such combinations are contained in the following table.

Exponent of 2	Exponent of 3	Factors of 24
0	0	$2^0 \times 3^0 = 1$
1	0	$2^1 \times 3^0 = 2$
2	0	$2^2 \times 3^0 = 4$
3	0	$2^3 \times 3^0 = 8$
0	1	$2^0 \times 3^1 = 3$
1	1	$2^1 \times 3^1 = 6$
2	1	$2^2 \times 3^1 = 12$
3	1	$2^3 \times 3^1 = 24$

Thus, 24 has 8 factors, namely 1, 2, 3, 4, 6, 8, 12, 24.

The technique used in Example 2 above can be used with any whole number that is expressed as the product of primes with their respective exponents. To find the number of factors of $2^3 \times 5^2$, a similar list could be constructed. The exponent of 2 would range from 0 to 3 (four possibilities) and the exponent of 5 would range from 0 to 2 (three possibilities). In all, there would be 4×3 combinations or 12 factors of $2^3 \times 5^2$ as shown in the following table.

Exponent of 2	Exponent of 5	Factors of $2^3 \times 5^2$
0	0	$2^0 \times 5^0 = 1$
1	0	$2^1 \times 5^0 = 2$
2	0	$2^2 \times 5^0 = 4$
3	0	$2^3 \times 5^0 = 8$
0	1	$2^0 \times 5^1 = 5$
1	1	$2^1 \times 5^1 = 10$
2	1	$2^2 \times 5^1 = 20$
3	1	$2^3 \times 5^1 = 40$
0	2	$2^0 \times 5^2 = 25$
1	2	$2^1 \times 5^2 = 50$
2	2	$2^2 \times 5^2 = 100$
3	2	$2^3 \times 5^2 = 200$

Thus $2^3 \times 5^2 (= 200)$ have 12 factors, namely 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200.

The method enables us to list the factors of a number and to tell the number of factors that the number has. This is summarized as follows.

> Suppose the natural number *n* is expressed as a product of distinct primes with their respective exponents, say $n = P_1^{e_1} \times P_2^{e_2} \times \ldots \times P_m^{e_m}$,then the number of factors of *n* is the product $(e_1+1)\,(e_2+1)(e_3+1) \times \ldots \times (e_m+1)$

Example 4 Find the number of factors of

a. 144 b. $2^3 \times 5^7 \times 7^4$ c. $9^5 \times 11^2$

Solution: a. $144 = 2^4 \times 3^2$. $\therefore$ the number of factors is $(4 + 1) \times (2 + 1) = 15$

b. $2^3 \times 5^7 \times 7^4$ has $(3 + 1) \times (7 + 1) \times (4 + 1) = 160$ factors

c. $9^5 \times 11^2 = 3^{10} \times 11^2$ has $(10 + 1)(2 + 1) = 33$ factors

After making sure that the students are capable of determining factors of a number, it will be possible to discuss greatest common factor of two or more numbers. To do so, we can first list all factors and then identify the common ones to determine the greatest common factor, or apply prime factorization method.

Example 5 Find the greatest common factor of 24 and 18

Solution: First, we need to list factors of each.

$F_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and $F_{18} = \{1, 2, 3, 6, 9, 18\}$

Second, we determine the common factors of both numbers.

Common factors of 24 and 18 = {1, 2, 3, 6}

Therefore, the greatest common factor is 6. That is GCF (24, 18) = 6

Afterwards, make sure that the students can find common factors and greatest common factor (GCF) of two or more natural numbers.

To this end, discuss thoroughly in the class, the definitions and illustrative examples given in the textbook.

Once the students are able to determine the GCF by listing the common factors, it will be helpful to proceed to using prime factorization method to determine the GCF. For this purpose, let the students discuss the activities given in Group Work 1.2 and make sure that they understand the phrases in 1d and 1e and conclude that they give the same result (GCF). Explain to the students that the definition of GCF of two or more numbers can be given either in terms of 1d or 1e as well. The prime factorization method to find GCF of two or more natural numbers is illustrated in Example 2 on page 8 of the student textbook. It is worth to write the example together with its solution on the board and discuss the steps elaborated. However, this by itself is not enough. It is necessary to assess whether the students have understood. You may assign as class work or homework Questions 9 and 10 of Exercise 1.1 on page 11 or similar questions. In addition to this you may assign to the fast learners Question 3 of review exercises on page 61. (Note that at this time you have to tell them to solve the questions using only the prime factorization method).

Answers to Group work 1.2

1. a. The prime factorization of a and b are $a = 1800 = 2^3 \times 3^2 \times 5^2$ and $b = 756 = 2^2 \times 3^3 \times 7$

   b. The prime factors that are common to both $a = 1800$ and $b = 756$ are 2 and 3

   c. The product of the lower powers of the prime that are common in the two prime factorization is $2^2 \times 3^2 = 36$

d. The highest powers of prime factor that are common to the two prime factorization are 2^2 and 3^2

e. The product of the highest powers of prime factors that are common to the two prime factorization is $2^2 \times 3^2 = 36$.

2. To answer Question 2 (a) and 2 (b) we have to find GCF (1800, 756). Here, we use the method used in Example 1, that is, first we list the factors of the numbers and then select the greatest common factor.

Thus; Set of factors of 1800 that is

$$F_{1800} = \begin{Bmatrix} 1,2,3,4,5,6,8,9,10,12,15 \\ 18,20,24,25,30,36,40,45 \\ 50,60,72,75,90,100,120 \\ 150,180,200,225,300,360 \\ 450,600,900,1800 \end{Bmatrix} \text{and that of 756 is } F_{756} = \begin{Bmatrix} 1,2,3,4,6,7,9 \\ 12,14,18,21,27 \\ 28,36,42,54,63 \\ 84,108,126,189 \\ 252,378,756 \end{Bmatrix}$$

$\therefore F_{1800} \cap F_{756} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$\therefore$ GCF (1800, 756) = 36

a. The result of (1c) as seen is 36. Therefore, GCF (1800, 756) is 36.
$\therefore$ The answer is yes they are the same. That is, the product of the common prime factors with the smallest respective exponent is the same with the GCF.

b. The result of (1e) as seen is 36 and GCF (1800, 756) is 36. We conclude that the product of the highest powers of primes common to the two factorizations is the same with the GCF of the numbers.

From the result of this Group work, lead the students to the following fact.

That is, "The GCF of two or more natural numbers is the product of the common primes with the smallest respective exponents in any of the numbers".

Notice that the set of common factors for some natural numbers may be only {1}. For example, 12 and 17. When the set of common factors of two numbers is only {1}, the numbers are called relatively prime. That is, two numbers are relatively prime if their GCF is 1.

Right after students have captured the idea of greatest common factor you can proceed to discuss the least common multiple.

B. Common multiples and the least common multiple

Students must recall the concept of multiples of a natural number and common multiples of two or more natural numbers which they have discussed in the previous subsection 1.1.1, in particular, question number 1 (g) and 2 (c) of Activity 1.2. Write these questions once again on the blackboard and let them discuss in groups. Make sure that students can say in their own words what is meant by multiples of a number and common multiples of two or more natural numbers to understand more about these concepts. Group the students and let them do Group Work 1.3 as enlisted. Encourage and assist them to solve the problems as well. Let some of the groups present their findings to the class.

Finally, discuss the findings of the group and lead the discussion to Definition 1.3. Make sure that students understand the definition. You let them tell the definition orally or restate it. To this end, discuss in the class Example 3 and Example 4 given in the student text on page 9. To assess whether the students understood the different methods illustrated in examples 3 and 4 you may ask the students, for example to solve questions of the following form.

Using the different methods shown in examples 3 and 4, for each question, find

1. LCM (18, 15)
2. LCM (10, 12, 14)

For fast students you can give them more numbers such as

3. LCM (180, 270)
4. LCM (68, 120, 144, 168)

Answers to Group work 1.3

1	2	3	4	5	6	7	~~8~~	9	~~10~~	11	12	13	14	15
~~16~~	17	18	19	~~20~~	21	22	23	~~24~~	25	26	27	28	29	~~30~~
31	~~32~~	33	34	35	36	37	38	39	~~40~~	41	42	43	44	45
46	47	~~48~~	49	~~50~~	51	52	53	54	55	~~56~~	57	58	59	~~60~~
61	62	63	~~64~~	65	66	67	68	69	~~70~~	71	~~72~~	73	74	75
76	77	78	79	~~80~~	81	82	83	84	85	86	87	~~88~~	89	~~90~~
91	92	93	94	95	~~96~~	97	98	99	~~100~~					

While you guide the group work, you can do the following by using colored chalks or white board markers:

1. Cross out multiples of 10 by red color. Again, cross out multiples of 8 by blue color. From this, assist the students to realize that:
2. 40 and 80 are crossed by both colors.

3. These two numbers that are crossed out by the two colors are common multiples of 10 and 8.

4. The least number crossed out by both colors is the number 40.

5. This number, that is the number 40, is the least common multiple of 10 and 8.

Following the discussion conducted above, you can give various examples or exercises and you may inquire students to imagine any possible relationship between GCF and LCM of two numbers, and between LCM of numbers whose GCF is 1. It is easy to discover the relationship between the GCF and LCM of two natural numbers. Activity 1.4 helps the students to discover the relationship between the GCF and LCM of two natural numbers. Group the students and let them do the activity in the class. Let a group present the result of the activity. Discuss the result presented by the group. Encourage and assist the students to reach the following generalization.

> For any natural numbers a and b, GCF $(a, b) \times$ LCM $(a, b) = a \times b$

Answer to Activity 1.4

1. a. GCF (36, 48) = 12, LCM (36, 48) = 144

   b. GCF (36, 48) × LCM (36, 48) = 1728

   c. 36 × 48 = 1728

2. From 1 (b) and 1 (c) we see that

   GCF (36, 48) × LCM (36, 48) = 12 × 144 = 36 × 48

From this we see that for natural numbers a and b,

GCF $(a, b) \times$ LCM $(a, b) = a \times b$

Using this fact let the students check

i. GCF (8, 18) × LCM (8, 18) = 2 × 72 = 8 × 18

ii. GCF (36, 42) × LCM (36, 42) =6 × 252 =36 × 42

Finally, ask the students whether the given generalization works for more than two numbers.

To this end, let them calculate:

i. GCF (12, 60, 90) ii. LCM (12, 60, 90)

iii. 12 × 60 × 90 iv. GCF (12, 60, 90) × LCM (12, 60, 40)

and let them compare the results of (iii) and (iv)

1.1.3 Rational Numbers

You may start the lesson by asking the students whether the set

$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$ is closed under division. In particular, you may ask the students whether the division 2÷3 has an answer on the set of integers. Make them understand the fact that the division 2÷3 has no answer on the set of integers leads us to define a new number denoted by the fraction $\frac{2}{3}$. You discuss that this new number is called a rational number and hence you write or discuss Definition 1.4 which is given in the student text on page 10. At this juncture, it may be useful to explain the use of rational numbers in our daily life. We share resources from which we may take some part of it. We also deal with measurements that we may not always express them with integers. You can practically show the students some examples for which rational numbers are useful. Example: proportion of female students in the class. After explaining the use of rational numbers and their representation, you can discuss with the students the important property of rational numbers. That is, many rational numbers may represent the same number concept. For example, the rational numbers $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{12}{18}, \frac{30}{45}$ represent the same number concept of these rational numbers. Mention that $\frac{2}{3}$ is a rational number in its lowest term.

In relation to this property, you have to discuss the closure property of the set of rational numbers with respect to addition, subtraction, multiplication and division excluding division by zero. To this end, discuss question number 20 of Exercise 1.1 given in the student text page 12.

To show that the set of rational number is closed under addition,

Consider $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ (addition of rational numbers).

Since $\mathbb{Z}$ is closed under addition and multiplication, it follows that $ad \in \mathbb{Z}$, $bc \in \mathbb{Z}$ and hence $ad + bc \in \mathbb{Z}$

This shows that the sum of any two rational numbers is a rational number. In the same way, discuss the other properties. To answer question 20 (b) of Exercise 1.1, you have to be sure that students know how to find a rational number between two given rational numbers; namely

> If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers such that $\frac{a}{b} < \frac{c}{d}$ then the average of these numbers $\left(\text{which is equal to } \frac{ad + bc}{2bd}\right)$ is a rational number between $\frac{a}{b}$ and $\frac{c}{d}$.

In addition to the given problem in question 20 (b), you may ask additional questions such as “write 3 rational numbers between $\frac{1}{3}$ and $\frac{3}{4}$”. Using this additional question lead the students to conclude that there are infinite rational numbers between two given rational numbers. Due to this, we say that the rational numbers are dense. Here, you may ask the students the question “Is any number between two rational numbers a rational number?”. Since there are irrational numbers as well whose discussion will be on the next section, the answer is actually “No”. However, you do not need to tell them the answer. Leave them questioning.

Assessment

Always think of the minimum learning competencies that are expected at the end of the section. For this purpose, the syllabus is attached at the end. Use different formal and informal assessment techniques to get feedback about the level of student understanding of the topic.

Ask them to define the terms: prime number, composite number, prime factorization, multiples and factors of numbers. Ask them also to find: the prime factorization of a composite number, common multiples and common factors, greatest common factor and least common multiple of two or more natural numbers.

Oral questions, group work, class activities, quizzes, homework and assignments will help you as formative assessment techniques to collect relevant data about the performance of the students so that you can assist individual students during instruction. Have the habit of asking 2 or 3 questions at the end of every class in written form. This will motivate students to attentively listen to the daily lesson and read the topic in advance.

Answers to Exercise 1.1

1. a. 45 is a composite number
   b. 23 is a prime number
   c. 91 is a (composite number because $7 \times 13 = 91$)
   d. 153 is a composite number (because $153 = 17 \times 9$)
2. i. a and c are twin primes
   ii. 3 and 5, 5 and 7, 11 and 13, 17 and 19
3. a. i. The one digit, 8, is divisible by 2. So, 48 is divisible by 2
      ii. 4 + 8 or 12 is divisible by 3. So, 48 is divisible by 3
      iii. 48 is divisible by 4 because the number formed by the last two digits which is 48 itself is divisible by 4
      iv. The unit’s digit is neither 0 nor 5. So, 48 is not divisible by 5.
      v. 48 is divisible by 2 and 3. So, 48 is divisible by 6
      vi. 48 is divisible by 8
      vii. 4 + 8 or 12 is not divisible by 9. So, 48 is not divisible by 9
      viii. The last digit is 8. So 48 is not divisible by 10.
      So, 48 is divisible by 2, 3, 4, 6, and 8
   b. i. The unit’s digit, 3, is not divisible by 2. So, 153 is not divisible by 2

ii. 1 + 5 + 3 or 9 is divisible by 3. So, 153 is divisible by 3
iii. 53 is not divisible by 4. So, 153 is not divisible by 4
iv. The ones digit, 3, is neither 0 nor 5. So, 153 is not divisible by 5
v. 153 is not divisible by both 2 and 3. So, 153 is not divisible by 6.
vi. 153 is not divisible by 8
vii. 1 + 5 + 3 or 9 is divisible by 9. So, 153 is divisible by 9.
viii. The last digit is not 0. So, 153 is not divisible by 10.
Thus, 153 is divisible only by 3 and 9

c. 2, 470 is divisible only by 2, 5 and 10 (for the same reason in (a) and b)
d. 144 is divisible by 2, 3, 4, 6, 8, 9 (for the same reason in (a) and (b))
e. 12, 357 is divisible only by 3 and 9 (for the same reason in (a) and (b))

4. a. Yes: (7 + 7 + 7 or 21 is divisible by 3) $3 \times 259 = 777$.
So, 3 is a factor of 777.
b. No, because 9 + 8 + 9 or 26 is not divisible by 9
c. Yes, because 48 is divisible 4

5. i. $2 \times 42 = 84$ ii. $4 \times 21 = 84$ iii. $3 \times 28 = 84$

6. a. $25 = 5^2$ b. $36 = 2^2 \times 3^2$
c. $117 = 3^2 \times 13$ d. $3{,}825 = 3^2 \times 5^2 \times 17$

7. When $a = 11$ and $b = 7$, the expression $2a + 3b$ becomes 43 which is a prime number.

8. Let F_{30} be the set of factors of 30 and F_{42} the set of factors of 42.
Thus, $F_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and $F_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$
$\therefore F_{30} \cap F_{42} = \{1, 2, 3, 6\}$
Thus, all common factors of 30 and 42 are 1, 2, 3 and 6.

9. a. $24 = 2^3 \times 3$
$36 = 2^2 \times 3^2$ $\therefore$ GCF (24, 36) = $2^2 \times 3 = 12$

b. $35 = 5 \times 7$
$49 = 7^2$ $\therefore$ GCF (35, 49, 84) = 7
$84 = 2^2 \times 3 \times 7$

10. Set of factors of $2 \times 3^3 \times 5^2$ is
$\{1, 2, 3, 5, 6, 9, 10, 45, 50, 75, 90, 150, 275, 450\}$
and set of factors $2^3 \times 3 \times 5^2$ is
$\{1, 2, 3, 4, 5, 6, 8, 10, 60, 75, 100, 120, 150, 200, 300, 600\}$
$\therefore$ All common factors of $2 \times 3^3 \times 5^2$ and $2^3 \times 3 \times 5^2$ are
$\{1, 2, 3, 5, 6, 10, 75, 150\}$ and hence GCF $(2 \times 3^3 \times 5^2, 2^3 \times 3 \times 5^2) = 150$

11. 98, 70 and 21 are three numbers whose GCF is 7. You have to explain to the students that the answer to this question can be any other three numbers that are multiples of 7. Where
i. one of them has only one 7 as a factor
ii. all the three should not have other common prime factor

12. a. 7, 14, 21, 28, 35 , 42 b. 5, 10, 15, 20, 25, 30
c. 14, 28, 42, 56, 70, 84 d. 25, 50, 75, 100, 125, 150
e. 150, 300, 450, 600, 750, 900

13. a. LCM (12, 16) = 48 c. LCM (15, 18) = 90
    b. LCM (10, 12, 14) = 420 d. LCM (7, 10) =70
14. When the two numbers are relatively prime.
15. a. $-\frac{1}{3}$ b. $\frac{1}{5}$ c. $\frac{2}{3}$ d. $\frac{36}{49}$
16. a. 6, they are 1, 2, 3, 4, 6 and 12
    b. 6, they are 1, 2, 3, 6, 9 and 18
    c. 8, they are 1, 2, 3, 4, 6, 8, 12 and 24
    d. 12, the factors are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72
17. 175
18. 59
19. The other number is $12 = \frac{3\times180}{45}$
20. a. i. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ (Addition of rational numbers since the set $\mathbb{Z}$ is closed under multiplication and addition it follows that $ad\in\mathbb{Z}$, $bc\in\mathbb{Z}$, $bd\in\mathbb{Z}$ and hence $ad + bc\in\mathbb{Z}$

       $\therefore \frac{ad+bc}{bd} \in\mathbb{Q}$, where a, b, c and d are nonzero integers.

       ii, iii and iv are done similarly.

       Therefore, addition, multiplication and division (by nonzero) are closed on the set of rational numbers.

    b. Consider the average of $\frac{1}{3}$ and $\frac{3}{4}$ that is

       $$\frac{\frac{1}{3}+\frac{3}{4}}{2} = \frac{1}{2}\left(\frac{1}{3}+\frac{3}{4}\right) = \frac{1}{2}\left(\frac{13}{12}\right) = \frac{13}{24}$$

       $\therefore \frac{1}{3} < \frac{13}{24} < -\frac{3}{4}$(the average of two numbers is always between the numbers)

       $\therefore \frac{13}{24}$ is one rational number between $\frac{1}{3}$ and $\frac{3}{4}$

       Similarly, consider $\frac{\frac{1}{3}+\frac{13}{24}}{2}$ which is $\frac{7}{16}$

       $\therefore \frac{1}{3} < \frac{7}{16} < \frac{13}{24}$

       So we have $\frac{1}{3} < \frac{7}{16} < \frac{13}{24} < \frac{3}{4}$

       $\therefore \frac{7}{16}$ and $\frac{13}{24}$ are two rational numbers between $\frac{1}{3}$ and $\frac{3}{4}$

1.2 THE REAL NUMBER SYSTEM

Periods allotted: 30

Competencies

At the end of this sub unit, students will be able to:

- *show that repeating decimals are also rational numbers.*
- *identify irrational numbers.*
- *locate some irrational numbers on a number line.*
- *define real numbers.*
- *describe the correspondence between real numbers and points on a numbers line.*
- *realize the relationship between a power with fractional exponent and a radical form.*
- *convert powers with fractional exponent to radical form and vice-versa.*
- *perform any one of the four operations on the set of real numbers.*
- *use the laws of exponents to simplify expression.*
- *give appropriate upper and lower bounds for a given data to a specified accuracy (e.g. rounding off).*
- *express any positive rational number in its standard form.*
- *explain the notion of rationalization.*
- *identify a rationalizing factor for a given expression.*
- *use the Euclid's division algorithm to express given quotients of the form* $\frac{p}{q}$ *where* $p > q$.

Vocabulary: Terminating decimal, Repeating decimal, Radical, Radicand index, Power, Exponent, Upper bound, Lower bound, Accuracy, Significant figure, Decimal point, Scientific notation, Rationalization, Division algorithm, Exponent, Base, Square root, Perfect power, Principal square root, Principal n^{th} root, Radical and radical sign, Terminating decimal, Repeating decimal.

Introduction

The main task of this sub-unit is to make students familiar with the notion of real numbers and their properties systematically. In the process of doing this, first it will be useful to discuss the representation of a rational number by a decimal. You can also inquire the students whether it is possible to locate all rational numbers on a number line and vice-versa. After discussing these and necessity of irrational numbers to make a complete set of real numbers, the set of irrational numbers will be defined. In relation to the irrational numbers, the concept of radicals and their simplification will be considered. Finally, after defining the set of real numbers as the union of the set of rational and irrational numbers, some related concepts such as approximation, accuracy and scientific notation will be considered.

Teaching Notes

To address this sub-unit, the presentation is classified into several sessions. Each of them are discussed and guided in this sub-unit. Participation of students is required in each discussion.

1.2.1 Representation of a Rational Number by Decimal

Although considerably more useful than whole numbers, fractions are not always adequate and many times lead to awkward manipulative procedures. For example, suppose you were asked to add $\frac{4621}{7839}+\frac{5641}{8441}$, with a hand calculator. These can be easily converted to decimals and the answer will be given to the accuracy of the calculator. This shows that decimals are a convenient numeration system for fraction. Thus, in this lesson, students will learn how to convert fractions to decimals and vice-versa. Especially, for converting fractions to decimals students might be inclined to use calculators. Though it is good to have such an option for converting fractions to decimals, it is advisable to know how the mathematics helps doing so. Thus, students need to know why they have to study this sub-section. But, you should make the students realize that the use of calculators will enable them to easily determine whether a number is rational. They will discuss this issue in subsequent lesson.

To start the lesson, first group the students and make them do Activity 1.5. The purpose of this activity is to help the students realize the concept that every fraction can be represented by either a terminating or repeating decimal and vice-versa. To lead the students to this generalization, you divide the lesson into subtopics as follows.

A. Fractions with terminating decimal representation

You many start this subtopic by grouping your students and asking to do the following activity.

Activity: - Rewrite the following fractions as decimals.

i. $\frac{1}{10}$ ii. $\frac{1}{100}$ iii. $\frac{7}{100}$ iv. $\frac{123}{10,000}$ v. $\frac{7}{8}$

1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths

To motivate them, you may remind them our usual place-value base ten notation by drawing the above table.

Remind the students that the places to the right of the ones place are called tenths, hundredths and thousandths.

Encourage and assist some of the groups to present their answers to the class. Finally, discuss the answers and write on the blackboard and discuss that

i. $\frac{1}{10} = 0.1$ ii. $\frac{1}{100} = 0.01$ iii. $\frac{7}{100} = 0.07$

iv. $\frac{123}{10,000} = \frac{100+20+3}{10,000} = \frac{100}{10,000} + \frac{20}{10,000} + \frac{3}{10,000} = \frac{1}{100} + \frac{2}{1000} + \frac{3}{10,000} = 0.0123$

v. $\frac{7}{8} = \frac{7}{2\times2\times2} = \frac{7\times5\times5\times5}{2\times2\times2\times5\times5\times5} = \frac{875}{1000} = \frac{800+70+5}{1000} = \frac{800}{10} + \frac{7}{100} + \frac{5}{1000} = 0.875$

Remind the students that if the denominator is 10, there is one digit to the right of the decimal point. If the denominator is 100, there are two digits to the right of the decimal point. If the denominator is 1000, there are three and so on.

Write the following on the blackboard and let the students read each.

0.004 (four thousandths)
0.006 (six thousandths)
0.016 (sixteen thousandths)
0.369 (three hundred and sixty-nine thousandths)

Next, ask the students if the fraction $\frac{7}{8}$ can be converted to a decimal without writing its denominator as a power of 10. Simply ask them to divide 7 by 8. Thus, when 7 is divided by 8, the result is 0.875. So you may generalize this result, that is, a fraction can be converted to a decimal form by dividing the numerator by the denominator.

Make sure that students noticed that a fraction can be represented by a terminating decimal if its denominator can be written only in the form $2^m \times 5^n$ where m and n are non-negative integers. In other words, fractions whose denominators have only factors 2 or 5 or both 2 and 5 can always be expressed in decimal form since they have equivalent fractional form whose denominators are powers of 10.

To check their level of understanding regarding this concept, assign them to do question number 4 of Exercise 1.2 in the student text.

B. Non terminating repeating decimal

To start the discussion of this concept, first you have to discuss fraction of the form $\frac{11}{80}, \frac{3}{4}, \frac{7}{20}$ whose denominators are of the form $2^m \times 5^n$ where m and n are non negative integers and can be changed to decimal in two methods.

The 1st method is by finding its equivalent fractional form whose denominator is a power of 10. For example $\frac{3}{4} = \frac{3\times25}{4\times25} = \frac{75}{100} = 0.75.$

The other method is using the long division algorithm for decimals. After you give this brief discussion, you start changing fractions whose denominator have a prime factor other than 2 or 5. Thus you may discuss the one given in Example 1, namely, $\frac{7}{12}$ or any other example whose denominator contains a prime factor other than 2 or 5.

Finally, you are expected to discuss the generalization given namely, "Any rational number $\frac{a}{b}$ can be expressed as a terminating or non terminating repeating decimal by dividing the numerator *a* by the denominator *b*." As explained above to convert a fraction to a decimal we divide the numerator by the denominator. This is elaborated in example 1 on page 13 of the student textbook. You write the example on the board and discuss the outcome of the division. The activity given on page 24 of the Teachers Guide is meant for the slow learners. So you are expected to discuss the given problems and also give some similar questions. To assess student level of understanding the concept you may give question 1 of Exercise 1.2 on page 17 and other similar questions as class work. Examples 2 and 3 on page 14 and 15 of student text elaborate the reverse process. That is, expressing terminating and repeating decimals as fractions. So you have to discuss the workout of the given example. Make sure that students understand the formulas $d = \frac{10^n \times d}{10^n}$ and $d = \frac{d\ \ 10^{k+p} - 10^k}{10^{k+p} - 10^k}$ which are the results of the examples of 2 and 3 respectively. You may assess the students by asking them to generalize the outcomes of examples 1, 2, 3 and 4. In addition, you may assign Question 5 of Exercise 1.2 as class work. In addition to these for fast learners you may ask the derivation of the formula $d = \frac{d\ \ 10^{k+p} - 10^k}{10^{k+p} - 10^k}$. You may also ask them to write the result of, for example, $\frac{0.\overline{31}}{0.\overline{6}}$, $2.\overline{13} \times 0.\overline{3}$ as fractions.

Answers to Activity 1.5

1. a. It is a way of writing fractions. b. 3.001, 4.501
2. $\frac{3}{4} = 0.75$; $\frac{1}{3} = 0.\overline{3}$
3. $0.4 = \frac{4}{10} = \frac{2}{5}$; $1.34 = \frac{134}{100} = \frac{67}{50}$

Assessment

You can assess you students by giving them various exercises of converting fractions form into decimals and decimals into fractions. You can let students do these as homework and present their work.

Answers to Exercise 1.2

1. a. $0.\overline{4}$ b. 0.12 c. $1.\overline{571428}$ d. $-5.\overline{6}$
   e. 37.06 f. $3.\overline{142857}$

2. a. $0.3 = \frac{3}{10}$ b. $0.004 = \frac{1}{250}$ c. $0.12 = \frac{3}{25}$ d. $0.369 = \frac{369}{1000}$

3. a. 4mm in fraction is $\frac{1}{250}$m and in decimal it is 0.004m
   b. 6cm 4mm when given in meters as a fraction it is $\frac{64}{1000}$ m or $\frac{8}{125}$ m and when given as a decimal it is 0.064m.
   c. 56cm 4mm as a fraction is $\frac{56}{100} + \frac{4}{1000}$ or $\frac{141}{250}$ m and 0.564m as a decimal.

4. a ,d, g and h are non-terminating decimals where as b ,c ,e and f are expressed as terminating decimals

A fraction $\frac{a}{b}$ (in its lowest form) has a terminating decimal representation if *b* can be written as $2^m \times 5^n$ where m and n are non-negative integers. That is, if *b* is a multiple of 2 or 5 or both 2 and 5. Thus, for example

b) $\frac{7}{10}$ has a terminating decimal representation because $10 = 2^1 \times 5^1$ similarly

c) $\frac{69}{64}$ has terminating decimal representation because $64 = 2^6 \times 5^o$ and soon.

Similarly e) and f) has terminating decimal representations.

5. a. $\frac{22}{25}$ b. $\frac{7}{9}$ c. $\frac{5}{6}$
   d. $\frac{177}{25} = 7\frac{2}{25}$ e. $\frac{52}{99}$ f. $-\frac{3311}{3300} = -\left(1\frac{11}{3300}\right)$

6. a. $0.\overline{45}$ b. $0.1\overline{345}$

7. a. 0.1313131313 . . . b. –0.3053053053 . . .
   b. 0.3818181818 . . .

8. a. $0.\overline{275} = \frac{275}{999}$, $0.\overline{714} = \frac{714}{999}$

   $0.\overline{275} + 0.\overline{714} = \frac{275}{999} + \frac{714}{999} = \frac{989}{999}$. But $\frac{989}{999} = 0.\overline{989}$

   b. $0.\overline{6} = \frac{6}{9}$ and $1.\overline{142857} = \frac{1142856}{999999}$

   $0.\overline{6} - 1.\overline{142857} = \frac{6}{9} - \frac{1142856}{999999} = -\frac{476190}{999999} = -\overline{0.476190}$

   This shows that $0.\overline{6} - 1.\overline{142857} = -0.\overline{476190}$

1.2.2 Irrational Numbers

You may start the lesson by asking the students to indicate which of the following are rational numbers (or you may use any other numbers you feel convenient)

a.	0.36	b.	0.846333	c.	0.333…
d.	0.1454545…	e.	$\sqrt{4}$	f.	$\sqrt{0.04}$
g.	0.73773777377773…	h.	0.4271271	i.	$\sqrt{2}$
j.	$\sqrt{5}$	k.	$\sqrt[3]{0.027}$	l.	$\sqrt[3]{9}$
m.	3.2020020002…				

Give them about 10 minutes to discuss in pairs. You may list on the blackboard those numbers on which the students agree to be rational numbers. Then you start discussing their answers and lead them to generalize. You remind them what they have learned in the previous lesson; that terminating and non terminating repeating decimals are rational numbers: moreover, perfect roots such as $\sqrt{9}, \sqrt{0.04}$ and $\sqrt[3]{0.027}$ are rational numbers. In this way, you lead the students to conclude that numbers like those indicated in (g) (i) (j) (l) and (m) are not rational numbers and hence are called irrational numbers. So encourage the students to reach the following generalization.

i) A decimal that is neither a terminating nor a repeating decimal is an irrational number.

ii) Roots that are not perfect are irrational numbers.

Make sure that students can distinguish between a non-terminating repeating decimal such as 0.16666… and a non-terminating, non-repeating decimal such as 0.737737773 … where the number of 7's between successive 3's increases by 1 each time. This number is an irrational number because it is a non-terminating, non-repeating decimal. It is non-repeating because each pair of consecutive 3's has one more 7 between them than the preceding pair.

Discuss with the students that roots that are not perfect are also irrational numbers. Because, by using scientific calculators, they can see that such numbers have decimal representations that are non-terminating and non-repeating. To this end, using calculator, you may discuss that $\sqrt{5}$ = 2.236067977… which is non terminating and non repeating. Thus, it is irrational number.

The most famous irrational number is Pi (π). Some students take π as $3\frac{1}{7}$ or 3.1416.

You have to discuss and emphasize π is a non-terminating and non repeating decimal whose first few digits are given by 3.14159265358979323846.

1. Locating irrational numbers on the number line

You may start this lesson by asking the students whether the rational numbers "fill up" the number line or whether there are points on the number line that are not occupied by rational numbers. You may remind the student the dense property of rational numbers which states that there are infinite rational numbers between two given rational numbers.

Because of this property, it seems that the rational numbers may "fill up" the number line. But this is not correct. To help the students capture the idea, you can guide them to do Group Work 1.4. The purpose of Group Work 1.4 is to demonstrate that the irrational numbers are also represented by points on the number line.

In other words, the purpose of this Group work is to justify that the rational numbers don't "fill up" the number line, because there are points on the number line that corresponds to irrational numbers. Thus, make sure that students understand not only the geometrical skill of locating the irrational numbers on the number line, but equally important students should understand that, the irrational numbers fill up the holes on the number line, that the rational numbers could not and hence make the number line complete in the sense that for each point on the number line there is a number be it rational or irrational that corresponds to it and vice versa.

In order for students to do Group Work 1.4, you are supposed to tell the students to bring to the class a pair of compass and ruler. After students bring those necessary equipment to the class, group the students and let each group perform the group work. You are supposed to monitor and assist each group to do the group work in accordance with the steps enlisted in the student text. Let at least one group present its performance of the given Group Work on the black board.

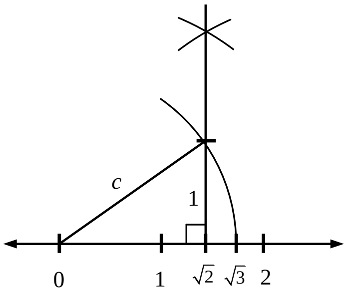
Answers to Group Work 1.4

1. By following the steps mentioned in the student textbook you will get the result to be Figure 1.5 of the textbook.
2. Follow the same procedure and get your answer to be figure 1.7 of the student textbook.

Once they have discussed the topic of locating irrational numbers in group, they need to be individually able to do the same. For this purpose, you can guide them to do Activity 1.6. Explain that the purpose of Activity 1.6 is the same as Group work 1.4; but here some of the questions may require the application of the $\sqrt{2}$ which they have located, in Group Work 1.4. So, you are expected to assist the students in handling this question. At this moment, you need to limit the activity to questions 1 and 2. You may use question 3 of the activity for fast students. After they finalize the activity, however, you need to support all students to realize how question 3 is located.

Answers to Activity 1.6

1. a. b. c.



$\therefore\ 1^2 + (\sqrt{2})^2 = 3$

Figure1.1

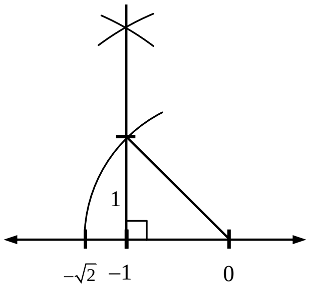


Figure1.2

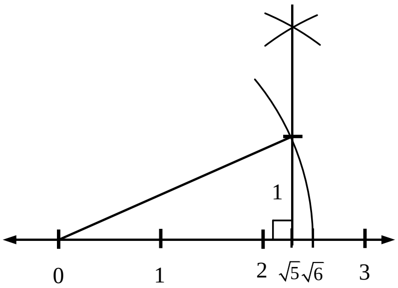


Figure1.3

2. a) look at 1) (a).

   b) since $\sqrt{2}^{\,2}$ + $2^{\,2}$ = 6, we can use $\sqrt{2}$ in locating $\sqrt{6}$.

3. a. b. c.

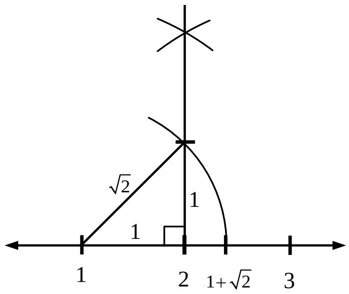


Figure 1.4

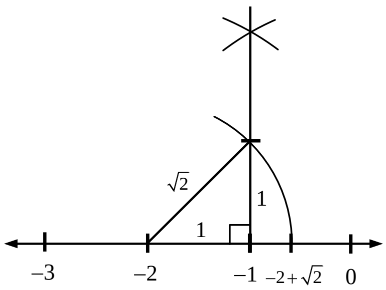


Figure 1.5

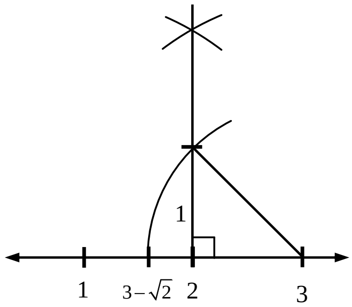


Figure 1.6

After having made sure that they have acquired the ability to locate irrational numbers, you can guide them to do Activity 1.7 for conducting operations on numbers. The purpose of Activity 1.7 is to lead students to reach the conclusion that the set of irrational numbers is not closed under addition, subtraction, multiplication and division. So make sure that students get this idea. On the other hand, discuss with students the objective of Example 2 given in student text page 20; that the sum of a rational number and an irrational number is an irrational number. In the same way, you may encourage the students to show that for example say

i) $3 - \sqrt{2}$ ii) $3\times\sqrt{2}$ and iii) $3\div\sqrt{2}$ are irrational numbers and hence lead them to conclude that the difference, product and quotient of a rational number and an irrational number are also irrational numbers.

Answers to Activity 1.7

1. $0.\overline{4}$ 2. $0.\overline{3}$ 3. 7 4. 2

Assessment

Here you can assess students through several approaches. You can give the students irrational numbers and ask them to locate each on the number line. You can also ask them to describe the properties like closure on the set of irrational numbers. You may also ask them to identify rational and irrational numbers.

Answers to Exercise 1.3

1. a. $\frac{5}{6}$ is a rational number b. $2.\overline{34}$ is a rational number
   c. – 0.1213141516 . . . is an irrational number
   d. $\sqrt{0.81}$ is a rational number because 0.81 is a perfect square.
   e. 0.121121112 . . . . . is an irrational number
   f. $\sqrt{5}-\sqrt{2}$ is an irrational number
   g. $\sqrt[3]{72}$ is an irrational number because 72 is not a perfect cube.
   h. 1 + $\sqrt{3}$ is an irrational number.
2. Any radical $\sqrt[n]{a}$ which defines as an irrational number if *a* is not a perfect n^{th} power. Thus i. $\sqrt[3]{9}$ is an irrational number
   ii. 0.3131131113 . . . is an irrational number.
3. a. False. Let x = 0.3131131113 . . .
   and y = 0.4646646664 . . .
   Now $x + y$ = 0.777 . . . = $0.\overline{7}$ which is a rational number.
   b. True
   c. True
   d. False. take 0 the rational number and $\sqrt{2}$ the irrational number, then $0\times\sqrt{2}=0$ which is a rational number.

1.2.3 Real Numbers

You may start the lesson by asking the students

i. To list the sets of numbers they have learned so far
ii. To tell the set(s) of numbers that each of the following numbers belongs to

a. 23	b. –14	c. $\frac{3}{7}$	d. $-7\frac{2}{3}$
e. $0.6\overline{15}$	f. –1.75	g. $\sqrt{14}$	h. 0.130330333…
i. $2+\sqrt{5}$	j. 2 – 0.1010010001…		

Discuss the answers of these two questions with the students and, in particular, assist the students to notice the answers of (ii) will be either rational or irrationals number and hence lead them to define by themselves that a real number is either a rational number or an irrational number.

Based on this fact, guide them to realize that the number line will be filled up by real numbers. In other words, discuss that every real number, rational or irrational can be associated with a unique point on the number line and conversely that every point on the number line can be associated with a unique real number (rational or irrational).

Explain to the students that this relation is expressed by saying that there is a one-to-one correspondence between the set of points on the number line and the set of real numbers.

Since every real number can be located on the number line, make them notice that the number line can be used to compare and order all real numbers. Based on this concept, discuss example 1 given on page 23 of student text. You may give similar questions. For example, let the students arrange the following numbers in order from the smallest to the largest.

i. 0.58, 0.085, 0.85

ii. 781.345, 781.354, 780.999

iii. 4.9, 4.09, 4.99, 4.099

iv. 0.45, $0.\overline{4}$, 0.44, 0.45445445..., $0.\overline{45}$, 0.455

These will point out some idea about order property of numbers. As a follow up it is advisable if students can discuss the Trichotomy property and Transitive property of real numbers and reach at concluding the statements of these properties.

Assessment

To assess the students level of understanding you can ask through oral question and answer about the number systems. You can then give to the students some numbers and ask them to locate each into some number system, and can ask them to tell about the properties of order, trichotomy and transitivity.

Answers to Exercise 1.4

1. a. $a = \frac{\sqrt{6}}{4}$, $b = 0.\overline{6} = \frac{2}{3}$

Since $a^2 = \frac{6}{16} = \frac{3}{8}$ and $b^2 = \frac{4}{9}$

We have $a^2 = \frac{3}{8} = \frac{3 \times 9}{8 \times 9} = \frac{27}{72}$

$b^2 = \frac{4}{9} \times \frac{8}{8} = \frac{32}{72}$

$a^2 = \frac{27}{72} < \frac{32}{72} = b^2$ which implies $a < b$ i.e $\frac{\sqrt{6}}{4} < 0.\overline{6}$

b. $a = 0.432$ and $b = 0.437$

Since 4 = 4, 3 = 3 but 2 < 7

it follows that $a = 0.432 < 0.437 = b$

c. $a = -0.128$ and $b = -0.123$

Since 1 = 1, 2 = 2 but 8 > 3 and since the numbers are negatives, it follows that $a = -0.128 < -0.123 = b$

2. a. $\mathbb{N}$ is closed under addition and multiplication.

b. $\mathbb{Z}$ is closed under addition, subtraction and multiplication.

c. $\mathbb{Q}$ is closed under addition, subtraction, multiplication and division excluding division by zero.

d. The set of irrational numbers is not closed under any of the operations.

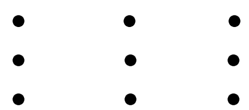
e. $\mathbb{R}$ is closed under addition, subtraction, multiplication and division excluding division by zero.

1.2.4 Exponents and Radicals

A. Roots and Radicals

This sub-unit is concerned largely with the meaning of square root and more generally the n^{th} root of a real number whenever it is defined and the manipulation of radicals. Moreover, the subunit deals with the relationship between a power with a fractional exponent and radical form. The important points in the subunit are the definitions of square root, cube root and n^{th} root; and multiplication and division properties of radicals. So, make sure that the students understand these important points.

You may start the lesson by discussing a number that can be pictured in squares of dots as shown in the student text. For example, you may take the number 9 pictured by nine dots as follows



9

3×3

Using this picture, discuss with the students by asking them the relationship between the total number of dots and the number of dots in each row or column. They are supposed to answer that the total number of dots is the square of the number of dots in each row or column. Or the number of dots in each row or column is the square root of the total number of dots. If they are not clear about the relationship, assist them to understand.

Tell them that this relation can be written as $9 = 3^2$ similarly $16 = 4^2$, $25 = 5^2$, etc.

Next, ask them how to express 3 in terms of 9 and how to write 3 symbolically in terms of 9. In this way, you may lead the discussion to the definition of square root. After making sure that the students grasped the concept of square root, you extend your discussion to the cube root and then to the n^{th} root as discussed in the student text. Make sure that students understand the definitions and the examples given in their text.

B. Meaning of Fractional Exponents

Previously students are aware of squares and square roots. The way these roots can be expressed will drive us into fractional exponents. For the purpose of explaining this concept, you may start the lesson by asking the students about the meanings of integral exponents that they learned in lower grade mathematics, namely, ask the meanings of

i. $2^2, 2^3, 3^2$ ii. $3^{-3}, 5^2$, etc

iii. When you are convinced that they answer such questions easily, let the students discuss the following Activity.

Activity I

Group the students in pairs. Then using scientific calculator let them find the values of

a. $4^{\frac{1}{2}}$ and $\sqrt{4}$ (using square root definition they may find $\sqrt{4}$ without calculator)

b. $5^{\frac{1}{2}}$ and $\sqrt{5}$

c. $81^{\frac{1}{4}}$ and $\sqrt[4]{81}$

a. Assist the students how to use the calculator to find $4^{\frac{1}{2}}$ pressing the buttons in the following sequence

$\boxed{4} \to \boxed{x^y} \to \boxed{2} \to \boxed{Inv} \to \boxed{\frac{1}{x}} \to \boxed{=}$ we obtain $4^{\frac{1}{2}} = 2$.

To find $\sqrt{4}$ press the button $\boxed{4}$ then the square button $\boxed{\sqrt{\ }}$ we find $\sqrt{4} = 2$.

$\therefore$ we see that $4^{\frac{1}{2}} = \sqrt{4}$

b. To find $5^{\frac{1}{2}}$ press the buttons in the following sequence $\boxed{5} \to \boxed{x^1} \to \boxed{2} \to \boxed{Inv} \to \boxed{\frac{1}{x}} \to \boxed{=}$ we obtain $5^{\frac{1}{2}} = 2.236067978$

To find $\sqrt{5}$ press the button $\boxed{5}$ then the square button $\boxed{\sqrt{\ }}$ we find that $\sqrt{5} = 2.236067978$.

$\therefore 5^{\frac{1}{2}} = \sqrt{5}$

c. To find $81^{\frac{1}{4}}$ press the buttons in the following sequences

$\boxed{81} \rightarrow \boxed{x^y} \rightarrow \boxed{4} \rightarrow \boxed{Inv} \rightarrow \boxed{\frac{1}{x}} \rightarrow \boxed{=}$ we obtain 3.

To find $\sqrt[4]{81}$ press the buttons in the following sequence

$\boxed{81} \rightarrow \boxed{INV} \rightarrow \boxed{\sqrt[x]{y}} \rightarrow \boxed{4} \rightarrow \boxed{=}$ you obtain 3. That means, $\sqrt[4]{81} = 3$

We see that $81^{\frac{1}{4}} = \sqrt[4]{81}$

Finally, after they compare the results, encourage them to do Activity 1.8 individually and give generalization. Thus, you may lead the students to state the related definition, namely, Definition 1.10.

To stabilize this definition, discuss the examples given in the student text. Make sure that students understand Definition 1.10.

Answers to Activity 1.8

1. Another name for $2^{\frac{1}{4}}$ is $\sqrt[4]{2}$, i.e. the fourth root of 2.
2. By $2^{\frac{1}{2}}$ it meant that $\sqrt{2}$, the square root of 2 and similarly $2^{0.5}$ means $\sqrt{2}$.
3. Since 32 = 2 × 2 × 2 × 2 × 2 = 2^5, we see that $\sqrt[5]{32} = 2$ i.e. the number is 32.

Once the students make generalization of type definition 1.10, to enable them to see various generalizations, you can let them do Group Work 1.5 and write down their observations.

Answers to Group work 1.5

i. a. $(8\times 27)^{\frac{1}{3}} = [(2\times 2\times 2)\times(3\times 3\times 3)]^{\frac{1}{3}} = [(2\times 3)\times(2\times 3)\times(2\times 3)]^{\frac{1}{3}}$

$= 2 \times 3 = 6$

b. $8^{\frac{1}{3}}\times 27^{\frac{1}{3}} = (2\times 2\times 2)^{\frac{1}{3}}\times(3\times 3\times 3)^{\frac{1}{3}} = 2\times 3 = 6$

ii. a. $\sqrt[3]{8\times 27} = \sqrt[3]{(2\times 2\times 2)\times(3\times 3\times 3)} = \sqrt[3]{(2\times 3)\times(2\times 3)\times(2\times 3)}$

$= 2\times 3 = 6$

b. $\sqrt[3]{8}\times\sqrt[3]{27} = \sqrt[3]{2\times 2\times 2}\times\sqrt[3]{3\times 3\times 3} = 2\times 3 = 6$

iii. a. $(36\times 49)^{\frac{1}{2}} = [(6\times 6)\times(7\times 7)]^{\frac{1}{2}} = [6\times 7] = 42$

b. $36^{\frac{1}{2}}\times 49^{\frac{1}{2}} = (6\times 6)^{\frac{1}{2}}\times(7\times 7)^{\frac{1}{2}} = 6\times 7 = 42$

iv. a. $\sqrt{36\times 49} = \sqrt{(6\times 6)\times(7\times 7)} = \sqrt{(6\times 7)\times(6\times 7)} = 6\times 7 = 42$

b. $\sqrt{36}\times\sqrt{49} = \sqrt{6\times 6}\times\sqrt{7\times 7} = 6\times 7 = 42$

From the above, you might have observed that

i. $(8\times 27)^{\frac{1}{3}} = 8^{\frac{1}{3}}\times 27^{\frac{1}{3}}$ and ii. $\sqrt[3]{8\times 27} = \sqrt[3]{8}\times\sqrt[3]{27}$, and

iii. $(36\times 49)^{\frac{1}{2}} = 36^{\frac{1}{2}}\times 49^{\frac{1}{2}}$ and iv. $\sqrt{36\times 49}=\sqrt{36}\times\sqrt{49}$ are equal.

After discussing the equality of such expressions, you can ask the students to imagine their generalization. This generalization will lead them to stating theorem 1.2. After you established Theorem 1.2, the next step is to discuss Theorem 1.3 which is helpful to simplify division of radicals. To do so, it will be useful if students can do Activity 1.9 in group. The purpose of Activity 1.9 is to lead the students to state Theorem 1.3 which states that for any two real numbers a and b where $b \neq 0$ and for all integers $n \geq 2$, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$.

From the above results it is clear that

i. $\frac{64^{\frac{1}{5}}}{2^{\frac{1}{5}}} = \left(\frac{64}{2}\right)^{\frac{1}{5}}$ ii. $\frac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \left(\frac{8}{2}\right)^{\frac{1}{2}}$ iii. $\frac{27^{\frac{1}{3}}}{729^{\frac{1}{3}}} = \left(\frac{27}{729}\right)^{\frac{1}{3}}$

Discuss that the formulas given in theorem 1.2 and 1.3 are helpful in simplifying radicals. To this end, you may discuss Example 7 and 8 given right below Theorem 1.2 and Theorem 1.3 in the student text. Pursuant to this, you may also give Activity 1.10 to students to do it in class and lead them to getting the idea of definition 1.11. To check whether the students have understood theorems 1.2 and 1.3 and their applications in simplifications which are illustrated in examples 7 and 8 you may assign Questions 7 and 9 of Exercise 1.5 on page 32 of the student textbook. For slow learners you may ask the same questions repeatedly by varying the radicand and the indices. For example, questions of the form simplify $\sqrt{8}\times\sqrt{2}$, $\sqrt[3]{16}\times\sqrt[3]{4}$, $\frac{\sqrt[3]{81}}{\sqrt[3]{3}}$, $\frac{\sqrt[4]{32}}{\sqrt[4]{2}}$, etc and for fast learners you may ask questions similar to question 7 (g), 7 (h) of Exercise 1.5.

Answers to Activity 1.9

i. a. $\frac{64^{\frac{1}{5}}}{2^{\frac{1}{5}}} = \frac{(2\times 32)^{\frac{1}{5}}}{2^{\frac{1}{5}}} = \frac{2^{\frac{1}{5}}\times 32^{\frac{1}{5}}}{2^{\frac{1}{5}}} = 32^{\frac{1}{5}} = (2\times 2\times 2\times 2\times 2)^{\frac{1}{5}} = 2$

b. $\left(\frac{64}{2}\right)^{\frac{1}{5}} = 32^{\frac{1}{5}} = 2$

ii. a. $\frac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{(2\times 4)^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2}}\times 4^{\frac{1}{2}}}{2^{\frac{1}{2}}} = 4^{\frac{1}{2}} = (2\times 2)^{\frac{1}{2}} = 2$

b. $\left(\frac{8}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = (2\times 2)^{\frac{1}{2}} = 2$

iii. a. $\dfrac{27^{\frac{1}{3}}}{729^{\frac{1}{3}}} = \dfrac{27^{\frac{1}{3}}}{(27\times27)^{\frac{1}{3}}} = \dfrac{27^{\frac{1}{3}}}{27^{\frac{1}{3}}\times27^{\frac{1}{3}}} = \dfrac{1}{27^{\frac{1}{3}}} = \dfrac{1}{(3\times3\times3)^{\frac{1}{3}}} = \dfrac{1}{3}$

b. $\left(\dfrac{27}{729}\right)^{\frac{1}{3}} = \left(\dfrac{27}{27\times27}\right)^{\frac{1}{3}} = \left(\dfrac{1}{27}\right)^{\frac{1}{3}} = \left(\dfrac{1}{3}\times\dfrac{1}{3}\times\dfrac{1}{3}\right)^{\frac{1}{3}} = \dfrac{1}{3}$

Answers to Activity 1.10

1. a. $2^{\frac{7}{2}}$ means $\left(2^{\frac{1}{2}}\right)^{7}$ b. $2^{\frac{9}{2}}$ means $\left(2^{\frac{1}{2}}\right)^{9}$

2. $5^{\frac{3}{2}} = \left(5^{\frac{1}{2}}\right)^{3}$

Assessment

To make sure that students have understood the statements of the theorems and their applications you can give them exercises as homework or assignment that require application of the theorems and check their answers.

Answers to Exercise 1.5

1. a. $64^{\frac{1}{3}} = 4^{3}\,^{\frac{1}{3}} = 4$ or using the radical notation we have

$64^{\frac{1}{3}} = \sqrt[3]{64} = 4$ because $4^3 = 64$

b. $256^{\frac{1}{8}} = 2^{8}\,^{\frac{1}{8}} = 2$ or $256^{\frac{1}{8}} = \sqrt[8]{256} = 2$ because $2^8 = 256$

c. $125^{\frac{1}{3}} = 5^{3}\,^{\frac{1}{3}} = 5$ or $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$ because $5^3 = 125$

2. a. 3 b. 3 c. 2 d. $\dfrac{3}{2}$ e. 0.2

f. 0.2 g. 3

3. Commutative and associative property of multiplication.

4. a. $[(5\times5)\times(11\times11)]^{\frac{1}{2}} = [(5\times11)\times(5\times11)]^{\frac{1}{2}}$

$= [(5\times11)^{\frac{1}{2}}]^{\frac{1}{2}} = 5\times11 = 55$

b. $[(25\times25)\times(4\times4)]^{\frac{1}{4}} = [(5\times5\times5\times5)\times(2\times2\times2\times2)]^{\frac{1}{4}}$

$= [(5\times2)\times(5\times2)\times(5\times2)\times(5\times2)]^{\frac{1}{4}}$

$= [(5\times2)^{2}]^{\frac{1}{4}} = 10$

c. Similarly as (a) and (b) above, $(1024 \times 243)^{\frac{1}{5}} = 12$.

5. $\sqrt[n]{ab} = \sqrt[n]{a} \;.\; \sqrt[n]{b}$ for $a \geq 0$, $b \geq 0$ and $n \geq 2$.

6. Since

a. $\left(7^{\frac{1}{4}} \times 5^{\frac{1}{4}}\right)^4 = \left(7^{\frac{1}{4}}\right)^4 \times \left(5^{\frac{1}{4}}\right)^4$

$$= \left(7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}}\right) \times \left(5^{\frac{1}{4}} \times 5^{\frac{1}{4}} \times 5^{\frac{1}{4}} \times 5^{\frac{1}{4}}\right) = 7 \times 5$$

It follows by definition of nth that $7^{\frac{1}{4}} \times 5^{\frac{1}{4}}$ is the fourth root of 7×5 that is

$\sqrt[4]{7\times5} = 7^{\frac{1}{4}} \times 5^{\frac{1}{4}}$ or ${7\times5}^{\frac{1}{4}} = 7^{\frac{1}{4}} \times 5^{\frac{1}{4}}$

b. Since ${\sqrt{5} \times \sqrt{3}}^{\,2} = {\sqrt{5}}^{\,2} \times {\sqrt{3}}^{\,2} = 5 \times 3$

$\therefore \sqrt{5} \times \sqrt{3} = \sqrt{5 \times 3}$ by definition.

c. Since $\sqrt[3]{7} \times \sqrt[3]{9} = 7^{\frac{1}{3}} \times 9^{\frac{1}{3}} = (7 \times 9)^{\frac{1}{3}} = \sqrt[3]{7 \times 9}$

d. By definition, $11^{\frac{1}{7}} \times 6^{\frac{1}{7}} = \sqrt[7]{11 \times 6} = (11 \times 6)^{\frac{1}{7}}$

7. a. 2 b. 3 c. 2 d. 2 e. 4

f. 2 g. 3 h. 2

8. For any two real numbers a and b ($b \neq 0$) and for all integer

$n \geq 2, \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$. That is, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

9. a. 2 b. $\frac{1}{3}$ c. $\frac{2}{3}$ d. $\frac{2}{3}$ e. 2

f. 2 g. 2 h. 5

10. a. $13^{\frac{9}{5}}$ b. $12^{\frac{11}{5}}$ c. $11^{\frac{5}{6}}$

11. a. ${3^{\frac{1}{5}}}^{\,7}$ b. ${5^{\frac{1}{3}}}^{\,6}$ c. ${64^{\frac{1}{6}}}^{\,5}$ d. ${729^{\frac{1}{3}}}^{\,2}$

12. a. ${\sqrt[5]{13}}^{\,9}$ b. ${\sqrt[5]{12}}^{\,11}$ c. ${\sqrt[6]{11}}^{\,5}$

13. a. ${\sqrt[5]{3}}^{\,7}$ b. ${\sqrt[3]{5}}^{\,6}$ c. ${\sqrt[6]{64}}^{\,5}$ d. ${\sqrt[3]{729}}^{\,2}$

14. a. 243 b. 243 c. 2

15. a. 2 b. 729 c. 2

d. 729 e. 64 f. 64

C. Simplification of Radicals

In addition to the above formulas given in Theorem 1.2 and 1.3 to compute and simplify expressions involving radicals, it is often necessary to distinguish between roots with odd indices and those with even indices. The purpose of Activity 1.11 is to motivate the students to state in their own words the rules related to even and odd indices. To discuss this, group the students in pairs and let them do and discuss the problems given in the activity. Encourage and assist them to state the rules required. Finally, you may explain the details of such indices and help them understand by doing all of the examples from the student text. Right after this explanation you can assign Question 1(c) ,1(f), 2(d), and 4(g) of Exercise 1.6,

Answers to Activity 1.11

1. a. $\sqrt[3]{(-2)^3} = \sqrt[3]{(-2)(-2)(-2)} = -2$

   b. $\sqrt{(-3)^2} = |-3| = 3 \therefore \sqrt{(-3)^2} = \sqrt{9} = 3$

   c. $\sqrt[4]{(-5)^4} = \sqrt{(-5)(-5)(-5)(-5)} = \sqrt[4]{625} = 5$

   d. $\sqrt[5]{4^5} = \sqrt[5]{4\times4\times4\times4\times4} = 4$

   e. $\sqrt{2^2} = \sqrt{2\times2} = 2$

   f. $\sqrt[7]{(-1)^7} = \sqrt{(-1)(-1)(-1)(-1)(-1)(-1)(-1)} = (-1) = -1$

2. Yes, For any real number a and a positive integer $n \geq 1$.

   i) $\sqrt[n]{a^n} = a$ when n is odd.

   ii) $\sqrt[n]{a^n} = |a|$ when n is even.

Assessment

Here you can give to students exercises of simplification that consist of different indices. Or assess while students work Activity 1.11.

Answers to Exercise 1.6

1 a. $2\sqrt{2}$ b. $20\sqrt{2}$ c. $6\sqrt{2}\ |x|$ d. $11\sqrt{3}$

e. 8 f. $\sqrt{3x}\ |xy|$ g. $3\sqrt[4]{5}$

2. a. $\sqrt{50} = \sqrt{5\times5\times2} = 5\sqrt{2}$ b. $2\sqrt{36} = 2\sqrt{6\times6} = 2\times6 = 12$

   c. $\frac{1}{3}\sqrt{72} = \frac{1}{3}\sqrt{6\times6\times2} = \frac{1}{3}\times6\times\sqrt{2} = 2\sqrt{2}$

   d. $3\sqrt{8x^2} = 3\sqrt{4\times2\times x^2} = 6\sqrt{2}\,|x|$ e. $\sqrt{a^3} = |a|\sqrt{a}$

   f. $\sqrt{0.27} = \sqrt{\frac{27}{100}} = \sqrt{\frac{9}{100}\times3} = \frac{3}{10}\sqrt{3}$

g. $-\sqrt{63} = -3\sqrt{7}$

h. $\frac{\sqrt{180}}{9} = \frac{\sqrt{36\times5}}{9} = \frac{6}{9}\sqrt{5} = \frac{2}{3}\sqrt{5}$ i. $\sqrt[3]{16} = 2\sqrt[3]{2}$

j. $\sqrt[3]{-54} = -3\sqrt[3]{2}$

3. a. The error is that the student takes $\sqrt{25+3} = \sqrt{25}\times\sqrt{3}$ which is wrong

$\sqrt{25+3} \neq \sqrt{25}\times\sqrt{3}$

$\therefore \sqrt{28} = \sqrt{4\times7} = \sqrt{4}\times\sqrt{7} = 2\sqrt{7}$

b. Simplifying $\sqrt{72}$ to $\sqrt{4}\sqrt{18}$ is correct but the mistake of the student is taking $2\sqrt{18}$ as the simplified answer. $2\sqrt{18}$ can be simplified further to $2\sqrt{9\times2}$ to $2\sqrt{9}\ \sqrt{2}$ to $6\sqrt{2}$ i.e. $\sqrt{72} = \sqrt{36\times2} = \sqrt{36}\times\sqrt{2} = 6\sqrt{2}$

c. The mistake of the student is applying the square root to the exponent of the radicand i.e taking $\sqrt{x^9} = x^{\sqrt{9}} = x^3$ which is wrong.

The correct simplification is $\sqrt{7x^9} = \sqrt{7x^8x} = \sqrt{7}\ x^4\sqrt{x} = x^4\sqrt{7x}$

4. a. $40\sqrt{10}$ b. $2\sqrt[3]{10}$ c. 5 d. 2

e. 3 f. 16 g. $|x|\sqrt{2y}$ h. $24\sqrt{6}$

5. a. 9600 units b. The production will be doubled i.e.

$12\sqrt{2K\times2L} = 24\sqrt{KL}$

D. Addition and subtraction of radicals

You may start by discussing how problems of the following types were simplified, namely

i. $\sqrt{2}\times\sqrt{16}$ ii. $\sqrt[3]{5}\times\sqrt[3]{25}$, etc

Then you ask the students to simplify the following types of problems:

i) $\sqrt{2}+\sqrt{18}$ ii) $\sqrt[3]{81}-\sqrt[3]{24}$ iii) $\sqrt{6}+\sqrt{5}$ iv) $\sqrt{98}+\sqrt[3]{16}$

After the students try to respond to the questions, you discuss the answers that students have given to these problems by explaining to them the radicals that can be added or subtracted as not every radical is possible to add or subtract. You may also discuss the example from the student text or any similar example to elaborate the concept.

Finally, you select related problems from Exercise 1.7 question number 2 and assign them as class work or home work.

Assessment

You can assess by giving class work or homework on simplifying radicals with same index and same radicand.

Answers to Exercise 1.7

1. a. $\sqrt{10}$ b. $3\sqrt{2}$ c. $\sqrt{105}$

   d. $4x,\ x \geq 0$ e. 1 f. $\frac{1}{4}\sqrt{\frac{10}{3}}$

   g. $5y,\ (y > 0)$ h. 6 i. 4 j. $\frac{2\sqrt{6}}{75}$

2. a. $7\sqrt{3}$ b. $4\sqrt{2}$ c. $3\sqrt{3}$

   d. $\sqrt{7}$ e. $\sqrt{3}$ f. $3\sqrt{2}$

   g. $-4\sqrt{2}\,|x|$ h. $13\sqrt[3]{2}$ i. $10\sqrt{6}$

   j. 1 because k. $\frac{a-b}{\sqrt{ab}}, b > 0, a > 0$

$$\frac{\sqrt{a+2\sqrt{ab}+b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{\left(\sqrt{a}+\sqrt{b}\right)^2}}{\sqrt{a}+\sqrt{b}} \qquad a > 0,\ b > 0$$

$$= \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = 1$$

3. a. $89 - 28\sqrt{10}$

   b. i. let $x = \sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}$, then

   $x^2 = 8 \Rightarrow x = 2\sqrt{2}$

   $\therefore \sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}} = 2\sqrt{2}.$

   ii. Similar to (i), let $x = \frac{\sqrt{7+\sqrt{24}}}{2} + \frac{\sqrt{7-\sqrt{24}}}{2}$

   $\Rightarrow 4x^2 = 24 \Rightarrow x = \sqrt{6}$

   $\therefore \frac{\sqrt{7+\sqrt{24}}}{2} + \frac{\sqrt{7-\sqrt{24}}}{2} = \sqrt{6}.$

   iii. 2; $p^2 \geq 1$

4. $d = 0.215\text{m}$

1.2.5 The Four Operations on Real Numbers

In this subsection, students are reminded of the four operations they studied on the set of rational numbers. To this end, you group the students and ask them to do questions of the following types and Activity 1.12.

1. Indicate whether each of the following statements is true or false. In each case, justify your answer.

a. i) $\frac{2}{3} + \frac{5}{4}$ is a rational number.

ii) $\frac{2}{3} - \frac{5}{4} \in \mathbb{Z}$ where $\mathbb{Z}$ is the set of integer

iii) $0.45 \div 0.19 \in \mathbb{Q}$ (where $\mathbb{Q}$ is the set of rational numbers)

iv) $\frac{2}{5} \times 0.\overline{25} \in \mathbb{Q}$

b) $0.3 + 0.25 = 0.25 + 0.3$

c) $\frac{1}{3}\left(\frac{2}{7} + \frac{1}{3}\right) = \frac{-2}{21} + \frac{1}{24}$

2. State the property that is used in each of (a) – (g)

a) $\frac{3}{7} + \frac{9}{14} < \frac{4}{7} + \frac{9}{14}$

b) $\frac{3}{7} + \left(\frac{3}{8} + \frac{1}{4}\right) = \frac{3}{7} + \left(\frac{1}{4} + \frac{3}{8}\right)$

c) $\frac{5}{13} \times \frac{4}{9} < \frac{6}{9} \times \frac{4}{9}$

d) $\frac{1}{18} + \left(\frac{3}{11} + \frac{5}{9}\right) = \left(\frac{1}{18} + \frac{3}{11}\right) + \frac{5}{9}$

e) $\frac{5}{9} \times \frac{-3}{4} > \frac{6}{9} \times \frac{-3}{4}$

f) $\frac{2}{3} \times \frac{1}{4} + \frac{3}{11} \times \frac{1}{4} = \left(\frac{2}{3} + \frac{3}{11}\right) \times \frac{1}{4}$

g) $\frac{5}{9} \times \left(\frac{1}{9} \times \frac{1}{7}\right) = \frac{5}{9} \times \left(\frac{1}{7} \times \frac{1}{9}\right)$

The purpose of Activity 1.12 is to help students recall and understand the operations on rational numbers.

Answers to Activity 1.12

1. a. $\frac{2}{9} + \left(\frac{3}{5} + \frac{7}{9}\right) = \frac{2}{9} + \left(\frac{7}{9} + \frac{3}{5}\right) = \left(\frac{2}{9} + \frac{7}{9}\right) + \frac{3}{5} = \frac{9}{9} + \frac{3}{5} = \frac{9}{9} + \frac{3}{5} = 1 + \frac{3}{5} = \frac{8}{5} = 1.6$

b. $\frac{3}{7} \times \left(\frac{-11}{21}\right) + \left(\frac{-3}{7}\right)\left(\frac{-11}{21}\right) = \frac{-11}{21}\left(\frac{3}{7} + \frac{(-3)}{7}\right)\frac{-11}{21} \times 0 = 0$

c. $\frac{3}{7} + \left(\frac{5}{6} + \frac{(-3)}{7}\right) = \frac{3}{7} + \left(\frac{-3}{7} + \frac{5}{6}\right) = \left(\frac{3}{7} + \frac{(-3)}{7}\right) + \frac{5}{6} = 0 + \frac{5}{6} = \frac{5}{6}$

d. $\left(\frac{-9}{7} \times \frac{23}{-27}\right) \times \left(\frac{-7}{9}\right) = \left(\frac{23}{-27} \times \frac{-9}{7}\right) \times \left(\frac{-7}{9}\right) = \left(\frac{23}{-27}\right) \times \left(\frac{-9}{7} \times \frac{-7}{9}\right) = \frac{23}{-27} \times 1 = \frac{23}{-27}$

2. a. Associative property of multiplication in $\mathbb{Q}$.

b. Commutative property of addition in $\mathbb{Q}$.

c. Adding or subtracting the same number to both sides of an inequality.

After you discuss questions of the above type on the rational numbers and those given in Activity 1.12, you extend this discussion to the set of real numbers by considering radicals and the decimal representation of real numbers.

To do this, let students do Group Work 1.6. The purpose of Group Work 1.6 is the same as that of Activity 1.12 in the sense that it helps students to understand the four operations and their properties on the set of real numbers. After doing Group Work 1.6, it is hoped that students understand that the four operations and their properties hold also in the set of real numbers.

To effect the intention of doing Group Work 1.6, let each student do and discuss the problems in the Group Work 1.6 with a partner and finally let some of the groups present their work to the class. From the answers of the questions given in the group work, assist the students to state in their own words: the laws of exponents, the properties of the four operations in the set of real numbers; the results that can be obtained when the four operations are operated on the set of irrational numbers, and on a rational and irrational number. In particular, assist the students to fill in the table for Question 7 and encourage them to generalize their observation.

Answer to Group work 1.6

1.

Factors	Product	Product written as a power
$2^3 \times 2^2$	32	2^5
$10^1 \times 10^1$	100	10^2
$\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)^3$	$\frac{1}{625}$	$\left(-\frac{1}{5}\right)^4$

2.

Division	Quotient	Quotient written as a power
$10^5 \div 10^1$	10,000	10^4
$3^5 \div 3^2$	27	3^3
$\left(\frac{1}{2}\right)^4 \div \left(\frac{1}{2}\right)^2$	$\frac{1}{4}$	$\left(\frac{1}{2}\right)^2$

3.

a. True b. True c. True d. False

Because

i. let $a = 3+\sqrt{2}$ and $b = 3-\sqrt{2}$ both are irrational numbers then $a+b = 3+\sqrt{2} + 3-\sqrt{2} = 6$ is a rational number and $a \cdot b = 3+\sqrt{2} \quad 3-\sqrt{2} = 7$ is a rational number . This shows that the set of irrational numbers is not closed under addition and multiplication.

ii. Consider $x = -1+\sqrt{2}$ and $x = 1+\sqrt{2}$ where both are irrational numbers then $x - y = \ -1+\sqrt{2} \ - \ 1+\sqrt{2} \ = -2$ a rational number.

∴ The set of irrational numbers is not closed under subtraction.

iii. Let $a = \sqrt{8}$, $b = \sqrt{2}$ where both are irrational numbers but $a \div b = \sqrt{8} \div \sqrt{2} = 2$ which is a rational number.

i.e the set of irrational numbers is not closed under division.

Questions 4, 5 and 6 can be handled as the justification given for question 3(d).

7.

	Number	Rational number	irrational number	Real number
a	2	Yes	No	Yes
b	$\sqrt{3}$	No	Yes	Yes
c	$\frac{-2}{3}$	Yes	No	Yes
d	$\frac{\sqrt{3}}{2}$	No	Yes	Yes
e	$1.2\overline{3}$	Yes	No	Yes
f	1.20220222. . .	No	Yes	Yes
g	$-\frac{2}{3} \times 1.2\dot{3}$	Yes	No	Yes
h	$\sqrt{75} + 1.2\dot{3}$	No	Yes	Yes
i	$\sqrt{75} - \sqrt{3}$	No	Yes	Yes
j	1.20220222. . . + 0.13113111. . .	Yes	No	Yes

From the results given in the table we can give the following generalization.

i. A real number is either a rational or an irrational number.

ii. The product of two rational numbers is a rational number.

iii. The sum of a rational and an irrational number is an irrational.

iv. The sum of two irrational numbers can be a rational or irrational number. (Example. $-\sqrt{2} + \sqrt{2} = 0$ is rational and $\sqrt{2} + \sqrt{3}$ is irrational)

etc.

Following such an effort, you can assign groups of students and guide them to do Activity 1.13 which is supposed to help them see if they have any missing idea during their generalization.

To be sure that the students can perform the four operations and their properties on the set of real numbers, you may ask the students questions which enable them to apply the four operations and their properties as indicated below from Activity 1.14.

Question: Explain each step in the following simplification.

$$\sqrt{6}-2\sqrt{15}\ \times\frac{\sqrt{3}}{3}+\sqrt{20}=\frac{\sqrt{3}}{3}\times\ \sqrt{6}-2\sqrt{15}\ +\sqrt{20}\ \ldots\text{Commutative}$$

$$=\left(\frac{\sqrt{3}}{3}\times\sqrt{16}-\frac{\sqrt{3}}{3}\times 2\sqrt{15}\right)+\sqrt{20}\ \ldots\text{Distributive of multiplication over subtraction}$$

$$=\left(\frac{\sqrt{18}}{3}\times\sqrt{6}-\frac{\sqrt{3}}{3}\times 2\sqrt{15}\right)+\sqrt{20}\ \ldots\text{Theorem 1.2}$$

$$=\left(\frac{\sqrt{9}\times\sqrt{2}}{3}-\frac{2\sqrt{9}\times\sqrt{5}}{3}\right)+\sqrt{20}\ \ldots\text{Theorem 1.2}$$

$$=\left(\frac{3\times\sqrt{2}}{3}-\frac{2\times 3\times\sqrt{5}}{3}\right)+\sqrt{20}\ \ldots\text{Simplification of radicals}$$

$$=\ \sqrt{2}-2\sqrt{5}\ +\sqrt{20}\ \ldots\text{Cancellation}$$

$$=\ \sqrt{2}-2\sqrt{5}\ +2\sqrt{5}\ \ldots\text{Simplification}$$

$$=\sqrt{2}+\ \ -2\sqrt{5}\ +2\sqrt{5}\quad\ldots\text{Associativity}$$

$$=\sqrt{2}\ldots\ \text{Additive inverse}$$

After you discuss problems of the above type, then, finally the intention is to state and discuss the properties of the operations given on page 42 of the student text. But, to verify the failure of the closure property of the set of irrational numbers under the operations of addition, subtraction, multiplication and division, you can let them work on Activity 1.13. You can also give them some other additional examples.

Answers to Activity 1.13

1.	a.	6	b.	$5+2\sqrt{3}$
2.	a.	0	b.	$\sqrt{5}-\sqrt{2}$
3.	a.	2	b.	$6\sqrt{6}$
4.	a.	$\frac{5}{3}$	b.	$3\sqrt{2}$

When students discuss and get a better understanding, you can state the common laws of exponents and guide them to realize that these laws also hold on real numbers. You can also use the activities given in Group Work 1.6 to help them realize the laws.

1. $a^n \times a^m = a^{n+m}$
2. $\frac{a^n}{a^m} = a^{n-m}$, $(n > m)$
3. $(a^n)^m = a^{n\times m} = (a^m)^n$
4. $(a \times b)^n = a^n \times b^n$

Finally you give them Activity 1.14 either as class work or homework to help them do more on identifying additive and multiplicative inverses and use each in simplification.

Answers to Activity 1.14

1. a. -5 b. $\frac{1}{2}$ c. $-\sqrt{2}-1$

   d. $-2.4\overline{5}$ e. $-2.1010010001\ldots$

2. a. $\frac{1}{3}$ b. $\frac{1}{\sqrt{5}}$ c. $\frac{1}{1-\sqrt{3}}$ d. $2^{-\frac{1}{6}}$

   e. $\frac{1}{1.71}$ f. $\frac{\sqrt{3}}{\sqrt{2}}$ g. $\frac{1}{1.\overline{3}}=\frac{3}{4}$

3. $\sqrt{6}-2\sqrt{5}\ \times\frac{\sqrt{3}}{3}+\sqrt{20}=\frac{\sqrt{3}}{3}\times\ \sqrt{6}-2\sqrt{5}\ +\sqrt{20}\ =\sqrt{2}$.

Assessment

You can assess whether students have mastered simplifying operations involving radicals, you can let them do Exercise 1.8 either as homework or assignment.

Answers to Exercise 1.8

1. a. 2^{-9} b. $-3\sqrt{11}$ c. $\frac{6\sqrt{26}+20\sqrt{3}-\sqrt{2}}{2}$

   d. -0.1 e. 1.2

2. a. 6 b. $2^{\frac{19}{15}}$ c. $3^{\frac{5}{2}}$

   d. $7^{\frac{1}{4}}$ e. $15^{\frac{1}{4}}$ f. 1

   g. $\sqrt[12]{7}$ h. $\frac{2}{3}$

3. a. $\sqrt{3}$, $1+\sqrt{3}$ b. $\sqrt{5}$, $2+\sqrt{5}$

   c. – 0.383383338 . . ., 0.535535550 . . . d. – 0.123456 . . .,– 1. 123456...

   e. – 0.3030003 . . ., 0.0303330 . . .

1.2.6 Limits of Accuracy

In this subsection, we are trying to give students a feeling for numerical value of measurements and size to help them judge when to approximate and how to approximate. This is perhaps rather sophisticated and we do not expect full answers in every case.

In choosing suitable examples, an attempt has been made to make the work relevant to student's life. It is hoped that teachers will think of other projects similar to those in the textbook, projects better suited to the environment of their particular cases.

1. Counting and measuring

An important distinction is drawn between counting and measuring. The idea of approximation and the fact that many problems cannot have an exact answer are introduced. The inevitable inaccuracy of measuring is obvious as soon as different students measure the same thing.

The questions in the section are suitable for group discussion. The distinction between counting and measuring should be discussed so that all students see the differences. Counting answers are exact. If there is any disagreement, we can find who is correct by checking provided that we agree on what is to be counted. To help them get into the required aim, you can let them do Activity 1.15.

Answers to Activity 1.15

1. a. 30,000 b. 29,000 c. 28,600
2.
   a. i. 7.9 ii. 6.4 iii. 4.6
   b. i. 7.86 ii. 6.44 iii. 4.57
3. a. 43 b) 43.3
4. lower bound 5.35 and upper bound 5.45; $5.35 \leq 5.4 < 5.45$

Answers which are results of measurement are not exact and we may have disagreement. That is, we may have different measurement values that are more or less close to each other. The possible causes that impact difference in measurements can be attributed to rounding, consideration of decimal places, or significant figures. However, there is an important concept namely accuracy in measurement that can alleviate possible wider gaps in measurement.

2. Accuracy in measurement

This section will show students that, when asked to measure some things, we need to know

i. how accurately we are required to measure or
ii. to what accuracy it is sensible to try to measure. The idea of sensible answers is the more difficult to realize. But, students will actually have to measure to get this idea.

3. Giving the answer

Teach decimal places and significant figures separately as they are easily confused.

A. Decimal Places

Students should have come across the idea of correcting a given number of decimal places in their lower grades. However, they may not be able to give 5.02 to one decimal place.

The word "correct" in the instruction 'correct your answer to one decimal place' is unfortunate. The answer may be wrong. We really mean 'give your answer to one decimal place'.

There is emphasis on 4.3 (to one decimal place) to be a number between 4.25 and 4.35. This is where the word 'correct' interpreted as 'exact' worries the students.

In fact if d = 4.3 correct to 1 decimal place, then we mean

$$4.25 \leq d < 4.35.$$

It is helpful to use a number line to show that 4.3 correct to 1 decimal place means nearer to 4.3 than to 4.2 or 4.4.

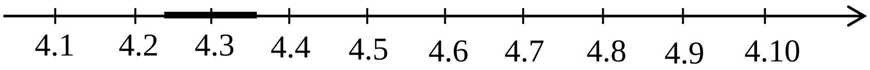


Figure 1.1

The dark black part on the number line shows possible values of d. One possible error by the students is to think that d = 4.3 (to 1 decimal place) means that

$$4.25 < d \leq 4.35.$$

This error occurs because students are often already working to 2 decimal places when they correct to one. This problem occurs when the maximum value of d is not given so that students can have a reference for approximation. For example, a discussion of this issue brings out the idea of limit so that d could be 4.349999 but not 4.35.

B. Significant figures

Significant figures may be more useful than decimal places as a means of specifying accuracy because they are independent of the unit used. For example 49m = 4900 cm = 0.049 km. In each case, it is implied that the length is accurate to two significant figures.

Assessment

Limits of accuracy and issues of estimation or approximation is one of the problems students need to clearly know and identify. In order to assess student understanding you can give them an assignment by setting questions or using Exercise 1.9 in group which may help them discuss one another and report their answers. You check their answers and conduct discussion to clear any observed gap.

Answers to Exercise 1.9

1.	a.	7000		b.	74000		c.	89000	d.	100,000
2.	a.	78500		b.	1000		c.	14100	d.	3000
3.	a.	490		b.	690		c.	8850	d.	0
	e.	80								
4.	i.	a.	5.6	b.	4.0	c.	157.4		d.	15.0
	ii.	a.	157.4	b.	12.0	c.	1.0		d.	3.0
	iii.	a.	6.47	b.	9.59	c.	0.01		d.	100.0
	iv.	a.	16.48	b.	3.00	c.	9.30		d.	12.05

5. a. 50000 b. 48600 c. 2.57 d. 2000
   e. 0.09 f. 0.95 g. 0.0031 h. 1.0

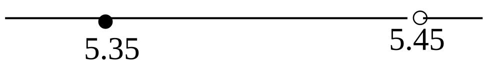
6. a. i. lower bound = 5.5
   upper bound = 6.5
   ii. $5.5 \leq x < 6.5$
   b. i. lower bound = 82.5
   upper bound = 83.5
   ii. $82.5 \leq x < 83.5$
   c. i. lower bound = 150.5
   upper bound = 151.5
   ii. $150.5 \leq x < 151.5$
   d. i. lower bound = 999.5
   upper bound = 1000.5
   ii. $999.5 \leq x < 1000.5$

7. a. i. lower bound = 3.75
   upper bound = 3.85
   ii. $3.75 \leq x < 3.85$
   b. i. lower bound = 15.55
   upper bound = 15.65
   ii. $15.55 \leq x < 15.65$
   c. i. lower bound = 0.95
   upper bound = 1.05
   ii. $0.95 \leq x < 1.05$
   d. i. lower bound = 0.25
   upper bound = 0.35
   ii. $0.25 \leq x < 0.35$
   e. i. lower bound = –0.25
   upper bound = –0.15
   ii. $-0.25 \leq x < -0.15$

8. a. i. lower bound = 4.15
   upper bound = 4.25
   ii. $4.15 \leq x < 4.25$
   b. i. lower bound = 0.835
   upper bound = 0.845
   ii. $0.835 \leq x < 0.845$
   c. i. lower bound = 415
   upper bound = 425
   ii. $415 \leq x < 425$
   d. i. lower bound = 4950
   upper bound = 5050
   ii. $4950 \leq x < 5050$
   e. i. lower bound = 0.0445
   upper bound = 0.050
   ii. $0.0445 \leq x < 0.050$

9. a. upper bound is 71.3475 and lower bound is 73.0575
   b. upper bound is 170.2725 and lower bound is 172.9325
   c. The quotient lies between 1.449 and 1.4569
   d. the quotient lies between 3.072 and 3.123
   e. the quotient lies between 4.23 and 4.47

10. a.



   b. $5.35 \leq M < 5.45$

11. lower bound is 3.90kg and upper bound is 4.10kg
12. 316.8m ≤ perimeter < 317.2m
13. 26.41cm ≤ x < 26.74cm

1.2.7 Scientific Notation

In this sub section, you are expected to assist students to practice writing standard notations of positive rational numbers and to recognize that this notation is useful in writing very small and very large positive numbers. For the students to have candid realization of this fact, you may group your students and encourage them to do Activity 1.16. Some examples that can help the students to realize usefulness of scientific notation is to guide them to note the following questions. What is the mass of the earth? How many neutrons does an atom have? What is the diameter of electron? etc.

Answers to Activity 1.16

1. a. 10^1 b. 10^2 c. 10^3

$13 = 1.3 \times 10^1$
$130 = 1.3 \times 10^2$
$1{,}300 = 1.3 \times 10^3$
$13{,}000 = 1.3 \times 10^4$
$1{,}300{,}000 = 1.3 \times 10^6$

2.

$13.0 = 1.3 \times 10 = 1.3 \times 10^1$
$1.3 = 1.3 \times 1 = 1.3 \times 10^0$
$0.13 = 1.3 \times \frac{1}{10} = 1.3 \times 10^{-1}$
$0.013 = 1.3 \times \frac{1}{100} = 1.3 \times 10^{-2}$
$0.0013 = 1.3 \times \frac{1}{1000} = 1.3 \times 10^{-3}$
$0.00013 = 1.3 \times \frac{1}{10000} = 1.3 \times 10^{-4}$
$0.000013 = 1.3 \times \frac{1}{100{,}000} = 1.3 \times 10^{-5}$
$0.0000013 = 1.3 \times \frac{1}{1{,}000{,}000} = 1.3 \times 10^{-6}$

When the students perform the Activity you can give them several examples that consist of expressing a number in scientific form and expressing a number given in scientific form in decimal form. You can also let them see the calculator representation of numbers given in scientific form. Here, you can form group of students and give them an assignment to collect different measurements expressed in scientific form such as speed of light.

Assessment

You can assess students understanding through the assignment or you can give them Exercise 1.10 for the purpose of assessment. You can ask them to bring examples that require scientific notation such as neutrons, protons, speed of light, etc and share.

Answers to Exercise 1.10

1. a. 7.67×10^{-3} b. 5.75×10^{9} c. 8.3×10^{-4}
   d. 4.004×10^{5} e. 5.4×10^{-2}
2. a. 488,200 b. 0.0000119 c. 202.1
3. 4×10^{-13}

1.2.8 Rationalization

Before starting this lesson, you may pose an inquiry by asking the students what we mean by rationalization and why we rationalize numbers. For this purpose, you may encourage students to do Activity 1.17. You may then proceed to the lesson by discussing the examples given in the student textbook. With active participation of the students, discuss the rules of rationalization with the help of these examples. In doing so, it is essential to give emphasis on how to determine the rationalizing factors. Emphasize on rationalizing the denominator as it is commonly used.

Answers to Activity 1.17

i. $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707107$ ii. $\frac{\sqrt{2}}{2} \approx 0.707107$

Following this Activity let students do more examples on rationalizing the denominator. Then, let them write their observation to lead them into the rule stated on student textbook on page 53.

Assessment

You can assess the students understanding by giving them exercises on rationalizing the denominator and checking their work. You can also use Exercise 1.11 for the purpose of assessment.

Answers to Exercise 1.11

a. $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$. Rationalizing factor $\frac{\sqrt{2}}{\sqrt{2}}$

b. $\sqrt{\frac{2}{6}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Rationalizing factor $\frac{\sqrt{3}}{\sqrt{3}}$, $\frac{\sqrt{6}}{\sqrt{6}}$ will also do, but it is not the simplest.

c. $\frac{5\sqrt{2}}{4\sqrt{10}} = \frac{5}{4}\sqrt{\frac{2}{10}} = \frac{5}{4} \times \frac{1}{\sqrt{5}} = \frac{5}{4} \times \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{4 \times 5} = \frac{\sqrt{5}}{4}$ Rationalizing factor $\frac{\sqrt{5}}{\sqrt{5}}$, $\frac{\sqrt{10}}{\sqrt{10}}$ will also do, but it is not the simplest.

d. $\frac{12}{\sqrt{27}} = \frac{12}{\sqrt{9\times3}} = \frac{12}{\sqrt{9} \times \sqrt{3}} = \frac{12}{3} \times \frac{1}{\sqrt{3}} = 4 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$.

Rationalizing factor $\frac{\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{27}}{\sqrt{27}}$ will also do

e. $\sqrt{\frac{5}{18}} = \frac{\sqrt{5}}{\sqrt{18}} = \frac{\sqrt{5}}{\sqrt{9} \times \sqrt{2}} = \frac{\sqrt{5}}{\sqrt{9} \times \sqrt{2}} = \frac{\sqrt{5}}{3\sqrt{2}} = \frac{\sqrt{5}}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{6}$.

Rationalizing factor $\frac{\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{18}}{\sqrt{18}}$ will also do

f. $\frac{3}{2\sqrt[3]{3}} = \frac{3}{2\sqrt[3]{3}} \cdot \frac{\sqrt[3]{9}}{\sqrt[3]{9}} = 3\frac{\sqrt[3]{9}}{2\sqrt[3]{27}} = \frac{3\sqrt[3]{9}}{2\times3} = \frac{\sqrt[3]{9}}{2}$. Rationalizing factor $\frac{\sqrt[3]{9}}{\sqrt[3]{9}}$

g. $\sqrt[3]{\frac{1}{4}} = \frac{1}{\sqrt[3]{4}} = \frac{\sqrt[3]{2}}{\sqrt[3]{4}\ \sqrt[3]{2}} = \frac{\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{\sqrt[3]{2}}{2}$. Rationalizing factor $\frac{\sqrt[3]{2}}{\sqrt[3]{2}}$

h. $\sqrt{\frac{9}{a^2}} = \frac{3}{a}$ $a > 0$ .No rationalizing factor required

i. $\frac{\sqrt[3]{20}}{\sqrt[3]{4}} = \sqrt[3]{\frac{20}{4}} = \sqrt[3]{5}$. No rationalizing factor required

j. $\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} . \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Rationalizing factor $\frac{\sqrt{5}}{\sqrt{5}}$

From the above discussion, students might be able to rationalize denominator where there is single rationalizing factor. But in practice we may need to rationalize where the rationalizing factor might be a combination. We may also need to apply the four operations of numbers that may need rationalization.

To give chance for the students to arrive at the required conclusion, you may guide them to do Activity 1.18.

Answers to Activity 1.18

1. 1 2. 7 3. $\frac{17}{4}$

When the students do this Activity you can lead them in to the generalization given on page 54 of the student textbook. Then, let them practice with more examples.

Assessment

You can use Exercise 1.12 for assessing students understanding. You can give them this Exercise as homework and check their work.

Answers to Exercise 1.12

a. $\dfrac{1}{3-\sqrt{5}} = \dfrac{1}{3-\sqrt{5}} \times \dfrac{3+\sqrt{5}}{3+\sqrt{5}} = \dfrac{3+\sqrt{5}}{3^2-\left(\sqrt{5}\right)^2} = \dfrac{3+\sqrt{5}}{4}.$

b. $\dfrac{18}{\sqrt{5}-3} = \dfrac{18\left(\sqrt{5}+3\right)}{\left(\sqrt{5}-3\right)\left(\sqrt{5}+3\right)} = \dfrac{18\left(\sqrt{5}+3\right)}{-4} = -\dfrac{9}{2}\left(\sqrt{5}+3\right).$

c. $\dfrac{2}{\sqrt{5}-\sqrt{3}} = \dfrac{2\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)} = \dfrac{2\left(\sqrt{5}+\sqrt{3}\right)}{2} = \sqrt{5}+\sqrt{3}.$

d. $\dfrac{\left(\sqrt{3}+4\right)\left(\sqrt{3}+2\right)}{\left(\sqrt{3}-2\right)\left(\sqrt{3}+2\right)} = \dfrac{3+2\sqrt{3}+4\sqrt{3}+8}{-1} = \dfrac{11+6\sqrt{3}}{-1} = -11-6\sqrt{3}.$

e. $\dfrac{10}{\sqrt{7}-\sqrt{2}} = \dfrac{10\left(\sqrt{7}+\sqrt{2}\right)}{\left(\sqrt{7}-\sqrt{2}\right)\left(\sqrt{7}+\sqrt{2}\right)} = \dfrac{10\left(\sqrt{7}+\sqrt{2}\right)}{5} = 2\left(\sqrt{7}+\sqrt{2}\right).$

f. $\dfrac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = \dfrac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} \cdot \dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} = \dfrac{18+6\sqrt{6}+3\sqrt{6}+6}{18-12}$

$= \dfrac{24+9\sqrt{6}}{6} = \dfrac{8+3\sqrt{6}}{2}.$

g. $\dfrac{1}{\sqrt{2}+\sqrt{3}-1} = \dfrac{1}{\sqrt{2}+\left(\sqrt{3}-1\right)} = \dfrac{\sqrt{2}-\left(\sqrt{3}-1\right)}{\left(\sqrt{2}+\left(\sqrt{3}-1\right)\right)\left(\sqrt{2}-\left(\sqrt{3}-1\right)\right)}$

$= \dfrac{\sqrt{2}-\sqrt{3}+1}{-2+2\sqrt{3}}$

$= \dfrac{\left(\sqrt{2}-\sqrt{3}+1\right)\left(-2-2\sqrt{3}\right)}{\left(-2+2\sqrt{3}\right)\left(-2-2\sqrt{3}\right)}$

$= \dfrac{-\sqrt{2}-\sqrt{6}+2}{-4} = \dfrac{\sqrt{2}+\sqrt{6}-2}{4}.$

1.2.9 Euclid's Division Algorithm

You may start this lesson by assisting students to express, state and generalize the Euclid's division algorithm, i.e, given two numbers p and d where $p > d$ then $p = q.d + r$, where q is the quotient and r is the remainder and $r \geq 0$. In doing so give emphasis on the nature of the numbers, i.e. all p, q, d and r are non-negative integers and $0 \leq r < d$. In addition to this, the students can also try to use Euclid's division algorithm to determine the greatest common factor of two numbers. You thus need to give exercises and problems on the application of the Euclid's algorithm like GCF (72, 12). However, to go about discussing each, it is recommended to group your students and give them Activity 1.19 to help them recall the concepts of factors, multiples and closure property under division. Dealing with the Euclid's division algorithm and some of the examples, you can give exercise 1.13 as a class work or homework.

Answers to Activity 1.19

1. No (example $\frac{2}{3} \notin \mathbb{W}$)
2. a. There is a non-negative integer c such that $a = bc$

   b. No (counter example: We can't find a number (non-negative integer) for 2 and 3 such that $3 = 2(c)$, $c \in \mathbb{Z}^+$)

Assessment

You can assess students while they do their class work on expressing two numbers in the form of $a = (q \times b) + r$ using Euclidean Division Algorithm.

Answers to Exercise 1.13

a.	72 = 11 (6) + 6	b.	16 = 9 (1) + 7
c.	11 = 18 (0) + 11	d.	106 = 13 (8) + 2
e.	176 = 21 (8) + 8	f.	25 = 39 (0) + 25

After deliberation of the Euclid's division algorithm, ask the students to repeat the algorithm and tell their observation. Example, Considering 64 and 13 we have:

$64 = 13 \times 4 + 12$

$13 = 12 \times 1 + 1$

$12 = 1 \times 12 + 0$

Simply leave them questioning themselves and proceed to give them Activity 1.20 which will guide them to do the tasks one by one and let them reach at their own conclusion. Finally, trace the ideas to arrive at how this repeated application of Euclid's division algorithm helps them to find the greatest common factor of pair of numbers.

Answers to Activity 1.20

1. 12   2. 12   3. 12
4. They are equal.
5. If a, b, q and r are positive integers, such that $a = q \times b + r$, then, GCF (a, b) = GCF (b, r).

Assessment

You can give to the students several pairs of numbers so that they can apply continued Euclidean Division Algorithm to determine their GCF. You can do this as class work or homework. You can also use the pairs of numbers in Exercise 1.14 for the purpose of assessment. As this is end of the unit you can also give test/quiz encompassing all parts of the unit.

Answers to Exercise 1.14

1. $224 = 2^5 \times 7$
   $84 = 2^2 \times 3 \times 7$
   $\therefore$ GCF (224, 84) $= 2^2 \times 7 = 28$

   $84 = 2^2 \times 3 \times 7$
   $56 = 2^3 \times 7$
   $\therefore$ GCF (84, 56) $= 2^2 \times 7 = 28$

   $6 = 2^3 \times 7$
   $28 = 2^2 \times 7$
   $\therefore$ GCF (56, 28) $= 2^2 \times 7 = 28$
   Hence, GCF (224, 84) = GCF (84, 56) = GCF (56, 28)

2. a. $18 = 1 \times 12 + 6$
   $\therefore$ GCF (18, 12) = GCF (12, 6)
   $12 = 2 \times 6 + 0$
   $\therefore$ GCF (12, 6) = 6
   $\therefore$ GCF (18, 12) = 6

   b. $269 = 3 \times 88 + 5$
   $88 = 5 \times 17 + 3$
   $17 = 3 \times 5 + 2$
   $5 = 2 \times 2 + 1$
   $2 = 1 \times 2 + 0$
   $\therefore$ GCF (269, 88) = 1

   c. $143 = 3 \times 39 + 26$
   $39 = 1 \times 26 + 13$
   $26 = 2 \times 13 + 0$
   $\therefore$ GCF (143, 39) = 13

   d. $1,295 = 3 \times 407 + 74$
   $407 = 5 \times 74 + 37$
   $74 = 2 \times 37 + 0$
   $\therefore$ GCF (1,295, 407) = 37

   e. $85 = 1 \times 68 + 17$
   $68 = 4 \times 17 + 0$
   $\therefore$ GCF (85, 68) = 17

   f. $7286 = 4 \times 1684 + 550$
   $1684 = 3 \times 550 + 34$
   $550 = 16 \times 34 + 6$
   $34 = 5 \times 6 + 4$
   $6 = 1 \times 4 + 2$
   $4 = 2 \times 2 + 0$
   $\therefore$ GCF (7286, 1684) = 2

Answers to Review Exercises on Unit 1

1. a. 533 is not divisible by any one of 2, 3, 4, 5, 6, 8, 9, and 10.
   b. 4,299 is divisible only by 3
   c. 111 is divisible only by 3
2. a. $150 = 2 \times 3 \times 5^2$ b. $202 = 2 \times 101$ c. $63 = 3^2 \times 7$
3. a. GCF (64, 16) = 16 b. GCF(480, 320, 160) = 160
4. a. $\frac{5}{8} = 0.625$ c. $5\frac{4}{9} = 5.\overline{4}$
   b. $\frac{16}{33} = 0.\overline{48}$ d. $3\frac{1}{7} = 3.\overline{142857}$

5. a. $0.65 = \frac{13}{20}$ d. $-24.\overline{54} = \frac{-2430}{99}$

b. $-0.075 = -\frac{3}{40}$ e. $-0.\overline{02} = \frac{-2}{99}$

c. $0.\overline{16} = \frac{16}{99}$

6. a. $\frac{-3}{2} < \frac{71}{100} < \frac{23}{30}$ c. $\frac{16}{27} < \frac{11}{18} < \frac{2}{3} < \frac{67}{100}$

b. $3.2 < 3.\overline{22} < 3.\overline{23} < 3.2\overline{3}$

7. a. $\sqrt{180} = 6\sqrt{5}$ c. $\sqrt[3]{250} = 5\sqrt[3]{2}$

b. $\sqrt{\frac{169}{196}} = \frac{13}{14}$ d. it cannot be simplified

8. a. $15^{\frac{1}{2}}$ c. $x-y\ ^{\frac{1}{3}}$

b. $a+b\ ^{\frac{1}{2}}$ d. $\left(\frac{13}{16}\right)^{\frac{1}{4}}$

9. a. $\sqrt{2}-1$ b. $\frac{\sqrt{15}}{3}$ c. $\frac{5\sqrt{3}-5\sqrt{7}}{4}$ d. $\frac{\sqrt[4]{13}}{2}$

10. a. $3\sqrt{7}-9$ b. 4 c. $\frac{2}{9}$ d. 6

11. 2.778

12. $x = 2,\ y = \frac{5}{6}.$

13. a. 7.41×10^{5} b. 6.48×10^{-5} c. 2.056×10^{-3} d. 1.24×10^{-5}

14. a. 2.7×10^{4} b. 7.96×10^{11} c. 6.4×10^{-7}

15. $3.56\sqrt{20}\,\text{km} \approx 16\,\text{km}$

UNIT 2 SOLUTION OF EQUATION

INTRODUCTION

This unit reviews students' previous concepts in regards to equations and their solutions. The unit gives much emphasis to equations involving exponents and radicals, systems of linear equation in two variables, equations involving absolute value, and quadratic equations. Different approaches to determining a solution to such types of equations is also elaborated with descriptive examples. Practical and application problems are also dealt to entail the fundamental use of the equations that model the problem and solving such problems is also addressed in the unit. Apart from these, some of the derivations of rules in solving equations and particularly that of quadratic equations are delivered in good detail in the unit. In general, the concepts discussed in this unit enable students to solve the types of equations stated above and to perform some of their application.

Unit Outcomes

After completing this unit, students will be able to:

- *solve problems on equations involving exponents and radicals.*
- *solve systems of simultaneous equations in two variables.*
- *solve simple equations involving absolute values.*
- *solve quadratic equations.*

SUGGESTED TEACHING AIDS IN UNIT 2

Although teaching aids may not be excessively exploited for this unit, you can present different charts that manifest graphical solutions to systems of equations. You can also encourage students to prepare different representative graphs of systems of linear equations by themselves.

Apart from use of the student textbook, you need to elaborate more application problems from your surrounding so that students can best appreciate and see how useful linear equations and systems of linear equations are.

You can group students, give them hints on a problem and let them assess such a problem from their daily life to develop their mathematical form, where the problems are either represented as a system of equations or can be reduced to a quadratic equation form.

2.1 EQUATIONS INVOLVING EXPONENTS AND RADICALS

Periods allotted: 3 Periods

Competency

At the end of this subunit, students will be able to:

- *solve equations that involve exponents and radicals by applying law of exponents.*

Vocabulary: Equations, Power, Exponents, Radicals, Bases

INTRODUCTION

Dealing with equations is a quite common experience in Mathematics. From the many different types of equations students dealt with in previous grade levels they are going to deal with those equations that involve exponents and radicals in this sub-unit. When you begin to explain equations with exponents and radicals it will be fundamental to revisit some of the laws of exponents discussed in unit 1. It is also recommended to look for practical applications whereby students can easily capture the meaning and get deeper understanding.

TEACHING NOTES

Students are expected to have some background on exponents and radicals. You may ask students to present and describe some of the rules of exponents they studied in unit one. After deliberation by students, you may start this lesson with introducing one of the rules for exponents that states "for $a > 0$, $a^x = a^y$, if and only if $x = y$". It is possible to encourage students, through question and answer, to revise the terms such as **power**, **base** and **exponent**, and illustrate with examples from real numbers which they have studied in unit one. For instance, 16 = 16 is always true. 16 can be expressed as a power as $2^4 = 2^4$ or $4^2 = 4^2$, in which case **if the bases are the same, then their exponent must also be the same**.

After pointing out these, let the students observe what the value of x must be if $2^4 = 2^x$. Following this, ask perhaps a question to the students to determine any relation between x and y, in the case $a^x = a^y$ where $a > 0$. Without mentioning the rule, give some exercises or the Activity 2.1 in the student textbook and group the students so that they can practice to solve. You need to facilitate their work. The purpose of this activity is to help the students to revise some of the rules of exponents, and to check whether they are capable of forming power form of numbers. At this moment if there are fast students to whom these exercises and the Activity could be easier, you can ask them to determine any relation between x and y, in the case $a^x = a^y$ where $a < 0$ which may help them to generalize the rule for any $a \neq 0$. It may also inquire them to think of the parity (being even or odd) of the exponent. The possible answers for Activity 2.1 are as follows.

Answers to Activity 2.1

1. a. False b. True c. True
   d. False e. True
2. a. $8 = 2^3$ b. $27 = 3^3$ c. $625 = 5^4$ or $625 = 25^2$
   d. $343 = 7^3$.

After ensuring the ability of the students in conducting Activity 2.1 select voluntary students and encourage them to do each of the equations presented as example 1 on the board. With students participation you need to assist them to tell the reasons for each of the steps when they solve the questions. You may then give Activity 2.2 so that each student will do by his/her own. This activity is prepared for the students to apply the rules of exponents in solving equations. Before they do, you can hint them to use example 3a as a guide for doing Activity 2.2. At this stage you may round to identify those who need further assistance and those who are fast enough to solve each, and put record. For those who are fast enough you can give them additional problems from the Exercise 2.1 or examples of type $\left(-9\right)^x = 27^{\frac{2}{3}x}$ so that they will think of what x must be and how such different bases may be equated. For all other students you can solve each of the questions of Activity 2.2 on the board by giving chance for each student to participate. You may then give chance for students to solve example 2 and example 3 on the board. The solutions of Activity 2.2 are the following.

Answers to Activity 2.2

a. $x = 2$ b. $x = -2$ c. $x = \frac{1}{8}$

Finally you can ask the students to state the rule of exponents they applied here and check if they have gained the insight.

You can add more exercises for consolidating the rules and their use in solving equations involving exponents and radicals.

At this stage, you may help students to realize the rule "for $a > 0$, $a^x = a^y$ if and only if $x = y$" and pose a question for them why is $a > 0$, in the rule? What will happen if $a < 0$ or $a = 0$?

You can give hint by asking them to solve $x^2 = 9$ which leads to $x^2 = 3^2$ in which $x = 3$ may be the immediate solution, as we had $a > 0$ in the rule, and leave them with the question what if we take $x = -3$? You can also add an example say $x^3 = 27$ which is expressed as $x^3 = 3^3$ in which the only solution is $x = 3$ so that they can develop more critical thinking on the rules of exponents and the restrictions.

You can also give application problems. For example, the area of a square with unknown length of sides is 16cm^2. Find the length of the side of the square whose solution can be as follows.

Area of a square is x^2 where x is the length of the side. Thus, $x^2 = 16$ implies $x^2 = 4^2$. From this we see that the length of the side of the square is 4cm by applying the rule. They need to reason out why $x = -4$ cannot be a solution, even if $(-4)^2 = 16$.

After delivering the lesson, since all students may not go in parallel, it is necessary to develop additional exercises of different capacity apart from the ones given in the student textbook that need to be solved by students themselves. You can also encourage students, either as a group work or as an assignment, to come up with different exercises and share each with one another.

For checking the student participation and understanding you can give Exercise 2.1 as homework, check their work and put a record.

Assessment

Apart from the details mentioned above, you can also use any one of the following for assessing students learning: class activities, group discussions, assignments, exercise problems on equations involving exponents and radicals similar to the problems in Exercise 2.1, and a quiz or a test.

Answers to Exercise 2.1

1. a. $x = 3$ b. $x = -2$ c. $x = \frac{-1}{12}$ d. $x = \frac{-13}{20}$

   e. $x = \frac{-3}{2}$ f. $x = -16$ g) $x = 1$ h. $x = \frac{4}{5}$

2. $x = 4$ or $x = -\frac{2}{5}$

3. a. $x = \frac{-1}{7}$ b. $x = -\frac{2}{5}$ c. $x = \frac{-10}{33}$

2.2 SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

Periods allotted: 8 Periods

Competencies

At the end of this subunit, students will be able to:

- *solve simultaneous equations (systems of equations in two variables).*
- *identify the three cases of solutions of simultaneous equations (a unique solution, no solution, infinitely many solutions).*

Vocabulary: Equations in two variables, System of equations, linear

INTRODUCTION

This sub-unit is devoted to discussing systems of linear equations particularly in two variables. Before beginning to discuss systems of linear equations in two variables and their solutions, students need to recall what a linear equation is and need to realize what happens if two or more equations are considered at a time. How to proceed to deal with this sub-unit may differ from one authority to another but some guide that you can use as a spring board is outlined hereunder.

TEACHING NOTES

You can introduce the unit by revising linear equations and solving linear equations. By forming groups of students, you can give them exercises on solving linear equations of type $2x + 3 = 7$ and $2x + 3y = 5$. Enable students to determine solution to a linear equation and see different equivalent equations, which is a basis for solving. Let them critically understand the difference between the solutions of a linear equation in one variable of type $2x + 3 = 7$ and linear equation in two variables of type $2x + 3y = 5$, as a single number and as infinitely many respectively.

For a better entry let each student do Activity 2.3. The purpose of this activity is to help students to revise solving linear equations in one variable and recognize that a linear equation in one variable has one solution. The following is a solution for Activity 2.3.

Answers to Activity 2.3

1. a. $x = 9$ b. $x = -4$ c. $x = 2$ d. $x = 6$ e. $x = 3$
2. One solution

Following this Activity, form group of students and let them perform Group Work 2.1. At this stage students are expected to reach at the conclusion that such equations may have exactly one solution, no solution or infinitely many solutions. When the students finish the group work, identify the better three groups and let them do their work on the board. With question and answer, assist students to arrive at the anticipated conclusion.

Answers to Group Work 2.1

1. a. $7x - 3 = 2(3x + 2) \Rightarrow 7x - 3 = 6x + 4 \Rightarrow x = 7$

   b. $-3(2x + 4) = 2(-3x - 6)$

   $\Rightarrow -6x - 12 = -6x - 12 \quad \Rightarrow 0 = 0$, which is always true for any x ($\mathbb{R}$).

   c. $2x + 4 = 2(x + 5)$

   $\Rightarrow 2x + 4 = 2x + 10 \Rightarrow 0 = 6$, which is always false ($\varnothing$).

2. (a) has only one solution, (b) has infinitely many solutions and (c) has no solution.
3. The conclusion is that a linear equation of such a type can have either one solution, infinitely many solutions or no solution.

After identifying these, give the students as many linear equations as possible to practice how to reduce and solve them. Here for a better understanding to solving, you may also need to give them equations to check if they are equivalent, and clarify how useful it is to form equivalent equations for solving as a revision from their previous grades. It is possible to give such equations as an assignment so that students will have ample options to approaching each question and discuss among one another.

While giving exercises or assignments you have to take care in selecting linear equations involving one variable so that the students can see the characteristics of a solution in that a linear equation may have no solution, may have only one solution, or may have infinitely many solutions.

So far students are aware of linear equations in one variable. At this stage, it is expected that students will be able to identify linear equations in two variables and determine the number of solutions such equations can have. This is done so to lay a foundation for discussing the concept of solving systems of linear equations.

In order to begin, let each student perform Activity 2.4 to identify linear equations in two variables and possible number of solutions for a linear equation in two variables. You can do the first question through oral question in class and giving chance for the students to give reasons. For questions 2 and 3, you can give chance for students to come out and answer each on the board. The answer to each of the questions in Activity 2.4 is as follows.

Answers to Activity 2.4

1. $2x - y = 5$ and $-x + 7 = y$ are linear equations in two variables.

   $2x + 3 = 4$ is linear equation but it has only one variable.

   $2x - y^2 = 7$ and $\frac{1}{x} + \frac{1}{y} = 6$ have two variables but they are not linear equations.
2. The number of solutions for each of the linear equations is infinitely many.
3. a. $y = 24000x + 400$

   b. $24000x + 2y = 106000$

Here, you may need to show to students that the geometric solution to linear equations is a point on the real line for linear equations of type $2x + 3 = 7$ and is a straight line for linear equations in two variables of type $2x + 3y = 5$, because this will help you deal with solution of a system of linear equations. You can elaborate these through the example in the student text.

From the example let the students discuss how to determine a value of one variable given a value of the other. They need also to discuss plotting the points they determined. From possible plots of some points encourage the students to discuss the number of solutions and point out why there will be infinitely many solutions for such equations in two variables if it has at least two pairs of numbers that satisfy the equation. At this moment you may ask fast learners to determine the equation of a line from its plot points. You can also give them arbitrary plot points, say (1, 0), (2, 1), (3, 4) and (4, 8) and ask them whether they represent a linear line or not. Its answer is “Not”. This will help them to visualize what plots of points of linear equation must look like.

Pursuant to the discussion and after ensuring the ability of plotting points on a coordinate plane, let students perform Activity 2.5 by forming groups. Through this activity students are expected to be able to determine coordinate points from linear equation in two variables and plot such coordinate points on a coordinate plane. They are also expected to make an understanding that a common point on both plots is a solution to a system of linear equations.

Answers to Activity 2.5

1. The value of each y based on each values of x gives the following list of ordered pairs (–2, –1), (–1, 0), (0, 1), (1, 2) and (2, 3) for $y = x + 1$ and (–2, 3), (–1, 2), (0, 1), (1, 0) and (2, –1) for $y = -x + 1$.
2. You are supposed to plot these points on a coordinate plane. You can draw the plot of these points on a flip chart before you come to class so that you can use that for display during discussion in class.
3. Form group of students and help each group of students to plot each of the points on one coordinate plane and report their observation.
4. Here you need to help them observe that the point (0, 1) satisfies both equations. You can now display the flip chart you drawn at and verify that that the point (0, 1) is the solution to both equations, and leave this for the students to arrive at the conclusion that the point (0, 1) is a solution to the system of the linear equations $\begin{cases} y = x + 1 \\ y = -x + 1 \end{cases}$ which simply can be considered as a joint consideration of these two linear equations.

Pursuant to this discussion, you can ask to students what a system of linear equations is all about and you may proceed to introducing the general form of a system of two linear equations (sometimes called as simultaneous equations) given in definition 2.2 and with the help of examples as described in the student text, and guide them to tell what a solution to a system of linear equations is. Considering that students would have a clue to these concepts, give them an activity to explain what a solution to a system of linear equations is and give them also some systems of linear equations as offered in the student textbook for them to try to solve, and note the solution they give. Here, you may not need to limit them in their approach to solving the systems of linear equations you give them. They can try to solve each in any of the approaches they know or they feel are suitable.

After their deliberations and observations you can state the formal definition of a solution to a system of linear equations. Here you can pose a question of how we solve a solution to a system of linear equations. Give chance for the students to discuss on this issue. From their previous observation or prior background, students can respond differently. Before telling the details of each, you can list some pairs of numbers that satisfy the equations in the system and check if they have a common one. At this moment students can think of listing the proper pairs, which may not be always easy. However, to give them another insight, it is advisable for you to let students do Group Work 2.2 so that they can discuss each question and reach at a conclusion that a system may have exactly one solution, no solution or infinitely many solutions irrespective of their approaches. They will also be sensitized with the applications of systems of linear equations in real life problems.

Answers to Group Work 2.2

1. From this group work please note that the lines from the system given in question 1 (a) are intersecting which means that they have one common solution, those from question 1(b) are parallel implying that they do not have a common point and thus has no solution, and those from question 1(c) are the same (one coinciding one over the other) implying that there are infinitely many solutions to the system.
2. The reply for question 2 will be that: the pairs in (a) intersect only at one point; the pairs in (b) do not intersect, and the pairs in (c) overlap one over the other to mean that there are infinite intersection points.
3. The conclusion from the tasks in (1) and (2) is that: If the lines of each of the systems of equations intersect at one point, then the point of intersection is the solution to the system. If they do not intersect at a point, however, either the system has infinitely many solutions or there is no solution.

   These three possibilities are listed in the student textbook as well immediate to the Group Work..
4. a. If (t, r) represents the pairing of time against rate then we have two ordered pairs at 1990 where t = 0, r = 5% and at 2002 where t = 12, r = 0.05% which are

$\left(0, \frac{5}{100}\right)$ and $\left(12, \frac{0.05}{100}\right)$. Applying two points equation we find that the equation of the straight line is $r = -0.00413t + 0.05$. From this equation if r = 0.001% then the time will be $0.00001 = -0.00413t + 0.05$ implying that required t will be more than 12 years.

b. Students can first identify possible pairs and then determine models to fit the reality. Example they can consider average number of patients visiting health center around them in two consecutive months and the average cost for medication in both months. This means that they will have the pairs (n_1, c_1) and (n_2, c_2) where n_1 is number in month 1 and c_1 is constant in month 1, and where n_2 is number in month 2 and c_2 is constant in month 2.

At this stage majority of the students may think of plotting as a way to solving systems of linear equations from the observation they made earlier. Acknowledging their possible feeling and expectation you first hint them that there are different approaches to solving systems of linear equations. You may tell them that the approach described earlier is not frequently used one. Instead there are other approaches that are commonly used in solving systems of linear equations that these include graphical method, substitution method, and elimination method. After finishing this, please inform your students to come to the next lesson with a ruler.

Giving them time to think and get prepared, in subsequent lesson you can start discussing the graphical method to solving system of linear equations. What you need to do this time is inform your students to do activity 2.6 individually and check if they can locate the solution to each of the systems of equations. You may then encourage students to come out and do each on the board. You need also assist them when essential for clarification purpose. You then help them do and discuss the example given in the student textbook. Here, there could be some students who are fast enough to get the solution easier since they are integer solution and may not consider it useful. For this purpose you can give additional exercises whose intersection point contains fraction so that they will be engaged in thinking of such a solution which you are going to discuss later.

Example: Solve the system $\begin{cases} x + 4y = 2 \\ 3x - 2y = 4 \end{cases}$ whose solution is $\left(\frac{10}{7}, \frac{1}{7}\right)$.

One of the points that need to be understood is that the graphical method may not always give us the exact point of intersection which makes it less used. What would happen if the pairs of lines intersect at a point which cannot be determined through observation? How can we determine the solution? You need to leave these questions for the students to discuss on. After discussing on these, it is expected that the students can understand the need for having other methods of solving systems of linear equations. You may elaborate this for better understanding and guide them that the points of subsequent discussion will be the other two methods of solution, namely substitution and elimination methods.

Here you need to give chance for students to discuss Group Work 2.3. The guidelines for dealing with this group work are offered in the student textbook. You need to facilitate helping the students to follow each of the steps outlined in the student textbook. This time you may record student abilities and their performance for the purpose of assessment.

Answers to Group Work 2.3

1. $\begin{cases} x+4y=2 \\ 3x-4y=6 \end{cases}$

By substitution method, we take one of the equations and express one of the variables in terms of the other.

Say we take $x + 4y = 2$ and we solve for x in terms of y which will be $x = 2 - 4y$. Substituting this in the second equation, we get $3(2-4y) - 4y = 6$

$\Rightarrow 6 - 12y - 4y = 6$

$\Rightarrow -16y = 0 \Rightarrow y = 0$

After finding this value of y, to find the value of x replace this value of y in any one of the equations which will give $x = 2$.

Therefore the solution set is {(2, 0)}

By elimination method, the interest is to make any of the coefficients of one variable (x or y) equal and opposite in sign. For this system the coefficients of y are equal and opposite in sign. We can add the equations and get $4x = 8$ which eliminates the variable y. (That is why it is called method of elimination).

From this we see that $x = 2$.

Now replace this value of x and find the value of y which is $2 + 4y = 2$ (if we replace

$x = 2$ in equation 1)

From this $4y = 0 \Rightarrow y = 0$

Therefore, the solution set is {(2, 0)}

2. $\begin{cases} -x+2y=4 \\ 3x-y=3 \end{cases}$ In the same way as we did above, the solution set is {(2, 3)}

Through the discussions delivered previously, the students may become capable of applying the three approaches to solving systems of linear equations. To enrich their understanding of each method you can let them discuss the examples presented in the students textbook. If there are fast students who finish the examples earlier than others you can add some more examples for them. From doing the examples each student can tell whether a system has one solution, many solutions or no solution. However, it will be after performing all those long steps that the students can tell about the solutions which may not be good to practice and it will be wastage when a system does not have a solution. Thus it is essential to be able to identify whether a system has a solution or not before performing the solution steps. The next section deals on this issue.

At this stage, you may need to facilitate discussion on Activity 2.7. On this activity students will be doing each of the leading questions and you will facilitate the discussion. After the students have tried, you can formulate some of the essential steps as outlined in the student textbook and give them additional examples with which the students will practice on.

This activity 2.7 is organized to help them relate the solution to a system of linear equations and equality of the ratios of corresponding coefficients.

Answers to Activity 2.7

1. a. has only one solution
   b. has no solution
   c. has infinite solutions
2. For (a) $\frac{3}{1} \neq \frac{1}{-2} \neq \frac{2}{3}$. For (b) $\frac{1}{2} = \frac{-2}{-4} \neq \frac{3}{5}$. For (c) $\frac{2}{4} = \frac{3}{6} = \frac{1}{2}$
3. The relation is:
   a. If the ratios of the coefficients of the variables are not equal then there is one solution.
   b. If the ratios of the coefficients of the variables are equal but are not equal with the ratio of the constants then there is no solution.
   c. If the ratios of the coefficients of the variables and the constants are all equal then there are infinite solutions.
4. The solution for 4 is outlined in the student text.

In doing so, help the students to reach at the conclusion that a system of linear equations may have no solution where $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, may have one and unique solution where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or may have infinitely many solutions where $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. You need to give more examples and exercises with which the students can tell whether they have one solution, many solutions or no solution, and help them understand that they only need to solve a system only if it has a solution.

After ensuring their ability of solving systems of linear equations, you can proceed to developing word problems that leads to a system of linear equations. You can let the students do Group Work 2.4 as given in the student textbook. The intention of this group work is to help students formulate the mathematical form of a problem and solve the problem. You may also need to deal with the examples from the students textbook and you can give an assignment (may be as a group work) to the students to come up with some real problems whose equation leads to a system of linear equations with their mathematical formulation. The assumption here is that the students are already capable of solving formulated system of linear equations and what is essential for them is to practice the formulation of problems through mathematical equations.

Answers to Group Work 2.4

1. i. Let x be the cost for a pencil and y be the cost for a rubber eraser

   ii. Now, Teshome's cost is $6x + 2y = 3$

   Meskerem's cost is $4x + 3y = 3$

   If we consider these two equations simultaneously we get the system of linear equations $\begin{cases} 6x+2y=3 \\ 4x+3y=3 \end{cases}$

   iii. The solution of this system of linear equations is {(0.3, 0.6)}

   That is the cost of pencil is 0.3Birr or 30 cents and the cost of rubber eraser is 0.6 Birr or 60 cents.

2. i. Let x be the cost for fertilizer brand A and y be the cost for fertilizer brand B

   ii. Now, Cooperative's cost is $10x + 27y = 20{,}000$

   Tolossa's cost is $15x + 9y = 14{,}250$

   If we consider these two equations simultaneously, we get the system of linear equations $\begin{cases} 10x+27y=20000 \\ 15x+9y=14250 \end{cases}$

   iii. The solution of this system of linear equations is {(650, 500)}

   That is the cost of fertilizer brand A is Birr 600 and the cost of fertilizer brand B is Birr 500.

After delivering the lesson, since all students may not go in parallel, it is important to develop additional exercises of different capacity apart from the ones given in the student textbook that need to be solved by students themselves. When students do certain activity or group work, if there are students who are in a better position, you may give them additional problems separately, and if you face difficulty with some students lagging behind it will be essential to organize them with the capable students so that they can get help from their peers, and you may also need to arrange some special program to assist them. You need also deliver additional exercises as deemed necessary for a better understanding.

Assessment

Apart from the assessments highlighted in each section; class activities, group discussions, homework/assignments, exercise problems on the application of each of the methods for solving systems of linear equations, and tests/quizzes can also be used based on your school situation.

Answers to Exercises 2.2

1. a and c are linear equations in two variables. Others are not.
2. Assuming the smaller number is x and the larger number is y, the two numbers are solved from the system $\begin{cases} x+y=64 \\ 5y+2y=20 \end{cases}$ and the two numbers are –36 and 100.
3. The numbers are obtained from the system $\begin{cases} x+y=14 \\ 2x-y=1 \end{cases}$ and the numbers are 5 and 9, meaning the number is 59.
4. a. one solution $\left\{\left(\frac{10}{3},3\right)\right\}$ b. one solution $\left\{\left(5,\frac{2}{5}\right)\right\}$
   c. one solution $\{(3,\ 2)\}$ d. one solution $\{(3,2)\}$
5. a. $\left(\frac{21}{13},\frac{16}{13}\right)$, because the lines intersect at one point.
   b. No solution, because the lines are parallel.
   c. No solution, because the lines are parallel.
   d. $\left(2,\ \frac{1}{2}\right)$, because the lines intersect at one point.
   e. (2, 0), the lines intersect at one point.
6. a. No solution or $\{\ \}$ b. No solution or $\{\ \}$ c. Infinite solutions
   d. $\left\{\left(1,\frac{5}{2}\right)\right\}$ e. $\{(1,\ 0)\}$
7. a. $\{(0,\ 5)\}$ b. $\{(3,\ 2)\}$ c. $\{(3,\ 0)\}$
   d. $\{(2,-2)\}$ e. $\{(1,\ 0)\}$
8. a. $\{(1,-6)\}$
   b. First replace the equation in terms of a and b as $\begin{cases} 2a+3b=-2 \\ 4a-5b=1 \end{cases}$.

After solving for a and b, we get $\left\{\left(\frac{-7}{22},\frac{-5}{11}\right)\right\}$ in terms of a and b which in turn gives

$\frac{1}{x}=a=\frac{-7}{22}$ from which $x=\frac{-22}{7}$ and $\frac{1}{y}=b=\frac{-5}{11}$ from which $y=\frac{-11}{5}$.

Therefore, the solution to $\begin{cases} \frac{2}{x}+\frac{3}{y}=-2 \\ \frac{4}{x}-\frac{5}{y}=1 \end{cases}$ is $\left\{\left(\frac{-22}{7},\frac{-11}{5}\right)\right\}$.

9. By substituting each we produce the system $\begin{cases} 3b + c = 5 \\ -4b + c = -9 \end{cases}$ whose solution is $\{(b, c)\} = \{(2, -1)\}$. Therefore $b = 2$ and $c = -1$.

10. Let x = the first solution and y = the second solution. From these conditions, we produce the system $\begin{cases} x + y = 100 \\ 0.2x + 0.45y = 30 \end{cases}$ and the solution is $\{(x, y)\} = \{(60, 40)\}$

Therefore, the student should mix 60ml of solution one and 40ml of solution two to obtain a 100ml solution whose 30% is acid.

2.3 EQUATIONS INVOLVING ABSOLUTE VALUE

Periods allotted: 3 Periods

Competency

At the end of this subunit, students will be able to:

- *solve equations involving absolute values.*

Vocabulary: Absolute value, Equations involving absolute values

INTRODUCTION

Some measurements such as distance, area and volume always assume positive magnitude. Pursuant to such considerations, discussion on absolute values is fundamental. In this sub-unit students need to discuss on absolute values first and then they need to proceed into discussing equations that involve absolute values. Some properties of absolute values are also highlighted in this sub-unit.

TEACHING NOTES

Students are expected to have some background about absolute value. You may start this subunit by asking students to state the meaning of absolute value. You can do this through question and answer. Depending on how they state absolute value, you may get various forms. You then need to give chance for the students to discuss on the possible different settings of the meaning of absolute value. After discussion, you can state the definition in the students textbook. You can also deliver the following as an alternative.

Absolute value of a number x means $|x| = \begin{cases} x; & x > 0 \\ 0; & x = 0 \\ -x; & x < 0 \end{cases}$

You may now give additional questions and the examples given in the student textbook to lead the students towards solving equations that involve absolute values of linear expressions. Based on their effort, you can ask them if they can state some of the observed properties of absolute values. After the students have tried the questions, you can add the critical notes such as:

1. For any real number x, $|x| = |-x|$.
2. For any real number x, $|x|$ is always positive.
3. For any non-negative number $a(a \geq 0); |x| = a$ means $x=a$ or $x=-a$.
4. For any non-negative number $a(a \geq 0); |x| = a; |x| = |a|$ means or $x = a$ or $x = -a$.

After discussing these points, you may give further activities for students to solve problems that involve absolute values similar to what is stated, as examples, in the student textbook. Finally, state properties of absolute values and give exercises that are solved by applying these properties.

To keep all students on board, offer additional exercises or Activities to support slow learners and enrich further notes of activities and/or exercises for gifted learners.

Assessment

It is possible to use any one of the following assessment techniques: class activities, group discussions, homework/assignments, exercise problems on solving equations involving absolute value of linear expression, and/or tests/quizzes.

Answers to Exercise 2.3

1. a. 5 b. 5 c. 7 d. 4
2. a. $\{\ \}$, since absolute value cannot be negative b. $\{0, 10\}$
   c. $\{-2, 5\}$ d. $\left\{\frac{-5}{4}, \frac{11}{4}\right\}$ e. $\{-9, 3\}$ f. $\{2, 8\}$
3. a. $\{1, 3\}$ b. $\left\{\frac{3}{2}\right\}$ c. $\{\ \}$
   d. $\left\{\frac{-11}{4}, \frac{3}{2}\right\}$ e. $\left\{\frac{-1}{2}, \frac{7}{4}\right\}$
4. a. $\left\{\frac{-3}{2}, \frac{15}{2}\right\}$ b. $\left\{-5, \frac{3}{2}\right\}$ c. $\{-3, 5\}$
   d. $\left\{-3, \frac{13}{5}\right\}$ e. $\left\{-2, \frac{4}{3}\right\}$ f. $\left\{\frac{5}{2}, \frac{11}{2}\right\}$
5. a. $|y - x| \leq |x| + |y|$, where $x = -2$ and $y = 3$ because,
   $|3 - (-2)| \leq |3| + |-2| \Rightarrow |3 + 2| \leq |3| + |-2| \Rightarrow |5| \leq |3| + |-2| \Rightarrow 5 \leq 5$ is true.
   b. $\sqrt{(3x - 7)^2} = |3x - 7|$ where $x = 5$ because,
   $\sqrt{(3 \times 5 - 7)^2} = |3 \times 5 - 7| \Rightarrow \sqrt{(15 - 7)^2} = |15 - 7| \Rightarrow \sqrt{(8)^2} = |8|$.

2.4 QUADRATIC EQUATIONS

Periods allotted: 8 Periods

Competencies

At the end of this subunit, students will be able to:

- *solve quadratic equations by using any one of the three methods.*
- *apply Viete's theorem to solve problems related to roots of a quadratic equation.*

Vocabulary: Quadratic equations, Factorization, Roots, Viete's theorem

INTRODUCTION

In previous sub-units students were discussing linear equations. In this sub-unit they are expected to discuss quadratic equations. For this purpose, initially students need to discuss factorization and then they will discuss on solving quadratic equations, whereby factorization is applied. As a means for generalizing solutions for quadratic equations, they will also discuss completing the square and apply this completing the square method for generalized quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ to derive the general solution for quadratic equation. Finally they will try to relate roots of quadratic equations with coefficients of the generalized form of a quadratic equation. Cognizant of this, they are also expected to discuss some of the applications of quadratic equations in modeling and solving real life problems.

TEACHING NOTES

You may start the lesson by recalling a linear equation and factors in a linear equation. You may also proceed to introducing the general form of a quadratic equation. You can do this by asking students to explain before you state it. After understanding the general form of a quadratic equation give the chance for students to discuss on "factorization: factorizing expressions, factorizing difference of two squares, and factoring trinomials". To let the students better understand factors and factorization, you can let them do Activity 2.8 so that they can multiply factors to get their products and practice going back from products to factorizing. Help them do the examples in the student textbook and more questions of such a type, and ensure that they are capable of factorizing expressions.

Answers to Activity 2.8

1. a. $x(x + 9) = x^2 + 9x$
   b. $(x + 3)(x - 3) = x^2 - 9$
   c. $(x + 2)(x + 3) = x^2 + 5x + 6$
2. a. $x^2 - 9 = (x + 3)(x - 3)$
   b. $x^2 + 9x = x(x + 9)$
   c. $x^2 + 5x + 6 = (x + 2)(x + 3)$

Following this, you can proceed to let the students perform Activity 2.9 to see difference of two squares and help them see how factorization makes things simple.

Answers to Activity 2.9

1. To solve $75^2–25^2$ we can apply even the direct method. But it seems cumbersome and unwise.
   We can apply difference of two squares and solve it easily as
   $(75 – 25)(75 + 25) = (50)(100) = 5000$.
2. $200^2 – 100^2 = (200 – 100)(200 + 100) = (100)(300) = 30000$.

You need here to give more examples for students to practice on factorization which can be in a form of homework. While they do factorization problems guide them to checkout if they can express the relation between sum of the factoring numbers with the middle coefficient and their product with the constant term. Before telling them anything, you can then proceed to factorizing more other forms by giving chance for the students to perform Activity 2.10. Give them further examples similar to the questions in the activity and hint them how useful factorization is for solving quadratic equations. You need also make sure that students can tell why they need to factorize expressions before proceeding to the next.

Answers to Activity 2.10

a. $2x^2 + 10x + 12 = 2x^2 + 4x + 6x + 12 = (2x^2 + 4x) + (6x + 12)$
$= 2x(x + 2) + 6(x + 2) = (2x + 6)(x + 2)$
$= 2(x + 3)(x + 2)$

b. $2x^2 – x – 21 = 2x^2 + 6x – 7x – 21 = (2x^2 + 6x) – (7x – 21)$
$= 2x(x + 3) – 7(x + 3)$
$= (2x – 7)(x + 3)$

c. $5x^2 + 14x + 9 = 5x^2 + 5x + 9x + 9 = (5x^2 + 5x) + (9x + 9)$
$= 5x(x + 1) + 9(x + 1) = (5x + 9)(x + 1)$

While you realize that the students can factorize expressions, you may raise a discussion point on how to solve a quadratic equation. Give them an activity to solve different quadratic equations that can be solved by factorization. For example, you can let students solve the problems from the student textbook on the board and allow a discussion so that students can tell something on each step. After ensuring their ability of solving quadratic equation by using factorization, give them an exercise to solve a quadratic equation that cannot be factored and let them discuss on the possibilities of solving such problems. This may lead to raising a notice for use of other approaches.

But to better understand solving quadratic equations using factorization you can give them Exercise 2.4 as homework.

Assessment

You can assess the understanding of your students by giving them quadratic expressions and quadratic equations and ask them to factorize and solve. They need to submit their work and you should correct and record points for each.

Answers to Exercise 2.4

1. a. $x = -4$ or $x = 3$ b. $x = 0$ or $x = 3$ c. $x = 0$ or $x = 3$

   d. $x = -2$ or $x = 2$ e. $x = 0$ or $x = \frac{6}{5}$ f. $x = 0$ or $x = 9$

   g. No real solution h. No real solution

2. a. $x = 1$ or $x = 5$ b. $x = -1$ or $x = \frac{5}{3}$ c. $x = -9$ or $x = 2$

   d. $x = -9$ or $x = 1$ e. $y = \frac{1}{5}$ or $y = 1$ f. $z = -4$ or $z = \frac{2}{3}$

3. a. $\left\{\frac{-1}{4}, \frac{-1}{2}\right\}$ b. $\left\{\frac{-5-5\sqrt{2}}{4}, \frac{-5+5\sqrt{2}}{4}\right\}$ c. $\{1, 3\}$

At this stage the students need to realize that factorization method does not work for all quadratic equations. Thus, there is a need for other method. Owing to the need for having other approaches to solving quadratic equations, you can proceed to completing the square method and deliberate on with various examples. Before giving examples, however, let the students do Group Work 2.5 and guide them how completing the square is performed. The detail of completing the square is to create a perfect square which can be easily factorized. Why we need to build a perfect square in completing the square? will then be important to be understood by the students. Following this, give as many examples as possible for the students to better understand the method of completing the square which will be the basis for deriving the general quadratic formula. This time you can give any of the assessment techniques such as test/quiz to check if the students have understood completing the square.

Answers to Group Work 2.5

1. Divide each coefficient by 2 gives $x^2 + \frac{5}{2}x - 2 = 0$
2. Transfer the constant term to the right side of the equation and get $x^2 + \frac{5}{2}x = 2$

3. Add the square of half of the middle term to both sides $x^2+\frac{5}{2}x+\frac{25}{16}=2+\frac{25}{16}$

4. Yes we have a perfect square because $x^2+\frac{5}{2}x+\frac{25}{16}=\left(x+\frac{5}{4}\right)^2$

5. From this we observe that $\left(x+\frac{5}{4}\right)^2=\frac{57}{16}$ from (3) and (4).

6. The solution will then be $\left(x+\frac{5}{4}\right)^2=\frac{57}{16}\Rightarrow\left(x+\frac{5}{4}\right)=\pm\sqrt{\frac{57}{16}}$

$$\Rightarrow x=\frac{-5}{4}\pm\sqrt{\frac{57}{16}}$$

Therefore, the solutions are $x=\frac{-5+\sqrt{57}}{4}$ or $x=\frac{-5-\sqrt{57}}{4}$ and the solution set is $\left\{\frac{-5-\sqrt{57}}{4},\frac{-5+\sqrt{57}}{4}\right\}$

Assessment

To check if the students have understood completing the square method and that they can apply it in solving quadratic equations, you can give them several quadratic equations or Exercise 2.5, solve each using completing the square method and submit their work. You then check and discuss the answers with students, and keep record.

Answers to Exercise 2.5

1. a. No real solution b. $x = 2$ or $x = 10$ c. $x = \frac{-3}{2}$ or $x = 2$

   d. $x = -2$ or $x = \frac{1}{2}$ e. No real solution f. No real solution

2. a. $\left\{\frac{-5-\sqrt{185}}{20},\frac{-5+\sqrt{185}}{20}\right\}$ b. $\{3, 5\}$ c. $\left\{\frac{-1}{2},\frac{2}{3}\right\}$

   d. $\left\{\frac{-5}{2},\frac{-4}{7}\right\}$ e. $\{-6, -5\}$ f. $\left\{\frac{-4-3\sqrt{2}}{2},\frac{-4+3\sqrt{2}}{2}\right\}$

3. a. $x^2-5x-7=0,\ x=\frac{5-\sqrt{53}}{2}$ or $\frac{5+\sqrt{53}}{2}$

   b. $x^2-6x-15=0, x=3-2\sqrt{6}$ or $x=3+2\sqrt{6}$

   c. $5x^2+5x-20, x=\frac{5-\sqrt{65}}{5}$ or $x=\frac{5+\sqrt{65}}{5}$

d. $2x^2 - 11x + 4 = 0,\ x = \frac{11-\sqrt{89}}{4}$ or $x = \frac{11+\sqrt{89}}{4}$

e. $10x^2 + 22x - 28 = 0, x = \frac{-11-\sqrt{401}}{10}$ or $x = \frac{-11+\sqrt{401}}{10}$

As a consequence, let the students try to apply completing the square method on the general form of a quadratic equation, $ax^2 + bx + c = 0$, $a \neq 0$. You can continue with this by letting students do Group Work 2.6. Let the students follow each of the steps outlined in the group work and reason out for each step. You may need to verify this in class as follows:

Answers to Group Work 2.6

1. Divide all by a and get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ (the reason why we divide by a could be: 1. $a \neq 0$, and 2. We want to create a perfect square which will be easy when the leading coefficient is 1.)

   2. Transfer the constant term to the right side of the equation and get $x^2 + \frac{b}{a}x = -\frac{c}{a}$

3. Determine half of the coefficient of x which is $\frac{b}{2a}$, add its square to both sides of the equation and solve.

$$x^2 + \frac{b}{a} = -\frac{c}{a}$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, the solution set is $\left\{\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}\right\}$

If you explain it this way, the students can capture the essence of completing the square and the general formula for solving any quadratic equation.

After deriving $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, raise a question for the students to characterize $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $\sqrt{b^2-4ac}$, and finally discuss with them that $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ stands for two roots.

After doing this you can give them an example, and pose a question "what will the solution be if b^2-4ac =0 and b^2 – 4ac < 0?. Students need to see that the value of $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ depends on the value of $\sqrt{b^2-4ac}$ in that there are three conditions. To help them capture the idea, you can let the students perform Activity 2.11.

Answers to Activity 2.11

a. when $b^2-4ac>0$ the solution x becomes $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ on which $\sqrt{b^2-4ac}$ is defined. Thus there will be two solutions namely $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$.

b. When $b^2-4ac=0$ the solution x becomes $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ on which $\sqrt{b^2-4ac}$ = 0. Thus there will be one solutions namely $x=\frac{-b}{2a}$.

c. When $b^2-4ac<0$, though the solution x is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, since $\sqrt{b^2-4ac}$ in $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is not defined, we conclude that the equation do not have a solution.

In order to do the Activity you need to let the students do the Activity being in pairs by themselves and help them reach at the three conditions that, $b^2-4ac>0$ (giving rise to two solutions), $b^2-4ac=0$ (giving rise to only one solution), $b^2-4ac<0$ (giving rise to no real solution). Show them how these conditions work by giving more examples. Note that question 4 in exercise 2.6 is offered for clever students.

Assessment

To assess whether students grasped the idea of general quadratic formula and possible conditions of the discriminant, you can give them quadratic equations and let them check if they have real roots by using the discriminant, and solve whenever there are roots.

Answers to Exercise 2.6

1. a. $x = -3$ or $x = -5$
   b. $x = \frac{6-\sqrt{30}}{3}$ or $x = \frac{6+\sqrt{30}}{3}$
   c. $x = \frac{1-\sqrt{2}}{2}$ or $x = \frac{1+\sqrt{2}}{2}$
   d. $x = \frac{-3-\sqrt{17}}{2}$ or $x = \frac{-3+\sqrt{17}}{2}$
   e. No real solution
   f. $x = \frac{2-\sqrt{10}}{3}$ or $x = \frac{2+\sqrt{10}}{3}$

2. a) $\{-4, -2\}$
   b) $\left\{-\frac{3}{5}\right\}$
   c. No real solution or $\{\ \}$
   d. $\left\{\frac{9}{2}\right\}$
   e. No real solution or $\{\ \}$
   f. $\left\{\frac{-4-\sqrt{14}}{2}, \frac{-4+\sqrt{14}}{2}\right\}$

3. a. $4x^2 - 55x - 7 = 0$; $x = \frac{5-\sqrt{137}}{8}$ or $x = \frac{5+\sqrt{137}}{8}$
   b. $x^2 + 2x + 8 = 0$; No real solution
   c. $x^2 - x + 6 = 0$; No real solution
   d. $7x^2 - x - 3 = 0$; $x = \frac{1-\sqrt{85}}{14}$ or $x = \frac{1+\sqrt{85}}{14}$
   e. $x^2 - x + 6 = 0$; $x = \frac{-1-\sqrt{41}}{10}$ or $x = \frac{-1+\sqrt{41}}{10}$

4. Current number of class rooms is 20 and the number of students per class is 60.

At this moment students are expected to realize any quadratic equation will either have one root, or two roots or no root. When they realize that there are roots for a quadratic equation, you need to relate the roots with the coefficients of the quadratic equation similar with the concept they saw during factorization. While they solve quadratic equation using factorization, they saw how the roots are related with the middle term and the constant term of a quadratic equation. i.e., the sum of the roots is equal to the middle term and the product is equal to the constant term. Recalling this it is also possible here to seek a relation between roots of quadratic equations and the coefficients in a quadratic equation. In order to deal with these it is advisable to encourage students to do Activity 2.12.

The purpose of this activity is to let students identify the relation between the roots of a quadratic equation $ax^2 + bx + c = 0$ and its coefficients a, b and c, by adding $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$, and by multiplying the same. This will lead to Viete's theorem stated in the student textbook. You can group the students and give one of the questions (adding roots) for one group and the other (multiplying) for the other group. Give them chance to present their work. You can also assess how they performed each task.

After dwelling with these points, give them more exercises for practice and encourage them to develop problems from their surrounding that lead to quadratic equations, so that they can solve the problems by using the discussion of the unit and appreciate application of quadratic equations.

Finally, diagnosing your students' level, add more exercises to help slow learners work hard, if any, and trigger further notes and more exercises for those gifted students.

Answers to Activity 2.12

1. a. The sum of the roots is $r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\Rightarrow \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \frac{-b-b}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

b. The product of the roots is $r_1 \times r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$

$$\Rightarrow \frac{(-b)^2-(b^2-4ac)}{4a^2} \Rightarrow \frac{4ac}{4a^2} = \frac{c}{a}$$

2. The relationship between the sum and product roots of a quadratic equation and its coefficients is stated as a theorem called Vieteis theorem given in the student text.

3. $r_1 + r_2 = \frac{7}{2}$ and $r_1 \times r_2 = \frac{5}{2}$

Assessment

At this stage you can give test/quiz representing the whole unit to assess student competencies.

Answers to Exercise 2.7

1. a. Sum of the roots = 9 b. Sum of the roots = $\frac{-11}{4}$

   c. Sum of the roots = –3

2. a. Product of roots = –9 b. Product of roots = $\frac{-3}{2}$

   c. Product of roots = $\frac{-1}{3}$

3. $k = -21$
4. $k = 3$
5. the two roots are 1 and 3 and the value of $k = 3$
6. $k = 2$

Up until now, the students discussed the various ways that we can use to solve quadratic equations. They also know how the general formula can help to determine whether a quadratic equation has a solution or not, and how many roots a quadratic equation has whenever it has a solution. After ensuring their ability of solving quadratic equations of any type, you can lead them to see how such an equation is useful in daily life problems.

Some examples of applications in daily life are given in the student textbook from page 99 -100. You may discuss each example in class. Afterwards, you can use Exercise 2.8 for further practice for the students.

Assessment

You can assess the students' ability in formulating and solving word problems by using Exercise 2.8. You can form multiple ability group and let them do each question in group and present their work.

Answers to Exercise 2.8

1. The dimensions of the rectangle are 3cm and 7cm.
2. The equilateral triangle has a side length of = $4\sqrt{3}$ units
3. $x = 13$ and $x = 16$
4. 12 and 13
5. The piece is 20 metre long and the original price is 10 Birr per metre.
6. 50 persons
7. 476 days
8. 3 hours
9. 60 km/hr

Answers to Review Exercises on Unit 2

1. a. $x = 6$ b. $x = \frac{3}{2}$ or $x = \frac{-5}{2}$ c. $x = \frac{2}{3}$

   d. $x = 7$ e. $x = 0$

2. a. $x = 1$ b. $x = \frac{8}{7}$ c. $x = \frac{7}{19}$

   d. No solution e. All real numbers or $\mathbb{R}$

3. a. One solution b. No solution c. Infinite solutions

4. a. All pairs of numbers that satisfy the line $2x + 3y = -5$ or

   $SS = \{(x, y): 2x + 3y = -5\}$ b. $\left\{\left(\frac{50}{13}, \frac{-5}{13}\right)\right\}$ c. $\left\{\left(\frac{110}{19}, \frac{35}{19}\right)\right\}$

5. a. $x = 0$ or $x = 3$ b. $x = \frac{-4}{3}$ or $x = \frac{10}{3}$

   c. $x = -1$ or $x = \frac{4}{3}$ d. No solution

   e. $x = -15$ or $x = 35$ f. $x = 0$ or $x = \frac{3}{4}$

   g. $x = -8$ or $x = \frac{2}{3}$ h. All real numbers or $\mathbb{R}$

   i. $x = \frac{-10}{7}$ or $x = \frac{14}{5}$ j. $x = 3$ or $x = \frac{3}{2}$

   k. No solution

6) a. $x^2 - 16x = x(x - 16)$

   b. $4x^2 + 16x + 12 = 4(x^2 + 4x + 3) = 4(x + 1)(x + 3)$

   c. $1 - 4x^2 = 1^2 - (2x)^2 = (1 - 2x)(1 + 2x)$

   d. $12x + 48x^2 = 12x(1 + 4x)$

   e. $x^2 + 11x - 42 = (x + 14)(x - 3)$

7) a. $x = 8$ b. $x = -2 - 2\sqrt{2}$ or $x = -2 + 2\sqrt{2}$

   c. No solution d. $x = -7$ or $x = -6$

   e. $x = \frac{-1 + \sqrt{141}}{14}$ or $x = \frac{-1 - \sqrt{141}}{14}$

8) a. Two solutions b. Two solutions

   c. No solution d. One solution

9. $x^2 - x - 6 = 0$

10. Let the numbers be x and y. We are given $x + y = 13 \Rightarrow y = 13 - x$ and $xy = 42$. By substituting this y in $xy = 42$, we get $x(13-x) = 42$. This is quadratic equation $x^2 - 13x + 42 = 0$ whose solution gives the numbers to be 6 and 7.

11. $\frac{x+y}{2} = 7 \Rightarrow x + y = 14 \Rightarrow y = 14 - x$, and $xy = 45 \Rightarrow x(14 - x) = 45$ whose solution gives

$x = 5$ or $x = 9$. Thus, the scores are 5 and 9.

12. Let $a = \frac{3-\sqrt{3}}{3}$ and $b = \frac{3+\sqrt{3}}{3}$. Then,

a. $a + b = 2$ b. $ab = \frac{2}{3}$ c. $\frac{1}{a} + \frac{1}{b} = 3$

d. $\frac{1}{a+2} + \frac{1}{b+2} = \frac{9}{13}$ e. $a^2 + b^2 = \frac{8}{3}$ f. $a^3 - b^3 = -\frac{20\sqrt{3}}{9}$ or $a^3 - b^3 = \frac{20\sqrt{3}}{9}$ when $b = \frac{3-\sqrt{3}}{3}$ and $a = \frac{3+\sqrt{3}}{3}$

13. $p = 5$ and $q = 2$

14. t will be the solution to the quadratic equation $15 = -16t^2 + 64t$ which is the same as $16t^2 - 64t + 15 = 0$

a. $t = \frac{1}{4}$sec or $t = 3\frac{3}{4}$sec b. $t = \frac{7}{4}$sec or $t = \frac{9}{4}$sec

15. $k = \frac{1}{2}$ or $k = \frac{3}{2}$

16. 6 km/hour

UNIT 3 FURTHER ON SETS

INTRODUCTION

This unit has two main tasks. The first one is to systematically review all those elements of set theory that had been implicit components of mathematics instruction in Grades 1 to 8 and the second aim is to make students familiar with operations on sets and enable them to perform such operations independently.

In the introduction, we discuss the essential and fundamental notions of set theory for revision. We define the fundamental notions in succession. The set operations will be defined and illustrated through various forms. These also include use of Venn diagram. Using the defined operations, students must be able to solve simple problems independently. Solving problems regularly deepens the basic knowledge on set theory. Thus, discussion on word problems related to students' daily life need to be conducted, especially those that can be solved with the help of Venn diagrams should be discussed.

Unit Outcomes

After completing this unit, students will be able to:

- *understand additional facts and principles about sets.*
- *apply rules of operation on sets and find the result.*
- *demonstrate correct usage of Venn-diagram in set operations.*
- *apply rules and principles of set theory to practical situations.*

SUGGESTED TEACHING AIDS IN UNIT 3

This unit is meant for developing the concepts on sets and operations on sets. Since sets are quite essential for the development of modern mathematics, letting students practice in good details about sets will be of paramount importance. For this purpose, it is essential to deliver this unit by offering practical examples that represent sets from our surroundings. Different associations, student community, football teams, collection of animals, etc are among the practical examples that you can use to support teaching sets. Different drawings of Venn diagrams that represent sets are also useful as a teaching aid for this unit.

3.1 WAYS TO DESCRIBE SETS

Periods allotted: 2 periods

Competency

At the end of this sub-unit, students will be able to:

- *describe sets in different ways.*

Vocabulary: Set, Element. Describing set

INTRODUCTION

Set theory is one of the basis for studying mathematics. Students have been studying sets and elements since primary grade level. In this sub-unit, an attempt will be made to discuss ways to describe sets. With this in mind, sets and elements, and description of sets that include the verbal method, listing method and set-builder methods will be discussed in this sub-unit.

TEACHING NOTES

To introduce this sub-unit, you need to revise important points about sets and their description from previous grades. For doing so, you can give chance to the students to do activity 3.1 whose answer is outlined below. The purpose of this activity is to encourage students recall the concepts about sets they have discussed in previous grades. Since students may have varying backgrounds, it will be better if the students can do the activity by forming groups. Approaching this activity as a group work can bring various options to one idea and students can discuss each option.

Answers to Activity 3.1

1. A set is any well-defined collection of objects. An element of a set is that which belongs to the set. In other words, an element satisfies the characterization of the set.
2. a. All the possible two numbers that belong to this set that are composite.
   These may include 4 and 6, 4 and 8, 4 and 9, 6 and 8, 6 and 9, and 8 and 9.

b. There are many some of which are:3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48. Of these you are expected to give only two say 12 and 15.

c. There is no whole number between 0 and 1.

d. There are infinite real numbers between 0 and 1. Some of them can be 0.5, 0.6; 0.7, 0.8; etc.

e. 1,2; 4, 6; 100, 107; 0, 1; etc.

f. We can consider 0 and 2; there are many other such numbers. We can consider any two integers.

3. i. a. {4, 6, 8, 9}.

b. {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48} or $\{3n / n \in \mathbb{N} \text{ and } n \leq 16\}$.

c. $\varnothing$ or { }.

d. $\{x \in \mathbb{R} | \ 0 < x < 1\}$.

e. {0, 1, 2, . . .}or the set of whole numbers.

f. {. . . , –2, –1, 0, 1, 2, . . .} or the set of integers.

ii. a. 4 b. 16 c. 0

d. infinite e. infinite f. infinite

iii. In 3 ways (complete listing method, partial listing method and set builder method).

But you do not need to mention them this time.

4. a. c b. a, b, c c. d, e and f

3.1.1 SETS AND ELEMENTS

This lesson deals with a number of important ideas concerned with sets and elements. The words 'set' and 'element' do not have formal definitions. The definition presented in the student textbook is an intuitive definition. However, given any element and any set, we can be interested in what relations they have. We would like to describe or explain whether a given element belongs to a given set. After discussing this fact, you may ask the students oral questions such as, give elements that belong to the following sets.

1. A = {positive factors of 72}
2. M = {All natural numbers less than 100 that are divisible by 3}

Whose answers are:

A = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72} and

B = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69,72, 75, 78, 81, 84, 87, 90, 93, 96, 99}

You can also give additional other questions with which students will be familiar with possible relation between a set and an element. This time, you need to make sure that students understood the concept of set. After such deliberations on expressing a set and an element and prevailing relation between them, it is possible to proceed to ways of describing a set.

3.1.2 DESCRIPTION OF SETS

You may start the lesson by reminding the students some of the sets of numbers they learned in previous grade levels.

Example:

i. The set of prime numbers less than 20.
ii. The set of positive factors of 72.
iii. The set of positive multiples of 4.

After demonstrating the examples stated above, you can give chance for the students to look for possible options of representing the sets. One way, for example, to represent the last set by using mathematical formula is as follows, i.e. $\{4n : n \in N\}$.

For clarity and better understanding, it is worth discussing all the examples given in the student textbook. Pursuant to this discussion, assign some of the questions in Exercise 3.1 as a class work and the remaining ones as homework. Before you give feedback, pick some students at random and ask them some questions from the exercise. You can make them do that on the board.

Assessment

Apart from the details mentioned above, you can also use any one of the following for assessing students learning: class activities, group discussions, assignments, exercise problems from Exercise 3.1.

Answers to Exercise 3.1

1. a. The set of natural numbers between 4 and 10 or the set of integers between 4 and 10.
   b. The set of all prime numbers between 1 and 14.
   c. The set of natural numbers greater than 7 or the set of integers greater than 7.
   d. The set of all odd natural numbers less than 100 or the set of odd integers between 0 and 100.
2. a. $\{2, 3\}$
   b. $\{5, 10, 15, 20, \ldots, 110\}$
   c. $\{0, 1, 2, 3, \ldots\}$
   d. not possible to list the elements
   e. $\{2, 4, 6, \ldots\}$
   f. $\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$
   g. not possible to list the elements

After revising what the students know about set description including verbal method and complete listing method, introduce the new way of describing sets called set builder method by using several examples. In order to introduce this way of describing sets, you can give activity 3.2 as a class work with which students can match between the sets given in column A with those sets in column B, because this activity is intended to guide students into the new way of describing sets, set-builder method. By doing some more examples of describing sets using set-builder method, you can proceed to exercise 3.2. This time you can record some score for student's effort and ability to perform the activity.

Answers to Activity 3.2

1. d   2. c   3. b   4. A

Assessment

To make sure that students have understood sets and their descriptions, you can give them Exercise 3.2 as homework and let selected students present it to class. You check their work and the overall discussion during presentation. You can also give them several sets and let them represent each set using several ways.

Answers to Exercise 3.2

1. c and d are well defined.
2. a. False   b. True   c. True
3. a. i. A = { m, a, t, h, e, i, c, s}.
   ii. A = { x: x is a letter in the word "Mathematics"}.
   b. i. B = {Amahara, Tigrai, Oromia, Afar, SNNP, Gambela, Harari, Benshangul Gumuz, Somali}.
   ii. B = {x: x is a regional state in Ethiopia}.
   c. i. A = { 6, 7, 8, 9, 10, 11, 12}.
   ii. $A = \{x: x \in$ and $5 < x < 13\}$.
   d. i. if we consider integers, C = { 2, 3, 6, 8, 10, 12, 14, 16, 18}.
   ii. $C = \{x: x = 2n < 19$ for some $n \in \mathbb{N}\}$.
   e. i. It can be listed.
   ii. {x : x is a student in Ethiopia}.
   f. i. {1, 3, 5, 7, 9 . . .}.
   ii. $\{x : x = 2n - 1$ for some $n \in \mathbb{N}\}$.
4. a. i. The set of all Natural numbers less than 11, or {all natural numbers less than11}.
   ii. $N = \{x : x \in \mathbb{N}$ and $x \leq 10\}$.
   b. i. The set of all odd natural numbers, or {all odd natural numbers}.
   ii. $A = \{x: x = 2n - 1$ for some $n \in \mathbb{N}\}$.
   c. i. The set of all natural numbers that are multiples of 5, or {all multiples of 5}.
   ii. $R = \{x: x = 5n$ for some $n \in \mathbb{N}\}$.
   d. i. The second and the fourth days of a week.
   ii. { x : x is a second or fourth day of a week}.
   e. i. The set of all prime natural numbers.
   ii. { x : x is a prime natural number}.

3.2 THE NOTION OF SETS

Periods allotted:4 periods

Competencies

At the end of this subunit, students will be able to:

- *identify the elements of a given set.*
- *explain the notion "empty set" and "universal set"*
- *determine the number of subsets of a given finite set and list them.*
- *give the power set of a given set.*
- *determine the number of proper subsets of a given finite set and list them.*
- *distinguish between equal sets and equivalent sets.*
- *find equal sets and equivalent sets to a given set.*

Vocabulary: Set, Empty set, Universal set, Proper subset, Power set, Finite set, Equal sets, Equivalent sets

INTRODUCTION

Once students study sets, elements and different ways of describing sets, it will be essential to characterize sets. In this sub-unit, emphasis will be given to the notions of sets that include empty sets, finite and infinite sets, subset and proper subsets. Universal sets, equal and equivalent sets and Venn-diagram representation of sets will also be discussed in this sub-unit.

TEACHING NOTES

You may start this sub-unit by encouraging and assisting students to name some elements of a given set and encourage them to explain whether a given object/number belongs to the set or not and to use the appropriate symbol accordingly. By doing this, when the students are able to describe different sets (using either word description or set builder method) that they have discussed in the previous sub-unit, let the students identify which of these set(s) is/are empty set/s. So, to do each of these, follow the sub topics discussed in this sub-section.

3.2.1 EMPTY SET, FINITE SET, INFINITE SET, SUBSET, PROPER SUBSET

Pursuant to the discussion above, allow students to compare and characterize different sets by ways of determining number of elements, and descriptive relations. For these purposes you may start this lesson by letting students discuss Activity 3.3 as a class activity. First, group the students in pairs and let each pair discuss the problems in the activity. After a few minutes let each pair present its outcome to the class. At this stage, the activity is expected to help students to determine the number of elements of a given set and need to realize that

- a set may have no elements
- a set may have a limited number of elements
- a set may have an unlimited number of elements

From their results as well, they are expected to describe pair wise relations among the sets. For example, the elements in set in e) X = {2, 4, 6, . . .} are all contained in the set in d) E = {*x*: *x* is an integer}. The other observation is that the number of elements in set D = {*x*: $x \in$ {1, 2, 3}} is 3 and hence finite and that of X = {2, 4, 6, . . . } is infinite.

Answers to Activity 3.3

1. a. 0 b. 8 c. 3
   d. infinite e. infinite
2. D, C, X, ⊆ E
   E contains C, D and X

You may also proceed to Activity 3.4 with which students will be able to determine the number of elements in a given set. Based on this activity, students will be able to proceed to defining finite and infinite sets.

Answers to Activity 3.4

1. It has infinite number of elements (i.e. unlimited number of elements).
2. It has finite (limited) number of elements. In fact the set is {8, 9, 10, . . .,7^{100}-1}.
3. It has infinite (unlimited) number of elements. The set is {3, 6, 9, …}.
4. It has no element. Therefore we say it has limited (finite) number of elements.
5. It has finite (limited) number of elements. The set is {5, 10, 15, …, (101^4-1)}.

Right after identifying a number of elements of a set, you may proceed to Activity 3.5 so that students can do in pairs with which they can describe relationship between pairs of sets. At the end of this activity, make sure that the students have noticed that a set may be contained in another set. From this activity, students are required to brainstorm ideas that lead them into the concepts of subset, power set and proper subset. For discussion, let the students do Group Work 3.1 in order to be able to list all subsets of a set and determine the number of subsets a given set may have. This time, you can give a test with which you can record students' ability and understanding.

Answers to Activity 3.5

1. All elements of N are contained in M
2. B contains all elements of A
3. E contains all elements of F and F contains all elements of E.

Answers to Group Work 3.1

1. ∅, {*a*}, {*b*}, {*c*}, {*a*, *b*}, { *a*, *c*}, { *b*, *c*}, {*a*, *b*, *c*}.
2. 8

Following the discussion on subsets, it will be essential to check if students can list all possible proper subsets of a given set. For realizing this, you can proceed to Activity 3.6 so that students can practice identifying proper subsets of a given set.

Answers to Activity 3.6

i. ∅, {–1}, {0}, {1}, {–1, 0}, {–1, 1} and { 0, 1}

ii. 7

Finally, before reaching conclusion on the number of subsets and proper subsets of a given set, please let the students perform Activity 3.7 and then generalize the rule for determining the number of subsets and proper subsets. For the generalization on number of subsets and proper subsets of a set, refer page 114 of the student textbook.

Following these activities, you can give exercise 3.3 as homework to the students with which you can assess students understanding.

Answers to Activity 3.7

1. a. *n*(subsets of A) = 1

   n(proper subsets of A) = 0

   b. *n*(subsets of B) = 2

   n(proper subsets of B) = 1

   c. *n*(subsets of C) = 4

   n(proper subsets of C) = 3

   d. *n*(subsets of D) = 8

   n(proper subsets of D) = 7

2.

	Set	No of elements	Subsets	No of Subsets	Proper Subsets	No of Proper Subsets
a	∅	0	∅	$1 = 2^0$	-	$0 = 2^0 – 1$
b	{0}	1	∅, {0}	$2 = 2^1$	∅	$1 = 2^1–1$
c	{–1, 0}	2	∅, {–1}, {0}, {–1. 0}	$4 = 2^2$	∅, {-1}, {0}	$3 = 2^2–1$
d	{–1, 0, 1}	3	∅, {–1}, {0}, {1}, {–1. 0}, {–1, 1}, {0, 1}, {–1, 0, 1}	$8 = 2^3$	∅, {–1}, {0}, {1}, {-1. 0}, {–1, 1}, {0, 1}	$7 = 2^3–1$

Assessment:

For assessing students learning apart from class activities, group discussions and assignments, you can give test/quiz.

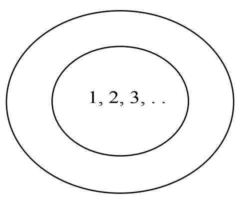
Answers to Exercise 3.3

1. i. a ii. a, c iii. a, c iv. a, d, e
2. a. $\varnothing$, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}.
   b. $\varnothing$, {0}, {{1, 2}}, {0, {1, 2}}.
3. a True b. False because for any set A , $A \subseteq A$
   c. False because $\{4\} \in \{\{4\}\}$ d. True

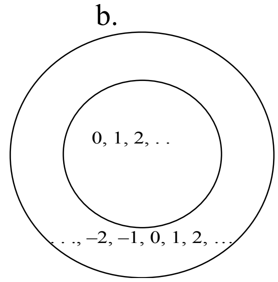
In the previous discussion, students were acquainted with notions of sets that include number of elements of a set, finite set, infinite set, subset and proper subset. Now, the students can relate sets based on the number of elements they have. Beyond the equality of the number of elements two sets may have, they will also relate sets by comparing their elements that will lead them to determining equality and equivalence of sets. For deliberation on these concepts, you can give Activity 3.8 to students as a class work.

Answers to Activity 3.8

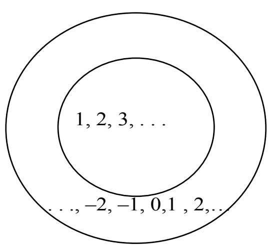
1. a. $\subseteq$ c. $\subseteq$
   b. $\subseteq$ d. $\subseteq$
2. a.



b.



c.



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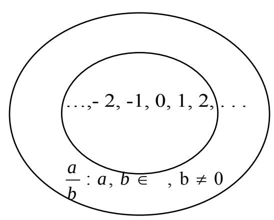
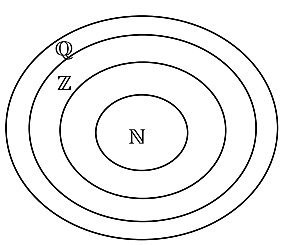


3.



UNIVERSAL SET AND COMPLEMENT

When we think and talk about sets, it is very helpful to have the members of a set from some specified "population". For example, if we want to talk about sets of students, it is helpful if we have some general population members of our set. We might want to focus our attentions on the students in a single school or we may want to consider all grade nine students in Ethiopia if we focus on grade nine students. If we specify a particular set of students to which we shall limit ourselves in drawing members for other sets to be discussed, then this specified set is called the **universal set,** or simply the universe of our discussion.

Thus, in any particular discussion involving sets, every set in the discussion is a subset of the universal set. A universal set is denoted by a symbol similar to the capital letter U.

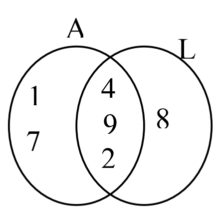
If U={People in Wereda 1 in Addis Ababa}and A= {People in Kebele 05 of Wereda 1} The set of people which are not members of A but members of U is called the **complement** of set A and is denoted by A' or U - A. Before directly writing the definition of a universal set and complement of a set, you may relate two sets by way of pictorial representation called Venn diagram and illustrate when and how to use it, especially for illustration of relations among sets.

Assessment

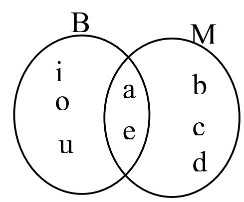
To assess students understanding on using Venn diagram to represent sets and describe relationships between sets you can give them Exercise 3.4 as homework and check how they do.

Answers to Exercise 3.4

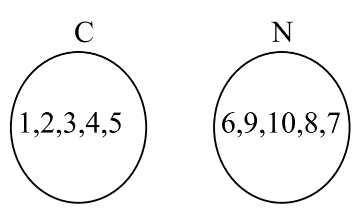
1. a.



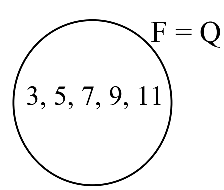
b.



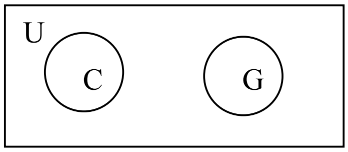
c.



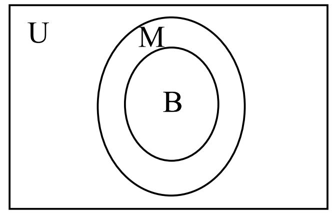
d.



2.



a.



b.

EQUAL AND EQUIVALENT SETS

Up until now, students were able to determine the number of elements of a set, and the relationship of two sets in a form of one as subset or proper subset. This time, students are expected to be capable of determining a relationship between two or more sets based on the number of members of a set and the members themselves. This will lead to the concepts of equality and equivalence of sets. You may start this subunit by encouraging students to do Activity 3.9 that requires them to match those sets that have equal number of elements and those that have identical elements. Pursuant to the discussion, you can define equivalence and equality of sets.

I. EQUAL SETS

Two sets are said to be **equal** if they have exactly the same elements. Thus, saying two sets A and B are equal is the same as saying $A \subseteq B$ and $B \subseteq A$.
Make the students discuss and investigate that the sets

$E = \{x \in \mathbb{R} : x^2 - 5x + 6 = 0\}$ and $F = \{x \in \mathbb{N} : 1 < x < 4\}$ satisfy the equality property of sets.

II. EQUIVALENT SETS

The notion of an exact match or one - to - one correspondence between two sets is basic to the process of counting. In order to say that two finite sets are equivalent, we should know that they have the same number of elements. In other words, we say two finite sets are equivalent when they have a one-to-one correspondence between them.

Make sure that the students can distinguish between equal sets and equivalent sets. After explaining the meaning of "equal sets" and "equivalent sets" by using several examples, assist the students to determine equal sets and equivalent sets to a given set.

In order to help them practice distinguishing equal and equivalent sets, you can give them exercise 3.5 as homework.

Answer to Activity 3.9

1. all the pairs in a, b, c, d and e have the same number of elements.
2. a, d and e have exactly the same elements.

Assessment

You can assess through oral question and answer to check if students can identify between equal sets and equivalent sets.

Answers to Exercise 3.5

1. Equivalent sets
2. Neither equal nor equivalent sets
3. Equivalent sets
4. Neither equal nor equivalent sets
5. Equal sets

3.3 OPERATIONS ON SETS

Periods allotted: 9 periods

Competencies

At the end of this subunit, students will be able to:

- *determine the number of elements in the union of two finite set.*
- *describe the properties of "union" and "intersection" of sets.*
- *determine the absolute complement of a given set.*
- *determine the relative complement of two sets.*
- *determine the symmetric difference of two sets.*
- *determine the Cartesian product of two sets.*

Vocabulary: Union, Intersection, Difference of sets, Complement

INTRODUCTION

This sub-unit is meant for discussing the different operations on sets (union, intersection, and difference of sets), and consequences including symmetric difference, De Morgan's Law and distributive property of the operations of sets. After discussing these ideas, finally, the students will gain necessary background for discussing relations and functions in the subsequent unit, by discussing the concept of Cartesian product of sets. The ordered pair form of representing members of product of sets will be the concern of the last section of this subunit.

TEACHING NOTES

Before being engaged in the discussion on operations on sets, it is advisable if we can give chance for students to tell, through question and answer, about the operations on numbers (addition and multiplication) and proceed to subtraction to describe it in relation to addition. In a similar way, give them chance to think of operations on sets (union and intersection) and to imagine how it is possible to express complement of a given set. Through this entry, you can then continue to discuss these operations one by one.

UNION, INTERSECTION AND DIFFERENCE OF SETS

A. UNION OF SETS AND THEIR PROPERTIES

Before starting this lesson, allow students to tell about sets described by listing elements of the following type. Example, if we have two sets presented as A = {1, 2, 3, 2, 4, 3, 1} and B = {1, 2, 3, 4}, the question What relations and differences do these sets have? can be a leading question for students to discuss.

Following some views forwarded by the students, give them two sets A and B which are disjoint, say A = {1, 2, 3} and B = {4, 5, 6} and ask them what the resulting set will be if we collect all of them in one set. The answer is obviously {1, 2, 3, 4, 5, 6}. Check if they can explain what this set is. In the same way, give them two sets which are not disjoint, for example, A = {1, 2, 3}and B = {2, 3, 4} and ask them the same question of listing them in one set. Students may frequently answer this by writing the list of elements of A followed by the list of elements of B. This seems quite correct. However, the students should be encouraged to observe that

$$\{1, 2, 3, 2, 3, 4\} = \{1, 2, 3, 4\}$$

because these two sets do indeed have exactly the same elements. In short, it is incorrect and also confusing to list some of the elements several times when listing a set. In this case, the resulting set is thus {1, 2, 3, 4}.

After discussing such representations, you can define union of two or more sets and enrich students understanding with several examples. You can give them Activity 3.10 as a group work in class so that they can further discuss it among themselves. The purpose of this activity is to motivate students to perform the operation of union and observe some of the properties of union. After they do their group work, ask them to present their observations in class. This will lead them to identifying commutative, associative and identity properties.

Before you state the properties of union of sets, it is worth discussing additional examples. After doing enough examples, encourage and lead the students to state the properties in their own words. Then you can finally list the formal statement of these properties.

Answers to Activity 3.10

1. a. $A \cup B = \{1, 2, 3, 4, 6, 8\}$ and b. $B \cup A = \{1, 2, 3, 4, 6, 8\}$
   The relationship between these two sets is $B \cup A = A \cup B$
2. a. $A \cup B = \{1, 2, 3, 4, 6, 8\}$
   b. $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$
   c. $B \cup C = \{2, 3, 4, 5, 6, 8\}$
   d. $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$
   $\therefore (A \cup B) \cup C = A \cup (B \cup C)$
3. $A \cup \varnothing = \{1,2,3,4\} = A$
   $\therefore A \cup \varnothing = A$ for any set A.

Assessment

In addition to the possible assessments mentioned earlier, you can also use anyone of the following for assessing students learning: class activities, group discussions, assignments and a quiz or a test.

Answers to Exercise 3.6

1. a. $A\cup B = \{1, 2, \{3\}, 3\}$ b. $B\cup C = \{2, 3, \{3\}, 4\}$

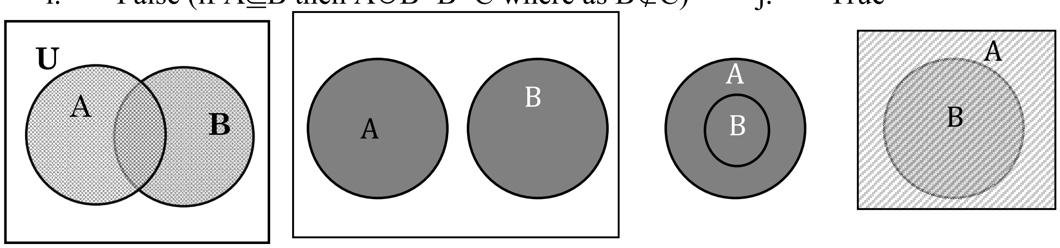
   c. $A\cup C = \{1, 2, \{3\}, 4\}$ d. $A\cup(B\cup C) = \{1, 2, 3, \{3\}, 4\}$

   e. $(A\cup B)\cup C = \{1, 2, \{3\}, 3, 4\}$

2. a. False b. True c. True d. True e. True
   f. True g. True h. False

   i. False (if $A\subseteq B$ then $A\cup B=B=C$ where as $B\not\subset C$) j. True

3.



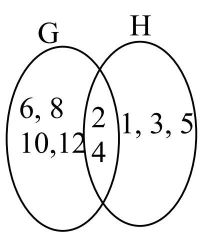
a b c d

B. INTERSECTION OF TWO SETS AND THEIR PROPERTIES

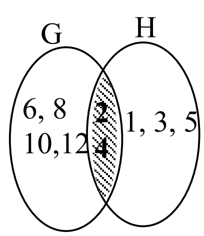
You can approach the intersection of two or more sets in a similar way with that of the union discussed previously. You may also give chance to students to discuss and draw the investigation given in the student textbook by doing activity 3.11. Then proceed with writing the definition of intersection of sets. Using the examples given in the student textbook, consolidate the understanding on intersection of sets.

Answers to Activity 3.11

a.



b.



$G\cap H = \{2, 4\}$

Pursuant to the definition of intersection of sets, it is likely possible to proceed into the properties of intersection of sets. For this purpose, lead your students to conclude that the commutative and associative properties of "intersection" of sets holds true by giving them chance to do activity 3.12.

Answers to Activity 3.12

1. a. $(A \cap C) = \{2, 3\}$
   b. $C \cap A = \{2, 3\}$
   $\therefore A \cap C = C \cap A$
2. a. $A \cap B = \{0,2\}$
   b. $(A \cap B) \cap C = \{2\}$
   c. $B \cap C = \{2, 6\}$
   d. $A \cap (B \cap C) = \{2\}$
   $\therefore (A \cap B) \cap C = A \cap (B \cap C)$
3. $A \cap U = A$

Assessment

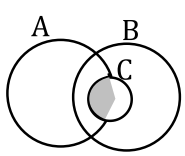
To assess students understanding on performing set operations and their properties, you can give exercises that the students must do individually and you check their work.

Answers to Exercise 3.7

1. a. $A \cap B = \{b\}$ c. $B \cap C = \varnothing$
   b. $A \cap C = \{\{c\}\}$ d. $A \cap (B \cap C) = \varnothing$

2. a. False b. True c. False
   d. True e. True f. True
   g. False (example A = {1, 2, 3} and B = {4, 5} but $A \cap B = \varnothing$)
   h. True i. False j. False k. True

3.



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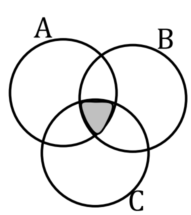
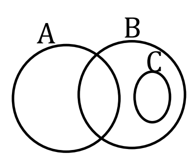
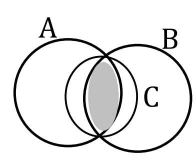


(b)



(c)



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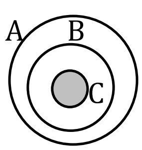
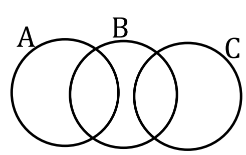


(e)



(f)

C. DIFFERENCE AND SYMMETRIC DIFFERENCE OF SETS

Here, there are two concerns in discussing difference of sets, the relative and absolute difference. It is advisable if students discuss difference of two sets first and then proceed to the absolute difference.

1. THE RELATIVE COMPLEMENT (OR DIFFERENCE) OF TWO SETS

To start this lesson, first revise the concept of complement of a set through question and answer. It is expected that students can have prior knowledge of complement of a set and particularly absolute complement that was discussed before. To this end, you can give a quick revision activity such as:

If $A = \{2, 4, 6, 8\}$ and
$B = \{1, 3, 6, 10\}$, find

a. The elements in A that are not in B (denoted as $A \backslash B$ or $A - B$)

b. The elements in B that are not in A (denoted as $B \backslash A$ or $B - A$)

$$A - B = \{2, 4, 8\} \text{ and } B - A = \{1, 3, 10\}$$

You can give them additional examples and proceed to let them do activity 3.13 which will help them realize that relative complement is neither commutative nor associative.
After they do activity 3.13, you can give them the following example as an exercise
If $U = \{x \mid x \text{ is a natural number less than } 10\}$, $A = \{1, 3, 4, 5, 6, 7\}$ and
$B = \{2, 3, 6, 8, 9\}$ then find

a. $U - A$ b. $U - B'$ c. $U - (A \cap B)$ d. $U - (A \cup B)'$

This will help you to proceed into discussing absolute complement.

Answers to Activity 3.13

a. $\{3, 5, 7\}$ b. $\{4, 6, 8\}$ c. $\{5, 7\}$ d. $\{2, 3, 5, 7\}$

2. THE COMPLEMENT OF A SET

After the students have discussed the quick revision activity and the exercise outlined above, it will be easy for them to see the difference between $A - B$ and $U - B$. Although both seem to be relative complements, the second set $(U - B)$ which we see here as a relative complement of B with respect to U is of interest here. Such a set is what we call the absolute complement (or simply the complement) of B and we denote it as B'.

In another way, we can see the absolute complement to be a relative complement with respect to a universal set.

To help students reach an understanding of relative and absolute complements, you can ask them by means of a group work to do some examples and draw their own conclusion of what a complement (difference of sets) is all about, and finally give them the definition stated in the student textbook.

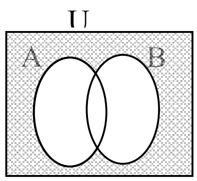
Once they capture the concept of relative and absolute complements, you can proceed to expressing complements in terms of the set operations. For this purpose, you can give example 6 of the student text as a class activity so that students can realize the relation $A \backslash B = A \cap B'$.

In a similar way, you can ask them to present any further observation they may have. In this case $(A \cap B)' = A' \cup B'$. For clarity and purpose of generalizations, you can let them do activity 3.14 so that they will be able to put their own generalization.

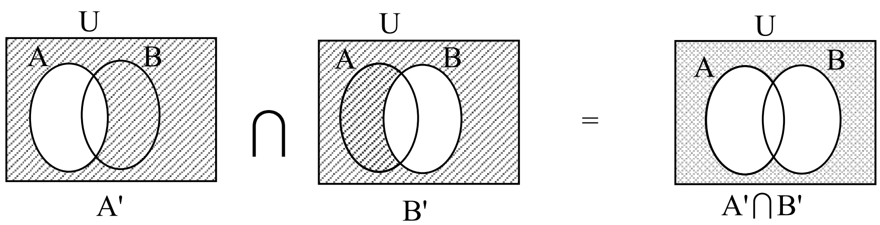
When you feel they have done so, please let them discuss in group and do group work 3.2 so that they can reach at the statements of De Morgan' laws.

Answers to Activity 3.14

a.



b.

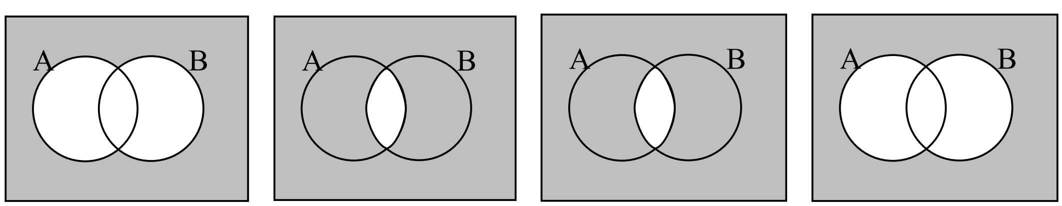


2. From a) and b) above, $(A \cup B)' = A' \cap B'$

Answers to Group Work 3.2

After shading the regions for the sets described, the students need to reach at the conclusions usually known as De Morgan's law.

1.



a. $(A \cup B)'$ b. $A' \cup B'$ c. $(A \cap B)'$ d. $A' \cap B'$

2. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

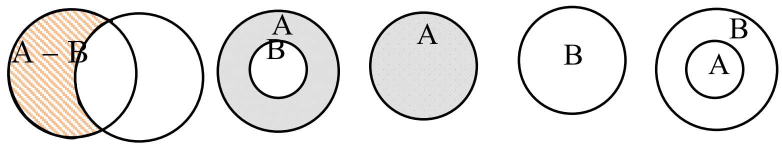
Assessment

You can give several problems that require use of Venn diagram to perform set operations or give them Exercise 3.8 for assessing your students.

Answers to Exercise 3.8

1. a. $B - A = \{d, e\}$ b. $A - B = \{a\}$ c. $B - A = \{d, e\}$

2.



a. b. c. d.

3. a. False b. True c. True d. True
   e. False f. True g. True h. True i. True

4. a. $A' = \{5, 6, 7, 8, 9\}$ b. $B' = \{1, 3, 5, 7, 9\}$
   c. $(A \cup C)' = \{7, 8, 9\}$ d. $(A - B)' = \{2, 4, 5, 6, 7, 8, 9\}$
   e. $A' \cap B' = \{5, 7, 9\}$ f. $(A \cup B)' = \{5, 7, 9\}$
   g. $(A')' = A = \{1, 2, 3, 4\}$ h. $B - C = \{2, 8\}$
   i. $B \cap C' = \{2, 8\}$

3. THE SYMMETRIC DIFFERENCE BETWEEN TWO SETS.

Up until now, students are aware of complements of a set (both relative and absolute). They also know that such relative complements are sets. If we have two sets A and B, we have two possible relative complements A – B or B – A. From previous discussions, they saw that $A - B \neq B - A$. After giving this as an introduction, you can leave them with the questions:
i. What will be $(A - B) \cap (B - A)$?
ii. What will be $(A - B) \cup (B - A)$?
iii. What will be $(A \cup B) - (A \cap B)$?

You can then proceed to giving them Activity 3.15 as a class work so that it will guide them to defining and understanding of symmetric difference of sets. It will be better if students do this activity individually. You can use this activity for assessment purpose because it also requires the use of previous discussions.

Once students discuss the activity, you can give the formal definition of symmetric difference and enrich their understanding by doing several examples.

Answers to Activity 3.15

a. $A \cap B = \{b, d\}$
b. $A \cup B = \{a, b, d, e\}$
c. $A \setminus B = \{a\}$
d. $B \setminus A = \{e\}$
e. $(A \cup B) \cup (B \setminus A) = \{a, e\}$
f. $(A \setminus B) \cup (B \setminus A) = \{a, e\}$
$\therefore (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Right after ensuring the understanding of the students on issues of De Morgan's laws and symmetric difference, you may proceed to Group work 3.3 which is intended to give chance for the students to shade parts of the given sets and realize the property of distributive property of intersection over union and equally distributive property of union over intersection.

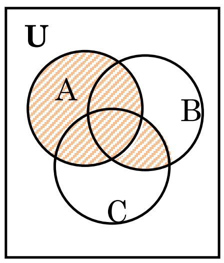
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \text{ and}$$

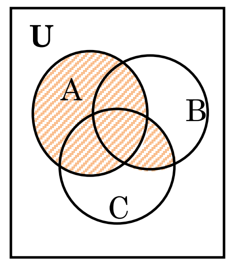
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ holds true.}$$

You can let some students draw Venn diagrams for this group work before they come to class on a flip chart so that you can use it as a teaching aid right after the students discuss the group work. After their Group work, you can let some group members present their findings in front of other students.

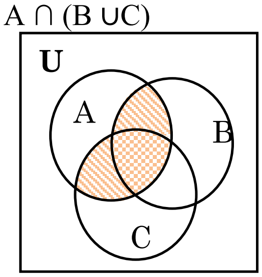
Answers to Group work 3.3

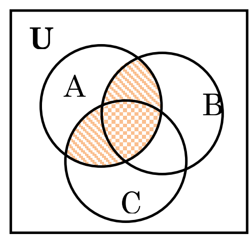
1. A. $A \cup (B \cap C)$ b. $(A \cup B) \cap (A \cup C)$





c. $A \cap (B \cup C)$ d. $(A \cap B) \cup (A \cap C)$





2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Assessment

You can assess your students by giving them several exercise problems that require application of distributive property of set operations similar to those given in Exercise 3.9.

Answers to Exercise 3.9

1. $A \cap (B \cup C) = \{1, 0, -1, 2, 3\}$
2. a. $A \cap (A \cup B) = A$ b. $P' \cap (P \cup Q) = P' \cap Q$

   c. $A \cap (A' \cup B) = A \cap B$ d. $P \cup (P \cap Q) = P$

THE CARTESIAN PRODUCT OF SETS

This subunit is devoted to introducing how to form a new set of ordered pairs from two given sets by taking the cross product and proceed to discussing the Cartesian product named after Rene Descartes. You may start this lesson by discussing the activity in Group Work 3.4 given in the student textbook. The purpose of this group work is to introduce to students the concept of ordered pairs and introduce sets whose elements are ordered pairs. After discussing this group work, you can ask students if they can construct sets with ordered pairs as elements. You can give them example 1 as hint for constructing sets of this kind. Then, write the definition of an ordered pair and discuss it using examples given in the student textbook. In addition to these, you can let your students determine cross product of sets and solve problems that involve determining number of elements of sets constructed to have elements as ordered pairs. For this purpose, you can let them do Activity 3.16 as a class activity whose answers are as follows.

Answers to Activity 3.16

1. a. $A \times B = \{(2, 0), (2, 1), (2, 2), (3, 0)\ (3, 1), (3, 2)\}$
   b. $B \times A = \{(0, 2), (0, 3), (1, 2), (1, 3), (2, 2), (2, 3)\}$
   c. $n(A \times B) = n(A).\ n(B) = (2)\ (3) = 6$
   $\therefore n(A \times B) = 6$
2 a. $A \times (B \cap C) = \{(a, c), (a, e), (b, c), (b, e)\}$
   b. $A \times (B \cup C) = \{(a, f), (a, e), (a, c), (a, d), (b, f), (b, e), (b, c), (b, d)\}$
   c. $(A \times B) \cap (A \times C) = \{(a, c), (a, e), (b, c), (b, e)\}$
   d. $(A \times B) \cup (A \times C) = \{(a, c), (a, e), (b, c), (b, e,), (a, d), (a, f), (b, d), (b, f)\}$

Pursuant to their effort in doing activity 3.16, you can assess their understanding by giving exercise 3.10 as homework.

Assessment

In order to assess students understanding of the concept Cartesian product you can use the questions in Exercise 3.10. You can give this exercise as homework or assignment.

Answers to Exercise 3.10

1. a. $A \times B = \{(2,\ 1),\ (2, 5)\}$ b. $B \times A = \{(1,\ 2),\ (5, 2)\}$
   c. $B \times C = \{(1,\ -1),\ (1, 1), (5,\ -1),\ (5, 1)\}$ d. $A \times (B \cap C) = \{(2, 1)\}$
   e. $(A \cup C) \times B = \{(-1, 1),\ (-1, 5),\ (1, 1),\ (1, 5),\ (2, 1),\ (2, 5)\}$
   f. $(A \times B)\ \cup\ (A \times C) = \{(2, 1),\ (2, 5),\ (2,\ -1)\}$
   g. $B \times B = \{(1,\ 1),\ (1, 5),\ (5, 1),\ (5, 5)\}$
2. a. $B = \{1,\ 4\}$ b. $C = \{1,\ 2,\ 3\}$

c. $C\times B = \{(1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4)\}$

3. $n(B) = 6$

4. a. $A'\times C' = \{(1, 1), (1, 6), (1, 7), (1, 8), (1, 9), (3, 1), (3, 6), (3, 7), (3, 8), (3, 9), (5, 1), (5, 6), (5, 7), (5, 8), (5, 9), (7, 1) (7, 6), (7, 7), (7, 8), (7, 9)\}$

   b. $B\times A' = (1, 1), (1, 3), (1, 5), (1, 7), (3, 1), (3, 3), (3, 5), (3, 7), (6, 1), (6, 3), (6, 5), (6, 7), (8, 1), (8, 3), (8, 5), (8, 7)$

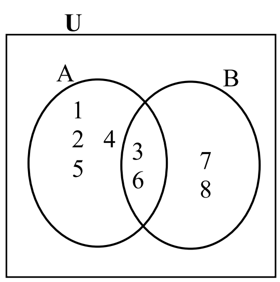
   c. $B\times (A' - C) = \{(1,1), (1,7), (3,1), (3,7), (6,1), (6,7), (8,1), (8, 7)\}$

5. $x = 2$ and $y = 2$

Owing to the fact that students have gained enough of sets and their operations, it is time to discuss problems involving sets. Focus of such problems that involve sets revolves around determining number of members of a set. From the outset, it seems that number of members of a set, say $A\cup B$ is the sum of a number of elements of set A and of set B, but the reality is not. For describing this you can use Venn diagram to list elements of sets A and B and determine the total number of members of $A\cup B$.

Example: Let A = {1, 2, 3, 4, 5, 6} and B = {3, 6, 7, 8}

To determine the number of elements of $A\cup B$, first we can construct the Venn diagram of these sets.



n (A) = 6 and n (B) = 4, but n ($A\cup B$) = 8 and not n (A) + n (B).

Encourage your students to discuss how they can determine such number of sets and help them reach at a generalization that $n(A\cup B) = n(A) + n(B) - n(A\cap B)$.

You can then give them additional illustrative examples. Finally, give them exercise 3.11 as group assignment because it will help them practice more and you can use this for assessment as well.

Assessment

As this is end of the unit, you can give quiz/test to assess students understanding.

Answers to Exercise 3.11

1. a. Since $n(A \cup B) = 9,\ n(A) = 5,\ n(B) = 5$

$n(A \cap B) = 1$ we have $n(A) + n(B) - n(A \cap B) = 5 + 5 - 1 = 9$

b. $A \times B = \{(2, 6), (2, 7), (2, 8), (2, 9), (2, 10), (3, 6), (3, 7),$

$(3, 8), (3, 9), (3, 10), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10)\ (5, 6),$

$(5, 7), (5, 8), (5, 9), (5, 10), (6, 6), (6, 7), (6, 9), (6, 10), (6, 8)\}$

$\therefore\ n(A \times B) = 25,\ n(A) = 5,\ n(B) = 5$

$\therefore\ n(A \times B) = n(A).n(B)$

c. $n(A \times A) = 25,\ n(A) = 5,$

$\therefore\ n(A \times A) = n(A).\ n(A)$

2. $n(C - D) = 6,$ but $n(C - D) = n(C) - n(C \cap D)$

$\therefore n(C) - n(C \cap D) = 6$

But, $n(C \cap D) = 8$

$\therefore n(C) - 8 = 6$

$\therefore n(C) = 8 + 6 = 14$

3. a. Given $n(Q - P) = 4,\quad n(P - Q) = 5$ and $n(P) = 7$

Now, $n(Q - P) = n(Q) - n(Q \cap P) = 4$

$n(P - Q) = n(P) - n(P \cap Q) = 5$

i.e, $n(P) - n(P \cap Q) = 5$

$\therefore\ 7 - n(P \cap Q) = 5$

$\therefore\ n(P \cap Q) = 2$

$\therefore\ n(Q) = 4 + n(Q \cap P) = 4 + 2 = 6$

b. From the hypothesis, we have

$n(R' \cap S') + n(R' \cap S) = 3$ so $n((U) - (R \cup S)) + n(S - R) = 3.$

i.e, $n(U) - n(R \cup S) + n(S) - n(R \cap S) = 3$

$n(U) - n(R) - n(S) + n(R \cap S) + n(S) - n(R \cap S) = 3$

$n(U) - n(R) = 3$

$n(U) = 3 + n(R)$

But, $n(S' \cap R) = n(R - S) = n(R) - n(R \cap S) = n(R) - 4 = 7$

So, $n(R) = 7 + 4 = 11$

$\therefore\ n(U) = 3 + n(R) = 3 + 11 = 14$

4. a. False, If $A = \{1, 2, 3\},\ B = \{2, 4, 5, 6\}$

Then $n(A \cup B) \neq n(A) + n(B)$

b. False Let $A = \{1, 2, 3\}$ $B = \{2, 4, 5, 6\}$

then, $n(A - B) = n(A) - n(A \cap B) = 2$

but $n(A) - n(B) = 3 - 4 = -1$

c. False Let $A = \{a, b\}$; $B = \{-3, 4\}$

$n(A) = 2,\ n(B) = 2$

$\therefore\ n(A) = n(B)$ but $A \neq B$

d. True e. True

f. False Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 5, 6\}$

$n(A) = 3$ and $n(B) = 4$

$n(A \cup B) = 6,\ n(A \cap B) = 1$

$\therefore\ n(A) + n(B) \neq n(A \cup B) - n(A \cap B)$

g. False Let $A' = \{2, 7, 9\}$ and $B' = \{7, 4, 1\}$

$n(A' \cup B') = n(A') + n(B') - n(A' \cap B')$

But, $n((A \cup B)') = n(A' \cap B') = 1$

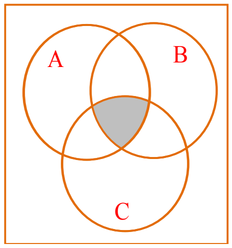
h. True i. True

5. a. $n(A \cup B) = 29$ b. $n(A \setminus B) = 6$
   c. $n(A \Delta B) = 25$ d. $n(B \setminus A) = 19$
6. 3
7. a. 15 b. 50 c. 25
8. a. 16 b. 20 c. 3 d. 8

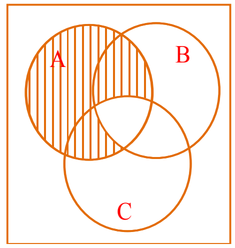
Answers to Review Exercises on Unit 3

1. b, c and e
2. a. $B = \{x, y, z, w\}$ b. $3 \not\in B$
   c. $D = \left\{x \in \mathbb{Q}: \sqrt{2} < x < \sqrt{5}\right\}$ d. $H = \{3n: n \in \mathbb{N}\}$
3. a. $\{1, 2, 3, 4, 5\} \leftrightarrow \{m, n, o, p, q\}$
   b. They are not equivalent.
   c. $\{a, b, c, d, e, f \ldots m\} \leftrightarrow \{1, 2, 3, 4 \ldots 13\}$
4. A = C, D = F, E = G
5. a. $A' = \{a, c, e, g\}$ b. $B' = \{c, d\}$ c. $A \cap B = \{b, f, h\}$
   d. $(A \cap B)' = \{a, c, d, e, g\}$ e. $A' \cap B' = \{c\}$
6. i. I = A – B ii. $A \cap B$ iii. B – A iv. $(A \cup B)'$

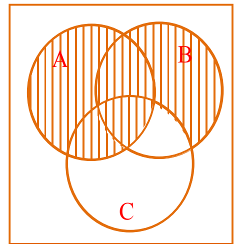
7. a. b. c.



$A \cap (B \cap C)$



$A - (B \cap C)$



$A \cup (B - C)$

8. a. $A' = \{0, 6, 7, 8, 9\}$ b. $B - A = \{0, 6, 8\}$

c. $A \cap C' = \{1, 2, 4, 5\}$

d. $C \times (A \cap B) = \{(3, 1), (3, 2), (6, 1), (6, 2), (9, 1), (9, 2)\}$

e. $(B - A) \times C = \{(0, 3), (0, 6), (0, 9), (6, 3), (6, 6), (6, 9), (8, 3), (8, 6), (8, 9)\}$

9. a. 7 b. 0

10. a. $n(A') = 9$ b. $n(B') = 4$ c. 7 d. 12

11. n (U) – (P$\cup$C)) = 5, where P and C represent for students who study physics and chemistry respectively.

$n(U) - n(P \cup C) = 31 - n(P \cup C) = 5$

$-n(P \cup C) = 5 - 31$

So, $n(P \cup C) = 26$

$= n(P) + n(C) - n(P \cap C)$

$= 22 + 20 - n(P \cap C)$

$n(P \cap C) = 42 - 26 = 16$

$n(P \cap C) = 16$

$\therefore$ $n(P \cap C) = 16$ = the number of students who study both subjects.

12. Given that

$n(A \cup B) = 20$, $n(A \cap B) = 7$, and $n(B) = 2n(A)$, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = n(A) + 2n(A) - n(A \cap B)$

$\Rightarrow$ $20 = 3n(A) - n(A \cap B) = 3n(A) - 7$ So, $n(A) = \dfrac{27}{3} = 9$

$\therefore$ $n(A) = 9$ and $n(B) = 18$

13. a, b and c are infinite whereas d, e and f are finite sets.

14. 21

15. 77

16. a. maximum $n(A \cup B) = 27$ (possible number)

b. minimum $n(A \cup B) = 15$ (possible number)

c. maximum $n(A \cap B) = 12$ (possible number)

d. minimum $n(A \cap B) = 0$ (possible number)

UNIT 4 RELATIONS AND FUNCTIONS

INTRODUCTION

This unit, among others, is expressed by its essence in our daily life. It is common to express different types of relations in every aspect: such as fatherhood, brotherhood, nationality, neighborhood, etc. It is also common to speak of a relation between any two things that possess some relating phrase. We may say: Abebe is taller than Habte. Here, Abebe and Habte are those that relate one another, and “is taller than” is a relating phrase.

A relation in mathematics has many fundamental details and uses for developing other consequences. It is quite common to talk about functions in mathematics which are types of relations with some peculiarities. In spite of its simplicity, a relation is thus crucial to discuss. In this unit, students are expected to see relations and relations in their mathematical sense. After discussing relations, they need to proceed to functions that are types of relations with determined properties.

In regard to the mathematical sense of a relation and a function, notions of Cartesian product, notions of relations and their graphs, functions and their combinations, and their graphs will be deliberated upon in this unit. Involvement of students in various aspects of this unit is sought to help for a better realization of the concepts as well. Considering this fact, try to explore local issues that can best describe a relation and a function along with the discussions of the ideas and examples delivered in the student textbook.

Unit Outcomes

After completing this unit, students will be able to:

- *know specific facts about relation and function.*
- *understand basic concepts and principles about combination of functions.*
- *sketch graphs of relations and functions (i.e. of linear and quadratic functions).*

SUGGESTED TEACHING AIDS IN UNIT 4

It is expected that all students are aware of a relation in its meaning from daily life. So as to be able to make their understanding up to standard, enabling students to participate in conception of relations and functions is of importance. The discussions hold in class along this line may not be exhaustive in themselves. Thus, it may be essential to look for various inputs via teaching aids and active participation of students.

Therefore, constituting different groups, students can develop local examples which will help as an additional teaching aid for a better and easy understanding of the notions of a relation and a function. You can also use charts that describe relations of different type, and also graphs of relations. You can also use software(s) such as Geometers' Sketchpad, Mathematica, Matlab, etc.

4.1 RELATIONS

Periods allotted: 7 Periods

Competencies

At the end of this subunit, students will be able to:

- *define the notions "relation", "domain" and "range''.*
- *draw graphs of relations.*
- *use graphs of relation to determine domain and range.*

Vocabulary: Cartesian product, Relation, domain, range, graph

INTRODUCTION

Relation is one of the commonly used descriptions in daily life of every one. We relate two or more things somehow and in some way. Trying to represent such relations mathematically is important. For this purpose students need to get acquainted with the concept of a relation. In this sub-unit, students need to get ideas on notions of relations, mathematical definition of a relation, domain and range of a relation, and graphs of relations.

TEACHING NOTES

You may start the lesson by giving chance to the students to explain their understanding about relation from their daily life. For this purpose, you can let the students perform group work 4.1 so that they can

1. explain and discuss the meaning of "relation" in their daily life.
2. give some examples of relations from their daily life.
3. discuss how they understand relations in mathematical language.

You can restate some explanatory relations which can be easily understood by the students. For example, we usually say many issues related to relations among and between human beings, our surroundings, and many others, such as someone is a father of some others, 5 is greater than 3, Addis Ababa is the capital of Ethiopia, Wallia Ibex is endemic to Ethiopia, etc.

You can consider a relation as in its dictionary meaning "the existence or effect of a connection, correspondence, contrast, or feeling prevailing between persons or things, especially when qualified in some way"; you can take some examples such as "teacher-student relation, friendship, neighborhood", etc as examples in their daily life. Mathematically, you can explain a relation as a connection between two sets.

Finally, you need to ensure that in any relation, there are two concepts, namely "those two things that relate one another" and, "a relating phrase". Here you need to give them more examples apart from the examples given in the student textbook. You can ask students to give more examples from their own understanding as an activity. You can also let them do activity 4.1. Encourage them to give as many examples of relations from their daily life and guide their view of how ordered pair representation helps in describing a relation.

Answers to Activity 4.1

1. a. {(4, 3), (6, 3), (6, 5), (7, 3), (7, 5)}

   b. {(1, 5), (1, 12), (2, 12), (4, 12), (6, 12), (1, 7), (7, 7), (1, 9), (1, 8), (2, 8), (4, 8), (1, 3)}

   c. {(1, 12), (1, 8), (2, 5), (2, 7), (2, 9), (2, 3), (4, 5), (4,, 7), (4, 9), (4, 3), (6, 5),

   (6, 7), (6, 9), (6, 3), (7, 12), (7, 8)}

   d. {(4, 8), (6, 12)}

2. a. {(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}

   b. {(1, 1), (4, 2), (9, 3), (0, 0)}

   c. {(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 5), (4, 6), (4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)}

   d. {(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (5, 5), (7, 7)}

3. This problem depends on each section. You may not feasibly do this for all students in a section. What is possible in this case is selecting some students, encourage them to stand in front of the students and ask each student to create possible pairing of the students based on the questions in the students textbook.

While students discuss the activity, you may need to help students realize that, in a relation, there are two fundamental conceptions: the related objects and the relating phrase. In a relation, the issues of order and the establishment of pairing between objects are fundamentals that every student needs to underscore.

After deliberating on a relation, you may need to formally define a relation and give more examples which the students need to do in class. You can also give a chance for students to do exercise 4.1 either individually or in group.

Assessment

For the purpose of assessing students understanding, you can give them various exercises of relations and let them describe relating phrase or describe them in the form of ordered pairs.

Answers to Exercise 4.1

1. a. "is taller than" b. "is the square root of"
   c. "is twice of" because R can be expressed as R = {(x, y): y is twice of x}. It can also be "is two times of"
2. a. R = {(x, y): x is equal to y; $x, y \in$ A}
   b. No. Because in a relation from A to B all the first coordinates must befrom set A. For example (1, 2) $\in$ R, but 1 $\notin$ A. The same is true for (3, 4) and (5, 6).
   c. R = {(2, 1), (4, 3), (6, 5)}
3. R = {(1, 3), (2, 5)}
4. R = {(2, 1), (3, 1), (3, 2), ...}

You may also elaborate the foundation of coordinate plane as a Cartesian product. For this purpose, you can proceed to deliver the lesson by revising the Cartesian product of two sets which have been discussed in unit one. You can also explain about the famous mathematician René Descartes (1596 –1650) who greatly contributed to the development of Cartesian product that is named after his candid and unreserved contribution. You can begin by giving an activity to students to develop some relation from which they will try to identify the members in each relating sets, and help them to recapitulate their understanding of a Cartesian product from the first unit. Then you can proceed to the concepts of domain and range. However, before determining domain and range of a certain relation, it will be better for the students to practice on examples of relations from their daily life and try to give domain and range of such relations. After this practice, it could be possible to go into a relation expressed as an ordered pair in a Cartesian plane for a relation is mathematically, a set of ordered pairs. This is so, because, if A and B are two sets, then the relation from A to B is the set of order pairs from A $\times$ B or simply any subset of A $\times$ B. This time, it is possible to describe a relation in the following way: If A and B are any two sets and R $\subseteq$ (A $\times$ B), we call R a binary relation from A to B or a binary relation between A and B. A relation R $\subseteq$ (A $\times$ A) is called a relation in or on A.

When students realize what a relation is from their daily life experience and some of the mathematical presentations, you can dwell upon domain and range in that domain is simply the set of the first coordinates and the range is the set of the second coordinates in a Cartesian product. So as to deal on this, you can let the students do Activity 4.2 individually.

Answers to Activity 4.2

a. D = {4, 6, 7} b. D = {4, 6}

c. R = {3, 5} d. R = {8, 12}

This activity is meant to support students understand the concept of domain and range. The set of the first coordinates satisfying a certain relation being domain and the set of the second coordinates being range. You can give further examples and state definition 4.2.

Assessment

You can present similar examples to those given in the student textbook and assess students understanding through oral question and answer, or give them exercise 4.2 as homework and record their achievement.

Answers to Exercise 4.2

1. Domain = {–2, 5} and Range = {2, 3, 4}
2. R = {(1, 2), (2, 3), (3, 4)}; Domain = {1, 2, 3} and Range = {2, 3, 4}
3. a. Domain = $\{x: x \geq 0\}$ and Range = $\{y: y \geq 0\}$
   b. Domain = the set of all real numbers and Range = $\{y: y \geq 0\}$
   c. Here domain can be expressed in many different ways, some of which can be "all sections in grade nine or all sections in a school" and range is set of all mathematics teachers teaching in section 9.
4. Domain = {4, 6, 8} and Range = {4, 6, 8}

4.1.3 Graphs of Relations

Up until now, students have discussed meaning of a relation and its representation in a form of ordered pairs. They also have discussed domain and range. Pursuant to the meaning of a relation, and domain and range, it is now possible to discuss how to sketch a graph of a relation. Graphs are alternative representations of relations. In order to proceed, you can let students do activity 4.3 so that they can discuss coordinate plane, a point on a coordinate plane and a region on a coordinate plane. For this purpose, you can prepare a flipchart that consists of a coordinate plane and points and a region on it. Encourage students to discuss plots of points on a coordinate plane and their representation as an ordered pair. You can also ask them to list the pairs of numbers represented as a point on a coordinate plane. Examples of other forms of representations such as arrow diagrams are also given in the student textbook.

After you do these, group students (different ability groups) and assign them to do Group Work 4.2 so that they can help each other. The purpose of this group work is to enable them determine a region representing a relation.

Answers to Group Work 4.2

1. The graph of the line $y = x$ is given below. We use broken line because the given relation is inequality (strict less than)
2. Any two ordered pairs can be (0, 4) and (4, 0).
3. The region that contains the ordered pair satisfying the relation is the solution.
4. The shaded part is the one given on the graph below.
5. The domain and range are both sets of real numbers.

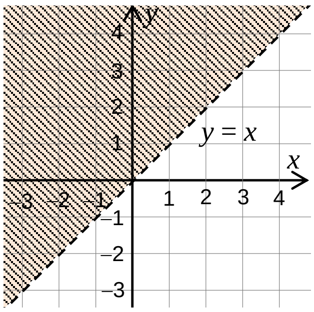


Figure 4.1

Following this group work, ask each student to do activity 4.4 individually and by rounding help each. You can assess how much they have done properly. You can also give additional exercise for those who are fast to sketch the graph until others finish the activity.

Answers to Activity 4.4

1. First they need to draw the line $y = 2x$ with solid line because it is not strict inequality.
2. Select two points on both sides, say (0, 4) and (4, 0).
3. The part that satisfies the relation is the one which shaded in the figure below.

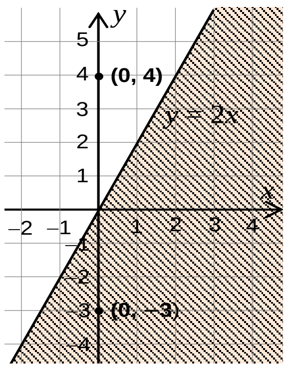


Figure 4.2

You can then give additional examples for practice and understanding and let them do each of the examples. Once students have become capable of sketching graphs of relations with inequalities, especially with one inequality, you can proceed by giving them the chance to sketch graphs of relations with mixed inequalities. For this purpose, group students and ask them to do activity 4.5.

Answers to Activity 4.5

1. The sketches of the relations are:

a

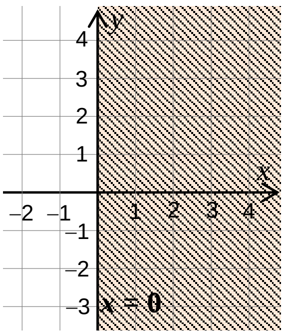


Figure 4.3

b

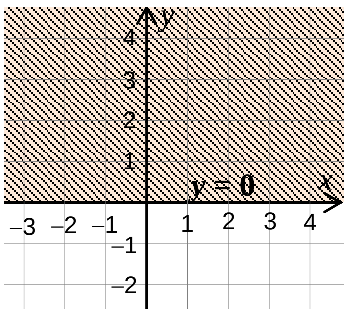


Figure 4.4

c

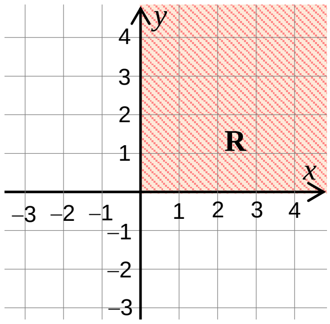


Figure 4.5

2. The possible observation is that the region sketched for R_3 is the intersection of the regions sketched from R_1 and R_2.

Pursuant to this, encourage students to sketch the graphs of the relations given as examples and let them describe why they do each step.

It is now time to let the students determine the domain and range of relations with mixed inequalities from their graphs. Before proceeding into the group work presented for this purpose, you can ask students to explain how they can determine domain and range for brainstorming. Then ask them to do Group Work 4.3.

Answers to Group Work 4.3

1. Students can come up with varieties of replies for this question. Entertain each and guide them towards the following possibility stated as a short cut for determining domain and range.

2. In order to determine domain and range from graph of a relation, they need to notice that they use vertical and horizontal lines. The region that intersects with a vertical line passing through the *x*-axis stands for domain and the region which intersects with a horizontal line passing through the *y*-axis stands for range of the relation.

You enrich their understanding by giving more examples and give them exercise 4.3 as an assignment for each student so that they can practice sketching graphs of relation and determine domain and range. While they do examples in class you can add exercises of the following type for fast learners.

Sketch the graph of the following relations:

a. R = $\{(x, y): y < x \text{ and } y > x\}$ whose graph is empty and

b. R = $\{(x, y): y \geq x \text{ and } y \leq x\}$ whose graph is the line $y = x$.

Assessment

You can give them different relations and ask them to sketch their graphs and determine domain and range for each graph. Let them also submit their work and you check and keep record. You can post the best works in class.

Answers to Exercise 4.3

1. a.

Domain = {5}

Range = {15}

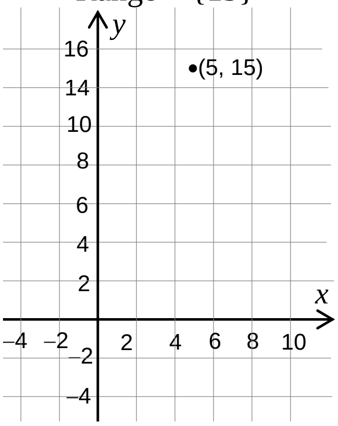


Figure 4.6

b.

Domain = {2, 3, 5}

Range = {6, 10, 15}

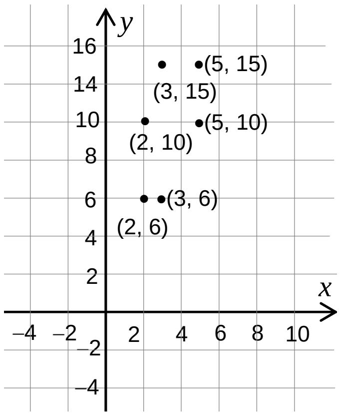


Figure 4.7

2. a. b). c.

Domain= ℝ Domain = $\{x: x \le \frac{1}{2}\}$ Domain = ℝ

Range = ℝ Range = ℝ

Range = ℝ

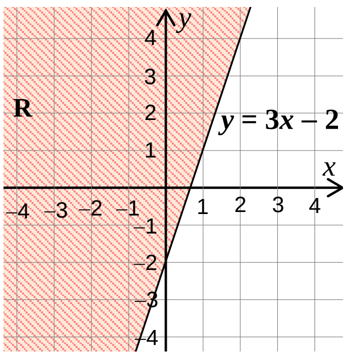


Figure 4.8

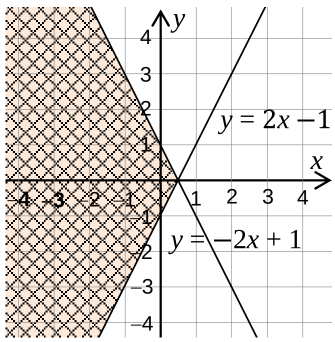


Figure 4.9

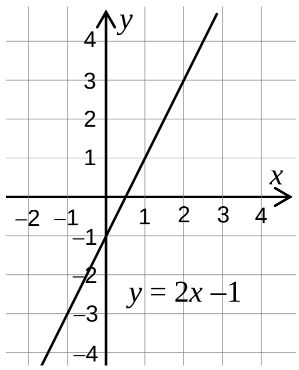


Figure 4.10

3. a. R = $\{(x, y): y \ge x+1 \text{ and } x \ge 2\}$; Domain = $\{x: x \ge 2\}$ and Range = $\{y: y \ge 3\}$

b. R = $\{(x, y): y \le\ x, x \le 4 \text{ and } y \ge 0\}$; Domain = $\{x: 0 \le x \le 4\}$ and Range = $\{y: 0 \le y \le 4\}\}$

4.2 FUNCTIONS

Periods allotted: 6 Periods

Competencies

At the end of this subunit, students will be able to:

- *define function.*
- *determine the domain and range of a given function.*
- *determine the sum, difference, product and quotient of functions.*
- *evaluate combination of functions for a given value from their respective domain.*

Vocabulary: Function, Equations in two variables, System of equations, linear function

INTRODUCTION

As a particular type of relation which students have discussed in the previous sub-unit, in this sub-unit they will discuss functions. First they will see ordered pair representations of functions and arrow diagram representations. Second they will see domain and range of functions. After discussing these representations, domain and range; they need to realize that value of a function at a point is a real number. Pursuant to these they need to discuss the possibility of applying operations on such values of functions which will lead them into combinations of functions.

TEACHING NOTES

This subunit seeks further look at relations that fulfill certain characteristics and called functions.

4.2.1 Functions

You can introduce the unit by revising different types of relations that are expressed by using set of ordered pairs as a result of Cartesian product. You also need to let students perform group work 4.4 to recall concept of a relation, describe relations in a form of ordered pairs and characterize possible observations on such ordered pair representation which will help them in constituting functions.

While giving examples of such ordered paired relations, you need to assist students to identify the representations of one-to-one, many-to-one and one-to-many, and let them discuss each if they can see their characteristics.

Answers to Group Work 4.4

1. a. The observation could be that all are given in ordered pairs. However, the setting of the pairs are not all the same in each relation.

   b. In R_1, each first coordinate is paired with unique second coordinate.

   In R_2, although each first coordinate is paired with unique second coordinate, two different first coordinates are paired with one second coordinate (1, 2) and (3, 2).

In R_3 each first coordinate is not paired with unique second coordinate. i.e., one first coordinate is paired with two second coordinates (1, 2) and (1, 4).

c. Such ordered pairs are found only in R_3.

2. There is change in the fatherhood. The pairings are also interpreted differently. For example, the R_1 one father may have one child or may have more than one child. This can be one-to-one or many-to-one. In R_2 many x cannot be fathers of y.

After doing so, you can proceed to the formal definition of a function. From the Group Work students are capable of identifying the order pair form of relations and those that are functions. As an alternative form of representing relations you can also discuss representations of such relations in form of Venn diagram. You then lead the students to do several examples so that they can assimilate their understanding.

Function is a (special) type of a relation that does not pair (or map) elements of the first coordinate to two or more elements of the second coordinate. That is, every element of the first coordinate is paired (or mapped) with exactly one and unique element in the second coordinate. In this regard, you can help students to identify that a relation is a function if it is either one-to-one or many-to-one. In both cases the second coordinate is unique. At this stage, you may need to give chance for the students to give examples of relations which are functions.

Once students become able to recognize a function, you can proceed to the concepts of domain and range of functions defined by the set of ordered pairs. At this stage, students need to recapitulate that domain and range of functions are determined analogous to determining domain and range of a relation. i.e. domain is the set of the first coordinates and range is the set of all second coordinates.

For practice you can let students discuss the examples on the students textbook. You can also give them additional examples.

Cognizant of students understanding, you can proceed with the lesson by considering functions defined by formulas like $f(x) = x + 2$ and $g(x) = 3 - 3x$ and guide students to determine domain and range. Before you do so, you can engage students by letting them do Activity 4.6. The purpose of this activity is to help students find algebraic rule for functions from their arrow diagram representation (you can also add an example for ordered pair form representation) and determine domain and range of functions from their algebraic rule.

Answers to Activity 4.6

1. The algebraic rule for $f(x)$ is $f(x) = 3x$ where $x \in \{0, 1, 2, 3\}$.

2. a. domain = $\mathbb{R}$ and range = $\mathbb{R}$

   b. domain = $\mathbb{R}$ and range = $\{y : y \geq 0\}$

   c. domain = $\{x : x \leq -\sqrt{3} \cup x \geq -\sqrt{3}\}$ and range = $\{y : y \geq 0\}$

Following their effort in doing the activity, you can let them discuss the examples given in the student textbook. While they discuss you can round and check their work. If there are fast learners who can do each easily, you can add some exercises to find domain and range of the following type.

a. $f(x)=\sqrt{(1-x)}$ whose domain is $\{x: x \le 1\}$ and range = $\{y : y \ge 0\}$

b. $f(x)=-x^2$ whose domain is $\mathbb{R}$ and range = $\{y : y \le 0\}$

c. $f(x)=\sqrt{x^2}$ whose domain is $\mathbb{R}$ and range = $\{y : y \ge 0\}$

At this stage, it will be essential and helpful to encourage students to do exercise 4.4 in group as an assignment. It is also possible to arrange presentation so that they can well prepare themselves, conduct discussion and get what is anticipated.

Assessment

You can assess the students understanding by asking them to write down algebraic rule for functions written in the form of ordered pairs, and giving exercise problems that the students should identify as relations or functions or both. You can also let them give their reason for their answers and let them give the domains and ranges as well. You can also ask students to evaluate the value of a given function at a given value from its domain. These can also be administered by class activities, group discussions, homework/assignments, and/or tests/quizzes as situations permit in your school.

Answers to Exercise 4.4

1. a. Not function because –1 is paired with 2 and 3 (One –to-many)
   b. Not function because 1 is paired with 1 and 3 (One –to-many).
   c. Yes
   d. Not function because one area can be mapped with more than one triangles (i.e. different triangles can have the same area)
   e. Not function because one *x* can have many multiples
   f. Yes
   g. Not function because for one *x* there are many y such that $y < x$
   h. Yes.
2. Yes, because a function is a special type of relation.
3. a. Domain = $\mathbb{R}$ and Range = {3}
   b. Domain = $\mathbb{R}$ and Range = $\mathbb{R}$
   c. Domain = $\{x: x \ge -4\}$ and Range = $\{y: y \ge 0\}$
   d. Domain = $\mathbb{R}$ and Range = $\{y: y \ge -1\}$
   e Domain = $\mathbb{R}\backslash\{0\}$ and Range = $\mathbb{R}\backslash\{0\}$
4. a. $f(-4) = -8$ b. $f(5) = 13$
5. 1. b 2. c 3. a
6. 1. a 2. b 3. a

4.2.2 Combinations of Functions

Once the students successfully accomplish the tasks outlined previously, this stage it may be fundamental for students to understand that functional values are numbers. And thus, if two or more functions are considered at a time, it is possible to do operations such as addition, subtraction, multiplication and division on those numbers as functional values. Such conception will lead into discussing combinations of functions. It may be better to give functions defined by formulas like $f(x) = x + 2$ and $g(x) = 3–3x$ and let students determine their sum, difference, product and quotient. For the purpose of engaging students in critical thinking about combinations, you can let them do group work 4.5 and help them understand how combinations of functions are determined and at the same time determine domain and range of combined functions. When they practice doing so, the formal rules of such combinations can follow. At this phase, it is good for students to identify the relationship between the domain of the component functions and the resulting function. Notice that ***domain of a combined function is the intersection of the domains of the component functions*** when the operations are addition, subtraction and multiplication. But during division, the domain of the resulting function is the intersection of the domains of the component functions provided that the denominator function is defined. Pursuant to the discussion of domain and range, it will be advisable for students to see the functional values of combined functions at points from the domain. To do so, you may proceed to assisting students to evaluate the sum, difference, product and quotient of functions at a given value of x from the domain.

Answer to Group work 4.5

a. $f + g = \sqrt{x-3} + \sqrt{10-x}$

$f - g = \sqrt{x-3} - \sqrt{10-x}$

$f.g = \sqrt{-x^2 + 13x - 30}$

$\frac{f}{g} = \sqrt{\frac{x-3}{10-x}}$

b. Domain of $f + g = \{x: 3 \leq x \leq 10\}$

Domain of $f – g = \{x: 3 \leq x \leq 10\}$

Domain of $f.g = \{x: 3 \leq x \leq 10\}$

Domain of $\frac{f}{g} = \{x: 3 \leq x < 10\}$

c. Domain of $f = \{x: x \geq 3\}$

Domain of $g = \{x: x \leq 10\}$

Domain of $(f + g) = \{x: 3 \leq x \leq 10\}$

Domain of f and g = domain of $f + g$

Following the group work, it will be advisable to practice more on combinations through the examples from the student textbook and more others. While students discuss the examples, you can give additional exercises of the following type for faster students.

a. $2(f - g) - 3f$
b. $(f + 2g)(2g)$
c. $2(f - g)(f + g)$
d. $\frac{f}{g} - \frac{g}{f}$ where $f(x) = 2 + 3x$ and $g(x) = x - 5$

Assessment

For the purpose of checking students understanding and assessing their level, you can give them Exercise 4.5 as homework and you then check and give them your feedback. You need also keep record.

Answers to Exercise 4.5

1. a. Domain of $f + g = \{1, 2\}$

   b. $f + g = \{(1, 7), (2, 9)\}$

   $f - g = \{(1, -3), (2, 1)\}$

2. a. $-2f = \{(2, -6), (4, -18), (3, 16)\}$

   b. $fg = \{(2, 15), (3, -80), (4, 153)\}$

   c. $fg(2) = 15$

   d. $g^2 = gg = \{(1, 4), (2, 25), (3, 100), (4, 289)\}$

3. a. Domain of $-2f = \{2, 3, 4\}$

   b. Domain of $fg = \{2, 3, 4\}$

   c. Domain of $fg(2) = \{2\}$

   d. Domain of $g^2 = \{1, 2, 3, 4\}$

4. a. $f + g = \frac{2}{x-1} + \frac{2x-2}{3x+3} = \frac{2x^2 + 2x + 8}{3x^2 - 3}$

   b. $fg = \left(\frac{2}{x-1}\right)\left(\frac{2x-2}{3x+3}\right) = \frac{4}{3x+3}$

   c. Domain of $f + g$ = Domain of fg = Domain of $f \cap$ Domain of $g = \mathbb{R}\backslash\{1\} \cap \mathbb{R}\backslash\{-1\} = \mathbb{R}\backslash\{-1, 1\}$

5. a. $2fg(2) = 12$ b. $\left(\frac{f}{g} - 2f\right)(3) = -6$ c. $(f - g)(4) = \frac{25}{3}$

6. Yes,
   i. Domain $(f + g)$ = Domain of $f \cap$ Domain of g.
   ii. Domain of $(f - g)$ = Domain of $f \cap$ Domain of g.
   iii. Domain of $(f.g)$ = Domain of $f \cap$ Domain of g.
   iv. Domain of $\frac{f}{g}$ = Domain of $f \cap$ Domain of g, $g(x) \neq 0$

4.3 GRAPHS OF FUNCTIONS

Periods allotted: 9 Periods

Competencies

At the end of this subunit, students will be able to:

- *sketch graphs of linear functions.*
- *describe the properties of the graphs of linear functions.*
- *sketch the graphs of a given quadratic function.*
- *describe the properties of the graphs of given quadratic functions.*
- *determine the maximum and minimum values of a given quadratic function.*

Vocabulary: Graph of linear function, graph of quadratic function, properties, maximum and minimum values

INTRODUCTION

In previous sub-unit functions, domain and range of functions, and combinations were discussed. In this sub-unit students will discuss graphs of functions (linear and quadratic) and will proceed to characterizing graphs and develop properties. Finally, they will discuss some of the applications in determining minimum or maximum values of quadratic functions.

TEACHING NOTES

Since students have discussed linear and quadratic equations in unit 2, you may start this subunit by asking students to state the definition of linear function and quadratic function. Following this discussion, you may proceed to writing down the following definition of a linear function:

Definition. If a and b are fixed real numbers, then $f(x) = ax + b$; $a \neq 0$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$ then $f(x) = b$ is called a constant function.

Before proceeding further, it may be important to let students revise evaluating values of functions and plotting some of the evaluated coordinate points on a coordinate plane to draw graphs of linear functions. As a consequence of this, it may be good to let students discuss some basic and important properties of linear functions and their graphs. For this purpose you can discuss, with active participation of the students, the examples given in the student textbook on page 159. You can then let students write down their observations from graphs of linear functions by giving them chance to do Activity 4.7. This helps them to make generalizations about increasing and decreasing nature of graphs in relation to the coefficient of the variable x. You also let them do more examples and exercise of these types. Whenever it is necessary, you can narrate all the properties at the end. At this moment, you can ask the following to the fast learners:

In $f(x) = 2x - 1$, if $f(x) = 3$ it true for all x, then

a. solve for x

b. Sketch the graph of $x = 2$

c. Characterize the behavior of the graph

d. Is the function increasing or decreasing?

The solution of this is: a) $x = 2$

b.

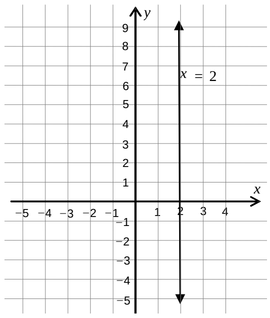


Figure 4.11

c. x-intercept is at (2, 0) and the graph is vertical line.

d. Neither increasing nor decreasing

Let the students do some of the examples in the students textbook first and then give them some more exercises from exercise 4.6 for additional practice as homework and keep records for assessment.

Assessment

You can assess students learning and understanding, you can give exercise problems on sketching the graph of linear functions and ask students to describe the properties of the graphs of linear functions. You can do these by giving class activities, group discussions, homework/assignments, and/or giving tests/quizzes.

Answers to Exercises 4.6

1. a, b, c are linear because all can be expressed in the form of $y = ax + b$. d is not linear

2. a.

x	–1	1	2	3
$f(x)$	–3	1	3	5

b.

x	–6	–3	0	3	6
$f(x)$	–3	–2	–1	0	1

c.

x	–3	–2	–1	0	1	2	3
$f(x)$	10	7	4	1	–2	–5	–8

3.

	Slope	*y*-intercept	*x*-intercept
A	–1	(0, 1)	(1, 0)
B	3	(0, –4)	$\left(\frac{4}{3}, 0\right)$
C	1	(0, 3)	(–3, 0)
D	3	(0, 5)	$\left(\frac{-5}{3}, 0\right)$

4. a. Increasing function, because its slope is positive.
   b. Increasing function, because its slope is positive.
   c. Constant function (Neither increasing nor decreasing) (Its slope is zero)
   d. Constant function (Neither increasing nor decreasing) (Its slope is zero)

5. a. $y - 3x - 5 = 4$

x	–3	–2	–1	0	1	2	3
$f(x)$	0	3	6	9	12	15	18

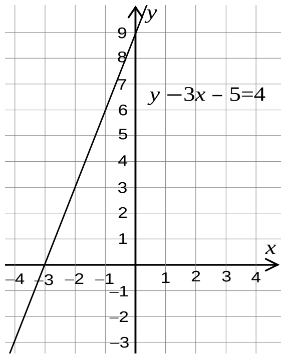


Figure 4.12

b. $4 = 4x - 2y$

x	–3	–2	–1	0	1	2	3
$f(x)$	–8	–6	–4	–2	0	2	4

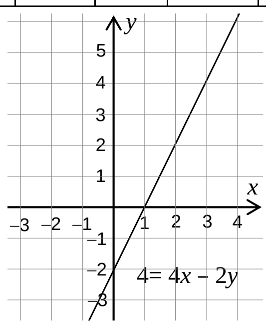


Figure 4.13

c. $f(x) = 1 - 7x$

X	–3	–2	–1	0	1	2	3
$f(x)$	22	15	8	1	–6	–13	–20

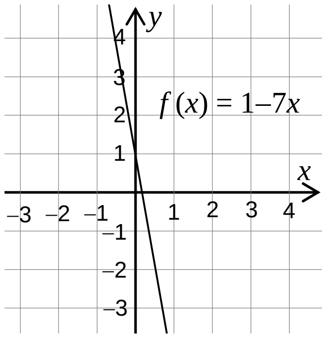


Figure 4.14

d. $y = 1$

x	–3	–2	–1	0	1	2	3
$f(x)$	1	1	1	1	1	1	1

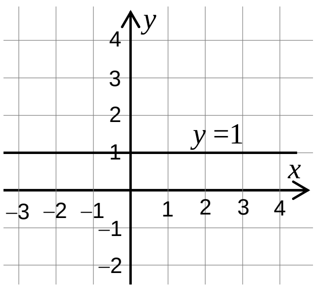


Figure 4.15

6. a. y-intercept = $(0, -5)$ and

x-intercept = $\left(\frac{5}{3}, 0\right)$

Locate these intercepts and draw a line passing through them.

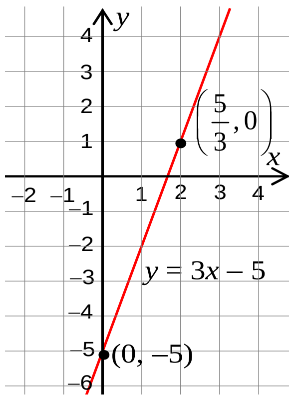


Figure 4.16

b. *y*-intercept = (0, –2) and
x-intercept = (1, 0)
Locate these intercepts and draw a line passing through them.

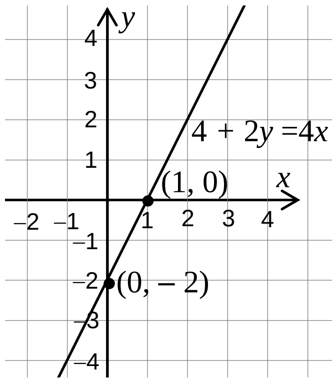


Figure 4.17

c. *y*-intercept = (0, –5) and
x-intercept = $(\frac{5}{3}, 0)$
Locate these intercepts and draw a line passing through them.

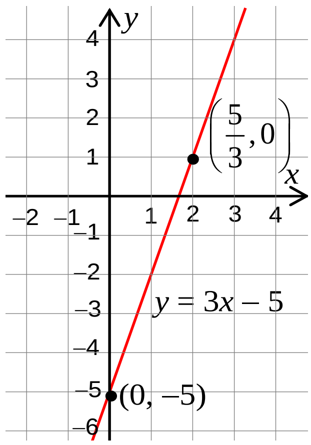


Figure 4.18

7. a. Slope = 1 and you can take any arbitrary point that satisfy the function say, (0, 3)
Draw a line parallel to $y = x$ and passing through the point (0, 3)

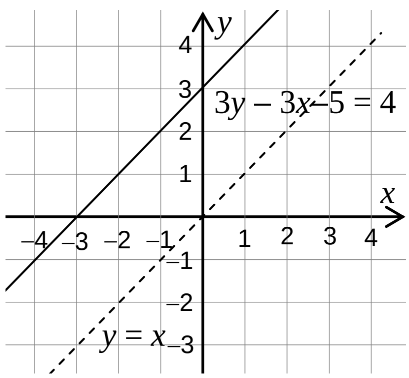


Figure 4.19

b. Slope = 4 and you can take any arbitrary point that satisfy the function say, (0, 2)
Draw a line parallel to $y = 4x$ and passing through the point (0, 2)

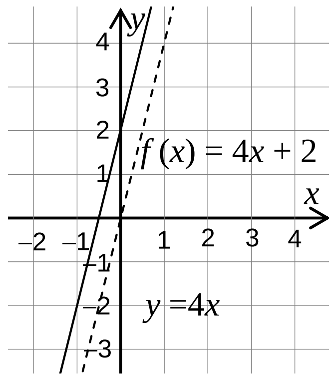


Figure 4.20

c. Slope = 1 and you can take any arbitrary point that satisfies the function say, (0, 2)
Draw a line parallel to $y = x$ and passing through the point (0, 2).

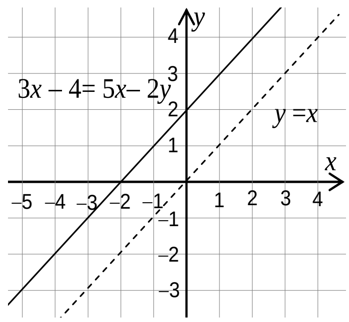


Figure 4.21

Following this analogy in linear functions, you can proceed to discuss quadratic functions. You can start this sub-unit by asking the students to write down the definition of a quadratic function. Based on the definition(s) students might give, you need to facilitate discussions and then you can give the following formal definition at the end.

Definition: A function defined by $f(x) = ax^2 + bx + c$; $a, b, c \in \mathbb{R}$ and $a \neq 0$ is called quadratic function.

At this stage, it may be good to let students discuss some basic and important properties of quadratic function such as $a \neq 0$. After considering these, you can proceed to discuss how table values are constructed and how these table values help us draw a graph of a quadratic function. You can let students do activity 4.8 and assist them how such table values help in drawing quadratic functions. You also need to help students take a notice that graphs of quadratic functions are curves (known as parabola).

Answer to Activity 4.8

1. a.

x	–3	–2	–1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

b.

x	–3	–2	–1	0	1	2	3
$f(x)$	2	0	0	2	6	12	20

c.

x	–3	–2	–1	0	1	2	3
$f(x)$	–25	–14	–7	–4	–5	–10	–19

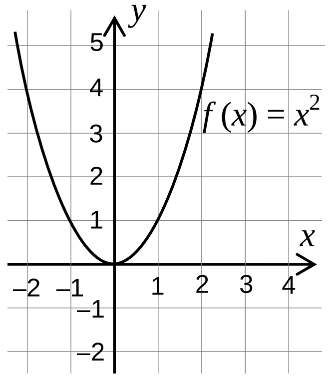


Figure 4.22

3. It is the graph of the parabola that opens upward. Has turning point (0, 0) and axis $x = 0$

When you ensure that students have captured computing table of values and can sketch the graph of a quadratic function, you can give them Activity 4.9 so that they can list their observation and understand the change in openness of the curve of the parabolas.

Answers to Activity 4.9

The graph of $f(x) = –x^2$

- opens downward
- has vertex (0, 0)
- has axis $x = 0$
- is symmetrical with respect to the y-axis.

Let them do other examples as well to determine vertex and the axis of a parabola. Cognizant of the fact that the students have reached at the required level of understanding of sketching the graphs of simple quadratic functions such as $f(x) = x^2$ and $f(x) = -x^2$, and that they can narrate on the behaviors of the graphs you can let them perform group work 4.6. The purpose of this group work is to give chance for students to practice drawing graphs of quadratic functions, write down their observations and help them seek an alternative way (if any) to drawing graphs of quadratic functions.

Answers to Group work 4.6

1. i.

	x	–2	–1	0	1	2
a.	$f(x) = 3x^2$	12	3	0	3	12
b.	$f(x) = 3x^2 – 1$	11	2	–1	2	11
c.	$f(x) = 3x^2 + 1$	13	4	1	4	13

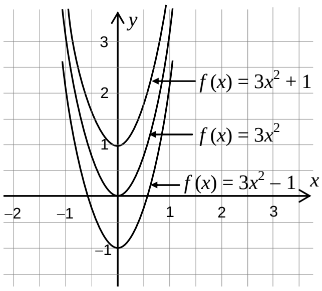


Figure 4.23

ii.

	x	–2	–1	0	1	2
a.	$f(x) = 3x^2$	–12	–3	0	–3	–12
b.	$f(x) = 3x^2 - 1$	–13	–4	–1	–4	–13
c.	$f(x) = 3x^2 + 1$	–11	–2	1	–2	–11

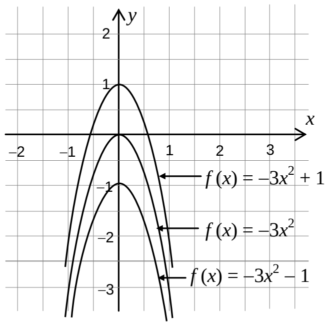


Figure 4.24

3. By using shifting methods (or rule)

This time, the students are expected to narrate different properties of quadratic functions and their graphs. Once students are able to draw graphs of quadratic functions, you may assist them to describe some more of the properties of the graphs of quadratic functions, the intercepts, the nature of the graphs in relation with the leading coefficients and the coordinates of the vertex of a parabola and symmetry. When they characterize openness of graphs of quadratic functions and their relation to the leading coefficient, you can proceed by letting students do some more examples similar to the group work 4.6 so that they can further characterize graphs of quadratic functions and take a leap into using ***shifting rule*** for drawing graphs of quadratic functions.

Some illustrative examples are delivered in the student textbook. You can give other exercises such as:

1. Sketch the graph of $f(x) = x^2$, $f(x) = (x-2)^2$, $f(x) = (x-2)^2 + 1$, $f(x) = (x-2)^2 - 1$ using the same coordinate system
2. Sketch the graph of $f(x) = x^2$, $f(x) = (x+2)_2$, $f(x) = (x+2)^2 - 1$, and $f(x) = (x+2^2) + 1$ using another coordinate system
3. Write down your observations on each.

These will help students practice more and support your assessment. You can also engage fast students with these exercises while others are doing the examples in the textbook.

Finally, when students can see that the graph of a parabola turns upward or downward depending on the leading coefficient, it will be good to let them see that a parabola has either a minimum or a maximum. In order to do this, you better let students do Group Work 4.7 and assist them understand the fact that a quadratic function has either minimum or maximum and you also help them seek ways of determining these. You can describe the following facts to students.

When the graph turns upward, the function has a minimum value, and when it turns downward it has a maximum value.

You can also encourage students to describe a quadratic function in the general form of

$f(x) = a(x + k)^2 + c$ so that they can characterize some properties. At last help them to understand the following.

- Change in the value of a affects the openness of the graph of the quadratic function. When $a > 0$ an increase in a widens the openness of the parabola.
- Change in k shifts the graph either to the left or to the right.
- Change in c effects in movement of the graph upward or downward.
- c is either the maximum or the minimum of a quadratic function.

Ask students to determine this minimum or maximum value for some quadratic functions. At last, you may give them a note that such a minimum or maximum value occurs at the vertex.

Answers to Group work4.7

1. For a quadratic function $f(x) = ax^2 + bx + c$
   i. If $a > 0$, $f(x)$ has a minimum value of the vertex. Or
   ii. If $a < 0$, $f(x)$ has a maximum value at the vertex.

   Therefore, the maximum value or minimum value is $f\left(\frac{-b}{a}\right)$

Assessment

Since it is the end of the unit you can give a comprehensive exam/test to assess students understanding.

Answers to Exercise 4.7

1. a. $a = -2$, $b = 3$ and $c = 2$ b. $a = 3$, $b = -4$ and $c = 1$
   c. $a = -1$, $b = 5$ and $c = -6$
2. a.

x	–3	–2	–1	0	1	2	3
$f(x)$	–36	–16	–4	0	–4	–16	36

   b.

x	–3	–2	–1	0	1	2	3
$f(x)$	29	14	5	2	5	14	29

c.

x	–3	–2	–1	0	1	2	3
$f(x)$	29	16	7	2	1	4	11

3. a. $f(x) = -3x^2$

x	–2	–1	0	1	2
$f(x)$	–12	–3	0	–3	–12

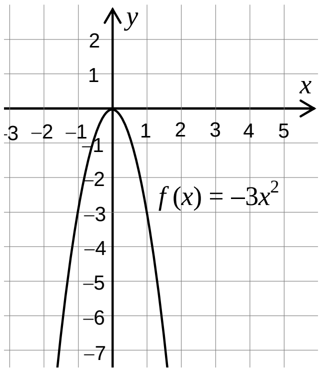


Figure 4.25

b. $f(x)=7x^2-3$

x	–2	–1	0	1	2
$f(x)$	25	4	–3	4	25

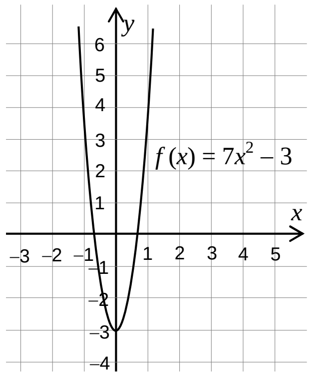


Figure 4.26

c. $f(x)=2x^2+6x+1$

x	–2	–1	0	1	2
$f(x)$	–3	–3	1	9	21

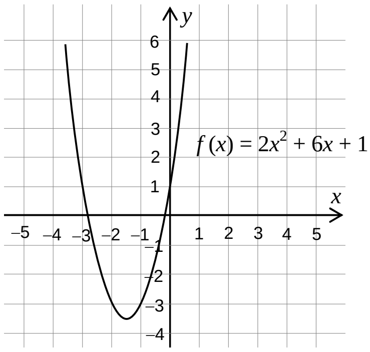


Figure 4.27

4. a. Domain = $\mathbb{R}$ and Range = $\{y: y \leq 7\}$

b. Domain = $\mathbb{R}$ and Range = $\{y: y \geq 0\}$

c. Domain = $\mathbb{R}$ and Range = $\{y: y \geq -\frac{1}{4}\}$

d. Domain = ℝ and Range = {y: $y \le -2$}

e. Domain = ℝ and Range = {y: $y \ge 2$}

5. a. $f(x)=9x^2+1$

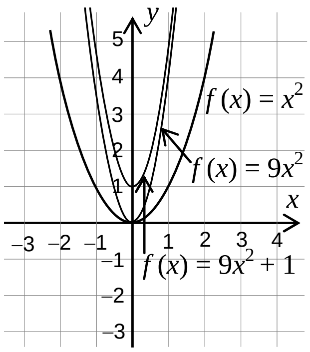


Figure 4.28

b. $f(x) = x^2 - 3$

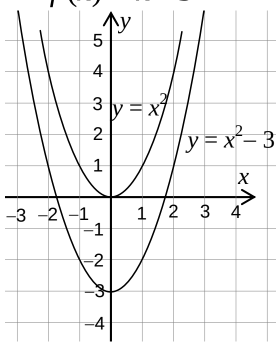


Figure 4.29

c. $f(x)=(x-5)^2$

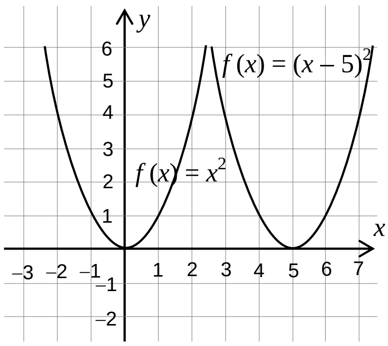


Figure 4.30

d. $f(x)=(x-2)^2+13$

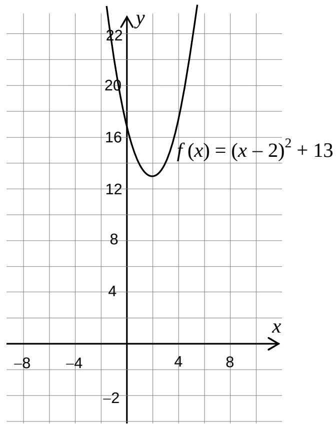


Figure 4.31

e. $f(x)=(x+1)^2-7$

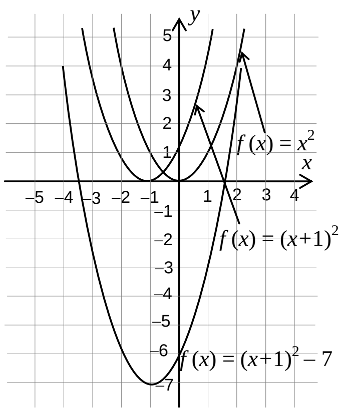


Figure 4.32

f. $f(x)=4x^2+7x+3$

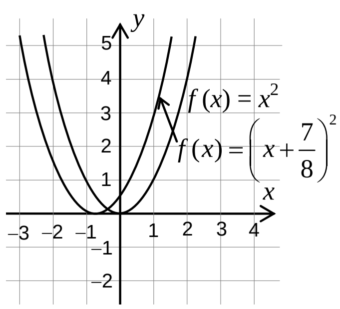


Figure 4.33

6. a. Vertex = $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) = \left(\frac{5}{2}, \frac{7}{4}\right)$ and Axis of symmetry: $x = \frac{5}{2}$

b. Vertex = $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) = (4, -3)$ and Axis of symmetry: $x = 4$

c. Vertex = $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) = (4, -13)$ and Axis of symmetry: $x = 4$

7. a. Since the leading coefficient = 1 > 0.

Thus the function has minimum value at $f\left(\frac{-b}{2a}\right) = f\left(\frac{-7}{2}\right) = -\frac{89}{4}$

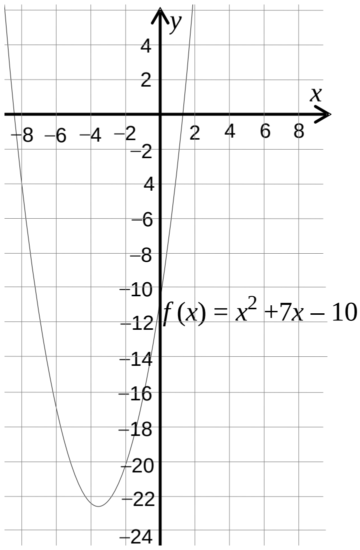


Figure 4.34

b. The function has minimum at $f\left(\frac{-b}{2a}\right) = f(-2) = -3$

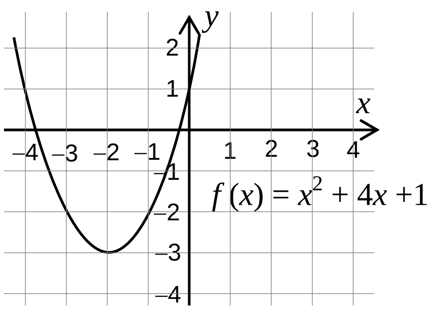


Figure 4.35

c. The function has minimum at $f\left(\frac{-b}{2a}\right) = f(1) = 1$

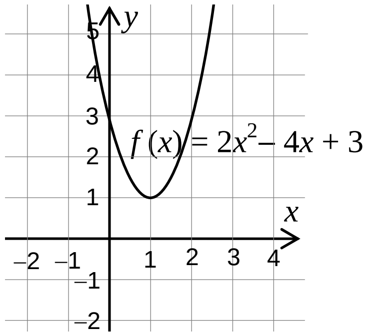


Figure 4.36

d. f has minimum value at $f\left(\frac{-b}{2a}\right) = f\left(\frac{-1}{4}\right) = \frac{15}{4}$

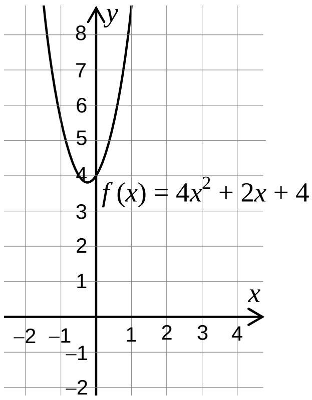


Figure 4.37

e. the function $f(x) = -x^2 - 4x$

has maximum value at $f\left(\frac{-b}{2a}\right) = f(-2) = 4$

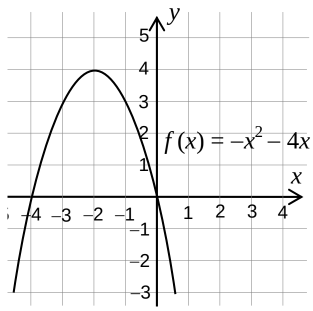


Figure 4.38

f. The function $f(x) = -6 - x^2 - 4x$ has maximum value at

$f\left(\frac{-b}{2a}\right) = f(-2) = -2$

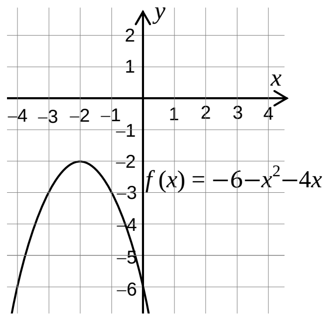


Figure 4.39

Answers to Review Exercises on Unit 4

1. Domain = {1, 2, 3, 4, 5} and Range = {2, 3, 4, 5, 6}
2. R = {(1, 4), (2, 5), (3, 6), (4, 7)}
   Range = {4, 5, 6, 7}
3. a. A×B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c), (5, a), (5, b), (5, c)}
   b. i. R_1 = {(1, a), (1, b), (1, c), (3, a), (3, b), (3, c), (5, a), (5, b), (5, c)}
   ii. R_2 = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)}
4. a. No. Because R from A to B is R = {(1, 2), (1, 4), (1, 5), (2, 2), (2, 4), (2, 5), (3, 2), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5)} where as R^* from B to A is R^* = {(2, 1), (2, 2), (2, 3), (2, 4), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)} which are not the same.
   b. Yes, because (2, 4), (2, 2), (4, 4), and (4, 2) belong to both relations from A to B and from B to A.
   c. If R is a relation from A to B, then a relation from $A \cap B$ to $A \cap B$ is always a relation from A to B and from B to A.
5. a. No. Because *x* is taller than *x* is wrong.
   b. No. Because if *x* is taller than *y*; then *y* is taller than *x* is incorrect.
   c. Yes, because if *x* is taller than *y* and *y* is taller than *z* , then *x* is taller than is correct.
6. In R = {(*x*, *y*): *y* = *x*};
   a. $(x, x) \in$ R is true because *x* = *x* is always true.
   b. If $(x, y) \in$ R then *y* = *x*. But, it is also true that *x* = *y* which means $(y, x) \in$ R
   c. If $(x, y) \in$ R and $(y, z) \in$ R then $(x, z) \in$ R is true because, *x* = *y* and *y* = *z* implies *x* = *z*.
7. a. Domain = $\mathbb{R}$ and Range = $\mathbb{R}$
   b. Domain = $\mathbb{R}$ and Range = $\{y: y \geq 0\}$
   c. Domain = {1, 2, 3} and Range = {1, 3, 5}
   d. Domain = $(-\infty, -2] \cup [2, \infty)$ and Range = $\{y: y \geq 0\}$
8. a.

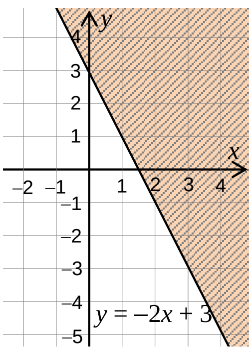


Figure 4.40

b.

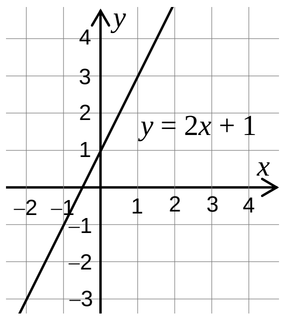


Figure 4.41

c.

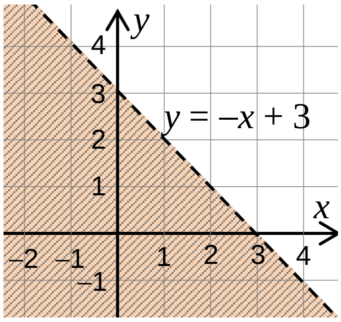


Figure 4.42

d.

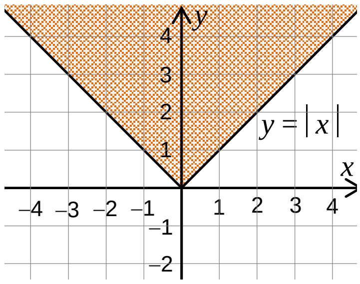


Figure 4.43

e.

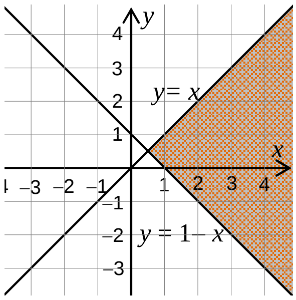


Figure 4.44

f .

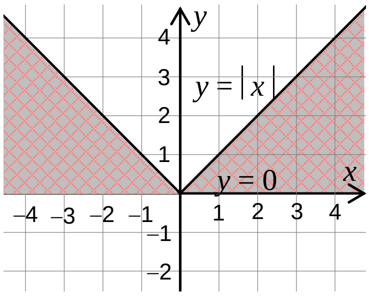


Figure 4.45

g.

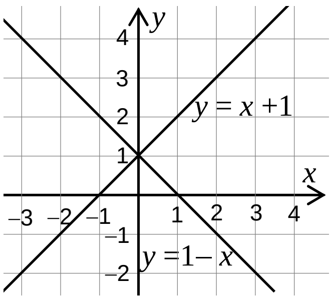


Figure 4.46

h.

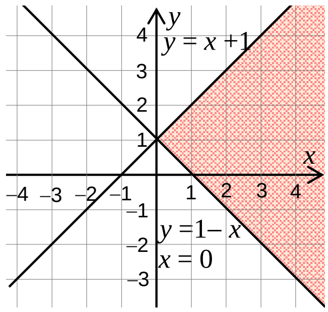


Figure 4.47

i.

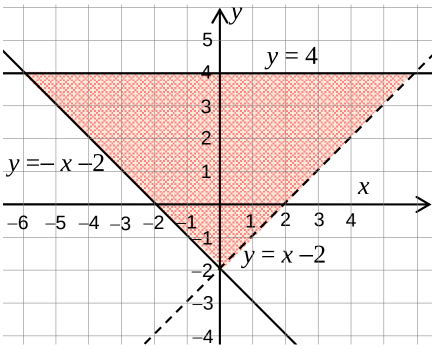


Figure 4.48

9. a. R = {(x, y): $y > x - 2$, $y > -x - 2$ and $y \leq 4$};
Domain = {x: $-6 < x < 6$} and Range = {y: $-2 < y \leq 4$}

10. a. function b. function c. Not function because a first coordinate is mapped with more than one second coordinate.

11. A×B = {(2, 2), (2, 3), (2, 4), (2, 6), (5, 2), (5, 3), (5, 4), (5, 6), (7, 2), (7, 3), (7, 4), (7, 6)} and hence it is not a function because a first coordinate is mapped with more than one second coordinate.

12. a. Domain = {1, 2, 5, 7} and Range = {2, 3, 6, 8}
b. $f(2) = 3$ and $f(5) = 6$

13. i. a. $f + g = -x - 3$ b. $f - g = 5x + 5$
c. $fg = -6x^2 - 11x - 4$ d. $\frac{f}{g} = \frac{2x+1}{-3x-4}$

ii. a. $(2f + 3g)(1) = -15$ b. $(3fg)(3) = -273$
c. $\frac{3f}{2g}(4) = \frac{-27}{32}$

iii. Domain of $\frac{f}{g} = \quad \setminus \left\{\frac{-4}{3}\right\}$

14. i. a. $fg = \frac{(x+4)(2x+4)}{2x(x+1)} = \frac{2x^2+12x+16}{2x^2+2x}; 2x^2+2x \neq 0$

b. $\frac{g}{f} = \frac{2x(2x+4)}{(x+4)(x+1)} = \frac{4x^2+8x}{x^2+5x+4}; x^2+5x+4 \neq 0$

c. $2f - \frac{f}{g} = \frac{2x+8}{2x} - \frac{x^2+5x+4}{4x^2+8x} = \frac{(4x^2+24x+32)-(x^2+5x+4)}{4x^2+8x}$

$= \frac{3x^2+19x+28}{4x^2+8x};\ 4x^2+8x \neq 0$

ii. a. Domain = $\mathbb{R} \setminus \{-1, 0\}$ b. Domain = $\mathbb{R} \setminus \{-4, -1, 0\}$
c. Domain = $\mathbb{R} \setminus \{-2, -1, 0\}$

iii. a. $(f - g)(1) = \frac{-1}{2}$ b. $\frac{g}{f}(2) = \frac{16}{9}$
c. $\left(2f - \frac{f}{g}\right)(3) = \frac{28}{15}$

15. a) $f(x) = 3x+2$

x	–2	–1	0	1	2
$f(x) = 3x+2$	–4	–1	2	5	8

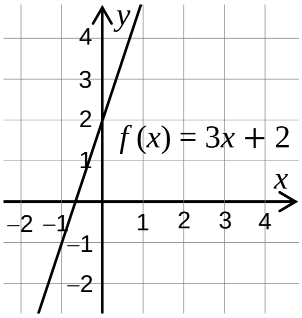


Figure 4.49

b. $x - 2y = 1$

	–2	–1	0	1	2
x–2y =1	$-\frac{3}{2}$	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$

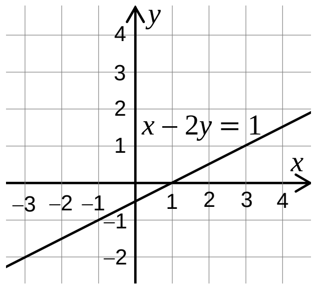


Figure 4.50

c. $f(x) = 2–7x$

X	–2	–1	0	1	2
$f(x) = 2–7x$	16	9	2	–5	–12

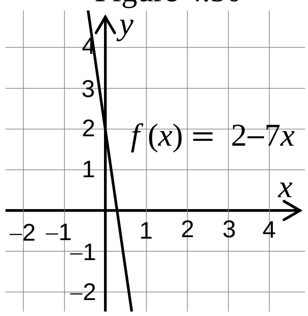


Figure 4.51

d. $f(x) = –3x^2–1$

x	–2	–1	0	1	2
$f(x) = –3x^2–1$	–13	–4	–1	–4	–13

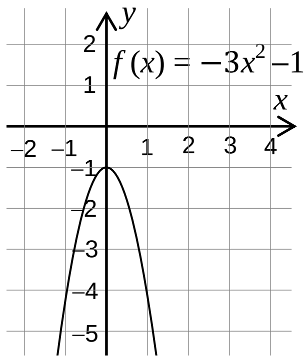


Figure 4.52

e. $f(x) = 3–2x+x^2$

x	–2	–1	0	1	2
$f(x) = 3–2x+x^2$	11	6	3	2	3

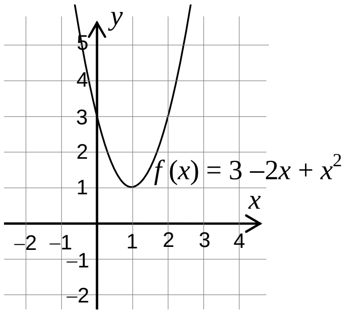


Figure 4.53

16. a. x-intercept = $\left(-\frac{7}{2},\ 0\right)$ and the y-intercept = (0, 7) and thus draw a line that passes through these points on the coordinate plane.

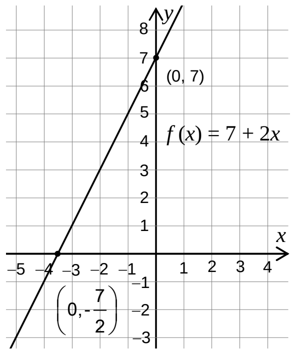


Figure 4.54

b. x-intercept $\left(\frac{5}{3}, 0\right)$ and y-intercept (0, –5)

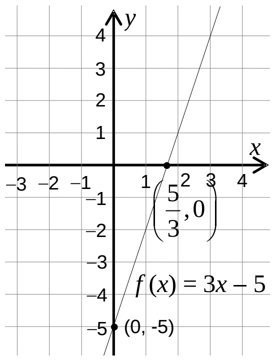


Figure 4.55

c. x-intercept $\left(\frac{4}{3}, 0\right)$ and y-intercept (0, –4)

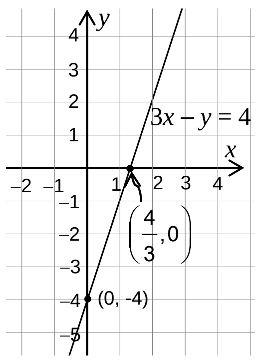


Figure 4.56

17. a.

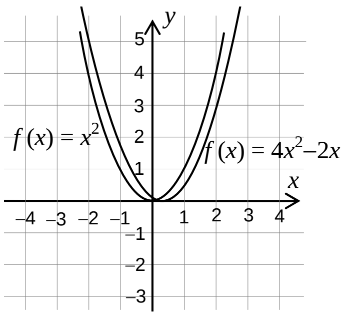


Figure 4.57

b.

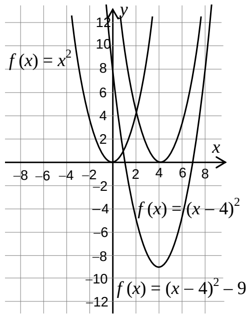


Figure 4.58

c.

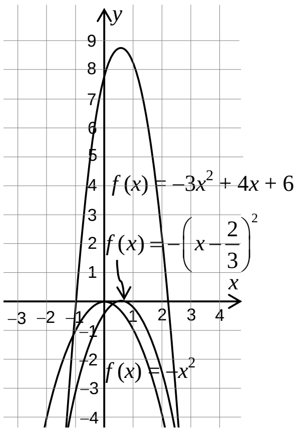


Figure 4.59

18. a. the graph turns upward because the leading coefficient 3 > 0.

b. the vertex $= \left(\frac{5}{6}, \frac{59}{12}\right)$

c. Axis of symmetry is the line $x = \frac{5}{6}$

19. a. the minimum is $f(4) = -5$

b. the minimum is $f\left(\frac{3}{2}\right) = \frac{5}{2}$

c. the minimum is $f\left(\frac{5}{6}\right) = \frac{71}{12}$

d. the maximum is $f(3) = 4$

e. the maximum is $f(1) = 0$

20. a. the range of $f(x) = \{y: y \geq 3\}$

b. the range of $f(x) = \left\{y : y \geq \frac{-41}{4}\right\}$

c. the range of $f(x) = \{y : y \leq 1\}$

d. the range of $f(x) = \{y : y \leq 5\}$

21. 21 Birr

22. a. $f(x) = 200{,}000 + 400x$

b. 232,000 Birr

UNIT 5 GEOMETRY AND MEASUREMENT

INTRODUCTION

The main task of this unit is to extend and deepen the knowledge and capability of the students about the basic concepts of geometry and measurement. The unit is subdivided into 5 topics in which each topic is in turn subdivided into subtopics. The topics dealt with in the unit are: regular polygons and their properties, extension of congruency and similarity, trigonometric ratios to solve a right-angled triangle, circles and angle properties of circles; measurements of parts of the circle such as arc length, perimeters, and areas of sectors and segments. The last topic deals with measurement, namely, areas of triangles and parallelograms, surface areas and volumes of cylinders and prisms.

Unit Outcomes

After completing the unit, students will be able to:

- *know basic concepts about regular polygons.*
- *apply postulates and theorems in order to prove congruence and similarity of triangles.*
- *construct similar figures.*
- *apply the concept of trigonometric ratio to solve problems in practical situations.*
- *know specific facts on circles.*
- *solve problems on areas of triangles and parallelograms.*

Suggested Teaching Aids in Unit 5

In addition to the student's textbook and the teacher's guide, you are advised to prepare and bring into the class the following materials whenever the topic requires.

Tools: pair of compass, ruler, protractor, scissors, thin card boards. It is also possible to use various mathematical softwares such as Geometer's Sketchpad, Mathematica, Matlab, etc whenever they are available.

Chart containing

- Set of various polygons including concave polygon like

Set of various polygons

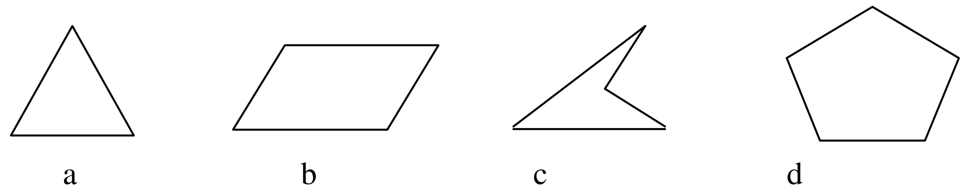


Figure 5.1

- Set of regular polygons such as

Set of regular polygons

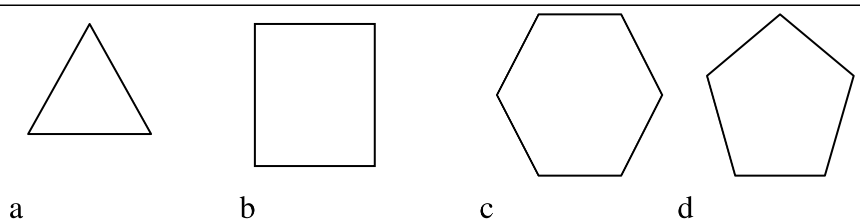


Figure 5.2

- Set of parallelograms such as

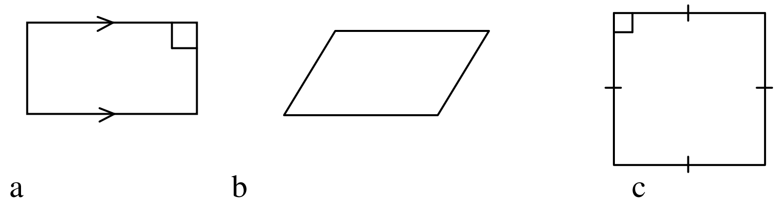


Figure 5.3

- Space (solid) figures such as prisms, cylinders, pyramids and cones
- Enlarged table of values of Trigonometric functions. (to be fixed on the wall)
- Scientific calculator

5.1 REGULAR POLYGONS

Periods allotted: 5 periods

Competencies

At the end of this subunit, students will be able to:

- *show that the sum of the measures of the interior angles of a triangle is* 180^{o}.
- *define a regular polygon and related terms.*
- *find the measure of each interior or exterior angle of a regular polygon.*
- *state properties and related terms of regular polygons.*
- *determine the lines of symmetry of regular polygons.*
- *determine the perimeter of a given regular polygon.*
- *determine the area of a given regular polygon.*

Vocabulary: Polygon, Concave polygon, Convex polygon, Regular polygon, Interior angle of a polygon, Exterior angle of a polygon, Diagonal of a polygon, Radius of a regular polygon, Apothem, Area of a polygon, Perimeter of a polygon.

Introduction

The main task of this subunit is to familiarize the students with the concepts of regular polygons and their properties. The subunit is subdivided into two main subtopics. The first subtopic deals with the measures of angles of a regular polygon. In this subtopic, you will discuss and state formulas for the measures of the central, interior and exterior angles of a regular polygon. The second subtopic deals with the properties of regular polygons. That is, you will discuss the symmetry of regular polygons and will define what is meant by radius, apothem of a regular polygon. Finally, you will derive formulas for the side, apothem, perimeter and area of a regular polygon.

Teaching Notes

This topic which deals with regular polygons encompasses various subtopics in it. Each of the sub-topics is treated with descriptive and illustrative examples. The following narrates those ideas that are useful for the delivery of this topic.

A. Revision on polygons

Draw various figures such as a triangle, rectangle, trapezium, and square. Dividing the class down in the middle, ask one side to think of what they can remember about the figures. Then ask the other side. Statements alternate between one side and another until one side loses because it cannot think of anything else. List all the answers that each side gives on the board/flipchart. Discuss the answers and lead the students to come to the concept of a polygon. You can then proceed to Activity 5.1. The purpose of Activity 5.1 is to help the students recall the several concepts they studied about polygons in previous grades. So, give the students about ten minutes to discuss in pairs and ask each side to give the answers to each question. List answers whether they are right or wrong. Finally, discuss their answers and give the correct answers. You also give definition 5.1. You must be sure that the students know the meaning of a polygon; they are also expected to tell the difference between convex polygon and concave polygon.

Answers to Activity 5.1

3. a. 180° b. 360° c. 540°

4. c and e

B. Interior and Exterior Angles of a polygon

To start the lesson, draw a polygon say, a triangle with extended sides and labelling the end points of a line segments and vertices as shown. Then ask the students to list all the interior and exterior angles by naming the angles using the three letter notation. Pick a student at random to write his/her list on the board. Pick another student at random to check and hence to agree or disagree with the list written on the board.

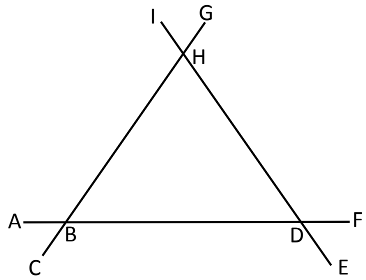


Figure 5.4

In this way, encourage and assist students to state the definition of interior and exterior angles of a polygon in their own words. Activity 5.2 will be helpful in enabling students acquire deeper knowledge about the interior and exterior angle. Group your students in pairs (or in any other convenient way) and let them discuss and do the activity in the class. In the mean time, go around and check how they do it. After a few minutes, let some groups orally present the answers to the class. Finally, give the correct answers, discuss the terms and concepts related to a polygon such as "interior angle", "exterior angle" and "diagonal". Furthermore, discuss in the class the following relationships:

a. the relationship between the number of sides of a polygon and the number of diagonals that can be drawn from one vertex.
b. the relationship between the number of sides of a polygon and the number of triangles into which the polygon is partitioned when diagonals are drawn from a vertex.
c. the relationship between the number of sides of a polygon and the sum of the measures of the interior angles of the polygon.

Explain to the students that having knowledge of these relationships will be of great help when deriving the formulas for the sum of the measures of interior angles and the exterior angles.

Answers to Activity 5.2

1. An interior angle of a polygon is that angle at a vertex on the inside of the polygon.
2. a) n -interior angles b) $n - 3$ diagonals c) $n-2$ triangles
3. Number of sides = number of vertices

   Number of interior angles = number of sides.

C. The sum of the measures of the interior angles of a polygon

To start this lesson, draw any polygon with number of sides greater than three on the board. For example, you may draw a quadrilateral. From one vertex, say A, draw the diagonal $\overline{AC}$.

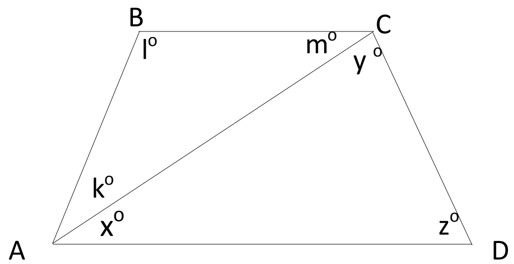


Figure 5.5

Let $k°$, $l°$, $m°$, $x°$, $y°$ and $z°$ be the degree measures of the angles as shown in the figure. Then ask the students to write $m(\angle A)+m(\angle B)+m(\angle C)+m(\angle D)$ (that is, the sum for the measures of the interior angles of the quadrilateral) in terms of $k°$, $l°$, $m°$, $x°$, $y°$ and $z°$ calling their name. Let them write their answers on the board. Then make sure that the students have noticed that the sum of the measures of the interior angles of the quadrilateral is the sum of the measures of the angles of the two triangles formed by drawing a diagonal from a vertex.

Thus, lead the students to discover "the angle sum theorem of a triangle" which in turn will be useful to find the sum of measures of angles of any polygon.

For this purpose Activity 5.3 is offered. However, for better understanding, you can guide the students to try various approaches, as outlined below, to finding the sum of measures of angles of a triangle.

Here, to teach this, we use the **Inductive Approach (experimentation)**. (The Inductive Approach is the technique of making the transition from particular facts to general knowledge about these facts).

In teaching the "**Angle sum theorem of a triangle**", the objective is to teach students that the sum of the measures of the three angles of any triangle whatever its shape or size, turns out to be always the same i.e. 180°.

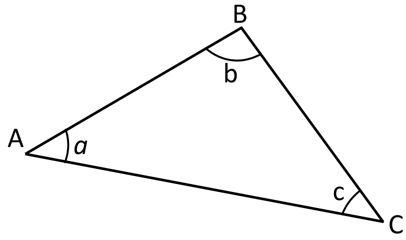


Figure 5.6

To start the teaching of the above objective, first form the students into three groups. Then, to each group, you may give one of the following activities. Activity 1 to Group 1, Activity 2 to Group 2 and Activity 3 to Group 3.

Activity I

Materials required: Ruler, Protractor

With your ruler draw a triangle, say nearly the size of your paper.

- Measure each angle with your protractor as carefully as you can.
- What is the sum of the measures of the three angles?
- Try the same again with another triangle.
- How does the sum this time compare with your first result?

Activity II

Materials required: Scissors, thin cardboard

(Note: This activity is the one given in the student text named Activity 5.3)

- Draw a fairly large triangle on a sheet of thin card board

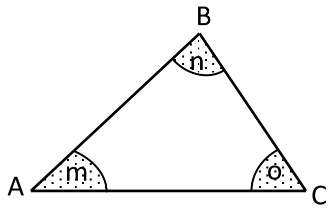


Figure 5.7

- Shade it as shown. Then cut or tear off the three "corners" as suggested by the shaded portion.
- Fit these three pieces together by laying them against the edge of your ruler (or the edge of your desk) as shown.

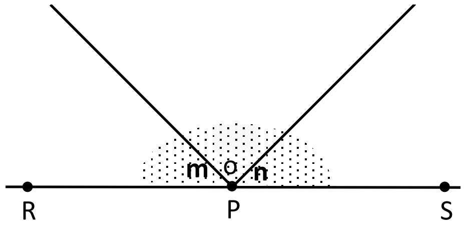


Figure 5.8

- What is the sum of the measures of all the angles having P as common vertex and lying in the half plane on one side $\overline{RS}$?
- What is the sum of the measures of the angle of ΔABC?

Activity III

Material required: Ruler, scissors

Draw ΔABC as shown in the figure below.

- Mark the midpoints P and Q of sides $\overline{AB}$ and $\overline{BC}$ respectively
- Join P and Q
- Draw $\overline{PS}$ and $\overline{QR}$ so that they are perpendicular to $\overline{PQ}$
- Carefully fold along $\overline{PS}, \overline{PQ}$ and $\overline{QR}$.

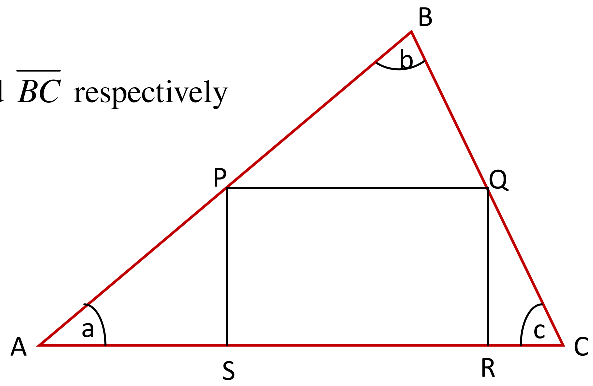


Figure 5.9

The three vertices A, B and C meet at point T on $\overline{AC}$.

What can you say about the sum of the measures of angles a, b and c? Look at the figure 5.10 below whenever necessary.

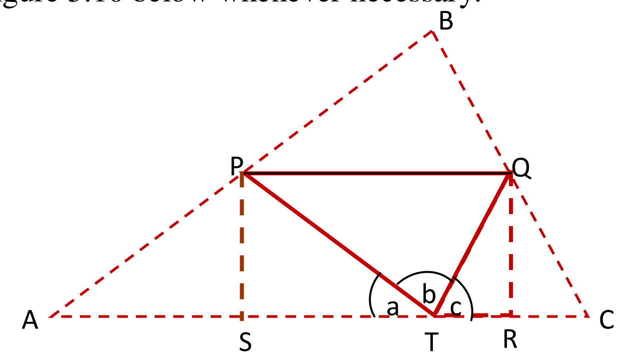


Figure 5.10

Finally, let each group present the outcome of its activity to the class. Discuss and comment on the outcome of the three activities.

In Activity I, some of the results may be 180.5°, 179.4° etc. Explain that this is due to the fact that measurements are always accompanied by errors.

The three methods used in the above activities to show the sum of the measures of angles of a triangle is an experimental exercise which is called Inductive approach.

There is also another method which is used to show that the sum of the measures of the three angles of a triangle is 180°. This method is deductive reasoning. The method does not depend on experiment. Rather, it depends on accepted properties of parallel lines, transversals and alternate angles.

The method in fact is a proof of theorem 5.1 given in the student textbook.

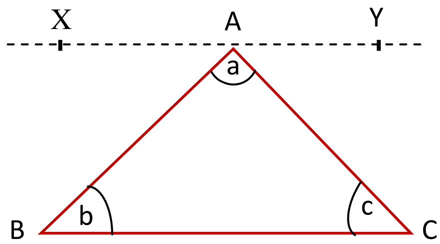


Figure 5.11

- Let ABC be a triangle with interior angles a, b and c.
- We want to show that $a + b + c = 180^{o}$.
- Draw a line $\overleftrightarrow{XY}$ through A, parallel to the opposite side $\overline{BC}$.
- $\overleftrightarrow{XY}$ // $\overline{BC}$. Hence $\overline{AC}$ and $\overline{AB}$ are transversals.
- $\angle ABC \cong \angle BAX$ (alternate angles)
- $\angle BCA \cong \angle CAY$ (alternate angles)
- Therefore, $m(\angle ABC) = m(\angle BAX) = b$ and $m(\angle BCA) = m(\angle CAY) = c$
- But, $m(\angle BAX) + m(\angle BAC) + m(\angle CAY) = a + b + c = 180^{o}$ (angles of a straight line)

$\therefore\ a + b + c = 180^{o}$

Answer to Activity 5.3

2) 180°

3) a) m(∠A) = 36°, m (∠B) = 78°; m (∠C) = 66°; = (36 + 78 + 66)° = 180° which makes a triangle.

b) m (∠A) = 124°; m (∠B) = 56°; m (∠C) = 20°; (124 + 56 + 20)° = 200°

∴ ABC do not make a triangle.

c) m (∠A) = 90°; m (∠B) = 74°; m (∠C) = 18°

⇒ (90 + 74 + 18)° = 182°

∴ ABC do not form a triangle. `

Discuss how this theorem helps to show the sum of the measures of angles of a quadrilateral and a pentagon is 360° and 540° respectively. You can also let students do Activity 5.4 to help them practice use of the angle sum theorem for various polygons. This time you can pose a question on how the number of sides and sum of the measures of angles of a triangle are related.

Answers to Activity 5.4

Number of Sides	Number of Triangles	Sum of Interior Angles
3	1	1 × 180° = 180°
4	2	2 × 180° = 360°
5	3	3 × 180° = 540°
6	4	4 × 180° = 720°
7	5	5 × 180° = 900°
8	6	6 ×180° = 1080°
.	.	.
.	.	.
.	.	.
n	$n - 2$	$(n - 2)180°$

After doing Activity 5.4 the students may have some understanding on relating the number of triangles that can be formed from a polygon and the sum of the measures of angles of a polygon. With this understanding you can state theorem 5.2 and proceed to giving Activity 5.5 so that students can draw a generalization on measures of angles of any polygon.

Answers to Activity 5.5

1. Given the heptagon *ABCDEFG*

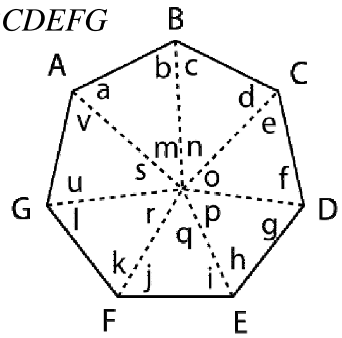


Figure 5.12

$(a+b+m)+(c+d+n)+(e+f+o)+(g+h+p)+(i+j+q)+(k+l+r)+(u+v+s)=7\times180$

but $m+n+o+p+q+r+s=360^{\circ}$ angles at a point.

$\therefore a+b+c+d+e+f+g+h+i+j+k+l+u+v=7\times180^{\circ}-360^{\circ}$

$=7\times180^{\circ}-2\times180^{\circ}=(7-2)\times180^{\circ}$

This shows that for 7 – sided polygon we have

$S=(7-2)\times180^{\circ}$

Similarly for n-sided polygon, S = $(n - 2) \times 180°$

So, S = $(n - 2) \times 180° = (7 - 2) \times 180° = 5 \times 180° = 900°$

Therefore, the sum of all interior angles of an n-sided polygon is S = $(n - 2) \times 180°$

2. a. Since for n-sided polygon we have $S = (n - 2) \times 180°$.

$S = (3 - 2) \times 180° = 180°$

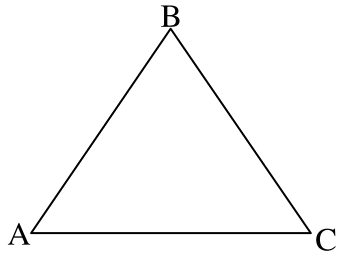


Figure 5.13

b.

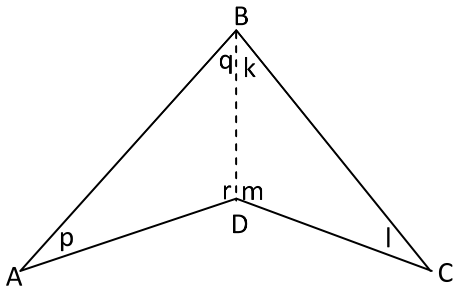


Figure 5.14

Join the point B to point D by drawing the line segment $\overline{BD}$ as shown. So the figure is divided into the two triangles ABD and CBD as shown.

Now $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = p + q + k + l + m + r$

$= (p + q + r) + (k + l + m)$

$= 180° + 180° = 360°$

Since the figure has 4 sides using the formula $S = (n - 2) \times 180°$. We see that $S = (4 - 2) \times 180° = 2 \times 180° = 360°$. This shows that the formula $S = (n - 2) \times 180°$ is valid for the given polygon.

c. To find the sum of the measures of the interior angles, we first divide the given polygons into the six triangles shown, namely, ΔABH, ΔBGH, ΔBFG, ΔBCF, ΔCDF and ΔDE

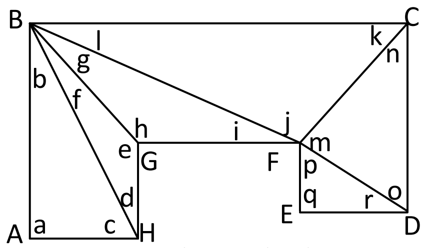


Figure 5.15

So, $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E) + m(\angle F) + m(\angle G) + m(\angle H)$

$= a + b + f + g + l + k + n + o + r + q + p + m + j + i + h + e + c + d$

$= (a + b + c) + (d + e + f) + (g + h + i) + (l + k + j) + (m + n + o) + (p + q + r)$

$= 180° + 180° + 180° + 180° + 180° + 180°$

$= 6 \times 180° = 1080°$

Since the polygon has 8 sides and since $(8 - 2) \times 180° = 1080°$, we have shown that the formula $S = (n - 2) \times 180°$ is valid for the given polygon.

d. Dividing the given polygon into 8 triangles as shown in figure 5.10, it can be shown that the formula S = $(n - 2) \times 180°$ is valid for this polygon also. Since the polygon has 10 sides the sum of all the measures of the interior angles is

$(10 - 2) \times 180° = 1440°$. On the other hand, using the triangles into which the polygon is subdivided you get that

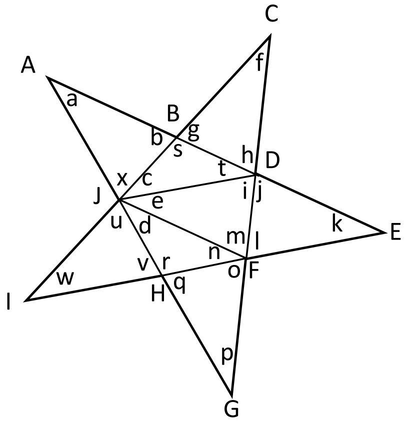


Figure 5.16

$(a + b + x) + (f + g + h) + (k + j + l) + (o + p + q) + (u + v + w) + (c + t + s) + (d + n + r) + (e + m + i) = 8 \times 180° = 1440°$

3. In a quadrilateral ABCD,

$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^o$

$80^o + 100^o + m\,(\angle C) + 110^o = 360^o$

$m(\angle C) = 360^o - 290^o$

$\therefore\ m(\angle C) = 70^o$

4. From the formula, $S = (n - 2)\times 180^o$ we have;

$S = (6 - 2) \times 180^o = 4 \times 180^o = 720^o$

So, $x^o + 2x^o + 60^o + (x + 30)^o + (x - 10)^o + (x + 40)^o = 720^o$

Solving the above yields

$x = 100°$

5. b. $e_1 + e_2 + e_3 + e_4 = 360^o$

c. $e_1 + e_2 + \ . . . \ e_n = 360^o$

6. Let e be exterior angle and a and b be the opposite interior angles.

1. $c + e = 180°$ (definition of straight line)
2. $a + b + c = 180°$ (sum of interior angles of a triangle)
3. $a + b = 180° - c = e$ (from 1 & 2)

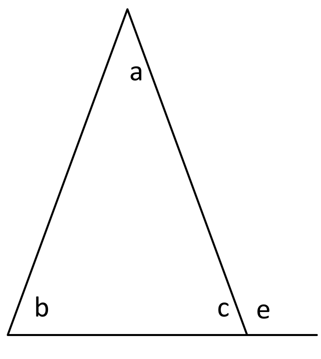


Figure 5.17

$\therefore\ a + b = e$

5.1.1 Measures of Angles of Regular Polygons

After you have finished the revision of polygons and their related terms, you start teaching the concept of a regular polygon.

To start the lesson, ask the students the following questions by writing them on the board.

1. What is a regular polygon?
2. Say all that you know about a square and an equilateral triangle.

Then, write on the board, all the answers that each student gives whatever the answer is. Following their effort, discuss with the students and sort out the answers given by the majority of the students for each question. Based on this, you comment on the answers and discuss the properties of a square and an equilateral triangle. Generalizing the outcome of this discussion, you give the definition of a regular polygon.

Combining the definition of a regular polygon and the formula $S = (n - 1) \times 180^{o}$, assist the students to arrive at the fact that each interior angle of a regular polygon is given by $\left(\frac{n-2}{n}\right)\times 180^{o}$.

Define exterior angle as "the angle formed on the outside of a polygon between a side and the extended adjacent side". Here, you should make the students be aware of the fact that the exterior angle and interior angle are measured from the same line so that they add up to 180^{o}. Therefore, the external angle is 180^{0} – (measure of interior angle) as elaborated in the student textbook. You can enrich their understanding by giving them several examples.

Assessment:

Always think of the minimum learning competencies that are expected of the students at the end of a section. Use different formal and informal assessment techniques to get feedback about their level of understanding of the topic.

Give exercise problems on calculations of interior angles and exterior angles of a polygon as well as a regular polygon for example.

- ask them to find the sum of the interior angles of a pentagon, a hexagon, a heptagon, etc.
- ask them to find the measure of each interior angle of a given regular polygon, say, an equilateral triangle, an octagon, a decagon, etc.
- ask them to find the measure of each exterior angle of a given regular polygon, say, a pentagon, a heptagon, or a square.

Oral questions, group work, class activities, quizzes, homework and assignments will help you as formative assessment techniques to collect relevant data about the performance of the students so that you can assist individual students during instruction.

Answers to Exercise 5.1

1. Because there is a vertex at which more than two sides meet.
2. Yes, it is a polygon. It has 12 sides and 12 vertices.
   The sum of the measures of all interior angels is 1800^o.
3. $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^o$
   i.e $m(\angle A) + 51^o + 75^o + 112^o = 360^o$
   $m(\angle A) + 238^o = 360^o$
   $\therefore\ m(\angle A) = 122^o$.
4. a) The measure of an interior angle of a 10-sided regular polygon is
   $$\frac{8\times180^o}{10} = 144^o$$
   b) The measure of an interior angle of a 20- sided regular polygon is
   $$\frac{18\times180^o}{20} = 162^o$$
   c) The measure of an interior angle of a 12- sided regular polygon is
   $$\frac{10\times180^o}{12} = 150^o$$
5. a) $150^o = \frac{(n-2)\times180^o}{n}$
   $\Rightarrow n\times150^o = n\times180^o - 360^o$
   $-30^o \times n = -360^o$
   $\therefore n = \frac{360^o}{30^o} = 12$
   Number of sides is 12

   b) $160^o = \frac{(n-2)\times180^o}{n}$
   $\therefore\ n\times160^o = n\ 180^o - 360^o$
   $-20^o\times n = -360^o$
   $\therefore\ n = \frac{360^o}{20^o} = 18$
   Number of sides is 18

   c. $147.\overline{27}^o = \frac{(n-2)\times180^0}{n}$
   $\Rightarrow 147.\overline{27}^o \times n = 180^o n - 360^o$
   $\Rightarrow n\times\frac{14580}{99} = 180^o n - 360^o$ (writing the repeating decimal as fraction)
   $14580\,n = 17820\,n - 35640$ (multiplying each term by 99)
   $\Rightarrow -3240n = -35640$
   $\Rightarrow n = \frac{35640}{3240} = 11$
   $\therefore$The number of sides is 11.
6. $18^o = \frac{360^o}{n}$

$\Rightarrow 18^\circ \times n = 360^\circ$

$\Rightarrow n = \dfrac{360^\circ}{18^\circ} = 20$

$\therefore$ The number of sides of the regular polygon is 20.

Thus, the measure of each interior angle = $\dfrac{(20-2)180^\circ}{20} = 162^\circ$

7. i. a. consider $20^\circ = \dfrac{360^\circ}{n}$

Then $20^\circ \times n = 360^\circ \Rightarrow n = \dfrac{360^\circ}{20^\circ} = 18$. The answer is yes.

Since $n = 18$ is a natural number, regular polygon can be drawn.

b) Consider $16^\circ = \dfrac{360^\circ}{n}$ from which $n = \dfrac{360^\circ}{16^\circ} = \dfrac{45}{2} = 22.5$

Since $n = 22.5$ is not a natural number, a regular polygon can't be drawn. Because there is no polygon with 22.5 sides.

c. $15^\circ = \dfrac{360^\circ}{n}$

$\therefore n = \dfrac{360^\circ}{15^\circ} = 24$

Since $n = 24$ is a natural number, it follows that a regular polygon with 24 sides can be drawn.

ii. The answer yes or no depends on whether the value of n from the equation $x^\circ = \dfrac{(n-2)\times 180^\circ}{n}$ is a whole number or not; where x° is the measure of an interior angle.

a. When $x^\circ = 144^\circ$, we have $144^\circ = \dfrac{(n-2)\times 180^\circ}{n}$

$\Rightarrow 144^\circ n = 180^\circ n - 360^\circ$

$144^\circ n - 180^\circ n = -360^\circ$

$-36^\circ n = -360^\circ \Rightarrow n = \dfrac{-360^\circ}{-36^\circ} = 10$

Since $n = 10$ is a whole number, a regular polygon can be drawn.

b. When $x^\circ = 140^\circ = \dfrac{(n-2)180^\circ}{n}$

$\Rightarrow 140^\circ n = 180^\circ n - 360^\circ$

$\Rightarrow -40^\circ n = -360^\circ \Rightarrow n = 9$

Since $n = 9$ is a whole number, a regular polygon can be drawn.

c. When $x^\circ = 130^\circ$, we have $130^\circ = \dfrac{(n-2)180^\circ}{n}$

$\Rightarrow 130^\circ n = 180^\circ n - 360^\circ$

$\Rightarrow -50^{o}n = -360^{o} \Rightarrow n = 7.2$

Since n is 7.2, there is no polygon whose number of sides is 7.2.

A regular polygon whose each of its interior angles measures 130^{0} does not exist. It is impossible to draw.

8. Since $\angle CBN$ and $\angle BCN$ are exterior angles of the regular octagon, we know that

$m(\angle CBN) = m(\angle BCN) = \dfrac{360^{o}}{8} = 45^{o}$

In ΔBCN we have

$m(\angle BNC) = 180^{o} - 2\times 45^{o} = 90^{o}$

$\therefore m(\angle AND) = 90^{o}$

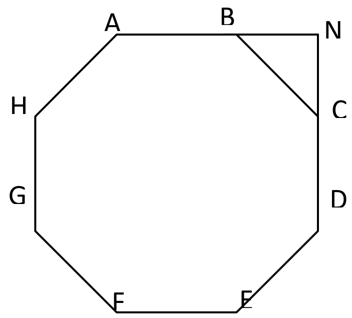


Figure 5.18

9.

(number of sides)	$m(\angle ARB)$ or $m(\angle BRC)$	$m(\angle ABR)$ or $m(\angle CBR)$	$m(\angle ABC)$
3	120^{o}	30^{o}	60^{o}
4	90^{o}	45^{o}	90^{o}
5	72^{o}	54^{o}	108^{o}
6	60^{o}	60^{o}	120^{o}
8	45^{o}	67.5^{o}	135^{o}
9	40^{o}	70^{o}	140^{o}
10	36^{o}	72^{o}	144^{o}
12	30^{o}	75^{o}	150^{o}
15	24^{o}	78^{o}	156^{o}
18	20^{o}	80^{o}	160^{o}
20	18^{o}	81^{o}	162^{o}

5.1.2 Properties of Regular Polygons

To start this lesson, take any regular polygon, say a square whose side is s and a circle of radius r and draw on the board. Then grouping the students in pairs you may give the following activity whose answer is also given below.

Activity I:

a. Does the given square have a line of symmetry? (i.e Can the square be divided exactly into two identical parts by drawing a line through it?)

b. What is the measure of the central angle?

c. Using the central angle and trigonometry, express the side in terms of the radius and the central angle.

d. Are the distances from the centre to a side equal? Justify your answer.

e. Express the distance OP in terms of the central angle and radius *r*.
f. Express the perimeter of the square in terms of the central angle and the radius *r*.
g. Express the area of the square in terms of the perimeter and the distance OP.

Answers to Activity 1 (in the Teachers Guide)

a. Yes it has. In fact it has a line of symmetry l_1, l_2 , l_3 and l_4 as shown in figure at the right.

b. ∠AOB is a central angle.

$\therefore$ m(∠AOC)=$\frac{360}{4}$=90°. Notice that the four triangles *AOB*, *BOC*, *COD* and *DOA* are congruent isosceles triangles.

c. In ΔAOP, $\text{m}(\angle\text{AOP}) = \frac{1}{2}\text{m}(\angle\text{AOB}) = \frac{1}{2}\times 90^\circ = 45^\circ$

$$\therefore \sin\left(\frac{1}{2}\text{m}(\angle\text{AOB})\right) = \frac{\frac{1}{2}s}{r}$$

i.e $\sin 45^\circ - \frac{s}{2r}$

$\therefore s = 2r \sin 45°$

d. Yes because the distances are the altitudes of the congruent isosceles triangles.

e. cosine of half of the central angle = $\frac{OP}{r}$

i.e $\cos\frac{1}{2}\text{m}(\angle AOB) = \frac{OP}{r}$

$\cos 45^\circ = \frac{OP}{r} \Rightarrow$ OP= $r\cos 45^\circ$

f. Perimeter = 45
but $s = 2 \times r \times$ sine of half the measure of each central triangle.
That is s = 2*r*sin45°
$\therefore$ Perimeter of the square = 45 = 2 × 4 × sin45° = 8sin45°

g. Area of the square = $4\times\frac{1}{2}OP\times(45) = \frac{1}{2}OP\times\text{perimeter}$

In addition to this activity, let the students do and discuss Activity 5.6 given in their text. While students are engaged in doing these activities go around the class, listen their discussion, encourage and assist them. During their discussion if there are fast learners who finish their work easily, you can let them do Question 6 of Exercise 5.2.

Finally, collect results of each pair and write them categorically on the board. Discuss the results by giving them the correct answers. Finally, give a generalization on the relation between the side *s* and radius *r* and the concepts discussed via the activities. For clarity purpose, you can state definition 5.4 and do the examples given in the student textbook. Furthermore, make sure that students understand and hence can apply the formulas stated in Theorem 5.3. To check this, assign Exercise 5.2 as class work or homework and as an assessment. Let students present their solutions to homework questions to the class.

Answers to Activity 5.6

1. a, b, c, d, e
2. a, c, d, e
3. n-lines of symmetry.

Assessment

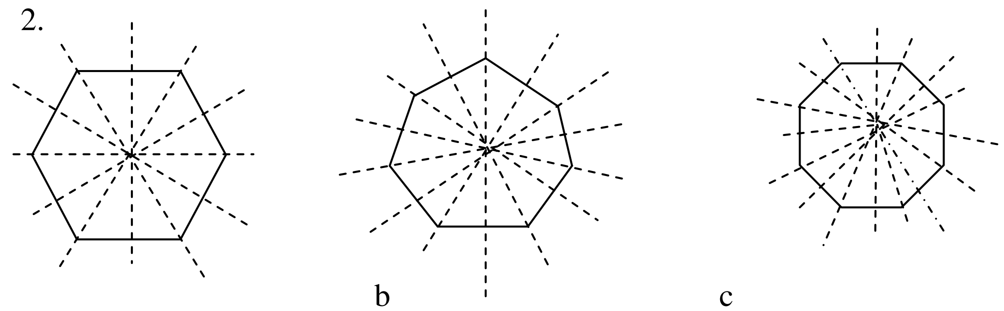
In order to assess students understanding, you can give exercise problems say those in Exercise 5.2 as a homework or assignment and correct their work.

Answers to Exercise 5.2

1. Letters of the English alphabet that have line of asymmetry are.

   A B C D E H I K M O
   U V W X.

2.



a b c

Figure 5.19

Each one has as many lines of symmetry as its sides

3. n is odd

4. a. False b. False c True d. False

5. We know that the length of a side s of an n-sided regular polygon is given by

   $S = 2r\sin\frac{180^0}{n}$ where r is the radius of the polygon.

   If $n = 6$ then $s = 2r\sin\frac{180^0}{6} = 2r\sin 30^0 = 2r \times \frac{1}{2} = r$

   Therefore, the length of a side s of a regular hexagon is equal to the radius r of the hexagon.

6. Using the formula $A=\frac{1}{2}aP=\frac{1}{2}\left(r\cos\frac{180^{o}}{4}\right)\left(2.4r\sin\frac{180^{o}}{4}\right)$

$$=\frac{1}{2}r\frac{\sqrt{2}}{2}.\ 8r\frac{\sqrt{2}}{2}\ =2r^{2}$$

7. a. b.

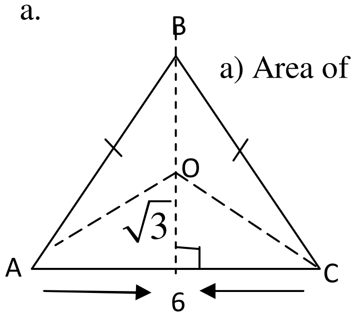


Figure 5.20

a) Area of $\Delta ABC = 3 \times \text{area}\,\Delta AOC$

$= 3\times\frac{1}{2}\sqrt{3}\times 6$

$= 9\sqrt{3}$ sq. cm

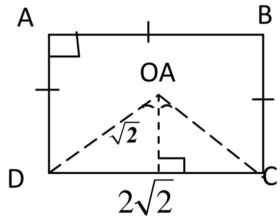


Figure 5.21

a. True

b. False, Area of $ABCD = 4$ area of ΔDOC

$=4\times\frac{1}{2}\times\sqrt{2}\times 2\sqrt{2}\ = 8$ sq.cm

8. $S= 10\sin 20^{o}$; $P = 90\sin 20^{o}$

9. $S= 6\sin 15^{o}$; $P= 72\sin 15^{o}$

10. $p=2\times 6r\sin 30^{o}\ =\ p=6r$

$\frac{P}{r}=\frac{6}{1}$ or P : r = 6 :1

11. $P = 2n\ r\sin\frac{180^{o}}{n}$

$24 = 2 \times 3 \times r\sin\frac{180^{o}}{3}$

$24 = 6r\sin 60^{o}$

$4\ =\ \frac{\sqrt{3}}{2}r\ ,\ r=\frac{8\sqrt{3}}{3}$

12. $P = 2n\ r\sin\frac{180^{o}}{n}$

$32 = 2 \times 4 \times r\sin\frac{180^{0}}{4}$

$4 = \frac{\sqrt{2}}{2}r$

$r = 4\sqrt{2}$ units

13. $P=2nr\sin\frac{180^{o}}{n}$

$48 = 2 \times 6 \times r\sin\frac{180^{o}}{6}$

$48 = 12r\sin 30^{o} = 48 = 6r \Rightarrow r = 8$ units.

14. a. $P = 2 \times 3 \times 12\sin 60^{o}$ b. $P = 2 \times 7 \times 12\sin\frac{180^{o}}{7}$

$P = 6 \times 12 \times \frac{\sqrt{3}}{2}$ $= 168 \sin \frac{180^o}{7}$ units

$P = 36\sqrt{3}$ units

c. $P = 2 \times 10 \times 12 \sin 18^o = 240 \sin 18^o$ units

5.2 FURTHER ON CONGRUENCY AND SIMILARITY

Periods allotted: 13 periods

Competences

At the end of this subunit, students will be able to:

- *use the postulates and theorem on congruent triangle in solving related problems.*
- *define similar plane figures and similar solid figures.*
- *apply the SSS, SAS and AA similarity theorems to prove similarity of triangles.*
- *discover the relationship between the perimeters of similar plane figures and use this relationship to solve related problems.*
- *discover the relationship between the areas of similar plane figures and use this relationship to solve related problems.*
- *enlarge and reduce plane figures by a given scale factor.*
- *solve real life problems using the concepts of similarity and congruency.*

Vocabulary : Congruency, Similarity, Enlargement, centre of enlargement, scale factor

Introduction

This sub-unit deals with further study on congruency and similarity. It is subdivided into six subtopics. The first subtopic that is 5.2.1 deals with the concept of congruency of triangles (congruency theorems). It also deals with the application of congruency theorem to solve problems. In subsection 5.2.2 definitions of similar figures and the concept of enlargement are treated. In the third subsection, 5.2.3, just like congruency of triangles, the similarity of triangles and tests for similarity of triangles are discussed. In the fourth subsection, 5.2.4, the ratio of perimeters and the ratios of areas of similar polygons are treated. In 5.2.5, the concept of drawing plane figures similar to a given one by multiplying each side by a given scale factor are treated. Finally, in subsection 5.2.6, application of congruency and similarity in solving problems is considered.

Teaching Notes

The discussion in each of the sub-topics assumes students to have some background on some of the concepts such as congruence, similarity, etc. Here, we expect the students to actively participate in the discussion and much of the delivery will base their discussion. The ways the subtopic may be treated are narrated as follows.

5.2.1 Congruency of Triangles

To start the lesson, first remind the students that they have studied congruency in their lower grade mathematics. Ask them what is meant by congruent figures. Encourage and assist them to answer the question.

Remind them that in their lower grade mathematics, they studied that figures having the same size and shape are called congruent figures. In other words, two figures are congruent, if they are exact copies of each other.

Thus, to make them recall this fact, you may start the lesson by giving Group Work 5.1 from their text which is supposed to be a revision activity. One of the activities in Group Work 5.1 involves the tracing of a given figure so as to explain to students that tracing is made by placing a thin transparent sheet of paper over a figure and moving a pencil over every line and curve of the figure so that an exact copy of the figure is made on the thin paper. Form groups and let each group perform the group work according to the steps listed in the student text.

Finally, let a group demonstrate how it worked out the group work on the board. Based on this, give corrections and your comments and lead the students to reach the conclusion that "Two plane figures are congruent, if they are exact copies of each other". What we mean by "exact copy" requires measuring sides and angles. To let them grasp these ideas begin with congruence of triangles.

To do so, revise the definition for congruence of two triangles. Given two triangles say ΔABC and ΔDEF, we say that ΔABC is congruent to ΔDEF (and we write $\Delta ABC \equiv \Delta DEF$), if and only if the three sides and the three angles of ΔABC are correspondingly congruent to the three sides and the three angles of ΔDEF i.e.

$$\Delta ABC \cong \Delta DEF \Leftrightarrow \begin{matrix} \overline{AB} \cong \overline{DE} & & \angle A \cong \angle D \\ \overline{BC} \cong \overline{EF} & \text{and} & \angle B \cong \angle E \\ \overline{AC} \cong \overline{DF} & & \angle C \cong \angle F \end{matrix}$$

Encourage and motivate the students to revise the tests for congruency of triangles. That is, SSS, SAS, ASA and RHS. Tests for congruence help students to realize that, in determining the congruency of triangles, they don't need to know all the six corresponding parts are congruent; rather it is enough to use the tests that involve only the three corresponding parts; i.e. SSS, SAS, ASA and RHS.

Make sure that students understand and hence can apply tests for congruency of triangles. Discuss in the class the three illustrative examples given in the student text on page 192. Assign Exercise 5.3 as class work, and home work. To check the students' level of understanding, make some students present and explain the result of their class work and home work to the class and have other group of students approve or reject the work of the presenters.

Answers to Group Work 5.1

1. a and k; b and i; c and *l*, d and j; e and o; f and m; g and n; h and p;

appear to be congruent.

Assessment

Assessing students understanding on congruence of triangles and polygons is essential to pass into discussing similarity. For this purpose you can give test/quiz that include possibly definitions of congruencies and all tests for congruencies.

Answers to Exercise 5.3

1. a. i. $\Delta PQR \cong \Delta CAB$ (by AAS)
   ii. $\Delta PQR \cong \Delta XYZ$ (by AAS)
   iii. $\Delta PQR \cong \Delta NML$ (by ASA).
2. i. $\Delta PQR \cong \Delta ZXY$ (by SAS) ii. $\Delta PQR \cong \Delta CAB$ (by SSS)
3. a. $\Delta ACB \cong \Delta PKQ$ (by SAS) b. $\Delta ACB \cong \Delta NRT$ (by SSS)
   c. The two triangles are not congruent.
4. a Consider ΔABM and ΔACM
   i. $\overline{AB} \equiv \overline{AC}$ (Given)
   ii. $\overline{AM} \cong \overline{AM}$ (Common)
   iii. $\overline{BM} \cong \overline{CM}$ (Given that M is midpoint of $\overline{BC}$)
   iv. $\Delta ABM \cong \Delta ACM$ (by SSS)

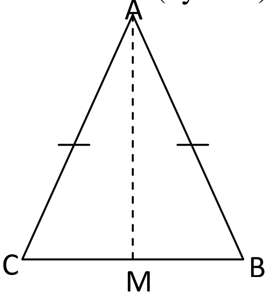


Figure 5.22

So $\angle ABM \cong \angle ACM$ (corresponding angles of congruent triangles).
$\therefore \angle ABC \cong \angle ACB$

b. Consider ΔFAB and ΔBCD
   i. $\overline{FA} \cong \overline{BC}$ (Given)
   ii. $\overline{AB} \cong \overline{CD}$ (Given)
   iii. $\angle FAB \cong \angle BCD$ (Given)
   iv. $\Delta FAB \cong \Delta BCD$ (by SAS)

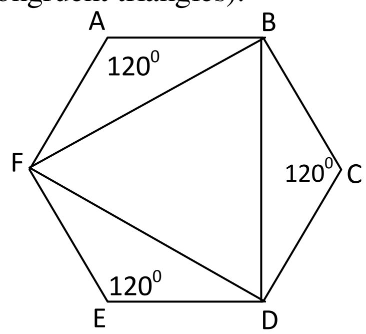


Figure 5.23

Therefore, $\overline{BD} \cong \overline{BF}$ (Corresponding sides of congruent triangles)……… (*)
Similarly, consider ΔFAB and ΔDEF. Following the same procedure as above, we get that $\Delta FAB \cong \Delta DEF$ (by SAS)
$\therefore \overline{BF} \cong \overline{DF}$ (Corresponding sides of congruent triangles)……..(**)
Combining (*) and (**) we obtain $\overline{DF} \cong \overline{BF} \cong \overline{BD}$
And this shows that ΔBDF is an equilateral.

c. Consider ΔRSQ and ΔQTR
   i. $\overline{RS} \cong \overline{QT}$ (Given)

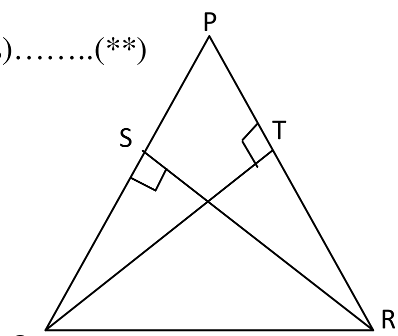


Figure 5.24

ii. $\angle RSQ \cong \angle QTR$ (right angles)

iii. $\overline{QR} \cong \overline{QR}$ (Common)

iv. $\Delta RSQ \cong \Delta QTR$. (By RHS)

So $\angle Q \cong \angle R$ (Corresponding angles of congruent triangles). But Since $\angle SQR$ and $\angle TRQ$ are two angles of ΔPQR, it follows that ΔPQR is an isosceles.

$\therefore \overline{PQ} \cong \overline{PR}$

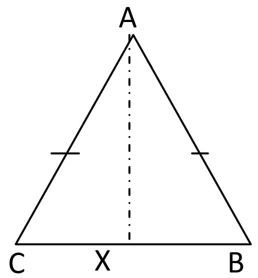


Figure 5.25

d. In Δs ACX and ABX

i. $\overline{AC} \cong \overline{AB}$ (Given)

ii. $\angle BAX \cong \angle CAX$ ($\overline{AX}$ bisects $\angle BAC$)

iii. $\overline{AX} \cong \overline{AX}$ (common side)

iv. $\Delta ACX \cong \Delta ABX$ (by SAS)

$\therefore \overline{CX} \cong \overline{BX}$ (Corresponding sides of congruent triangles).

This proves that X is the mid-point of $\overline{BC}$.

e. In $\Delta_s ADC$ and CBA

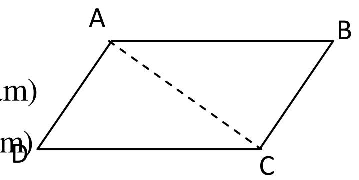


Figure 5.26

i. $\overline{AD} \cong \overline{CB}$ (opposite sides of a parallelogram)

ii. $\overline{DC} \cong \overline{AB}$ (opposite sides of a parallelogram)

iii. $\overline{AC} \cong \overline{AC}$ (common)

iv. $\Delta ADC \cong \Delta CBA$ (by SSS)

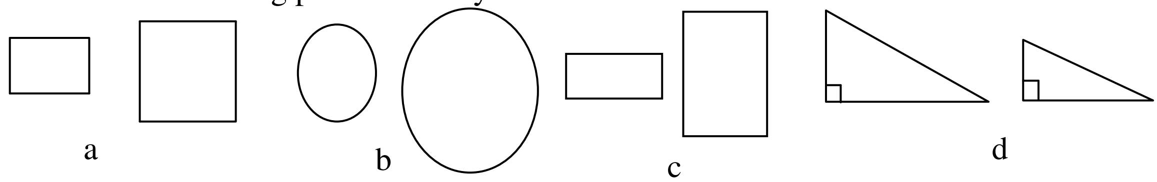
$\therefore \angle ABC \equiv \angle ADC$ (corresponding angles of congruent triangles)

5.2.2 Definition of Similar Figures

You may start the lesson by revising congruency of figures in that two figures are congruent if they have the same size and shape. Then you may ask the students to tell what happens if the figures do have the same shape but different size? Following their reply you can start discussion by defining similar figures as "similar figures are identical in shape but not necessarily in size". In other words, in a pair of similar figures, one shape is an enlargement of the other. Remind the students about the concept of enlargement that they studied in Grade 8 mathematics.

Explain the concept of enlargement which states that, when a figure is enlarged, the image is a figure similar to the object. After giving this preliminary information about the lesson, you proceed with your teaching by making the students participate in the teaching-learning process. In this particular lesson, you may use one of the active learning methods known as Drill Partners. You group the students and let them drill each other by first asking them questions of the following types.

Which of the following pairs are always similar?



Any two squares any two circles any two rectangles any two right triangles

Figure 5.27 Figure 5.28

Next to sharpen student intuition, let them do and discuss Group Work 5.2 given in the student text. Encourage each group to come up with answers and reasons.

Encourage and assist the students to come to the conclusion; "for any pair of similar figures, corresponding sides have the same ratio and corresponding angles are congruent" as given in the student textbook.

Answers to Group work 5.2

1. i. yes
   ii. All figures are similar
   iii. ∠B and ∠J are congruent
   iv. ∠G, ∠K and ∠O are congruent to ∠C
   v. ∠A ≡ ∠E ≡ ∠I ≡ ∠M;
   ∠H, ∠L and ∠P are congruent to ∠D ≡∠B ≡ ∠F ≡ ∠J ≡ ∠N;
   vi. Each pair of corresponding angles of two or more similar polygons work congruent.
2. $\frac{AB}{FG} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{EA}{JF} = 2$
3. Yes. Because the ratio of the corresponding dimensions is equal.
4. a. Yes. sides are proportional.
   b. Yes. there are three pairs of congruent angles.

Let some group present the answers of Group Work. Then let each group compare their answers and discuss any they have not answered similarly. You can also enrich their understanding by doing the examples given in the student textbook.

Assessment

You can assess students understanding while they do the group work. You can also give them Exercise 5.4 as homework and let them present their work.

Answers to Exercises 5.4

1. a. i. *A* and *D* are similar. ii. *B*, *E*, and *J* are similar.
   iii. *H* and *I* are similar. v. *F* and *L* are similar.
   vi. *C*, *G* and *K* are similar.
   b. Because, (i) their corresponding angles are congruent.
   (ii) their corresponding sides have the same ratio.

2. Because the ratios of any corresponding parts of the circle are the same. (These include radius, diameter, or circumference).
3. a. The two polygons namely, triangles *BED* and *BAC* are similar because
   i. $\angle E \equiv \angle A$, $\angle D \cong \angle C$ and $\angle DBE \equiv \angle CBA$
   ii. $\frac{DE}{AC} = \frac{DB}{BC} = \frac{BE}{BA}$

   b. $PQRS \cong LMNO$
   Because i. corresponding angles are congruent and
   ii. their corresponding sides have the same ratio.

   c. The two polygons are not similar because, even though their corresponding angles are congruent, their corresponding sides do not have the same ratio. $\left(\frac{4}{5} \neq \frac{5}{7}\right)$

5.2.3 Theorems on Similarity of Triangles

Before you discuss the theorems on similarity; first you may remind the students the definitions of similarity of triangles. Explain that two triangles are similar if their corresponding sides are proportional and their corresponding angles are congruent. That is, $\Delta ABC \sim \Delta DEF$ if and only if

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \text{ and } \angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$$

However, explain to the students that as with congruent triangle, there are special properties (tests) to use as shortcuts showing that two triangles are similar. In this lesson, these tests are known as theorems on similarity.

You may start the lesson by stating and discussing the SSS, SAS and AA similarity theorems one by one. You have to make sure that students understand and hence can apply the theorems to solving problem and solving similar triangle. For this purpose, group the students and ask questions of the following types.

1. In the adjacent figure of $\overline{HT} // \overline{AB}$, answer the following questions.
   a. Is ΔFHT ~ ΔFAB? Justify your answer.
   b. For each of the following fill in the blank space and justify.

   i. $\frac{FA}{FA} =$ _________ ii. $\frac{FA}{HA} =$ ________

   iii. $\frac{FH}{HA} =$ ________

   v. $\frac{FT}{FH} =$ ________ iv. $\frac{TB}{FT} =$ ________

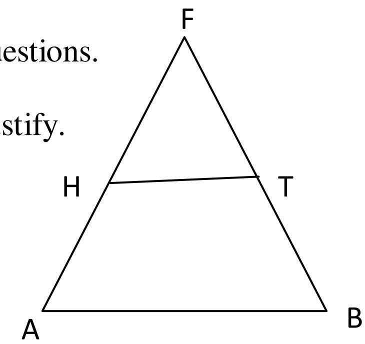


Figure 5.29

2. Given a right-angle triangle ABC with right angle at B if a perpendicular is drawn from B as shown. Prove that
   a. $\Delta ADB \sim \Delta ABC$
   b. $\Delta CDB \sim \Delta CBA$

c. $\Delta ADB \sim \Delta BDC$

While students are working on the answers, go round each group and finally let one group present its work. Discuss and give them feedback.

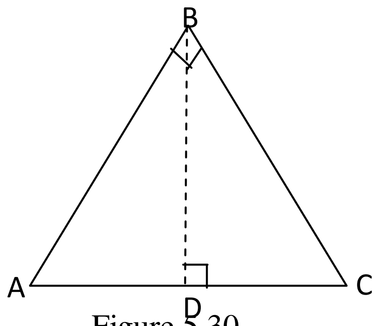


Figure 5.30

This moment, you can let some of the group present their work to the whole class and you facilitate their discussion. After making them warmed up with the questions of the above type, you can give them Group Work 5.3 for practicing and making generalizations on ratios of sides and congruence of angles.

Answers to Group work 5.3

1. It may not work when the numbers of sides are greater than three.
2. Yes

After they do the group work, you need to summarize the similarity theorems (tests) and enrich each with examples.

Assessment

To assess students understanding there could be various ways that you may follow. However, to suggest some, you can give them assignments to prove the tests. You can also give them exercise problems, say Exercise 5.5, of verifying similarity by applying the similarity tests.

Answers to Exercise 5.5

1. a. False: similar triangles may have sides with different lengths.
   b. True: by AA – similarity.
   c. False: the triangles may have sides with different lengths.
   d. True: by AA – similarity.
2. a. $\Delta FGH \sim \Delta IJK$ (SSS - similarity)
   b. $\Delta LNM \sim \Delta\ OQP$ (SAS-similarity)
3. a. Give $\Delta ABC \sim \Delta DEF$ $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ that is $\frac{3}{6} = \frac{AC}{10} = \frac{4}{EF}$

   From this, we find that $AC = 5$ and $EF = 8$

   c. Give $\Delta\ IHG \sim \Delta IFE$, $\frac{IH}{IF} = \frac{IG}{IE} = \frac{GH}{EF}$. That is $\frac{IH}{IF} = \frac{6}{6+GE} = \frac{3}{5}$, from this, $GE = 4$. Therefore, $IE = IG + GE = 6 + 4 = 10$

Using Pythagoras theorem in $\Delta\ IHG$, we have $IH = \sqrt{6^2\ -\ 3^2} = 3\sqrt{3}$

$$\text{Now, } \frac{IH}{IF} = \frac{GH}{EF} \quad \text{i.e } \frac{IH}{IH + HF} = \frac{3}{5}$$

$$\Rightarrow \frac{3\sqrt{3}}{3\sqrt{3} + HF} = \frac{3}{5}$$

$$\therefore HF = 2\sqrt{3}$$

$$\text{Thus, } IF = IH + HF = 3\sqrt{3} + 2\sqrt{3} = 5\sqrt{3}$$

The required sides are $EI = 10$, $HI = 3\sqrt{3}$, $FI = 5\sqrt{3}$

4. a. i. $\angle C = \angle C$ (common)

   ii. $\angle ADC \equiv \angle BEC$ (right angles)

   $\therefore \Delta ADC \sim \Delta BEC$ (By AA-similarity)

   b. i. $\angle FEA \equiv \angle FDB$ (right angles)

   ii. $\angle EFA \equiv \angle DFB$ (vertical opposite angles)

   $\therefore \Delta AFE \sim \Delta BFD$ (By AA-similarity)

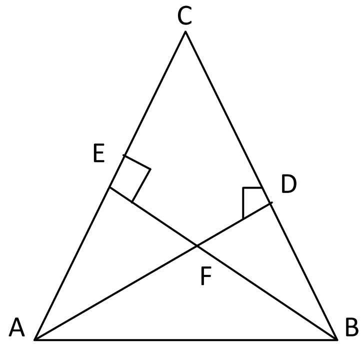


Figure 5.31

5. In the figure $m(\angle CGF) + 90^\circ + m(\angle CFG) = 180^\circ$ (angle sum of a triangle)

   i.e $m(\angle CFG) = 90^\circ - m(\angle CGF)$ ------------- (i)

   Also, $m(\angle CGF) + 90^\circ + m(\angle AGD) = 180^\circ$ (angles on a straight line)

   i.e. $m(\angle AGD) = 90^\circ - m(\angle CGF)$ ---------------- (ii)

   from (i) and (ii) we have $m(\angle CFG) = m(\angle AGD)$

   $\therefore \angle CFG \equiv \angle AGD$

   $\therefore$ In ΔADG and ΔGFC

   We have (i) $\angle AGD \equiv \angle CFG$ (shown above)

   (ii) $\angle ADG \equiv \angle GCF$ (right angles)

   $\therefore \Delta ADG \sim \Delta GCF$ (by AA- similarity). Thus, (b) follows.

   By similar arguments, we can show that:

$\Delta BEF \sim \Delta FCG$ (by AA- similarity) from this, $\angle BFE \equiv \angle FGC$ and

$\angle FBE \equiv \angle CF \equiv \angle AGD$

$\Rightarrow \Delta ADG \sim \Delta FEB$ (AA. Similarity). Thus, (a) follows.

5.2.4 Theorems on Similar Plane Figures

So far students have discussed similarity theorems that we can use to check whether two plane figures are similar or not (particularly triangles). Now, we are going to see properties that prevail similar figures in forms of ratios of sides, areas, and perimeters and possible relationship between each.

Ratio of perimeters and ratio of areas of similar plane figures

Before you start the teaching of the concept of the lesson, first you group the students and ask them to recall on their own for a couple of minutes and say all that they can remember about similar figures. Then ask them to discuss their response with students in their respective groups. After a few minutes, ask some of the groups to share their common answers with the whole class.

Having done this, you may start the lesson by asking the students to do Activity 5.7. Monitor and assist each group to do the Activity according to the steps enlisted in the student text. Let, at least a group, present the workout of the activity. Finally, let another group make a generalization about the outcome of the activity. Then discuss and give the correct generalization.

Make sure that the students have understood Theorem 5.8 and Theorem 5.9. Assign all problems given in Exercise 5.6 as class work and homework.

As an assessment, make some students from the different groups elaborate the answers on the blackboard.

Answers to Activity 5.7

a. Given the rectangles ABCD and PQRS

$$\frac{AB}{PQ}=\frac{9}{12},\ \frac{BC}{QR}=\frac{6}{8},\ \frac{DC}{SR}=\frac{9}{12} \text{ and } \frac{AD}{PS}=\frac{6}{8}$$

$$\Rightarrow \frac{AB}{PQ}=\frac{BC}{QR}=\frac{DC}{SR}=\frac{AD}{PS}=\frac{3}{4}$$

and $\angle A \equiv \angle P \equiv \angle B \equiv \angle Q \equiv \angle C \equiv \angle R \equiv \angle D \equiv \angle S$

Hence, the two rectangles are similar.

b. $\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CD}{RS}=\frac{DA}{SP}=\frac{3}{4}$

c. 1) i. Perimeter of the smaller rectangle is:

$P_1 = AB + BC + DC + AD$

$P_1 = (9 + 6 + 9 + 6)$ cm

$P_1 = 30$cm

ii. Area of the smaller rectangle is:

$A_1 = l \times w = 9\text{cm} \times 6\text{cm} = 54 \text{ cm}^2$

2) i. Perimeter of the larger rectangle is:

$P_2 = PQ + QR + RS + PS$

$P_2 = (12 + 8 + 12 + 8)$ cm

$P_2 = 40$ cm

ii. Area of the larger rectangle is:

$A_2 = l \times \text{w}$

$A_2 = 12 \text{ cm} \times 8 \text{ cm}$

$A_2 = 96\text{cm}^2$

d. $\frac{P_1}{P_2}=\frac{30}{40}=\frac{3}{4}$

e. $\frac{A_1}{A_2}=\frac{54\text{cm}^2}{96\text{cm}^2}=\frac{9}{16}=\left(\frac{3}{4}\right)^2$

f. In similar polygons, the ratio of the corresponding sides is equal to the ratio of their perimeters.

g. The ratio of the areas is the square of the ratio of the corresponding sides.

Assessment

You can assess students by giving them Exercise 5.6 so that they can apply the ideas of the theorems they discussed and then you check their work.

Answers to Exercise 5.6

1. Since $ABCD \sim EFGH$, we have

$$\frac{\text{Perimeter of } ABCD}{\text{Perimeter of } EFGH} = \frac{AB}{EF}$$

that is $\frac{40}{\text{Perimeter of } EFGH} = \frac{15}{18}$

$\therefore \text{Perimeter of } EFGH = \frac{18 \times 40}{15} = 48\text{cm}$

2. a. The ratio of their perimeters is $\frac{2}{5}$ or $\frac{5}{2}$

   b. The ratio of their areas is $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$ or $\frac{25}{4}$

3. 1 : 9

4. a. The ratio of their perimeters is $\sqrt{\frac{144}{81}} = \frac{12}{9} = \frac{4}{3}$ or $\frac{3}{4}$

   b. Let x unit be the corresponding side of the second triangle. Then,

   $\frac{6}{x} = \frac{4}{3}$

   $\Rightarrow x = \frac{3 \times 6}{4} = \frac{9}{2} = 4.5 \text{ units}$

   The corresponding side of the second triangle is 4.5 units.

5. Let the lengths of the corresponding sides of the other polygon be a, b, c, d and e.

$\therefore \frac{5+7+8+11+19}{75} = \frac{5}{a} = \frac{7}{b} = \frac{8}{c} = \frac{11}{d} = \frac{19}{e}$

i.e, $\frac{50}{75} = \frac{5}{a} = \frac{7}{b} = \frac{8}{c} = \frac{11}{d} = \frac{19}{e}$

Therefore, the lengths of the sides of the larger polygon are 7.5, 10.5, 12, 16.5, and 28.5 units.

5.2.5 Construction of Similar Figures

Once students discussed similarity, it is of interest to let them construct similar figures. One way to construct similar figures is to enlarge the figure by using some scale. The following discussion remits on enlargement.

Enlargement

You can start this lesson by encouraging and assisting the students to do the class discussion given in the student textbook and to come up with answers and reasons. After this, you can work with your students by defining the term "centre of enlargement" and letting them do Group work 5.4 and Example 2 of the student textbook. Questions 1 and 2 of Group Work 5.4 are practical problems and let students discuss them in the class. To this end, you can give Exercise 5.7 as a class work or homework. If mathematical softwares are available, it will be essential to help the students practice on identifying the centre of enlargement and do some of the constructions with the aid of those softwares.

Answers to Group work 5.4

3. $\Delta ABC \sim \Delta A'B'C'$
4. Yes, $\frac{OA'}{OA} = \frac{A'B'}{AB} = 2$
5. Their shape and the measure of the corresponding angles have not changed.

Assessment

If students can correctly enlarge figures then it will be a basis for understanding the concept of similarity. To assess student ability of enlarging figures, you can give them assignments consisting of different figures and ask them to enlarge and reduce the figures to construct similar figures.

Answers to Exercise 5.7

1. By drawing a line through A and A', and B and B'. The point of intersection is the centre of enlargement.

a. ii. 2 b. ii. $\frac{5}{3}$ c. ii. 2

5.2.6 Real Life Problems Using Congruency and Similarity

Once the students have discussed congruency and similarity of plane figures, they need to apply these concepts on real life problems. Some examples are discussed in the student textbook.

In this sub-section, you can encourage and motivate the students to relate the applications of congruency and similarity of geometric properties to their real life. You can explain to the students how carpenters use the result of Example 1 given in the student textbook. In addition to this, you may give more examples which are related to real life problems such a photograph presented with different size, prototype of buildings, etc.

Assessment

Use different formal and informal assessment techniques to get feedback about their level of understanding of the topic.

Ask them to apply the congruence or similarity on their daily life problems. Example you can let them measure dimensions of some given models that differ in size and get to conclude whether they are similar, congruent or not. You can also let them measure the sides of a wall in their classroom and the dimensions of the board and discuss on the ratio of their sides, perimeters, areas etc.

Answers to Exercise 5.8

1. $\frac{1}{30} = \frac{x}{120}$ implying $x = 4$. Therefore it takes him to cut the grass in a square field of 120m.
2. $\frac{20}{15} = \frac{h}{120}$ Implying the height of the cliff is $h = 160$m.
3. Assuming both the pole and the tree are vertical to the ground, we have equality of their ratios as $\frac{h}{10} = \frac{30}{12}$. Therefore, the height of the tree is 25m.

5.3 FURTHER ON TRIGONOMETRY

Period allotted: 7 periods

Competencies

At the end of this subunit, students will be able to:

- *describe radian measure of an angle.*
- *convert radian measure to degree measure and vice versa.*
- *use the trigonometric ratios to solve right angled triangles.*
- *find the trigonometric values of angles from trigonometric table.*
- *find the angle whose trigonometric value is given (using trigonometric table).*
- *determine the trigonometric values for obtuse angles using trigonometric table.*

Materials Required: Calculator, Compass, Ruler, Trigonometric table.

Vocabulary: Angle, Degree, Radian, Adjacent side, Opposite side, Hypotenuse, Special angles, Trigonometric values, Pythagoras theorem.

Introduction

We assume that the only previous knowledge of students about trigonometry is a brief contact with sine, cosine and tangent ratios in right-angled triangles and probably most of that has been forgotten. To proceed with the teaching of this sub-unit, we need to recall those earlier experiences, that is, the trigonometric ratios. However, before we do that, we will first introduce an entirely new view point related in measuring an angle that is, the **Radian measure of an angle**. Once this is introduced, the conversion of radian measure to degree measure and vice-versa will be dealt with. Moreover, by using trigonometric ratios, we do some revision of applications in solving right-angled triangles during which time you are supposed to discuss how to use trigonometric tables.

Teaching Notes

Trigonometry is one of the most applied part in mathematics which is useful in physics, Astronomy, Engineering, etc. In this subunit measures in degrees and radians and their corresponding trigonometric ratios will be discussed.

5.3.1 Radian Measure of an Angle

At this point, the students are quite used to measuring angles in degrees that some of them have difficulty with the idea that other units of measure are possible. You should point out to them that we use many different units of length: inches, centimetres, miles, kilometres and so on. Similarly, we use different units of weight.

It is true that one unit for angles is theoretically enough. Degree measure is all we need, and for many purposes it is satisfactory. In the same analogy, we would measure all lengths in inches. However, there are lengths such as the circumference of the earth that is not convenient to measure in inches. There are angles that are not convenient to measure in inches. There are also angles that are not convenient to measure in degrees. In advanced mathematics, radian measure is much more convenient than the degree measure. It is therefore important that, when the students study angles, they have some understanding of radian measure and degree measure.

You may start the lesson by introducing the concept of an angle and its measurement. Explain that the degree is used as the unit of measure to indicate the size of an angle. Discuss with the students that a degree is the measure of an angle formed by $\left(\frac{1}{360}\right)^{th}$ of a complete rotation (see the figure).

Explain to the students that the other unit of measurement of angle is the radian. Explain the radian in relation to the size of an angle subtended at the centre by an arc whose length is equal to the length of the radius *r*.

To determine the number of degrees in 1 radian, we recall the formula from plane geometry that the circumference of a circle is 2π times its radius. This means that the radius "fits into" the circumference 2π times. Hence in a complete rotation an angle of 2π radians is generated.

In using the degree units, a complete rotation represents an angle of 360°. This gives us the following relationships.

1 rotation = 360° = 2π radians

180° = π radians

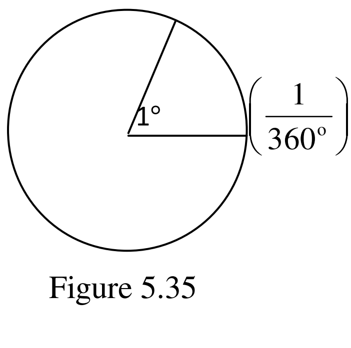


Figure 5.35

Based on these relations, encourage and assist students to guess formulas which help them convert degrees to radians and vice-versa. To this end, you may suggest to the students the following.

To change radians to degrees or degrees to radians

1. Write the equation: π radians = 180°
2. From the equation, obtain $1\text{radian} = \left(\frac{180}{\pi}\right)^{\circ}$ or $1^{\circ} = \left(\frac{\pi}{180}\right)\text{rad}$.
3. Multiply each side of the appropriate equation by the number of radians or the number of degrees given in a problem.
4. The result represents the number of degrees or radians in an angle whose measure was given.

To assess students' level of understanding of this concept, you may ask the following.

Convert

1. a. 90° b. 60° c. 30° d. 110° to radians.
2. a. $\frac{\pi}{5}$ b. $\frac{2}{3}\pi$ c. $\frac{3}{4}\pi$ d. $\frac{3}{2}\pi$ to degrees.

   In addition to these, you may ask the fast learners to convert:
3. a. 5 radians b. 1.5 radians to degrees.

5.3.2 Trigonometrical Ratios to Solve Right Angled Triangle

You may start the lesson by discussing Activity 5.8 given in the student textbook. Then assist the students to see the discussion of Activity 5.8 which leads them to grasp the concept of what is meant by Trigonometric ratio. Then given a right-angled triangle, define the sine, cosine and tangent of the acute angles of the right triangle in terms of the lengths of the sides of the triangle.

Answers to Activity 5.8

1. The fractions or ratios or proportions of the lengths of sides of a right triangle are called trigonometric ratios.
2. The two triangles are similar and hence their corresponding ratios of their sides are equal.

From this Activity, encourage students to summarize the trigonometric ratios and apply each in solving problems with examples as presented in the student textbook. Following this, as a group work, let students do Activity 5.9. Encourage and assist the students to find the trigonometric values of the angles 30°, 45° and 60° by drawing equilateral and isosceles triangles as indicated in the activity.

In finding the trigonometric values of angles 30° and 60°, instead of using an equilateral triangle, an alternative way is to use a right triangle whose acute angles are 30° and 60°.

In such a triangle, the length of the side opposite the 30° angle is always half the length of the hypotenuse, that is, if the length of the hypotenuse is r the length of the side opposite the 30° is $\frac{1}{2}r$.

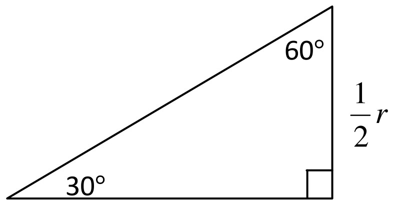


Figure 5.36

Explain how to apply the trigonometric ratio in solving problems. To this end, discuss Example 2 in the student textbook.

Answers to Activity 5.9

1.

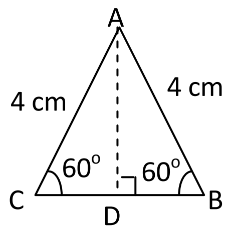


Figure 5.37

a. m(∠ABD) = 60°,

and m(∠BAD) = 30°

b. BD = 2 cm, AD = $2\sqrt{3}$ cm

c.

	30°	60°
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
tangent	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$

cos30° = sin60° = $\frac{\sqrt{3}}{2}$

2.

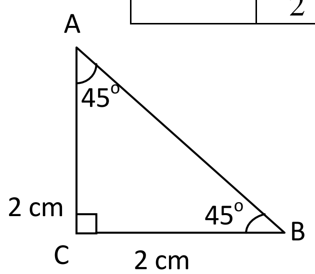


Figure 5.38

a. 45°

b. BC = 2 cm and AB = $2\sqrt{2}$ cm

c. $\sin 45° = \frac{\sqrt{2}}{2}$; $\cos 45° = \frac{\sqrt{2}}{2}$; $\tan 45° = 1$

Note: The angles 30^o, 45^o and their integral multiples are called special angles.

5.3.3 Trigonometrical Values of Angles from Table (sin θ, cos θ and tan θ for $0° \leq \theta < 180°$)

You may start the lesson by explaining how to construct and use Trigonometric tables. As a beginning to this subunit you can give Activity 5.10 and Activity 5.11 jointly. The purpose of Activity 5.10 and Activity 5.11 is to lead students to practice how to read Trigonometric tables and get the required construct for acute angles, and determine measures by using calculators. Explain to the students that, Trigonometric tables are constructed for acute angles.

Ask them to find the value of sine, cosine or tangent of a given angle. Similarly, given the value, say sine of an angle, ask them to get the degree measure of the angle. For this type of questions, you may use similar questions to Activity 5.10 and Activity 5.11.

Assist students how to read trigonometric table to find the values of the ratios and vice-versa. To this end, it is worth discussing all the examples and the activity given in the student textbook.

Encourage the students to use the Cartesian coordinate plane and trigonometric table to find the values for the ratios of obtuse angles as given in the student textbook.

Answers to Activity 5.10

a. $\cos 50^o = 0.6429$ b. $\sin 20^o = 0.3420$

c. $\tan 10^o = 0.176326$ d. $\sin 80^o = 0.9848$

Answers to Activity 5.11

1. a. 67^o b. 10^o c. 21^o
   d. 79^o e. 65^o f. 67.5^o
2. a. 66.92608193 b. 10.2630959 c. 20.81680359
   d. 78.8132612 e. 65.03907961 f. 67.46459221

Once students are aware of the ways of determining Trigonometric values or determining measures of angles, they need to see how similar ratios can be determined for triangles with arbitrary lengths of sides. For this purpose, encourage and assist the students to do Activity 5.12 in group. Let some of the groups present their discovery to the class. Finally, you are expected to make sure that students get the concept related to Activity 5.12. Pursuant to their discovery, you may give them the chance to see how trigonometric values of any acute angle can be determined by relating it with some other acute angle. That is, you have to help them discuss and summarize that for any acute angle A.

1. $\sin A = \cos (90° - A)$ 2. $\cos A = \sin (90° - A)$

3. $\tan A = \dfrac{\sin A}{\cos A}$ 4. $\sin^2 A + \cos^2 A = 1$

In the same way, encourage and assist the students to use Trigonometric table to find the value of sine, cosine or tangent of an angle whose measure is greater than 90°. Make sure that students understand and can apply the formulas that work for any obtuse angle θ,

$\sin\theta = \sin(180° - \theta)$
$\cos\theta = -\cos(180° - \theta)$
$\tan\theta = -\tan(180° - \theta)$

In order to check whether the student understands the above formulas, you may ask questions of the following type.

1. If A is an acute angle and $\sin A = 0.75$, then find
   a) Cos A b) tan A
2. If A is acute angle and $\tan A = \dfrac{5}{12}$, find
   a) cos A b) sin A
3. In each of the following, find the degree measure of the acute angle A.
   a) If $\cos A = \sin 40°$ then, m($\angle A$) = _________
   b) If $\cos A = \sin A$ then, m ($\angle A$) = ___________
   c) If $\sin A = \cos 2A$ then, m ($\angle A$) = __________
4. Find the exact values of each of the following leaving irrational results in radical form
   a) sin120° b) cos150° c) tan135°

Answers to Activity 5.12

1. a. $\sin(\angle A) = \dfrac{a}{c}$ b. $\cos(\angle A) = \dfrac{b}{c}$

   c. $\tan(\angle A) = \dfrac{a}{b}$ d. $\dfrac{\sin(\angle A)}{\cos(\angle A)} = \dfrac{a}{b} = \tan(\angle A)$

   e. $\sin(\angle B) = \dfrac{b}{c}$ f. $\cos(\angle B) = \dfrac{a}{c}$

   g. $\tan(\angle B) = \dfrac{b}{a}$ h. $\dfrac{\sin(\angle B)}{\cos(\angle B)} = \dfrac{b}{a} = \tan(\angle B)$

2. a. $(\sin(\angle A))^2 = \dfrac{a^2}{c^2}$ b. $(\cos(\angle A))^2 = \dfrac{b^2}{c^2}$

   c. $\sin^2(\angle A) + \cos^2(\angle A) = \dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{a^2+b^2}{c^2} = \dfrac{c^2}{c^2} = 1$

   (By Pythagoras theorem $a^2 + b^2 = c^2$)

   So, $\sin^2(\angle A) + \cos^2(\angle A) = 1$

   Note: $\sin^2(\angle A) = (\sin(\angle A))^2$

Assessment

Ask the students to describe the measurements of an angle; that is the radian measure and the degree measure. Ask them to convert radian measure to degree measure and the degree measure to radian measure. You can also form groups of students and assign them with some task related to trigonometric values for the purpose of assessing their understanding. To do this, you can assign Exercise 5.9 as homework.

Answers to Exercise 5.9

1. i. a. $\frac{\pi}{6}$ rad $= \frac{\pi}{6} \times \frac{180^o}{\pi} = 30^o$ b. π rad $= 180^o$

c. $\frac{\pi}{3}$ rad $= \frac{\pi}{3} \times \frac{180^o}{\pi} = 60^o$ d. 2 rad $= 2 \times \frac{180}{\pi} = \left(\frac{360}{\pi}\right)^o$

e. $\frac{3}{4}\pi = \frac{3}{4}\pi \times \frac{180^o}{\pi} = 135^o$ f. 5 rad $= 5 \times \frac{180}{\pi} = \left(\frac{900}{\pi}\right)^o$

ii. a. $270^o = 270^o \times \frac{\pi}{180^o} = \frac{3}{2}\pi$ rad b. $150^o = 150^o \times \frac{\pi}{180^o} = \frac{5}{6}\pi$rad

c. $225^o = 225 \times \frac{\pi}{180} = \frac{5}{4}\pi$ d. $15^o = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$ rad

2. a. $\sin\frac{\pi}{6} = \sin 30^o = \frac{1}{2}$

b. $\tan\frac{3}{4}\pi = \tan 135^o = -\tan(180 - 135^o) = -\tan 45^o = -1$.

c. $\cos 150^o = -\cos(180^o - 150^o) = -\cos 30^o = -\frac{\sqrt{3}}{2}$

d. $\tan\frac{2}{3}\pi = -\tan\left(\pi - \frac{2}{3}\pi\right) = -\tan\frac{\pi}{3} = -\tan 60^o = -\sqrt{3}$

3. $\tan 53^o = \frac{BC}{8.3}$

$\therefore BC = 8.3 \times \tan 53^o$

$= 8.3 \times 1.32 \simeq 11$cm

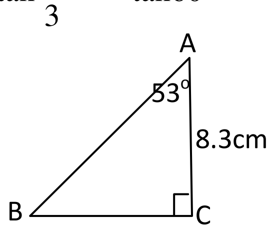


Figure 5.39

4. Let x be the height of the wall that the ladder reaches

then $\sin 65^o = \frac{x}{20}$

$\therefore x = 20 \times \sin 65^o = 20 \times 0.906 \simeq 18$

The height that the ladder reaches is 18ft

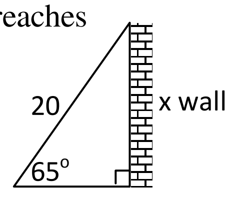


Figure 5.40

5. a. $\cos 165^o = -\cos(180 - 165^o) = -\cos 15^o$

b. $\sin 126^o = \sin(180^o - 126^o) = \sin 54^o$

c. $\cos\frac{3}{5}\pi = -\cos\left(\pi - \frac{3}{5}\pi\right) = -\cos\frac{2\pi}{5}$.

d. $\sin 139^o = \sin(180^o - 139^o) = \sin 41^o$

6. a. $\sin 64^o = \frac{8}{a}$ So, $a = \frac{8}{\sin 64^o} = \frac{8}{0.899} \simeq 9m$

b. $\cos 51^o = \frac{6.7}{a}$ So, $a = \frac{6.7}{\cos 51^o} = \frac{6.7}{0.629} \simeq 11cm$

c. $\sin 47^o = \frac{54}{a}$ So, $a = \frac{54}{\sin 47} = \frac{54}{0.731} \simeq 74mm$

d. $\cos 34^o = \frac{1.46}{a}$ So, $a = \frac{1.46}{\cos 34^o} = \frac{1.46}{0.829} \simeq 2km$

7. a. $\sin 25° = 0.4226$, $\cos 25° = 0.9063$, $\tan 25° = 0.4663$

b. $\sin 63° = 0.8910$, $\cos 63° = 0.4540$, $\tan 63° = 1.963$

c. $\sin 89° = 0.9998$, $\cos 89° = 0.0175$, $\tan 89° = 57.29$

d. $\sin 135° = \sin 45° = 0.7071$, $\cos 135° = -\cos 45° = -0.7071$, $\tan 135° = -\tan 45° = -1$

e. $\sin 142° = \sin 38° = 0.6157$, $\cos 142° = -\cos 38° = -0.7780$, $\tan 142° = -\tan 38° = -0.7813$

f. $\sin 173° = \sin 7° = 0.1219$, $\cos 173° = -\cos 7° = -0.9925$, $\tan 173° = -\tan 7° = -0.1228$

8. a. $m(\angle P) = 56^o$ b. $m(\angle P) = 62.5^o$ c. $m(\angle P) = 43^o$

d. $m(\angle P) = 14^o$ e. $m(\angle P) = 34.5^o$

5.4 CIRCLES

Periods allotted:5 periods

Competencies

At the end of this subunit, students will be able to:

- *discover the symmetrical properties of circles.*
- *use the symmetrical properties of circles to solve related problems.*
- *state angle properties of circles in their own words.*
- *apply angle properties of circles to solve related problems.*
- *find arc length, perimeters and areas of segments and sectors.*

Vocabulary: symmetry, diameter, chord, inscribe, subtend, arc, segment, sector.

Materials required:

Compass, Ruler, Protractor

Introduction

In this subunit, we consider the symmetry properties of circles. These properties are used to derive relations between circles and lines. Moreover, the theorems we will discuss will deal with the ways in which angles are measured by the arcs that they intercept. Finally, we will consider the concept of arc length, perimeter and area of segments and sectors of a circle.

Teaching Notes

To deal with this subunit there are subtopics classified in to different points of discussion that include symmetric property, angle property, arc lengths, perimeters areas which will be discussed one by one. Teaching notes on each are portrayed as follows.

5.4.1 Symmetrical Properties of Circles

Students are expected to have some background about the circle and its properties. Thus, group the students in pairs, and then, you may start the lesson by asking students to do Activity 5.13. That is, you may ask students to give the definition of a circle, radius of a circle, a diameter and an arc of a circle. You may also ask whether a circle is a symmetrical figure; if so, ask them to indicate the line of symmetry of a circle and the number of lines of symmetries that a circle can have. Let some of the groups present the answers to the class. Discuss their answers and give the correct answers to the questions. Encourage and assist the students to discover that a circle is symmetrical about its diameter. Based on this fact, discuss some properties of a circle that can be proved by using this fact.

Answers to Activity 5.13

1. A circle is the set of points in a given plane, each of which is at the same distance from a fixed point of the plane.
2. A line of symmetry is that line which divides a plane figure in to two identical parts.
3. a, b, c, e and f

Following these discussions, it is advisable to assist students understand some of the terminologies related to circle. After they discuss the ideas, it can be proved that the locus of a point equidistant from two fixed points on a circle is the perpendicular bisector of the line joining the two fixed points.

This property involves two distinct theorems

1. If $PA = PB$, then P lies on the perpendicular bisector of $\overline{AB}$
2. If Q lies on the perpendicular bisector of $\overline{AB}$,then $QA = QB$.

   Thus if A and B are any two points on the circumference of a circle, centre O, then OA = OB (radii).

Therefore, O lies on the perpendicular bisector of $\overline{AB}$

In other words:

The centre of a circle lies on the perpendicular bisector of any chord of the circle. Or the perpendicular bisector of a chord passes through the centre.

This statement is substantially equivalent to the facts expressed in Theorems 5.10, 5.11. After students have discovered that a circle is symmetrical about any diameter, make sure that students understand the properties of the circle that are stated as theorems. Theorems 5.10 and 5.11 are proved using this fact. Then you can proceed to group your students and encourage them to do Activity 5.14. In Activity 5.14, Question 1 suggests that the theorems can be proved through a different method. The purpose of Activity 5.14, is to enable students apply the properties that are already discussed and furthermore, to prepare students to study some more properties and present them to the class. So, after students are through with Activity 5.14, discuss in the class the outcomes that the groups have presented to the class.

Assist the students to construct circles and find out that:

i. A circle is symmetrical about every diameter. Hence, any chord $\overline{AB}$ perpendicular to a diameter is bisected by the diameter.

ii. In equal circles or in the same circle, equal chords are equidistant from the centre. Conversely chords which are equidistant from the centre are equal.

iii. Tangent line segments that meet outside the circle are equal in length.

Answers to Activity 5.14

2. 13 cm   3. $2\sqrt{5}$ cm   4. $\frac{\sqrt{91}}{2}$ cm

5. A tangent line to a circle is a line that touches the circle at one point only.
6. Two they have equal lengths from the external points their point of contact with the circle.

5.4.2 Angle Properties of Circles

You may start the lesson by revising some important terms of a circle. Using a circle, you can explain to the students what is meant by an arc, minor arc, semi circle, a central angle, angle inscribed in an arc, and angle subtended at the centre by an arc. In the given figure where *O* is the centre of the circle, make sure that students can indicate: the major arc *AXE* and the minor arc *AYE,* the angles subtended by the minor arc *AYE* at the centre and on the circle, and some other angles inscribed in some major arcs. Encourage and assist the students to discover the relationship between the measures of angles subtended by the same arc at the centre and at the circumference.

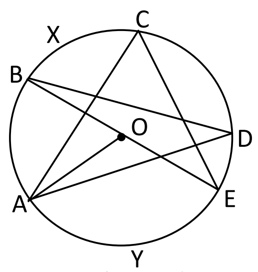


Figure 5.41

Using figure 5.41, you can explain to the students what is meant by an arc, minor arc, semicircle, a central angle, angle inscribed in an arc and angle subtended at the centre by an arc.

To this end, ask the students to use protractor and measure the central angle and the angle at the circumference given in **Figure 5.89 (c)** in the student textbook on **page 224**.

Let them repeat this experiment by drawing a figure of the same type by them and measure the angles. Make sure that students become familiar with the following angle properties of circles and restate the properties in their own words.

i. An angle at the centre of a circle is twice that angle at the circumference subtended by the same arc.
ii. Every angle at the circumference subtended by the diameter of a circle is a right angle.
iii. Angle in the same segment of a circle has equal measures.

Assessment

You can assess students understanding by following their work in each of the activities, and the class discussions. You can also give them exercises similar to the ones given in Exercise 5.10 and 5.11 as homework or assignment. It is also possible to give them a test/quiz which helps assess students.

Answers to Exercise 5.10

a. $x^o = \frac{1}{2}(80^o) = 40^o$ b. $x^o = 2 \times 40^o = 80^o$

c. $x^o = \frac{1}{2} \times 180^o = 90^o$ d. $x^o = 360^o - 2 \times 130^o = 100^o$

e. $x^o = 2 \times 40^o = 80^o$ f. $x^o = 200^o - 180^o = 20^o$

g.

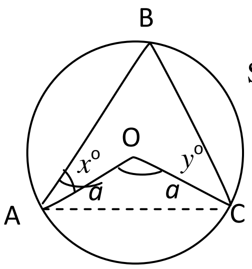


Figure 5.42

$m(\angle AOC) = 72^o$

So, $a^o = \dfrac{180^o - 72^o}{2} = 54^o$ and m $(\angle BAC)$

$= \dfrac{180^o - 36^o}{2} = 72^o$

So, $x^o = y^o = 72^o - 54^o = 18^o$

$\therefore x^o = 18^o$ and $y^o = 18^o$

Answers to Exercise 5.11

1. a. $x^o = 72^o$ b. $y^o = 33^o, x^o = 2 \times 33^o = 66^o$ c. $p^o = 48^o, q^o = 32^o$

2. a. $x^o = 180^o - (90^o + 30^o) = 60^o$ b. $x^o = 90^o + 45^o = 135^o$
   c. $x^o = 110^o - 90^o = 20^o$ d. $x^o = 10^o$

With specific interest you can group the students, assigning clever students in each group, and let them do Activity 5.15 so that they can discuss each other and get a better understanding of the ideas they discussed thereof.

Answers to Activity 5.15

a. $a^o = 80^o$, $b^o = 65^o$ b. $q^o = 98^o$, $p^o = 110^o$ and $r^o = 70^o$

c. $x^o = 75^o$, $y^o = 115^o$ d. $k^o = 98^o$, $l^o = 90^o$ and $h^o = 90^o$

5.4.3 Arc Length, Perimeter and Areas of Segments and Sectors

Previously students were acquainted with some of the terminologies related to a circle. You may start the lesson first by asking the meaning of arc, segment and sector of a circle. After you are sure that students can explain these concepts in their own words, encourage and help them to go through Group Work 5.5 and resume discussion on their work as given in the student textbook.

Afterwards, you can assist the students to calculate lengths of arcs, perimeters and areas of segments and sectors. After you discuss the examples given in their textbook, give Exercise 5.12 as class work and home work. Finally give feedback.

Answers to Group Work 5.5

1. 0.5 2. $\frac{\pi}{2}\text{cm}^2$ 3. $\frac{\pi}{4}\text{cm}^2$ 4. $\left(\frac{\pi}{4}-\frac{1}{2}\right)\text{cm}^2$

5. The areas can be determined as fractions of the areas discussed above. For example, to determine the area in (a), first we can determine area of the circle which is 100 π mm^2. Then, the shaded part is a fraction of full circle, in this case the fraction is $\frac{60^\circ}{360^\circ}=\frac{1}{6}$. Therefore, area of the shaded region is $\frac{50\pi}{3}\text{mm}^2$.

In the same way, area of (b) $=\frac{9\pi}{8}\text{cm}^2$, area of (c) = $\frac{3872\pi}{9}\text{mm}^2$, area of (d) = $\pi\,\text{cm}^2$

For the remaining exercises as well we can first find the areas of the sectors and subtract the areas of the triangles that are left un-shaded.

Example: To find area of (a) first we find area of the sector which is $\frac{9\pi}{4}\text{cm}^2$. Then determine area of the triangle that is left un-shaded which is $\frac{9}{2}\text{cm}^2$. The to get the area of the shaded region subtract area of the triangle from the area of the sector. That is, $\left(\frac{9\pi}{4}-\frac{9}{2}\right)\text{cm}^2$. In the same way, area of (b) = $\left(\frac{4\pi}{3}-\sqrt{3}\right)\text{cm}^2$ and area of (c) = $\left(\frac{20\pi}{3}-4\right)\text{cm}^2$

Assessment

You can assess students understanding by giving several exercises similar to the group work and discuss their answers to the questions in class. You then keep records.

Answers to Exercise 5.12

1. a. $P = (0.8\pi + 7.2)$ m, $A = (0.32\pi + 4.48)$ m^2

   b. $P = (50\pi + 136)$ cm, $A = (625\pi + 3400)$ cm^2

   c. $P = 10\pi$m, $A = \frac{75}{4}\pi$ m^2

   d. $P = (3.45\pi + 6.9)$ cm, $A = 3.9675\pi$ cm^2

   e. $P = 30\pi$cm, $A = 125\pi$cm^2

2. a. i. $PQ = 3\pi$cm
   ii. $A = 9\pi$cm^2

   b. i. $PQ = \frac{7\sqrt{2}}{3}\pi$ cm
   ii. $A = \frac{49\pi}{3}$ cm^2

3. $\left(\frac{40}{3}\pi + 4\sqrt{3}\right)$ cm^2

4. $(4\pi - 8)$ cm^2

5. a. i. $P = \left(\frac{85}{18}\pi + 34\right)$ cm
   ii. $A = \frac{845}{36}\pi$cm^2

   b. i. $P = (30\pi + 96)$ cm
   ii. $A = 468\pi$cm^2

5.5 MEASUREMENT

Periods allotted: 6 periods

Competencies

At the end of this subunit, students will be able to:

- *calculate areas of triangles using Heron's formula whenever only the lengths of the three sides are given.*
- *calculate areas of parallelograms.*
- *calculate the surface areas of cylinders and prisms.*
- *calculate volumes of cylinders and prisms.*

Vocabulary: Area, surface area, volume.

Materials required

Models of cylinders and prisms, ruler, protractor

Introduction

The purpose of this subunit is to state and apply area formula for any triangle whenever only the lengths of the three sides are given. This formula which is used for finding the area of a triangle that involves the lengths of the three sides is called Heron's formula. We apply area formula of triangles to state area formulas for parallelograms. Finally, in this subunit, we consider how to calculate the surface areas and volumes of cylinders and prisms.

Teaching Notes

In thus subunit students will discuss areas of triangles and parallelograms, surface areas and volumes of cylinders and prisms. These will help the students realize measurement and the ways they can be done along the discussions in this subunit.

5.5.1 Areas of Triangles and Parallelogram

A. Areas of triangles

Students are expected to recall area formulas of triangles that they studied in their lower grade mathematics. To this end, group the students and ask them to do Activity 5.16. In doing this activity, students may remember that they studied these formulas except (iii) in their lower grade mathematics. Encourage and assist students to derive formula (iii).

Answers to Activity 5.16

Let the students calculate the area by using the three formulas and compare their answers.

i. Area $= \frac{1}{2}ac = \frac{1}{2}(6\,\text{cm})(6\sqrt{3}\,\text{cm}) = 18\sqrt{3}\,\text{cm}^2$

ii. Area $= \frac{1}{2}bh = \frac{1}{2}(12\,\text{cm})(3\sqrt{3}\,\text{cm}) = 18\sqrt{3}\,\text{cm}^2$

iii. Area $= \frac{1}{2}bc\sin(\angle A) = \frac{1}{2}(12\,\text{cm})(6\sqrt{3}\,\text{cm})\sin(30^\circ) = 18\sqrt{3}\,\text{cm}^2$

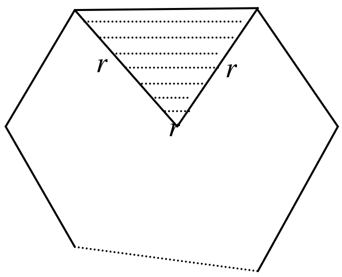
To make the students much aware of the use of Area formula (iii), let you group the students and do Group work 5.6. Area formula (iii) is useful in finding the area of a regular polygon. Group Work 5.6 is meant to apply this formula to derive the area of a regular polygon. Make sure that students have understood this point.

Answers to Group Work 5.6

1. Consider a polygon with n sides.

The central angle is $\frac{360^\circ}{n}$.

Area of the shaded region $= \frac{1}{2}r^2\sin\left(\frac{360^\circ}{n}\right)$



But, there are n such triangles in a polygon of n-sides

The area of the polygon is the sum of the areas of each triangle.

Figure 5.43

Therefore, Area of the polygon $= \frac{1}{2}nr^2\sin\left(\frac{360^\circ}{n}\right)$

2. Consider the following equilateral triangle inscribed in a circle of radius r.

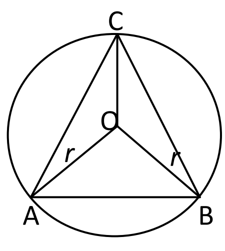


Figure 5.44

Area of the triangle (ΔOAB) = $\frac{1}{2}r^2\sin\left(60^\circ\right) = \frac{1}{2}r^2\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}r^2$

But there are such three triangles

Thus, area of (ΔABC) = $\frac{3\sqrt{3}}{4}r^2$

After you finish discussing the various cases indicated in finding the area for any triangle, make sure that the students can state and apply Heron's formula to find the area of a triangle whenever only the lengths of the three sides of the triangle are given. Discuss in the class Example 1 given on page 235 of student text right after Theorem 5.15.

B. Areas of parallelograms

To start this lesson, Activity 5.17 will help you to motivate the students. Discuss the activity. Explain that finding the area of a parallelogram involves dividing the parallelogram into two triangles.

Answers to Activity 5.17

1. A parallelogram is a quadrilateral which has both pairs of opposite sides parallel.
2. Given parallelogram ABCD and diagonal BD
   i. $\angle A \cong \angle C$ (definition of parallelogram)
   ii. $\overline{AD} \cong \overline{BC}$ *and* $\overline{AB} \cong \overline{CD}$

   ∴ ΔABD ≅ ΔCDB (by SAS)

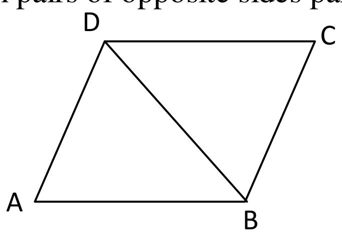


Figure 5.45

To this end, discuss the proof of Theorem 5.16 given in the student text on **page 236**. Since students know the area formula of a triangle when two sides and the measure of the included angle are given, encourage and assist the students to derive a similar formula for the area of a parallelogram.

Assessment

You can assess students by giving them an assignment to prove the theorems or by giving them exercise problems related to areas as homework. You can also use the questions in Exercise 5.13 for the purpose of assessment.

Answers to Exercise 5.13

1. $BE = \frac{24}{5}$ units
2. a. A = 126 sq. Units b. h = 12 units c. sin ($\angle D$) = $\frac{63}{65}$
3. a. 96 sq. Units b. 9.6 units
4. $\frac{18}{5}$cm and $\frac{9}{2}$cm respectively.

5. a. $A = 6\sqrt{6}\ \text{cm}^2$ b. $h = \frac{12}{7}\sqrt{6}$ cm

6. $h = \frac{3}{2}\sqrt{3}$ cm

7. a. 12 inches, 8 inches and 6 inches b. $A = \sqrt{455}$ sq. inches

8. 15 sq. inches.

5.5.2 Further on Surface Areas and Volumes of Cylinders and Prisms

Let the students start the lesson by doing Activity 5.18 given in their textbook. Give them about 10 minutes to do and discuss the problems of the activity. This will give the students the opportunity to revise the definitions of cylinders and prisms. Encourage and assist the students to make models of these solids before they come to class. In addition to that, assist them to formulate and use area formula for these solids. Make sure that students are able to state in their own words that the volume of solids are given as

Volume = Area of the base × Height

Volume of a cylinder = $(\pi r^2)h$ and

Volume of a prism = $(\ell \times w) \times h$

Answers to Activity 5.18

2. a, b and c are prisms
   f and g are cylinders
   d, e and h are neither prism nor cylinders.

3. a. $12\pi\ \text{cm}^2$ b. $20\pi\ \text{cm}^2$ c. $12\pi\ \text{cm}^3$

After the students do the questions in Activity 5.18, let them discuss the different types of prisms and let them do the examples given in the student textbook that describe surface area and volume of prisms. You can add as many examples as possible to help them understand better. Following this discussion, cylinders are also presented with which you may help the students relate prisms and cylinders and how their areas and volumes can also be related. For excellent students you can give them chance to prove surface area and volume of cylinder from that of prisms. You then discuss the examples and many others that are related to surface area and volume of cylinders. To strengthen their understanding, assign Exercise 5.14 as home work.

Assessment

You can give several exercises on finding surface area and volume of prisms and cylinders either as homework or as an assignment. Let some of them present their answers to the class and conduct discussion.

Answers to Exercises 5.14

1. For *Figure* (i)
   a. Total surface area = Lateral surface area + 2×Base area
   = 5 (26) +2 (4×9) = 202 sq. units
   b. Volume of the figure = height ×base area = 5 ×36 = 180 cubic units

In the *Figure* (ii) the length of the hypotenuse of the right triangle is $\sqrt{3^2 + 4^2} = 5$

Taking the right triangle as the base of the prism

We have;

a. Total surface area of *Figure* (ii) = Lateral surface area +2×base area

= perimeter of the base ×height + 2base area

$= 12 \times 8 + 2(\frac{1}{2} \times 4 \times 3) = 96 + 12 = 108$ sq. units

b. Volume of *Figure* (ii) = base area × height

$= \frac{1}{2} \times 4 \times 3 \times 8 = 48$ cubic units

Figure (iii) is a right circular cylinder.

a. Total surface area of the cylinder $= 2\pi rh + 2\pi r^2$

$= 2\pi \times 4 \times 7 + 2\pi \times 4^2 = 56\pi + 32\pi = 88\pi$ sq.units

b. Volume of the cylinder = Base area ×height

$= \pi r^2 \times h = \pi \times 4^2 \times 7 = 112\pi$ cubic units.

Figure (iv) is a triangular prism whose lengths of base edges are 4, 13 and 15

So, semi- perimeter S $= \frac{4 + 13 + 15}{2} = 16$

$\therefore$ Area of the triangle $= \sqrt{16(16-13)(16-15)(16-4)} = 24$ sq units

a. Total surface area of *Figure* (iv) = Lateral surface area +2×base area

$= p \times h + 2 \times$ base area

$= 32 \times 6 + 2$ base area = 240 sq units.

b. Volume of *Figure* (iv) = Base area ×height

= 24×6 =144 cubic units.

2. Perimeter of the isosceles triangle = 5 + 5 + 4 = 14 inches

Semi perimeter S $= \frac{14}{2} = 7$

$\therefore$ Area of the isosceles triangle $= \sqrt{7(7-5)(7-5)(7-4)}$

$= 2\sqrt{21}$ sq. inches.

So, a. Total surface area of the prism = Lateral surface area + 2×base area

$= P \times h$ + 2× base area

$= 14 \times 6 + 2(2\sqrt{21}) = (84 + 4\sqrt{21})$ sq. inches.

b. Volume of the prism = base area ×height

$= 2\sqrt{21} \times 6 = 12\sqrt{21}$ cubic inches

3. a. Lateral surface area $= 2\pi rh = 2\pi \times 4 \times 12 = 96\pi$ sq. ft

Total surface area = Lateral surface area + 2× base area

$= 96\pi + 2\pi \times 4^2 = 128\pi$ sq.ft

b. Lateral surface area $= 2\pi rh = 2\pi \times 6 \cdot 5\text{cm} \times 10 = 130\pi$ sq.cm.

Total surface area = Lateral surface area + 2× base area

$= 130\pi$ sq.cm $+ 2\pi \times (6.5)^2 = 214\pi$ sq.cm.

4. Perimeter of the regular hexagon $= 8 \times 6 = 48cm$

Length of a side of the rhombus $= \sqrt{3^2 + 4^2} = 5$ cm

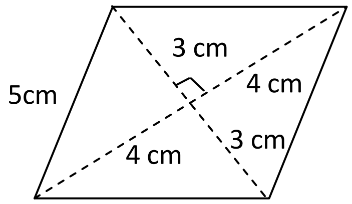


Figure 5.46

So, Perimeter of the rhombus = 4 ×5 = 20cm

- Area of the regular hexagon $= \frac{1}{2} 6 \times 8^2 \times \sin \frac{360^0}{6} = 96\sqrt{3}$sq.cm
- Area of the rhombus $= \frac{1}{2} \times 6 \times 8 = 24 sq.cm$
- Lateral area of the hexagonal prism $= p \times h = 48 \times 12 = 576$sq.cm
- Lateral area of the quadrangular prism $= p \times h = 20 \times 12 = 240$sq.cm

∴a) Total surface area of the remaining solid = lateral area of the hexagonal Prism + lateral area of the quadrangular prism +2(area of the hexagon-area of rhombus).

$= 576 + 240 + 2\left(96\sqrt{3} - 24\right)$

$= 816 + 192\sqrt{3} - 48$

$= (768 + 192\sqrt{3})$ sq. cm

b. Volume of the remaining solid = volume of the hexagonal prism – volume of the quadrangular prism $= 96\sqrt{3} \times 12 - 24 \times 12$

$= 12\left(96\sqrt{3} - 24\right) = 288\left(4\sqrt{3} - 1\right)$ cubic cm

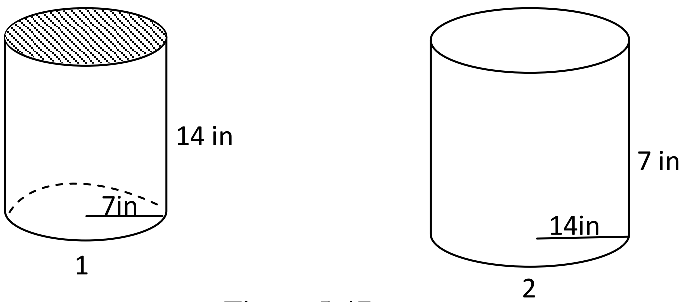


Figure 5.47

5.

$A_{T_1} = 294\pi \text{ in}^2$

$A_{T_2} = 588\pi \text{ in}^2$

∴ a. Container 2 requires more metal b. It requires 294π in^2 more metal.

Answers to Review Exercises on Unit 5

1. We know that the sum of the measures of the interior angles of a pentagon is $(5-2)\times 180^\circ = 3\times 180^\circ = 540^\circ$

$$\therefore \underbrace{m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D)} + m(\angle E) = 540^\circ$$

$$4\times 115^\circ + m(\angle E) = 540^\circ$$

$$460^\circ + m(\angle E) = 540^\circ$$

$$\therefore m(\angle E) = 540^\circ - 460^\circ = 80^\circ$$

2. i. The measure of an interior angle $= \dfrac{(20-2)\times 180^\circ}{20} = 162^\circ$

ii. The measure of an exterior angle $= \dfrac{360^\circ}{20} = 18^\circ$

iii. The measure of a central angle is $\dfrac{360^\circ}{20} = 18^\circ$

3. $\dfrac{(n-2)180^\circ}{n} = 150^\circ$

So, $150^\circ n = 180^\circ n - 360^\circ$

$150^\circ n - 180^\circ n = -360^\circ$

$\therefore n = \dfrac{-360}{-30} = 12$

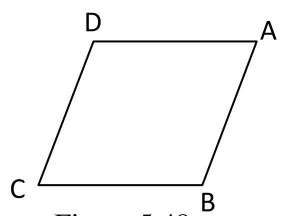


Figure 5.48

4. Let ABCD be a quadrilateral such that

$m(\angle A) = y^\circ$, $m(\angle B) = (3y^\circ)$

$m(\angle C) = (5y)^\circ$, and $m(\angle D) = (7y)^\circ$

$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = y^\circ + (3y)^\circ + (5y)^\circ + (7y)^\circ = (16y)^\circ = 360^\circ$

$\therefore y^\circ = 22.5^\circ$

Thus, $m(\angle A) = 22.5^\circ$

$m(\angle B) = 3\times 22.5 = 67.5^\circ$

$m(\angle C) = 5\times 22.5 = 112.5^\circ$

$m(\angle D) = 7\times 22.5 = 157.5^\circ$

Since $m(\angle A) + m(\angle D) = 180^\circ$ and $m(\angle B) + m(\angle C) = 180^\circ$

$\overline{CD} \parallel \overline{AB}$

5. Since the length of a side and radius of a regular hexagon are equal, we find the area of the hexagon using the formula $A = \dfrac{1}{2}nr^2 \sin\dfrac{360^0}{n}$

$\therefore A = \dfrac{1}{2}\times 6\times 8^2 \sin 60^\circ = 96\sqrt{3}$ sq. cm

6. Using the formula for the area A of an $n-$ sided regular polygon,

That is A $= \frac{1}{2} nr^2 \sin\frac{360^o}{n}$

We have $384\sqrt{3} = \frac{1}{2}\times 6\times r^2 \sin 60^o = 3\times r^2 \frac{\sqrt{3}}{2}$

So, $r^2 = \frac{2\left(384\sqrt{3}\right)}{3\sqrt{3}} = 256$

$\therefore\ r = \sqrt{256} = 16\text{cm}$

We know that for a regular hexagon r = s

a. Therefore, the length of each side of the hexagon is 16cm.

b. The radius r = 16 cm

c. For any regular n–sided polygon of radius r the apothem a is given by

$$a = r\cos\frac{180^o}{n}$$

So, $a = 16\times\cos\frac{180^o}{6} = 16\times\frac{\sqrt{3}}{2} = 8\sqrt{3}$

Therefore, the apothem of the hexagon is $8\sqrt{3}$cm.

7. $(3x)^o = 60^o \Rightarrow x = 20$

8. Joining B and K, Consider

ΔAKB and ΔCKB

i. $\overline{AB} \equiv \overline{BC}$ (Given) ii. $\overline{KA} \equiv \overline{KC}$ (Given)

iii. $\overline{KB} \equiv \overline{KB}$ (Common)

$\therefore \Delta AKB \equiv \Delta CKB$ (by SSS)

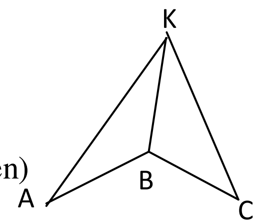


Figure 5.49

So, $\angle BAK \equiv \angle BCK$ (corresponding angles of congruent triangles).

$\therefore$m ($\angle BAK$) = m ($\angle BCK$)

9. Let x and y be the sides that correspond to 6 and 7 respectively; then,

$\frac{4}{10} = \frac{6}{x} = \frac{7}{y}$.So, x = 15cm and y = 17.5cm

10. Consider the triangles ABC and ADB

$\angle A \equiv \angle A$ (common)

$\angle ABC \equiv \angle ADB$ (right angles)

$\therefore \Delta ABC \sim \Delta ADB$ (AA similarity)

So, $\frac{AB}{AD} = \frac{BC}{DB} = \frac{AC}{AB}$

$\therefore BC = \frac{AB}{AD}\times DB = \frac{5}{3}\times 4 = \frac{20}{3}$ cm

and $AC = \frac{AB}{AD}\times AB = \frac{5\times 5}{3} = \frac{25}{3}$ cm

$\therefore DC = AC - AD = \frac{25}{3} - 3 = \frac{16}{3}$cm

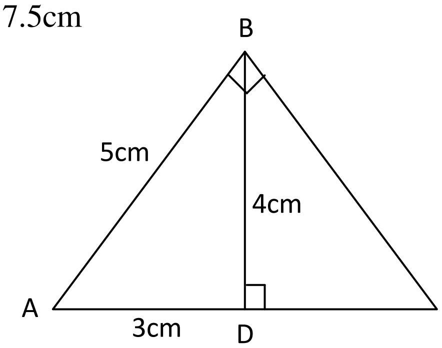


Figure 5.50

11. Let the length of the corresponding side of the later be x; then,

$$\frac{144}{81} = \left(\frac{6}{x}\right)^2 = \frac{36}{x^2}$$

i.e $x^2 = \dfrac{36\times81}{144}$

So, $x = \sqrt{\dfrac{36\times81}{144}} = \dfrac{6\times9}{12} = \dfrac{9}{2}$

12. Since a line segment drawn from the centre of the circle perpendicular to a chord bisects the chord, we have the distance

$d = \sqrt{6^2 - 4^2} = \sqrt{20} = 2\sqrt{5}$cm

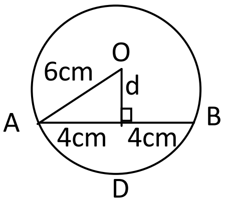


Figure 5.51

13. m ($\angle$ACD) = $180^o - 35^o - 90^o = 55^o$

and m ($\angle$ ACD) = m ($\angle$ABD) (angles on the circumference subtended by the same arc).

So, m ($\angle$ABD) = 55^o.

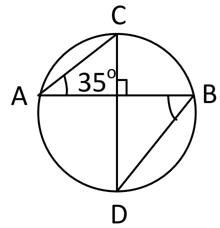


Figure 5.52

14. a. $\angle$ A and $\angle$C are supplementary angles, and $\angle$B and $\angle$D are also supplementary angles.
    b. $\angle$BAO and $\angle$BCO are right angles and $\angle CBO \cong \angle ABO$, $\angle AOB \cong \angle BOC$.
    c. $\angle TRS \cong \angle TQS \cong \angle TPS$, $\angle QTR \cong \angle QSR$

15. a. $\left(\dfrac{64}{3}\pi - 16\sqrt{3}\right)$ cm^2
    b. $\left(24\pi - 16\sqrt{2}\right)$ cm^2

16. V $= 0.49\pi$ m^3 , $A_T = 2.38\pi$ m^2
17. $h \approx 2.5$ m
18. The water level rises by 1 cm.
19. The level of the field increased by 76 cm.

UNIT 6 STATISTICS AND PROBABILITY

INTRODUCTION

Students are expected to have some of the basics about statistics from their primary grades mathematics. In this unit, they will get more familiarized with basic ideas of statistics and probability. In statistics, the students will be introduced to many new terminologies like, descriptive statistics, population, population function, primary data, secondary data, frequency distribution table, etc. They will also practice constructing frequency distributions and their Histograms.

Under this unit, students are expected to get familiar with measures of location such as Mean, Median and Mode, and some of the measures of dispersion such as Range, Variance and Standard deviation.

Finally, the students will be introduced to the notations of experiment, sample space (or possibility set), event and probability of an event.

Unit Outcomes

After completing this unit, students will be able to:

- *know methods and procedures in collecting and presenting simple statistical data.*
- *know basic concepts about statistical measures.*
- *understand facts and basic principles about probability.*
- *solve simple mathematical problems on statistics and probability.*

Suggested Teaching Aids in Unit 6

As far as statistics demand presentations and representations of different data under consideration, it is useful to have teaching aids that can either facilitate teaching-learning or simplify explanations. In this regard, the following teaching aids are considered to be essential that need to be done beforehand:

Graphs(specially histograms), and other graphs like bar charts, pie charts, line graphs which are also useful for comparison purpose, dice and different coins for probability. You can also use Ms-EXCEL or other statistical software for drawing different graphs and calculating various statistics.

6.1 STATISTICAL DATA

Periods allotted: 14 Periods

Competencies

At the end of this section a student will be able to:

- *differentiate primary and secondary data.*
- *collect data from their environment..*
- *classify and tabulate primary data according to the required criteria.*
- *construct a frequency distribution table for ungrouped data.*
- *construct a histogram for a given data.*
- *interpret a given histogram.*
- *determine the mean, median and mode(s) of a given data.*
- *describe the purposes and uses of mean, median and mode.*
- *identify the properties of the mean of a given data (population function).*
- *compute the measures of dispersion for ungrouped data.*
- *describe the purpose and use of measures of dispersion for ungrouped data.*

Vocabulary: Primary and secondary data, Data collection, Quantitative and Qualitative data, Descriptive and Inferential Statistics, Frequency, distribution, frequency distribution and Histogram, Mean, Median, Mode, Range, Standard deviation, and Variance.

Introduction

Students are expected to have some basics of statistics. With this background, in this sub-unit they will discuss on collection and tabulation of statistical data. They will also deal with basic terminologies in statistics that include quantitative and qualitative data, and primary and secondary data. They will also be introduced with the measures of central tendency and measures of dispersion.

Teaching Notes

This sub-unit is devoted to introducing statistics as a subject and its classification. In order to start from the background of the students and to help them relate every bit of discussion with their daily life, it is advisable to let the students get engaged in the activities outlined in each subtopic.

6.1.1 Collection and Tabulation of Statistical Data

The first phase in statistical process in collecting data (information) which must be collected systematically and scientifically to entail relevant and reliable information. Before getting into the concept of data collection and tabulation, it will be useful to discuss the overall idea of statistics and the ways these are practiced.

In order to start the lesson you can group the students and let them do Group Work 6.1. The purpose of this group work is enlightening and sensitizing students on the use and application of statistics. It will also give them the chance of practicing data collection, organization and some level of interpretation. The group work will also give ideas on how to collect data from our environment and make observations after the students collect data. They can discuss how to get information from the data collected and, in the mean time, they can discuss the importance of statistics in different fields.

But before students go to field to collect data some discussion could be done on identifying data to be collected and how to collect the data. This could include for example:

For group A, what are the expected grades in EGSECE like A, B, C, D and F?

For group B, Some hints on how to classify the type of disease like ENT (Ear, Nose and Throat), internal disease like Stomachache, Parasite, etc, and Injury, and how many patients of each have visited a health center around them.

For group C, the unit that could be used like cm can be informed.

The first part of this unit can be taken as a general overview of what we do in this section and the remaining parts of 6.1. It lays emphasis on understanding what statistics is, and helps the students observe the general statistical methods and uses of statistics.

Here are some of the suggested answers to Group Work 6.1

Answers to Group Work 6.1

1. The answer to question 1 of this group work depends on the situation around the students and depends on the data they collect.

2. **Statistics** - is used to present facts in a definite form.
   - facilitates comparisons.
   - gives guidance in the formation of suitable policies.
   - is useful for prediction.
   - is helpful in formulating and testing hypothesis and in developing new theories.
   - is used as a guide in capital programming.

Statistics has many other purposes which can be described.

3. The average birth rate or death rate varies from year to year. But according to the report based on the census in Ethiopia (2007G.C), the birth rate is 2.72% and the death rate is 1.7%. (you can refer updates if any). The government organization responsible for the preparation of such records is the Central Statistical Agency.
4. To design suitable polices and strategies for the holistic development of the country and for eradication of poverty, for various decisions, etc.

Pursuant to the group work, give chance for students to give their understanding and meaning to statistics. Finally you can tell them that there are varieties of meanings and definitions given to statistics by different scholars. The definition given in the student textbook is only one of the various definitions which we feel is suitable for our purpose.

After sensitizing students, our discussion will focus on the definition 6.1 given in the student textbook. Following this definition, students need to be familiar with the terminologies of data, and collection, organization, analysis and interpretation of data.

Under this topic, population and function on population need to be discussed. You can explain this with the use of the examples in the student textbook. You can give an assignment to students to collect data, identify the population of that data and determine the population function they used for. Emphasis should be made on the difference between the meaning of population in everyday life and its meaning in statistics. Students should realize that population in statistics means any finite or infinite set of objects under consideration.

After the students have a clear idea about the notion of population, the idea of population function as a means of collection of data has to be discussed. This time you can let them identify primary data and secondary data, and classification of statistics as descriptive or inferential.

From the data they collected, you can give them the chance to discuss the different steps stated in the text on how to get the right information from a raw data i.e.

Collection of data $\longrightarrow$ Organization of data $\longrightarrow$ Presentation of data $\longrightarrow$ Analysis of data $\longrightarrow$ Interpretation of data.

By using the examples given in the text (examples 1 – 8), students can discuss what population is and see the different areas in which statistics can be applied through which students will have a better understanding of statistics. They can also discuss primary and secondary data.

The final part of this section focuses on how to collect and tabulate data which will be useful for classification. In order to enrich this, you can form groups of students and let them do activity 6.1 and present it in class in different forms that can stimulate (arouse) discussion.

Before doing the activity a short discussion could be done on how to classify the employees. One method could be the one indicated in the example after the activity. In the second part hint could be given on how to classify the age group as 1 – 10, 11 – 20, 20 – 30, etc. Also discussion on how to use tally marks can be included so that they can easily put their findings in tabular form.

Answers to Activity 6.1

The solutions for this activity are different depending on the situation around your school. You need to do is, however, facilitate discussion on the results students bring.

The students can use the examples in the student textbook for classification and presentation of the data they collect.

In the part of this section, we have seen what statistics is and its use in different fields. The students have also practiced collecting data and tabulating the information, identifying primary and secondary data. What remains is describing collected and classified data in different forms such as Histogram, or by use of various measures of location or dispersion which will be discussed in subsequent sections.

Assessment

You can use the practical assignments and activities the students do for the purpose of assessment or also give exercise 6.1 as a group work so that students can present their work.

Answers to Exercise 6.1

1. Collection $\rightarrow$organization $\rightarrow$presentation $\rightarrow$analysis $\rightarrow$interpretation of data.
2. By organizing data, we mean the data must be edited and classified so that one can have a general understanding of the information gathered. By presenting data we mean illustrating data using tables or diagrams.
3. a. **Qualitative data** refers to data that are collected on the basis of attributes or qualities such as sex, religion, marital status, occupation, nationality, etc.
   b. **Quantitative data** refers to data that are collected on the basis of characteristics that can be measured or counted; characteristics such as height, weight, age, number of export or import items, volume, test scores, etc.
   c. Population refers to the complete collection of individuals, objects or measurements or totality of related observation in a given investigation.
   d. Population function is a rule which assigns a corresponding value to each member of the population.
   e. Parts (or subset) of a population is sample.
4. Statistics used to:
   - design the economic policy of a country.
   - judge the effectiveness of a program or strategy.
   - advance knowledge concerning economic and business behavior of an industry.
   - formulate plans and policies well in advance of the time of their implementation.
5. Descriptive statistics is a branch of statistics concerned with summarizing and describing data without drawing any conclusion about a given data.

6. Population refers to the number of people living in an area of country, where as sample is a small part of groups of population.
7. a. qualitative b. quantitative c. quantitative
   d. quantitative e. qualitative f. qualitative
   g. quantitative h. qualitative i. qualitative
8.
   - It enables required figures to be located more quickly.
   - It enables comparisons between different classes to be made more easily.
   - It reveals patterns within the figures which cannot be seen in the narrative form.
9. Because unorganized data is not capable of being rapidly or easily assimilated or interpreted.
10.

		2000				2001			
	Age group		Assistants	Clerks	Total	Supervisors	Assistants	Clerks	Total
Male	0 – 18 19– 49 50 and above								
	Total								
Female	0 – 18 19– 49 50 and above								
	Total								

6.1.2 Distribution and Histogram

From previous sub-unit students can explain the need for data collection. However, simply collecting data may not be sufficient to enable draw suitable and valid conclusion. This would require organizing statistical data in a suitable manner that helps manipulating the collected data. For this purpose, in this section students will practice how to prepare a frequency distribution table and present it using a histogram.

Students need to focus on the important point that it is not possible to draw important information from a raw data unless it is organized in a proper way through which the need for organizing data can be made clear.

To see the application of this section, you can let students visit different offices either in the school or outside of the school and see how the graphs and tabular information are necessary in day-to-day activity and observe how statistics is important in decision making.

If they go to a kebele office nearby, they can see different information. And this information can help the management for decision making and planning. They can ask the officers around the kebele about the way they use such presented information. You can ask your students questions such as: Suppose a kebele office decides to build new condominiums. What information do they need to collect? You can give hints like, "How many people do not have houses?" "What are the incomes of these people?", etc.

By using examples 1 and 2, you can guide the students on the procedures to prepare a frequency distribution and draw a histogram. You can then guide them to do Activity 6.2 making a group of students. While they do their activity, make sure that, in each group, there are clever students who can assist their group and help some students who may not par themselves with other students.

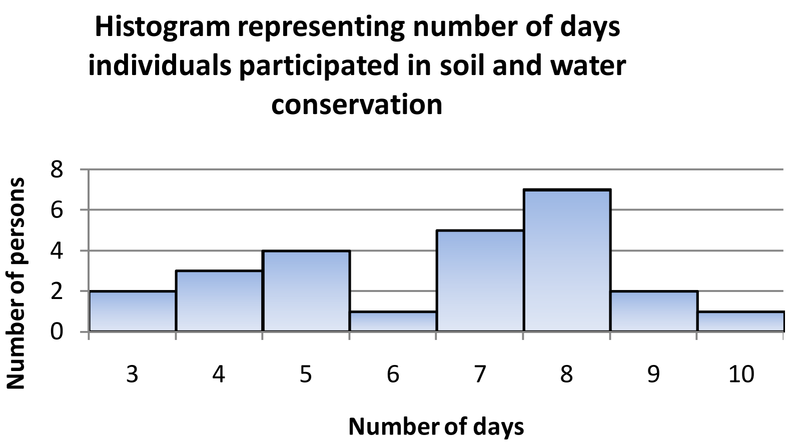
Following their discussion, you can formally define frequency distribution and histogram, and guide the students on how they can draw a histogram.

Answers to Activity 6.2

The possible frequency distribution for this data is:

Number of days	3	4	5	6	7	8	9	10	Grand Total
Frequency	2	3	4	1	5	7	2	1	25

The histogram for this data is:



After the discussion, you can give them more examples that include how they can draw information from a histogram. The students should do exercise 6.2 which could give enough practice to prepare frequency distribution table and construct a histogram.

Assessment

You can use the practical assignments and activities the students do for the purpose of assessment or also give exercise 6.2 as a group work so that students do each in group and present their work. You can assess how they did and present their work.

Answers to Exercise 6.2

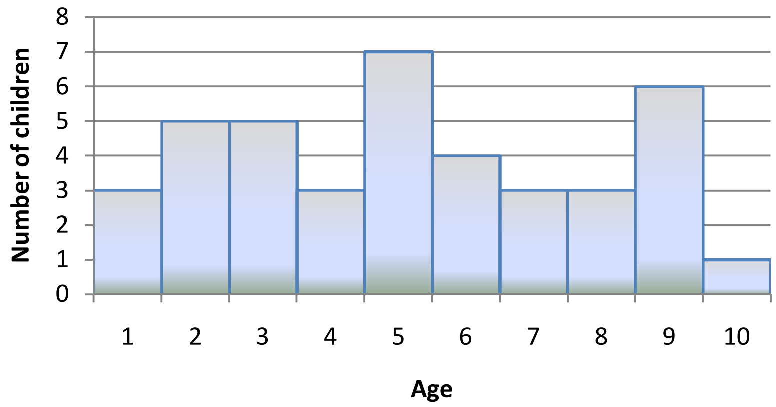
To summarize and organize a set of data

Allows quick overview of the information collected.

1. Frequency distribution is a distribution showing the number of observations associated with each value in the set of data while a histogram is a pictorial representation of a frequency distribution.

3.

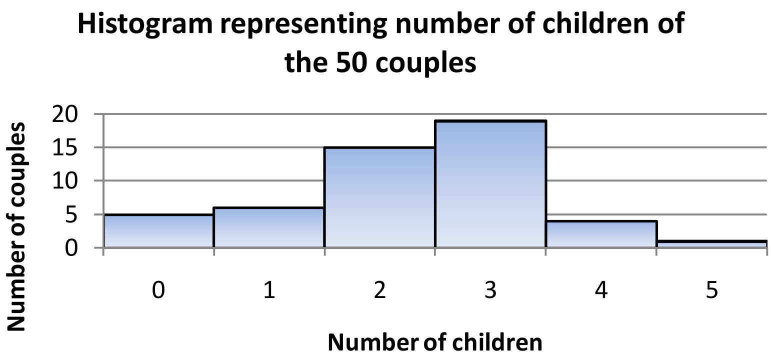
Age (v)	1	2	3	4	5	6	7	8	9	10
Frequency (f)	3	5	5	3	7	4	3	3	6	1



5. a.

v	0	1	2	3	4	5
f	5	6	15	19	4	1

b.



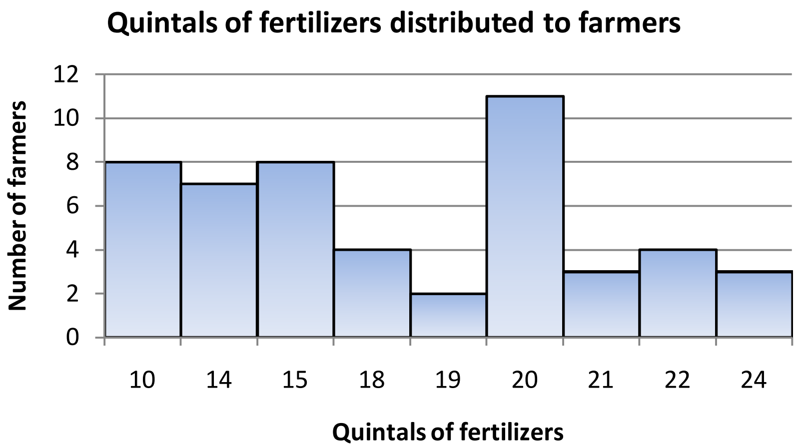
c. 30% d. 78%

6.

a.

v	10	14	15	18	19	20	21	22	24
f	8	7	8	4	2	11	3	4	3

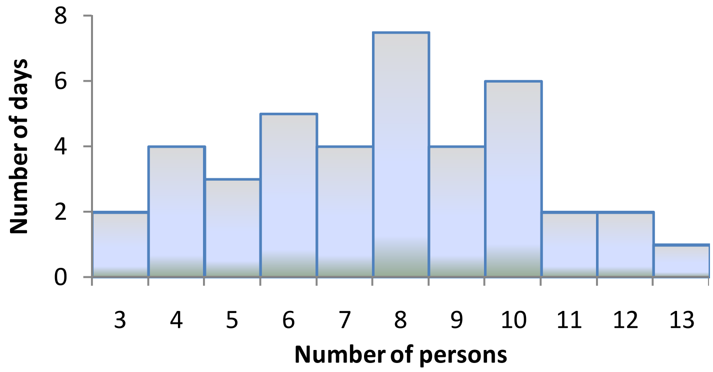
b.



7. a.

value	3	4	5	6	7	8	9	10	11	12	13
frequency	2	4	3	5	4	7	4	6	2	2	1

b.



c. $\frac{1}{8}\times 100\% = 12.5\%$

6.1.3 Measure of Location

In the last two sections, discussions were conducted on how to collect data. And, from the collected raw data the students have learned how to summarize them by preparing a frequency distribution and drawing a histogram that represents the data. Through this way, one can get summarized information and draw an important conclusion based on what one want to observe.

In this section, students need to see how to represent a data by using a single value and see this as the highest state of summarization of data.

To start the discussion of this section forming groups and letting them do Activity 6.3 will be appropriate. From their primary grades, students are familiar with the terms like mean, median and mode.

The purpose of activity 6.3 is to let students practice by responding to questions from various forms of data and see why they need discussing the measures of central tendency, and to help them realize that these measures are useful to reach a conclusion.

In activity 6.3, the first activity shows the scores of 22 students in certain test. The teacher gave the test just to see how far the students have understood the unit. By observing the raw data, if one simply wants to evaluate or say something about the performance of the students in the test, it could be difficult. Owing to such difficulty, now we have to guide our students to summarize the data and observe what the results of the 22 students look like. For example, from the data given in the first activity, the students can simply find the average from the raw data and can discuss what the result of the group is.

By doing the activities 2, 3, 4 and 5, the student will gain a lot on how a large group of the population could be represented by a single value. As a guide, you can see the possible answers for question 1 of Activity 6.3

Answers to Activity 6.3

1. a. It is not very easy to conclude about the group of students.

b.

Student	2	3	4	5	6	7	8	9	10	**Grand To tal**
frequency	2	3	2	2	5	2	3	2	1	22

c. The average score of the group = 5.77

d. Those who score above the average (5.77) = 13

e. Large group of students (59.1%) have scored above average. Therefore, we can conclude that the students are not weak.

f. The single value 5.77 estimates the average score of students (assuming) each student as close to 5.77 although this may not be exact. The single value summarized the data. We can consider this number 5.77 to represent the whole data.

2. The discussions for question number 2 can also be discussed in the same way depending on the actual data collected from the target group.

3. a. 87 b. 89

4. a. mean = 35.46, median = 25, and mode = 27

   b. mean (because it is affected by 98, the extreme value)

5. It is not possible to find the mean and the median of a qualitative data. But it is possible to find the mode.

After discussing the importance of measures of central tendency in Activity 6.3, we can directly move to the discussion of arithmetic mean. So far, the summation notation is not introduced. So it will be advisable to use the formula in the text.

The mean

In discussing this topic, we will first deal with mean from row data.

To help students find mean from frequency distribution table it will be advisable to follow the following method.

Example

Consider the following data 16, 13, 12, 18, 13, 18, 12, 18, 18, 13, 18, 16, 12, 13.

a. To find the mean from the raw data here, let the students add the 14 values and divide the sum by 14 to get

$$\text{mean} = \frac{16+13+12+18+13+18+12+18+18+13+18+16+12+13}{14} = \frac{210}{14} = 15.$$

To help students find mean from frequency distribution table it will be advisable to follow the following method.

Example

Consider the following data 16, 13, 12, 18, 13, 18, 12, 18, 18, 13, 18, 16, 12, 13.

b. To find the mean from the raw data here, let the students add the 14 values and divide the sum by 14 as presented earlier. However, you can ask the students to prepare a frequency distribution table which looks like

Value	12	13	16	18
Frequency	3	4	2	5

From which calculating the mean will be easier. Now you can help the students on how to find the mean from the frequency table which in this case will be

Mean $= \dfrac{3\times12+4\times13+2\times16+5\times18}{14} = 15$. Allow the students to compare the results obtained from the raw data and the frequency distribution table, and help them realize how useful frequency distribution tables are.

These guide the students how they can use a short cut method to calculate mean. This same idea can also be used to find median and the mode from frequency table which will be discussed later.

After the students attempted to do the activity 6.3 and become familiar with the approaches to finding the mean, median and mode, they can try to derive the way averages are calculated in this sense they discussed above. At this stage you can give the formal definition of computing the mean. When you ensure that students can calculate the mean, you can then encourage students to do Activity 6.4 individually. The purpose of this activity is to help students identify some of the properties of Arithmetic mean.

In this activity, there are five students considered and the amount of money they had in their pocket was asked and data is collected.

By now, the students can easily find the mean. After finding the mean help the students to observe the new data obtained by adding 2 on each value. Let the students find mean from the new data and compare this new mean with new number added on each value by trying to add 3 on each value or add 6 on each value. Help the students to compare the older mean with new mean and discuss the result and generalize. The same procedure could be followed to see the effect of multiplying each value by a fixed number.

Answers to Activity 6.4

a. Mean = 6

b. The new Mean = 8

c. The new Mean = 18

d. The sum of the differences from the Mean = 0

e. The observations goes to be:

i. If we add a constant number to each value, the new mean is the previous with the constant.

ii. If we multiply each value by a constant number, the new mean is the previous mean multiplied by the constant.

iii. The sum of the deviation of each value from the mean = 0.

Following these activities, encourage students to draw their own observations and help them understand the properties stated in the student textbook.

At this moment when all other students do examples you can ask the following questions for the fast/clever students.

What does it mean if mean = 0? whose answer is that: each value is 0 or some of the values are negative while others are positive so that their sum is 0.

Does mean belong to the set of data it is calculated from? Its answer is not necessarily.
Example: if our data is 2, 3, 5 them mean = 3.33 which is not in the data.

if our data is 2, 3, 4 then mean is 3 which belongs to the data.

Beyond the discussion made above, it is possible to ask any condition where the mean may not be good representative of a given data. When students try to look for, you can direct them by assigning extreme values which affect the mean in which case the mean will not be a good representative of the data.

Example: the mean of 3, 3, 3, 3, 3, 3, 3 is 3 but if by mistake the data was recorded as 3, 3, 3, 3, 333 then the mean will be 69 which is affected by recording 333 instead of 3.

Under such conditions where there is an extreme value an alternative to using mean is the median.

The median

In treating this part, the discussion will focus on the number of observation being odd or even. When you come to the problem of finding median from the frequency distribution table, you can guide the students what the values look when put in row, ask the students what the first value is if we write the values in a row, what will the second value be, what will the third value be, etc. . .

So as to attempt to deal with median, first give chance for the students to do Activity 6.5 and allow them to narrate their observation.

Answers to Activity 6.5

1. a. Median = 5 b. Median = 11

2. When the number of observations is odd the middle value is the median which is part of the data, whereas when the number of observations is even, the median is half way between the two middle values and may not be part of the data.

In Example 6 and 7, assist students to discuss how median is determined for values given as raw data or as frequency distribution, and let them identify some properties of the median.

Mode

The students can easily find mode from raw data and from frequency distribution table. Here you can discuss the unimodal and multimodal distributions.

For the purpose of discussion, you can let students do Activity 6.6 whose answer is given as follows.

Answers to Activity 6.6

1. a. mode = 7 b. mode = F
2. The mean and median for (a) can be evaluated but the mean and the median for (b) is not possible. Accordingly, (a) mean= 7.8, median = 7.5

   b. no mean , no median
3. From this, we can see that mode can be used for data types mean and median fail to represent. This is usually attached with qualitative data.

After discussing mode and its distinction from mean and median, ask your students if there are possibilities where mean= median = mode. What conditions does such equality tell about any data?

Similar other points will trigger interest to discuss. It is possible to give such problems as an assignment for the students so that they can refer further materials and consolidate their understanding. You can use this for assessment purpose.

After discussing the measures of central tendencies, the students can do Exercise 6.3 whose answers will be as follows.

Assessment

You can use a quiz/test for the purpose of assessment or also give Exercise 6.3 as an assignment so that students can do each in group and present their work.

Answers to Exercise 6.3

1. a. mean = 41.45, median = 11, mode = 11

   b. Median or mode
2. a. mean = 0.35 b. mode = 1

c. median = $\frac{10^{th}\text{ value } + 11^{th}\text{ value}}{2} = \frac{1+1}{2} = 1$

3. Since the sum of the four remaining numbers must be 30, the number 10 must be deleted.

4. First arrange the numbers 8, 9, 10, 12, 15.

median = $\frac{10+x}{2} = 11$

Therefore, the number 12 should be included.

5. Since the mode is 3, either x or y is 3.

Let $x = 3$

Now, the given data is 3, 4, 3, 5, y, 12.

Given that mean = 6,

$\Rightarrow \frac{27+y}{6} = 6$

$\Rightarrow y = 36 - 27 \Rightarrow y = 9$

Therefore, the value of x and y is 3 and 9 respectively. Similarly we can find the values of x and y assuming the value of $y = 3$ and getting $x = 9$.

6. a. $k + b$ b. kb

7. mean = 21 median = 20 mode = 20

8. a. 5 b. 10 c. 10

9. a. mean = 4.61, median = 6, mode = 1

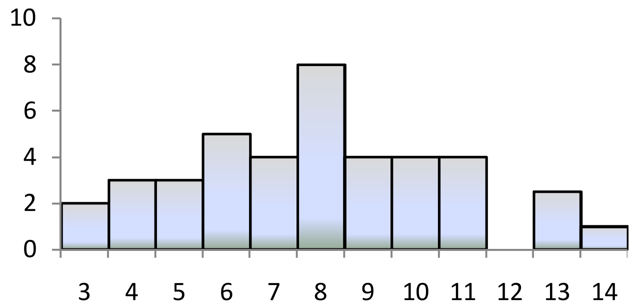
b. The arithmetic mean

10. a.

V	3	4	5	6	7	8	9	10	11	13	14
F	2	3	3	5	4	8	4	4	4	2	1

b. mean = 7.875, median = 8, mode = 8

c.



11. a. mean score for boys= 6.57

    b. mean score for girls = 5

    c. mean score for both= 5.6

    d. mean score of boys + Mean score of girls ≠ Mean score of both boys and girls.

12. The mode is also increased by 2, So the new mode is 22.

13. From the histogram we can find the following frequency distribution table.

V	1	2	3	4	5	6	7
F	1	2	3	2	2	2	2

$$\text{Mean} = \frac{1\times 1+2\times 2+3\times 3\ +4\times 2+5\times 2+6\times 2+7\times 2}{14} = \frac{58}{14} = 4.14$$

median = 4, mode = 3

14. a. mean = 74.5 cm, median = 91 cm

    b. Yes, because the average height of the plants after one year is expected to be 85 cm.

15. 96

6.1.4 Measure of Dispersion for Ungrouped Data

From previous sub-unit students can explain how a given collected data can be represented with a single value such as mean, median or mode. However, each may not be always suitable and useful to describe a give data. Trying to see the way each data is dispersed will be useful for sound investigation and description. In this sub-unit students are going to discuss measures of dispersion that are useful to determine the dispersion/variation each value possesses from the central value (mean, median or mode).

Before you move to the actual measures of dispersion, it will be helpful if you discuss the question: **why do we need the measures of dispersion**?

As it was discussed in the first part of this unit, we need statistics in different areas and we need to summarize data to help give decisions. If decisions are made based on the mean found from the population, how reliable is the mean to help us give decision on a certain issue? Let us illustrate this with the following example.

Example: In a room there are 5 people aged 5, 5, 6, 7, and 82. The mean age here is

$$\text{mean} = \frac{5+5+6+7+82}{5} = \frac{105}{5} = 21$$

If we inform somebody about the group by saying there are 5 people in the room and their average age is 21 then what can the same person understand about the people in the room? Can s/he understand that they are children? Or they are youngsters? Here the mean can mislead the conclusion. And in this case, it seems it is not a good measure of central tendency because, based on this mean, if we want to decide something our decision may not be good. This has happened because mean is affected by extreme values.

So before reaching at a conclusion by using the measures of central tendency we have to see how far the mean is reliable to represent a given data. For such cases we have to study the measures of dispersion which shows how far the data is scattered or how far each data is from the mean.

After making this discussion, it will be advisable to encourage students to do Group Work 6.2 which will help them understand the need for measures of dispersion.

Answers to Group Work 6.2

a.

Group	Values							Total	Mean	Mode	Median
A	7	7	7	7	7	7	7	49	7	7	7
B	4	5	6	7	7	9	1	39	5.57	7	6
C	1	7	12	7	2	19	1	49	7	1 and 7	7

b. The means and the medians are not equal. The modes are all the same.

c. i. The group that shows most variation is C.

ii. The group which shows no variation is A.

iii. The group which shows slight variation is B.

d. Table showing differences of each value from the mean

Group	Values						
A	0	0	0	0	0	0	0
B	–1.57	–0.57	0.43	1.43	1.43	3.43	–4.57
C	–6	0	5	0	–5	12	–6

i. The mean is closest to each value in group A.

ii. The group in which the difference between the mean and each value is the highest is C.

e. The range of group A = 7 – 7 = 0

The range of group B = 9 – 1 = 8

The range of group C = 19 – 1 = 18

To make things more clear you can give additional examples as situations may permit and proceed to discussing some of the measures of dispersion one by one.

Range

You can begin the discussion by asking the students if they can tell what range is. Following the reply of the students you can formally define range and discuss. After discussing Range, we can again discuss the example we used above on the age distribution of the people in the classroom to see how useful range may be. That is, let the students collect the age of each student in the classroom and calculate the measures of central location (mean, median and mode). Ask them if the mean represents the age of each student in the class. Ask them also to determine the range and discuss issues that you draw based on the collected data.

In that example, if we are informed that the mean age of the people in the room is 14 with a Range of (42 – 5) = 37 we can easily see that, if there is this much range, then our mean is not reliable and conclusions done based on the mean are not reliable. However, if the mean is 14 with a Range of 1, then the mean can be considered as a reliable measure to represent the age of each student in the class. But such a result depends on the representativeness of the range itself.

Here, notice need to be taken in that we are using Range to measure dispersion. However, Range is a very crude measure of dispersion which may not always be useful. Hence we need to discuss some more other measures of dispersion. (Consult statistics books!). For the time being, we will see only variance and standard deviation. But for talented students you can guide them to read and work on quartile deviation which considers the quartiles and mean deviation which will be a good input for them to understand well the concept standard deviation. The variance which is called the squared deviation is the mean of the squared deviations of each value from the mean. The principal square root of this variance is called standard deviation. In order to make things easier, you can let students do Activity 6.7. in groups.

Answers to Activity 6.7

a. The mean = $\frac{2+3+10+6+9}{5} = \frac{30}{5} = 6$

Values	(a) Mean	(b) $(x-\bar{x})$	(c) $(x-\bar{x})^2$
2 3 10 6 9	6	−4 −3 4 0 3	16 9 16 0 9

d. The mean of the squared deviations = $\frac{16+9+16+0+9}{5}=\frac{50}{5}=10$ and the principal square root of 10 = 3.16

After discussing the above activity, make sure that the students have captured the essence of variance and standard deviation and make sure also that they are able to follow each step to compute these values from any data. You can give them more examples and exercises. You can assess your students by giving them sort of a quiz or a test for both the measures of central location and measures of dispersion.

As you did on studying the properties of mean, here you can also discuss some of the properties of variance and standard deviation. Before dealing with the description of such properties, you can group students and encourage them to do Group Work 6.3. In this group work we can guide the students to see what will be the change on the standard deviation when a constant is added to each member, or each value is multiples by a constant number?

Answers to Group Work 6.3

i. mean $=\frac{6+4+8+9+3}{5}=\frac{30}{5}=6$

ii. variance = $\frac{26}{5}=5.2$ and stand deviation $\sqrt{\frac{26}{5}}=2.28$

iii. a. mean = 6+2 = 8

b. no change in variance and standard deviation

c. mean is increased by 2 but no change in variance and standard deviation

d. use (c) above and see property 1 next to this question for more discussion.

iv. a. the mean is twice the old mean i.e. $2 \times 6 = 12$

b. the variance is $2^2 \times$ old variance $4 \times \frac{26}{5}=\frac{104}{5}$

New standard deviation is $2 \times \sqrt{\frac{26}{5}}$

After this discussion, you can give Exercise 6.4 as an assignment for more practice and for the purpose of assessment.

Assessment

You can use quiz/test for the purpose of assessment or you can give Exercise 6.4 as an assignment.

Answers to Exercise 6.4

1. Range =6 – 1 = 5; variance: σ^2 = 1.8 and standard deviation: $\sigma = \sqrt{1.8} = 1.34$
2. Range = 2 – (–2) = 4; σ^2 = 1.29; and $\sigma = \sqrt{1.29} = 1.136$
3.

V	1	2	3	4
F	1	2	3	4

Range = 4 – 1 = 3; variance = σ^2 = 1; and 9.d: $\sigma = \sqrt{1} = 1$

4. 8
5. a. k b. square root of k
   c. $c^2 k$ d. $|c|k$
6. Variance = $M(x^2) - (M(x))^2 = 8 - 4 = 4$ and the standard deviation is square root of 4 = 2

6.2 PROBABILITY

Periods allotted: 8 Periods

Competencies

At the end of this section students will be able to:

- *determine the probability of an event from a repeated experiment.*
- *determine the probability of an event.*

Vocabulary: experiment, sample space, event, probability

Introduction

It is quite common phenomena to relate success or failure in one thing with luck or chance. Trying to determine how likely something can occur is useful in applications. For such a purpose, discussing probability is essential which will help represent such concepts as chance or luck mathematically. In this sub-unit students will discuss introductory concepts of probability starting from terminologies to evaluations of probability that include relative frequency approach and axiomatic approach to probability.

Teaching Notes

In this sub-unit, students will learn about experiment and associated sample spaces and events. After capacitating them with these terminologies and the way they can be established, more emphasis will be given to the different methods of finding the probability of an event.

While discussing an experiment, it is essential for the students to realize that the outcome of an experiment (if conducted randomly) is not known. Thus, the probability of an event under such a condition will be determined only after the experiment is conducted at full scale. As an implicit concern, however, after the students are made clear with the notions of experiment, sample space and event, the notion of probability of an event established under a random experiment with ***equally likely*** outcomes can be known before conducting the experiment or even without conducting the experiment. In order to assist students reach such an understanding, it is recommended to let them do Group Work 6.4 first.

Answers to Group Work 6.4

1. It is not possible because there are six different possibilities.
2. The set of all possible outcomes S = {1, 2, 3, 4, 5, 6}
3. E = The die shows up even.

   E = {x| x is less than 5}

   E = {2} and many others can be events in this experiment.
4. i. It is impossible to get a number 7 by throwing a die.

   ii It is always true that the number that shows up when we throw a die is an integer.

   a. ii. is certain event

   b. i. is impossible event
5. a. The probability that the number on the upper face of the die is 2 = $\frac{1}{6}$

   b. The probability that the number on the upper face of the die is 7 = 0

   (There is no chance of getting 0 in throwing a die)

   c. The probability that the number on the upper face of the die is less than 7 = 1 (We are certain that the number on the upper face of the die is less than 7 because the possible numbers are 1, 2, 3, 4, 5, and 6)
6. a. Experiment is any trial the outcome of which cannot be predicted in advance. Example tossing a coin.

   b. Possible set is the set of all possible outcomes of an experiment.

   c. Event is any subset of the set of possible set.

   d. Impossible event is an event that cannot occur as an outcome from an experiment.

   e. Certain event is an event whose occurrence is sure in an experiment.

After conducting the group work which could be replied through the prior knowledge of the students, you can now let them do practical experiment in groups. For this purpose, let them perform Activity 6.8. You need to assist them in registering the outcomes of each trial. Take a notice that tossing a coin 5 times is one experiment, tossing it again 10 times is another experiment.

Answers to Activity 6.8

These two questions can be considered as class activities to practice. The answers depend on the observations performed. As an example, however, if one group registered as follows,

1.

	Number of tosses			**Total**
Number of times a coin is tossed	5	10	15	30
Number of times the coin showed up Head	3	4	9	16
Number of times the coin showed up Tail	2	6	6	14

The proportion of head in the first trial is $\frac{3}{5}$.

The proportion of head in the second trial is $\frac{4}{10}$.

The proportion of head in the third trial is $\frac{9}{15}$.

Each of the proportions are probabilities that the coin shows up head. If we consider by the total, the probability that the outcome is head is $\frac{16}{30}$.

2.

Number of the upper face of the die	1	2	3	4	5	6
Number of times it shows up	3	5	1	6	2	3
Proportion of each number	$\frac{3}{20}$	$\frac{5}{20}$	$\frac{1}{20}$	$\frac{6}{20}$	$\frac{2}{20}$	$\frac{3}{20}$

a. The number of times 3 is on the upper face of the die = 1

b. The number of times 6 is on the upper face of the die = 3

c. The number of times 7 is on the upper face of the die = 0

d. given on the table

e. The probability that the number that shows up on the upper face of the die is $4 = \frac{6}{20} = 0.3$

After the activity, you can formally write the definition of an experiment, a sample space and an event as stated in the student textbook. One important point in the discussion made so far is that the proportions or the probabilities could be identified only after the experiment is conducted. Once an experiment is performed, it will be thus possible to determine the probability. For this purpose, let each student do activity 6.9.

Answers to Activity 6.9

a. The experiment was performed 1000 times.

b. The event Head occurs 495 times.

c. The probability of getting Head in this case is $\frac{495}{1000} = 0.495$

Following this activity, you can help them do as many examples as possible. Let them note also that under such an experimental meaning of probability we cannot have fixed value as a probability of an event. However, if an experiment is repeated a large number of times, then the probability will get closer and closer to a certain fixed value, which we call a theoretical probability. If we toss a coin a large number of times, it is expected that the number of heads and tails gets closer to each other and towards half of the total number of trials. If we toss a coin, say 1000 times it is most likely that the probability of head and the probability of tail are equal and is 0.5. To illustrate this concept you can let students do Activity 6.10 and assist them with the verification of the equality of outcomes.

Answers to Activity 6.10

Since this is an experiment to be conducted in class different groups can come up with different results. But, the final result is more or less the same which can be explained as follows.

Suppose one group responded as follows:

Number of throws	Number of heads	proportion of the number of throws landed as heads
10	6	0.6
20	9	0.45
30	14	0.467

c. From these proportions the result seems to get closer to 0.5 as the number of throws increases.

At this stage the concept of equally likely will come out and in an experiment with equally likely outcomes the probability of an event can be determined without conducting the experiment.

Example

In throwing a fair die each outcome is equally likely. This means that the numbers 1, 2, 3, 4, 5, or 6 have equal chance of showing up while throwing a die. Under such a circumstance probability of showing up 1 = $\frac{1}{6}$.

In determining probability an essential part is to know the number of total possible outcomes. As one means of knowing the number of total possible outcomes is use of factor tree as described on page 283-284 of the student textbook. You can enrich these with additional examples. When the students discussed all these, it is recommended for the students to understand the following important note:

For any even E, $0 \leq P(E) \leq 1$ where $P(E) = 0$ means it is impossible event and

$P(E) = 1$ means it is certain event.

Assessment

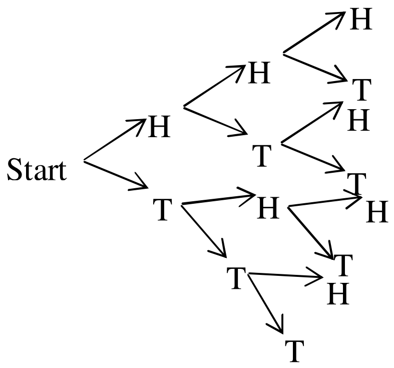
You can assess students while they conduct experiments by way of asking them different oral questions such as: If you keep on doing an experiment of tossing a coin one million times how many times do you expect head to show up? You can also give Exercise 6.6 as a group work so that students can present their work. Or as this is the end of the unit, you can give test/quiz to assess students learning and understanding.

Answers to Exercise 6.5

1. a. $A = \{11, 22, 33, 44, 55, 66\}$

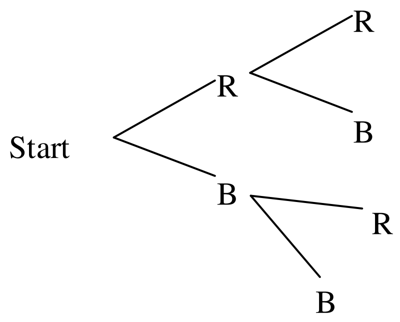
   b. $B = \emptyset$ c. $C = \{11\}$ d. $D = \emptyset$

2.



The possibility set: S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

3.



The possibility set is: {RR, RB, BR, BB} equivalent to {RR, RB, BB}

4. To be conducted as an experiment.

5. a. $\frac{1}{2}$ b. $\frac{1}{6}$ c. $\frac{1}{2}$ d. 0

6. The possibility set is {1, 2, 3, . . . , 20}

a. $\frac{1}{2}$ b. $\frac{3}{20}$ c. $\frac{3}{10}$

d. 1 e. $\frac{1}{20}$ f. 0

7. a. $\frac{4}{12}=\frac{1}{3}$ b. $\frac{5}{12}$ c. $\frac{3}{12}=\frac{1}{4}$

d. $\frac{8}{12}=\frac{2}{3}$ e. $\frac{9}{12}=\frac{3}{4}$

8. a. $\frac{50}{100}=\frac{1}{2}$ b. $\frac{50}{100}=\frac{1}{2}$ c. $\frac{14}{100}=\frac{7}{50}$

d. $\frac{20}{100}=\frac{1}{5}$ e. $\frac{33}{100}$ f. $\frac{75}{100}=\frac{3}{4}$

g. $\frac{68}{100}=\frac{17}{25}$ h. $\frac{8}{100}=\frac{2}{25}$

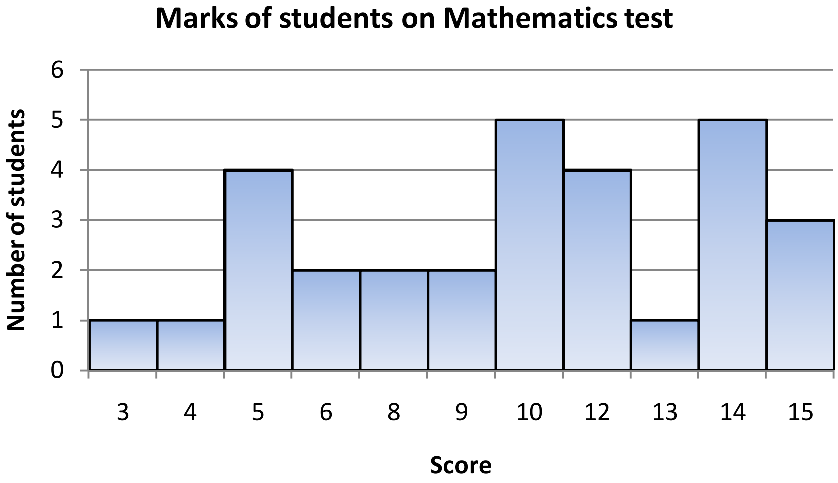
Answers to Review Exercises on Unit 6

1. Writing the data in to a manageable form by use of such as charts, tables or graphs.

2. a.

Score	3	4	5	6	8	9	10	12	13	14	15
No of students	1	1	4	2	2	2	5	4	1	5	3

b.



c. 90% of the students have scored less than 15

3. a.

v	1	2	3	4	5	6	7
f	10	15	5	25	0	20	5

b. the highest variable is 4

c. The highest frequency is 25

d. Two. i.e., 3 and 7

e. there is only one variate that have minimum frequency. That is 5.

4. mean = 4, median = 3.5, mode = 3

Range = 6 – 2 = 4, variance = $\frac{13}{6}$, standard deviation = $\sqrt{\frac{13}{6}}$.

5.

v	1	2	3	4	5	6	7	8
f	1	4	7	5	4	2	2	1

Mean = 4, median = 4, mode = 3

Range = 8 – 1 = 7, variance = 3, standard deviation = $\sqrt{3}$

6. Since for any event E,P(E) is between 0 and 1 inclusive. i.e. $0 \le P(E) \le 1$ and probability cannot be negative.

7. $\frac{1}{12}$

8.

v	1	3	4	5	6	7	9
f	1	2	1	4	1	2	1

$$\text{Mean} = \frac{1\times1 + 3\times2 + 4\times1 + 5\times4 + 6\times1 + 7\times2 + 9\times1}{12} = \frac{60}{12} = 5$$

a. mode = 5, $P(V = 5) = \frac{4}{12} = \frac{1}{3}$

b. $P(V < 5) = \frac{4}{12} = \frac{1}{3}$

c. $P(V = 1, 4, 6, \text{or } 9) = \frac{1+1+1+1}{12} = \frac{4}{12} = \frac{1}{3}$

d. $P(V > 5, V \in \text{odd}) = \frac{3}{12} = \frac{1}{4}$

9. a.

	Die 2						
	−	1	2	3	4	5	6
Die 1	1	0	**1**	2	3	4	5
	2	**1**	0	**1**	2	3	4
	3	2	**1**	0	**1**	2	3
	4	3	2	**1**	0	**1**	2
	5	4	3	2	**1**	0	**1**
	6	5	4	3	2	**1**	0

There are 6 × 6 = 36 sample spaces.

$$E = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,6),(6,5)\}$$

$$P(\text{diff } = 1) = \frac{n(E)}{n(S)} = \frac{10}{36} = \frac{5}{18}$$

10. a. $\frac{35}{50} = \frac{7}{10}$

b. $\frac{45}{50} = \frac{9}{10}$

c. $\frac{15}{50} = \frac{3}{10}$

d. $\frac{30}{50} = \frac{3}{5}$

11. $x-4$, x, $2x$, $2x+12$

Median is $\frac{x+2x}{2}=9$

$3x = 18 \Rightarrow x = 6$

12. $\frac{3\times3+4x+5\times4}{7+x}=4.1$

$\frac{29+4x}{7+x}=4.1 \Rightarrow 29+4x=28.7+4.1x$

$0.1x=0.3 \Rightarrow x=3$

13. Let the number of girls = x

$\therefore \frac{55(8)+48x}{x+8}=50.8$

$440+48x=50.8x+406.4$

$x=12$

There are 12 girls in the class.

14. Let ℓ = number of left-handed

$\therefore \ell = 30 - 24 = 6$

$\therefore$ P (left handed) = $\frac{6}{30}=\frac{1}{5}$

15. Probability of getting a day with the letter r in it = $\frac{3}{7}$

16. Let us refer to one die as the first die and the other as the second die. Using ordered pairs to represent outcomes as follows: (2, 3) denoted the outcomes of obtaining a 2 on the first die and a 3 on the second; (5, 1) represent a 5 in the first die and a 1 on the second and so on. Six different possibilities for the first number of ordered pairs and with each of these six possibilities for the second number, the total ordered pairs is 6 × 6 = 36.

a. $\text{E}=\{(3,6),(4,5),(5,4),(6,3)\}$

$\text{P}(\text{E})=\frac{n(E)}{n(S)}=\frac{4}{36}=\frac{1}{9}$

b. $E=\{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$

$P(E)=\frac{n(E)}{n(S)}=\frac{6}{36}=\frac{1}{6}$

c.

$$E=\left\{\begin{matrix}(1,7),(1,3),(1,5),(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(3,1),(3,3),(3,5),(5,1),(5,3),\\(5,5),(6,2),(6,4),(6,6)\end{matrix}\right\}$$

$$P(E)=\frac{n(E)}{n(S)}=\frac{18}{36}=\frac{1}{2}$$

d.

$$E=\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),\\(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),\\(4,5),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3)\end{matrix}\right\}$$

$$P(E)=\frac{n(E)}{n(S)}=\frac{30}{36}=\frac{5}{6}$$

e. $E=\{(4,6),(5,5),(6,4),(6,6)\}$

$$P(E)=\frac{n(E)}{n(S)}=\frac{4}{36}=\frac{1}{9}$$

f.

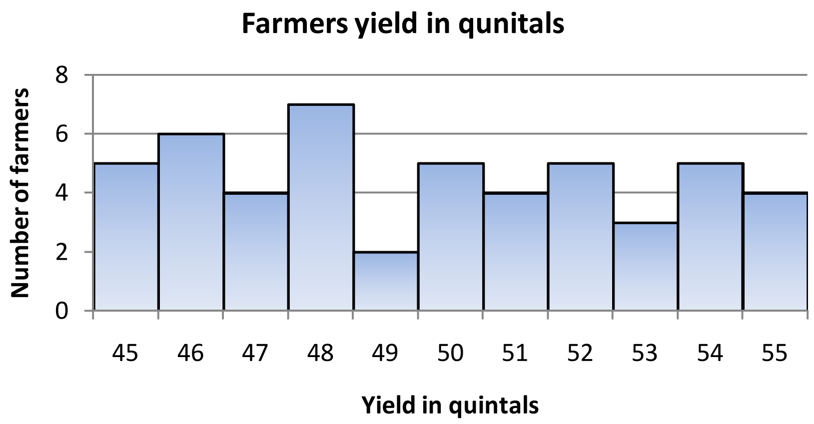
$$E=\left\{\begin{matrix}(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),\\(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),\\(5,3),(5,5),(6,2),(6,4),(6,6),(6,5),(5,6)\end{matrix}\right\}$$

$$P(E)=\frac{n(E)}{n(S)}=\frac{20}{36}=\frac{5}{9}$$

17. a.

Value	45	46	47	48	49	50	51	52	53	54	55
Frequency	5	6	4	7	2	5	4	5	3	5	4

b.



c. The mode is 48 (6 farmers harvested 48 quintals of wheat)

d. 12 farmers will get the prize.

18. a. False they can be equal. For example consider the following data

1, 2, 3, 3, 3, 4, 5

Mean = Median = Mode = 3

b. False. When the range is large, it is more likely that the standard deviation is also large.

c. True. Because range = maximum value – minimum value, and hence cannot be negative (non-positive)

d. True as $\sum(x-\overline{x})$

UNIT 7 VECTORS IN TWO DIMENSIONS

INTRODUCTION

This unit introduces vectors in two dimensions that have many applications in real life problem. In this regard, a closer look will be made on quantity measures classified as scalar and vector quantities. This enables students to understand such quantities as they may appear in real life problems. Here, students are expected to identify different measurements as scalar and/or vector quantities. After the students have clearly identified such scalar and vector quantities, discussions will be conducted on operations of vectors that include vector additions and scalar multiplication of vectors. At this juncture, students need to understand what a scalar multiplication of vectors is in terms of the length of the vector, dictated by the fact that, if the scalar multiplier is greater than one, it enlarges the vector and if the scalar multiplier is less than one, it shortens the vector. A discussion of position of a vector will be also put in this lesson to relate the concept of ordered pairs the students have discussed in unit three with components of a vector. They are also expected to discuss such concepts of a position of a vector in terms of the trigonometric values at a certain angle of reference of the position of a vector.

Unit Outcomes

After completing this unit, the students will be able to:

- *know basic concepts and specific facts about vectors.*
- *identify scalars and vectors.*
- *determine magnitude of a vector.*
- *perform scalar multiplication of a vector.*
- *determine a vector as a position depending on its initial and terminal points.*
- *perform operations on vectors.*

Suggested Teaching Aids in Unit 7

At this unit, it is expected that all students are not familiar with identifying scalar and vector quantities. Thus, a great deal is anticipated for the students to identify various measurements from their surrounding and try to categorize them as a scalar or vector quantity. Practical problems of measurements and some drawings can be taken as an additional teaching aid for this lesson. Some laboratory activities from physics can also be used as a means for better understanding. The difference between going up in a hill and coming down in the same hill can be used as a practical teaching aid in how it creates difference (although the length is the same). Such examples and related real life problems can be used as a teaching aid so that the students can have better understand of the concepts of scalars and vectors. Geometric drawings can also be used to describe operations on vectors.

For this unit, advise students to come to class with rulers and other mathematical instruments including protractors. You may also need to prepare flipcharts to draw vectors of different types.

7.1 INTRODUCTION TO VECTORS AND SCALARS

Periods allotted:2 Periods

Competency

At the end of this subunit, you will be able to:

- *differentiate vectors from scalar quantities.*

Vocabulary: Scalar quantity, Vector quantity

Introduction

This sub-unit is devoted to the introduction of vectors and scalars. In our surrounding there are many measurements some of which possess only magnitude while others have magnitude and direction. From this sub-unit, students are expected to gain the ability of differentiating measures as scalars or vectors.

Teaching Notes

As far as students may not have detail background about vectors, you may start the lesson by giving chance to the students to explain their understanding about scalars and vector quantities in its raw sense. Different measurements can be offered to the students such as weight, height, temperature, angle, velocity and acceleration with the use of which they can discuss the meaning of scalars and vectors. For a more clear understanding of the concept and for presenting these ideas from student real life situations, you can encourage students to do Group Work 7.1 from the student text that focuses on

1. discussing the concepts of vector quantities and scalar quantities.
2. give examples of scalar quantities and vector quantities.
3. discuss some quantities that require both size and direction.

Answers to Group Work 7.1

1. Some of the quantities that can be expressed completely using a single measurement (with units) include weight (x kg), height (x cm or x m), temperature (36^{o}C), etc
2. Some quantities that require size and direction are velocity (35m/h East), force (30N North-East), etc.

You may simply direct them to look for measures from their surrounding and to identify whether or not those measures have directions.

Help the students to understand that the height of a student as a measure does not have direction, and hence is scalar quantity whereas velocity is a vector since it has a direction. You also give them the chance to try to discuss the concepts of vector quantities and scalar quantities, and give examples of scalar quantities and vector quantities of measurements from their daily life. At this stage, students are expected to differentiate between those measures that have direction and those that do not have. For this purpose, help the students to identify that height, weight are scalar quantities and that velocity, acceleration, etc are vector quantities.

You can also use Activity 7.1 given in the student textbook for the students to begin a discussion as the activity demands them to identify whether the following quantities are scalar or vector quantities.

- Amount of rainfall in mm
- Temperature in a room
- Gravity
- Volume of a solid figure
- Area of a plane figure
- Force of water hitting a turbine
- Acceleration of a motor bicycle
- Speed of an airplane

Answers to Activity 7.1

The quantities described by Amount of rainfall in mm, Temperature in a room, Volume of a solid figure, Area of a plane figure, and Speed of an airplane stand for scalar quantities and the measurement quantities described by Gravity, Force of water hitting a turbine, and Acceleration of a motor bicycle are vector quantities.

After the student finishes the discussion, you can offer the formal definition of scalar quantities and vector quantities. The examples offered in the student textbook can also be used to identify the difference between scalar and vector quantities.

Assessment

From a given list of different quantities, ask students to list scalar quantities and vector quantities separately. You can do this in the form of class activity, group discussions, giving assignments or a test. You can also give them some questions from review exercises as homework.

7.2 REPRESENTATION OF A VECTOR

Periods allotted: 2Periods

Competencies

At the end of this subunit, students will be able to:

- *represent vectors pictorially.*
- *explain what is meant by magnitude and direction of a vector.*

Vocabulary: Vectors, Directed arrows

Introduction

Students are able to differentiate scalars and vectors, from the previous sub-unit. However, since vectors possess direction as well trying to identify the direction of a vector is useful. In this sub-unit students will discuss representing vectors which will help them for mathematical amenability. They will also see how easy it will be to determine magnitude and direction of a vector after it is represented pictorially.

Teaching Notes

Since the students are now expected to identify vectors it is advisable to start this lesson by encouraging them to do Activity 7.2.You can direct the students to:

1. discuss algebraic and geometric representations of vectors.
2. represent the vector $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ geometrically.
3. discuss the magnitude and direction of a vector.
4. find the magnitude and the direction of the vector $\overrightarrow{OP}$.
5. when are two vectors equal?

Here, what the students need to understand is that a vector can be represented either algebraically or geometrically, and how it may be represented.

Answers to Activity 7.2

1. A vector can be represented algebraically as an ordered pair or geometrically as a directed line segment.
2. The geometric representation uses a directed line segment to represent a vector. The representation of $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is

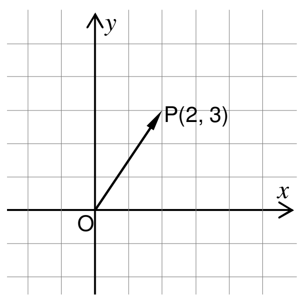


Figure 7.1

3. Magnitude is simply the length of the directed line segment and direction is the angle the directed line segment makes with the x-axis.

4. Magnitude $=\sqrt{13}$ and direction is $\tan\theta = \frac{3}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{2}\right)$. For this trigonometric value you need to express it the way they study them in unit 5.

5. Two vectors are equal when they have the same magnitude and same direction.

From above we can see that when a vector is given as an ordered pair or as a directed line segment where the initial is the origin it is possible to determine its magnitude and direction. When the ordered pairs are not offered, however, it may seem difficult for the students to determine the magnitude of a vector. In this case, students can use ruler or any other mathematical instrument.

You may not need to do Activity 7.2 but help students capture the overall vector related points mentioned above. However, let them sense that a vector quantity requires direction. Such a direction also matters in trying to represent a vector. At this stage, you may need to explain how we can represent a vector using a directed line segment. In using directed line segment for representing vectors, it may be useful to let students realize that the direction of the arrow stands for the direction of the vector and the representation of the directed line segment is proportional to the length of the vector itself. The fact that vectors are best represented geometrically as a directed arrow in a way as $\overrightarrow{OP}$, where the point O is called the initial point and P is called the terminal point, or sometimes as a single letter with a bar over it such as $\vec{u}$, $\vec{v}$, etc makes easy to characterize the vectors and their operations, one of the characterizations of a vector being its length (magnitude) and direction.

Before you proceed to define the magnitude and direction of a vector, let the students try to determine a length and direction of vectors that you give them in class. After you ensure their practice and discussion, give them the definitions of a magnitude and direction of a vector.

Magnitude (Length) of a vector $\overrightarrow{OP}$ or simply $\vec{u}$ is the length of the line segment from the initial point O to the terminal point P, or simply, the length of the directed line. Here, you need to help students to realize that the magnitude of a vector is proportion and represented by the length of the arrow. In order to do this, give them different vectors so that they can measure their lengths and also see the geometric representation of the vectors.

The direction of a vector is the angle that is formed by the arrow (that represents the vector) with the horizontal line at its initial point (or with the vertical line in case of compass direction). Here, you need to notice that a direction of all other vectors is measured relative to horizontal line but for compass direction we measure direction relative to a vertical line (As it is relative to North Pole in actual measures). After you give these definitions, give different vectors that have same direction, same length, opposite direction, different length, etc and allow the students to list the various differences and similarities they have observed on these given vectors in their own observation. And then, proceed to hold a discussion by raising the following questions of discussion. What do you observe? Do they have the same length? Do they have the same direction?, etc.

Ensuring the understanding of students following the discussion, you can state equality of vectors. Two vectors are said to be equal if they have the ***same length*** and the ***same direction***.

For synthesizing the overall discussions of this sub-unit, you can then continue giving Group Work 7.2 to the students so that they discuss the following points.

1 Suppose vectors $\vec{u}$ and $\vec{v}$ are equal,
   a. can we conclude that they have the same initial point? Why?
   b. do they have the same length? Why?
   c. do they have same direction? Why?
2 Suppose vectors $\vec{u}$ and $\vec{v}$ are opposite,
   a. can we conclude that they must start from the same initial point? Why?
   b. do they have the same length? Why?
   c. do they have same direction? Why?
3 They will write down their observations

Answers to Group Work 7.2

1. a. No. Equal vectors may or may not have same initial point. What they all need to have is ***same length*** and ***same direction*** even if they do not have same initial point. (Here, you may need to tell them that such equal vectors which may not have same initial point can be made to have a standard representation in which both will have same initial point O(0, 0) which they will discuss in subsequent sessions).

   b. Yes. They must have the same length.

   c. Yes. They must have the same direction.

2. a. No. Vectors that are opposite may or may not have the same initial point. What matters here is their direction. They must be in opposite.

   b. No. They may not have the same length. Irrespective of their length, they can be opposite in direction. But later on students will see opposite vectors that have equal length.

   c. No. They cannot have the same direction as they are opposite.

3. The observation could be different from student to student. However, the common ones are opposite vectors which may or may not have same initial point. They also may or may not have same length. But it is possible to make them have same initial point by expressing them in standard form.

Example

Consider the following vectors,

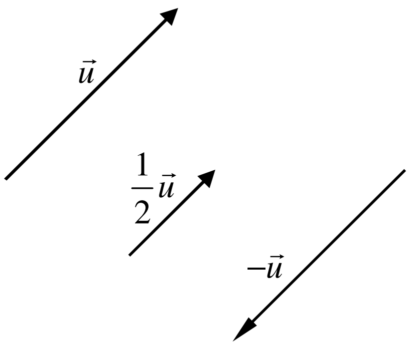


Figure 7.2

These vectors do not all have the same initial point. Here, notice can be taken, however, that if all are presented in standard form, they may have same initial point, but they differ in their direction and length.

Example

The vectors given above can be put as follows in standard form in which they may have the same starting point.

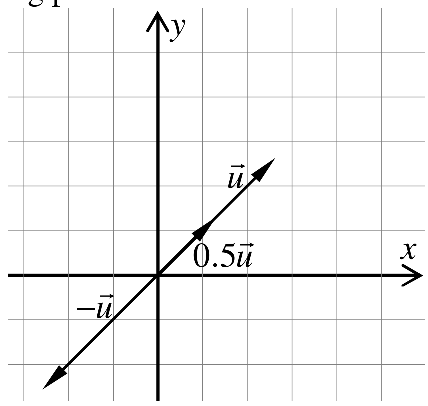


Figure 7.3

You can then give the Exercise 7.2 as a class activity with which you can assess your students understanding.

Assessment

As mentioned above you can give various class activities for the purpose of assessment. But it is also possible to assess students with group discussions, home-work, assignments or quiz/test.

Answers to Exercises 7.1

1. Using a ruler the magnitudes are 3 cm, 2 cm, 2.7 cm and 2.8 cm respectively.

   Using protractor the directions are 18^o, 0^o, 90^o and 146^o from the positive x direction.

2. a.

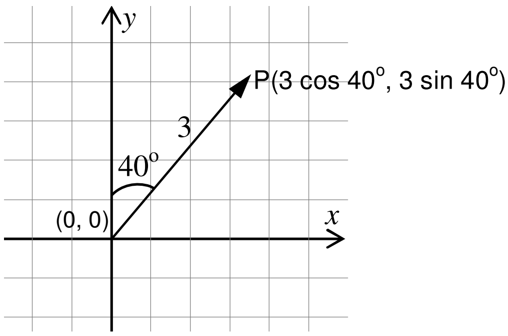


Figure 7.4

b.

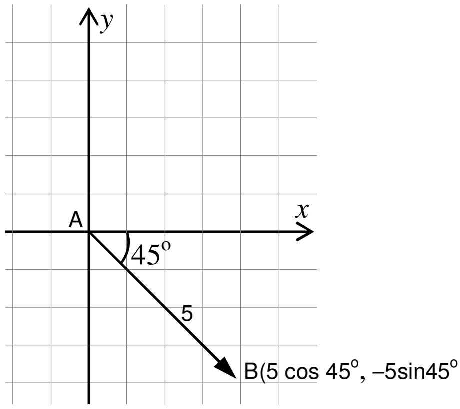


Figure 7.5

c.

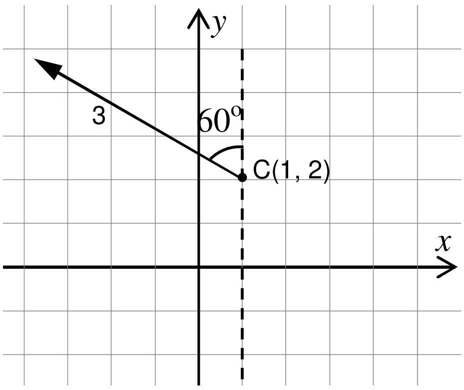


Figure 7.6

3. The paired vectors in a) and e) are equal vectors and the vectors in c) are opposite. The paired vectors in b) and d) have the same direction but are different in length.

7.3 ADDITION AND SUBTRACTION OF VECTORS AND MULTIPLICATION OF A VECTOR BY A SCALAR

Periods allotted: 6 Periods

Competencies

At the end of this subunit, students will be able to:

- *determine the sum of given vectors.*
- *determine the difference of two vectors.*
- *multiply a given vector by given scalars.*

Vocabulary: Addition of vectors, Subtraction of vectors, Scalar multiplication of a vector.

Introduction

Once magnitude and direction of a vector is determined, it is possible to discuss operations of addition (subtraction) and multiplication of a vector by a scalar. In this sub-unit students will discuss such operations on vectors and will also discuss the Triangle Law and Parallelogram Law for performing vector addition and extend these for scalar multiplication as well.

Teaching Notes

As far as vectors possess length and direction, it may seem trivial to talk about vector addition or scalar multiplication as a simple algebraic operation. When two vectors have the same length, then the sum of the lengths of these vectors is simply addition of their lengths. However, if there is a change in direction, it may not be direct to consider vector addition as addition of their lengths. In this subunit, we expect students to be familiar with these operations on vectors and as a consequence we expect them to capture the triangle inequality and parallelogram rules for addition of vectors, and the effect of scalar multiplication of vectors. For this purpose, you can start the subunit by giving an Activity 7.3 taking the leading questions of “What do we mean by an addition of vectors?” and “How do we add vectors?” with which students can discuss in groups and then proceed to giving examples of vector addition.

Answers to Activity 7.3

1. What is meant by addition of vectors?

   As vectors are measures, vector addition is simply addition of the measures that represent vectors. But consideration of the direction is important.

2. How would you add vectors?

   We add vectors in different ways. If they have same direction, then we simply add the measures, whereas, if they are opposite in direction, then the addition becomes subtraction. If they are neither having same direction or opposite we may use resultants to consider vector addition. We will see in answer for question 3.

3. Is the length of the sum of two vectors always equal to the sum of the lengths of each vector? Why?

   The sum is not always equal to the sum of the lengths of each vector.

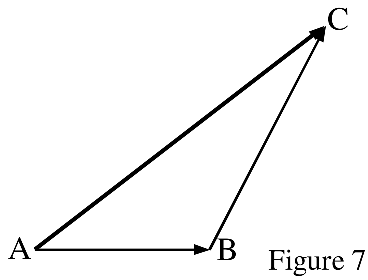


Figure 7.7

Here, do not give detailed explanation of the answers. Leave the students questioning so that they will best understand in the subsequent discussions when they see the laws of vector addition. However, let the students realize possible changes in the length and direction of the sum of two vectors in that;

If they are parallel, then the change in the sum is only length and there is no change in the direction. But, if the vectors are not parallel, then there is change in both the length and the direction which will lead in to the laws of vector addition "**The Triangle Law** and the **Parallelogram Law**".

Here, you can give chance for the students to discuss the points in Group Work 7.3 dealing with

a. The Triangle Law of vector addition
b. The Parallelogram Law of vector addition
c. What relations and differences do both Laws have?

Answers to Group Work 7.3

This is meant for students to discuss the ideas of triangle law and parallelogram law, and to get insight for the following discussion. They are not supposed to give an exact reply to each question.

After the students have discussed these points, you can formally state the Triangle Law and give them Activity 7.4 so that they will understand it through examples. When all students do the examples, you can give guiding questions for talented students to compare the triangle law with that of sum of two sides of a triangle is always greater than the third side. You can ask them a question"

In triangle law AB + BC = AC but from lengths of sides of a triangle AB + BC > AC. What do you think is the difference?

Answers to Activity 7.4

1. a. $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$ b. $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AB} + (-\overrightarrow{AB}) = \overrightarrow{AB} + \overrightarrow{BA} = 0$

   c. $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$ from parallelogram law or Triangle inequality but the length is that of the resultant vector AC which may require use of Pythagoras Theorem.

   d. $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$ from parallelogram law or Triangle inequality but the length is that of the resultant vector $\overrightarrow{AC}$ which requires use of trigonometry.

After the students discuss Activity 7.4, you can use Example 1 that is offered as an illustration in the students textbook. But since students might have been using ruler and protractor to find length and direction of a vector, you can now help them relate measuring length and direction with the applications of trigonometry. You can also give them additional examples as much as you can.

When you make sure that students have understood the triangle law, you can state the Parallelogram Law and give to the students Activity 7.5 so that they will understand the parallelogram law and identify the essence of triangle inequality and parallelogram laws. You can also use the examples in the student text for clarifications. You can give talented students to relate these two laws and recall the property of parallelogram stating that the opposite sides of a parallelogram are always equal, and how this law enforces the triangle law. You can also ask them: What will the parallelogram ABCD be if

AB + BC = AC, and what does this tell in vector addition? As a solution; the parallelogram will be a line segment AC. In this case the sum of AB and BC will have the same length with AC.

Answers to Activity 7.5

The answer to this activity is offered in the student text on page 303. But there are essential points that you need to help students realize.

1. Two or more vectors that have the same direction are parallel.
2. Any vectors positioned anywhere with same direction and length are equal.
3. Thus, Vectors $\overrightarrow{AD}$ and $\overrightarrow{CE}$ are equal.
4. Therefore, Adding vector $\overrightarrow{AD}$ to $\overrightarrow{AC}$ is the same as adding vector $\overrightarrow{AC}$ to $\overrightarrow{CE}$ which implies that $\overrightarrow{AD} + \overrightarrow{AC} = \overrightarrow{AC} + \overrightarrow{CE} = \overrightarrow{AE}$($\overrightarrow{AC} + \overrightarrow{CE} = \overrightarrow{AE}$ is a consequence of Triangle inequality)

Once you describe this, example 5 can be considered as support for their practice. You can give that example as a class activity and add more examples of similar type with which students can practice further.

After the discussion of the Laws, it will be essential to deliver scalar multiplication of vectors. For this purpose, you can give chance for the students to discuss in group subtraction and scalar multiplication of a vector. First you may encourage them to do Group Work 7.4 whose possible answers are as follows.

Answers to Group Work 7.4

a. $-\overrightarrow{AB}$ is the vector opposite to $\overrightarrow{AB}$ represented geometrically as

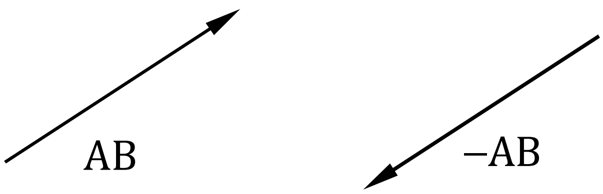


Figure 7.8

b.

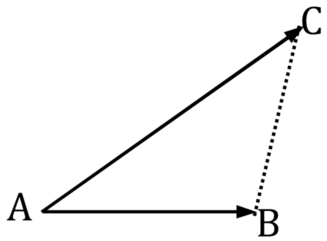


Figure 7.9

c. Vector subtraction is considered simply adding the opposite vector. For example, $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$ is the same as $\overrightarrow{AC} + (-\overrightarrow{AB}) = \overrightarrow{BC}$. In the same way, scalar multiplication of a vector is enlarging or shortening a vector. For example, if we consider vectors $\overrightarrow{AB}$ and $2\overrightarrow{AB}$, then the length of the vector $2\overrightarrow{AB}$ is twice that of the vector $\overrightarrow{AB}$.

d. Geometrically, subtraction of vectors and scalar multiplication of vectors can be represented as follows.

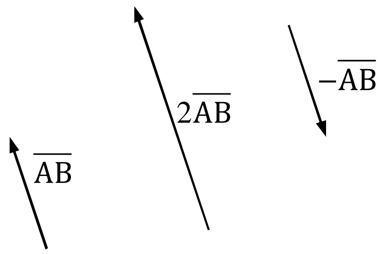


Figure 7.10

What we understand from these vectors is that: if we multiply a vector by a positive scalar greater than 1 its length increases (it becomes longer) maintaining its direction and if the scalar is less than 1 the length of the vector becomes smaller and the vector decreases still maintaining its direction. If we multiply by a negative number, however, the direction changes. The length can enlarge or get shorter depending on the scalar multiplier as well. If we multiply by a proper fraction scalar, then the length shortens where as if we multiply by a negative scalar less than -1 the length increases. In this case, as well, if the fraction is positive direction is preserved, whereas if the fraction is negative the direction changes.

After the group discussion, let the students do Activities 7.6. The purpose of this activity is to let students describe vector subtraction geometrically. Here, they need to identify that the longer diagonal of a parallelogram represents addition and the shorter diagonal represents vector subtraction.

Answers to Activity 7.6

a. $\overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$ b. $\overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD}$ c. $\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}$

After realizing the geometric meaning of vector subtraction through activity 7.6, you let them do Activity 7.7 to discuss and understand scalar multiplication algebraically from vector addition. Pursuant to the effort of the students, you can then give formal definition of scalar multiplication of a vector by a scalar. Let the students' notice how scalar multiplication either elongates or shortens a vector.

Answers to Activity 7.7

a. $2\overrightarrow{AC}$ b. $3\overrightarrow{AC}$ c. $-3\overrightarrow{AC}$

After discussing the concepts under this sub-unit, you may give an assignment to students to come up with practical and real problems that require performing operations on vectors. You can also give them such an assignment to refer physics textbooks and review additional discussions on vectors. Once they become able to add and subtract vectors, you can proceed to scalar multiplication of a vector and mention the fact that in scalar multiplication if the scalar multiplier is positive, it only makes change in the length of the vector while it maintains the direction, whereas if the multiplier is negative, it makes the direction opposite and changes the length of a vector.

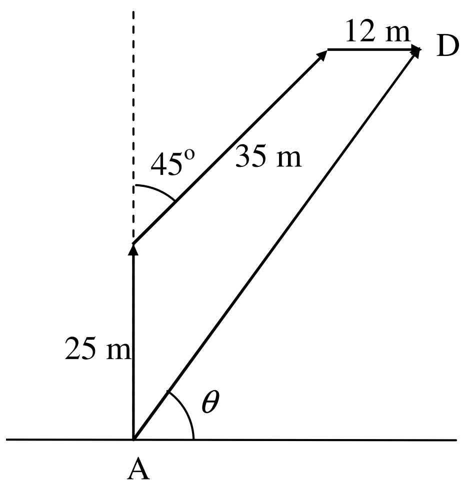
Assessment

Ask students to determine the sum and difference of some pair of vectors. You can also ask students to enlarge or shorten the pictorial representation of a given vector quantity and let them explain the physical interpretation of enlarging or shortening a vector. These can be made in forms of class activity, group discussions, assignments or quizzes/tests.

You can also give Exercise 7.2 as assignment in groups so that students can discuss each concept and get better understanding.

Answers to Exercise 7.2

1. 5.28m at S55.4°W
2. $\sqrt{29}$ kms at N22°W
3.



$\therefore$ $\overrightarrow{AD} = 60.21\text{m}$

$\theta = 54^{o}$

Figure 7.11

4. a) $\overrightarrow{AB}$ b. $\overrightarrow{CD}$

5. $\vec{u}+\vec{v} = 29.1\text{m}$ due N20°E and $\vec{u}-\vec{v} = 12.4\text{m}$ due N23°W

6. a. b. c. d.

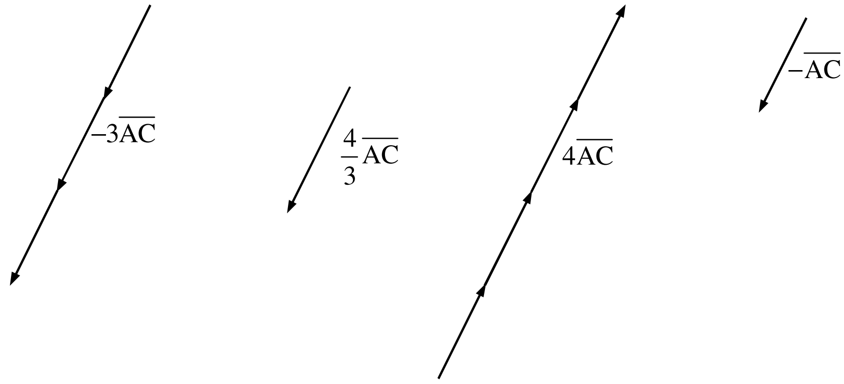


Figure 7.12

7.4 POSITION VECTOR OF A POINT

Periods allotted: 2 Periods

Competency

At the end of this subunit, students will be able to:

- *express any given vector as position vector.*

Vocabulary: Position vector, Coordinate representation of vectors

Introduction

From previous sub-units students can represent vectors in different forms and operate addition (subtraction) and scalar multiplication on vectors. All the vectors the students were discussing were position free. Another form of representing vectors which is mathematically easier is, however, to use the position vector. In this sub-unit they will discuss position vector of a point and will relate these with the discussions they had previously on representing vectors. They will also discuss determining magnitude and direction of vectors by using their position vector representation with the application of trigonometry that they have discussed in unit 5.

Teaching Notes

In the previous lessons students are already made familiar with vectors and vector operations. Here we expect them to see how vectors can be represented as position vectors. The vectors they were discussing were position free vectors. For this purpose, you can start this lesson by encouraging the students to do Group Work 7.5 and giving a brief revision about operations of vectors and then proceed into possibility of expressing vectors in their components. Giving a standard form of a vector (and later for any vector) you can ask students to determine the x-component and the y-component of the given vector. You can do this following their group work. The group work is designed to help the students see what it means to position a vector so that its initial point is the origin. With this attempt, they will also discuss the characteristics of the vectors after they are changed to have initial point at the origin.

After this, proceed to discussing presentation of vectors and operations on vectors that include determining the magnitude and direction by use of components of a vector which makes it easier. This is so because they can apply the trigonometric properties they discussed in unit 5. At this stage, you can make them try to determine the position of a point in a coordinate plane that represents a vector by providing its coordinates of the initial and terminal.

Answers to Group Work 7.5

1. a. The vectors do not change direction when they are moved to have the origin as their initial point. The direction of the vector a is 90° and that of vector b is 0° by simple inspection. But, we cannot decide the directions of c and d by inspection. Thus, we may need to apply trigonometry.

   b. If the initial point of the original vector is (x_1, y_1) and its terminal point is (x_2, y_2) then after shifting them to have initial point of (0, 0), the terminal point of this new vector will be $(x_2 - x_1, y_2 - y_1)$.

2. The terminal point of the moved vector will be (3 – 1, 4 – 2) = (2, 2)

3. The vector v can be expressed in terms of its components by using Triangle Inequality as $V = V_1 + V_2$.

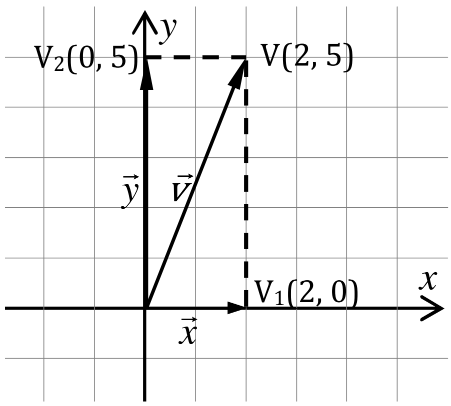


Figure 7.13

Following these discussions, the students need to see vectors that have different initial points but with equal magnitude and direction to get the idea of position free vectors. They also need to know that the position vector of these specific vectors is the same. To have a better understanding on position vectors you can give chance to the students to do Activity 7.8 dealing with the following.

Consider a vector $\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

1. Represent it geometrically
2. Find its position vector
3. Applying the Triangle Law of vector addition, determine the components of $\vec{u}$
4. Find the magnitude of $\vec{u}$
5. Determine the direction of the vector $\vec{u}$

Answers to Activity 7.8

1. We can simply take a directed line segment whose initial point is the origin and whose terminal point is the coordinate point (3, 1).We can represent the vector geometrically as

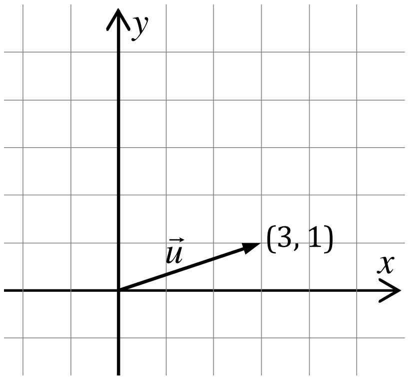


Figure 7.14

2. From Triangle Law, we can simply take the components as 3i and j so that it will be

$\vec{u} = 3i + j$

3. The magnitude is $|\vec{u}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

4. The direction of the vector $\vec{u}$ is $\tan\theta = \frac{1}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{3}\right)$

While discussing such analytic analysis of vectors and their representations, you need to underscore the use of standard form of vector representation. This will help to express

Any vector in standard form and then help to precede every discussion of such standard forms of vectors. This is made by analytically expressing vectors in component form. You can do this by considering the vector with the origin as its initial point and write the coordinates of its terminal point. For example, in two dimensions, you can consider a vector $\vec{u} = (x, y)$ which means that the initial point of the vector represented by $\vec{u} = (x, y)$ is O(0, 0) and the terminal point is P(x, y) whose geometric representation may be given below.

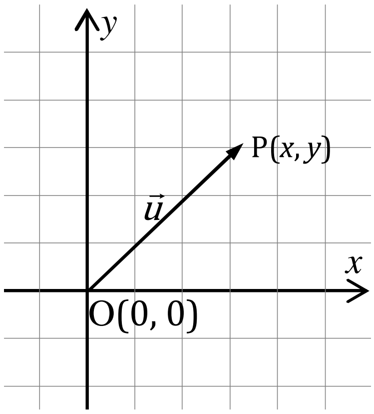


Figure 7.15

Note that the directed arrow depend on which quadrant P(x, y) lies. In this case for example, the point P(x, y) lies at the fourth quadrant.

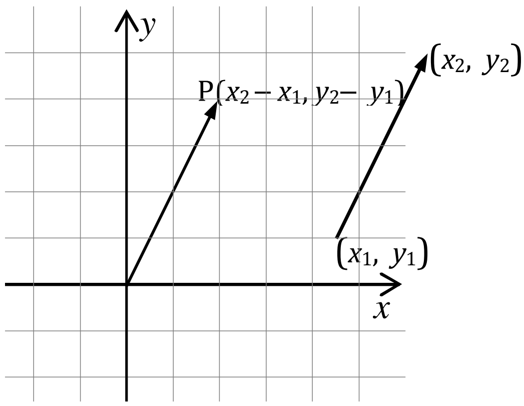


Figure 7.16

Given any vector with initial point (x_1, y_1) and terminal point (x_2, y_2), it is possible to express it in standard form of P ($x_2 - x_1$, $y_2 - y_1$) for which every operation can follow the discussion made earlier. After recognizing students understanding of expressing any vector in standard form, you can proceed to dealing with determining magnitude and direction of a vector under consideration. For this matter, if a vector $\vec{u} = (x, y)$ has O(0, 0) as its initial point and P(x, y) as its terminal point, its magnitude is $|\vec{u}| = \sqrt{x^2 + y^2}$ which is the length of the directed line from O(0, 0) to P(x, y) and its direction is simply an angle whose tangent is $\frac{y}{x}$. Which is the same as $\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$.

That is if the angle is θ then $\tan(\theta) = \frac{y}{x}$. To help the students practice with these ideas, you can give them additional examples. You can also give to students Exercise 7.3 as homework.

Assessment

Ask students to determine the coordinate representation of vectors, their lengths and directions. These can be made in forms of class activity, group discussions, assignments or quiz/test. Since this will be the end of the course, you need also consider each unit in you final examination.

Answers to Exercise 7.3

1. a. $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ b. $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

   c. $\overrightarrow{OC} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ d. $\overrightarrow{OD} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

2. a. $\begin{pmatrix} -3 \\ -7 \end{pmatrix}$ b. $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$ c. $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

3. a. $\overrightarrow{XY} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ b. (–3, 8)

   c. $\left(-1, \frac{5}{2}\right) = \begin{pmatrix} -1 \\ \frac{5}{2} \end{pmatrix}$ d. $\begin{pmatrix} -2 \\ \frac{11}{2} \end{pmatrix}$

4. a. b.

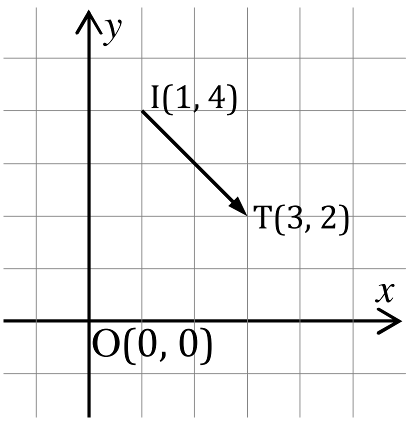


Figure 7.17

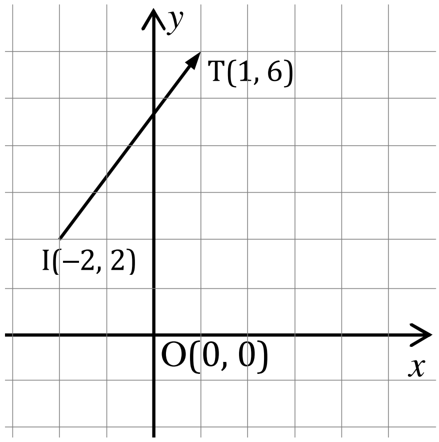


Figure 7.18

5. a.

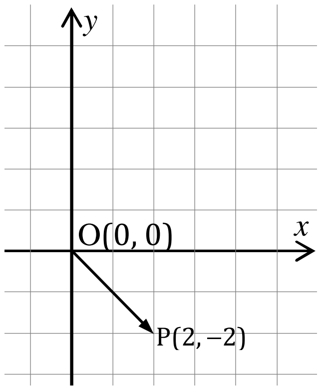


Figure 7.19

b.

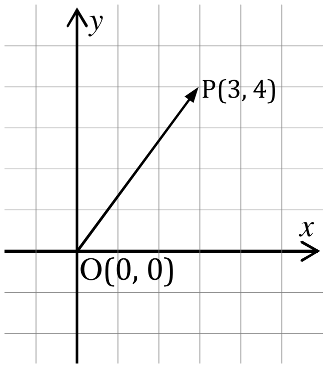


Figure 7.20

6. a. Magnitude OP = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{8} = 2\sqrt{2}$ and its direction is 320°.

b. Magnitude OP = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{25} = 5$ and its direction is 53°.

Answers to Review Exercises on Unit 7

1. Scalar quantities are those quantities of measures that have only magnitude and no direction. (Simply represented by a real number and a specific unit.

   Vector quantities are those quantities of measure that have both magnitude (length) and direction.

2. Scalar quantities: Weight, length, temperature etc. . .

   Vector quantities: Velocity, acceleration etc. . .

3. As a directed arrow.

4. a.

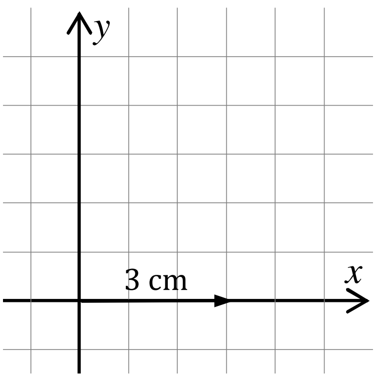


Figure 7.21

b.

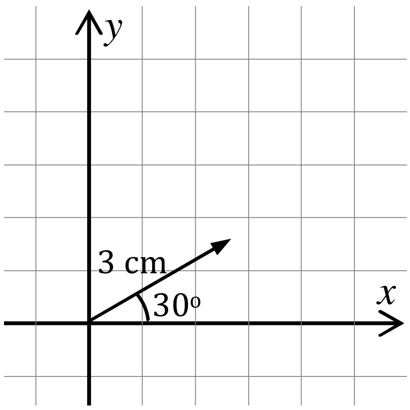


Figure 7.22

5. a.

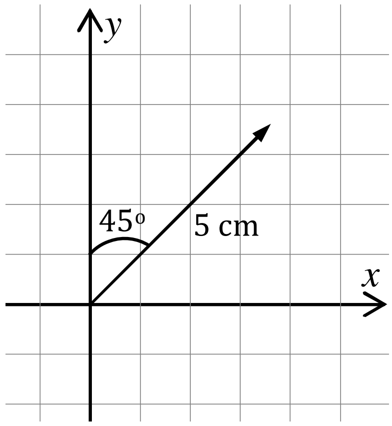


Figure 7.23

b.

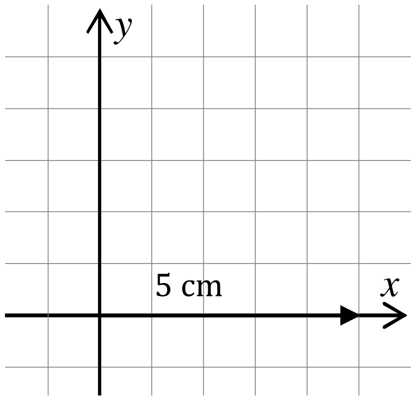


Figure 7.24

c.

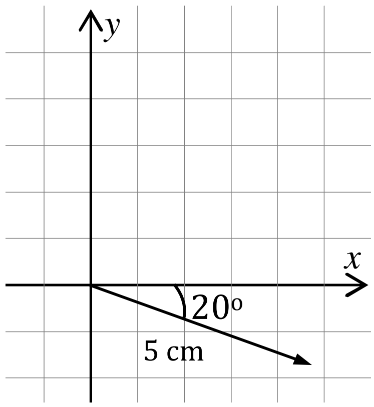


Figure 7.25

6. By the time they have the same direction or opposite direction.

7. **Proof:** Let the initial point and terminal point of $\overrightarrow{AB}$ be A (x_1, y_1) and B(x_2, y_2) respectively.

Let again, the initial point and terminal point of $\overrightarrow{BC}$ be B(x_2, y_2) and C(x_3, y_3) respectively, then

$\vec{v}_1 = (x_2 - x_1, y_2 - y_1)$ (the position vector of $\overrightarrow{AB}$)

$\vec{v}_2 = (x_3 - x_2, y_3 - y_2)$ (the position vector of $\overrightarrow{BC}$)

$\vec{v}_3 = (x_3 - x_1, y_3 - y_1)$ (the position vector of $\overrightarrow{AC}$)

Now, $\vec{v}_1 + \vec{v}_2 = (x_2 - x_1, y_2 - y_1) + (x_3 - x_2, y_3 - y_2)$

$= (x_3 - x_1, y_3 - y_1)$

$= \vec{v}_3$

$\Rightarrow v_1 + v_2 = v_3$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Geometrically,

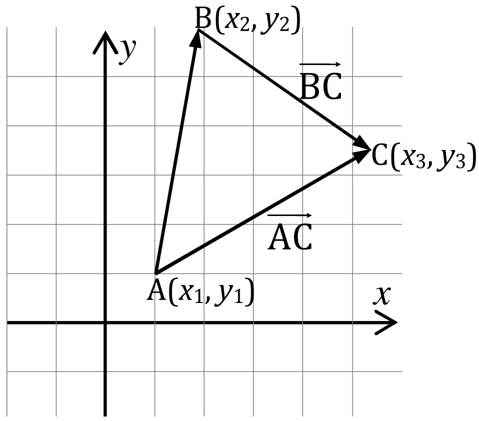


Figure 7.25

8. a. 12cm b. 1cm c. 4cm

9. $\overrightarrow{AB} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$ that $u = -7\mathbf{i} - 4\mathbf{j}$

10. The possible values of $\begin{pmatrix} x \\ y \end{pmatrix}$ are (0, 5), (5, 0), (0, –5), (–5, 0), (3, 4), (–3, 4), (–3, –4), (3, –4), (4, 3), (–4, 3), (4, –3), and (–4, –3).

Table of Trigonometric Functions

↓ →	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	.....	1.000	.....	**90°**
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	**89°**
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	**88°**
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	**87°**
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	**86°**
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	**85°**
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	**84°**
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	**83°**
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	**82°**
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	**81°**
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	**80°**
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	**79°**
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	**78°**
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	**77°**
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	**76°**
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	**75°**
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	**74°**
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	**73°**
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	**72°**
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	**71°**
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	**70°**
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	**69°**
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	**68°**
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	**67°**
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	**66°**
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	**65°**
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	**64°**
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	**63°**
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	**62°**
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	**61°**
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	**60°**
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	**59°**
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	**58°**
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	**57°**
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	**56°**
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	**55°**
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	**54°**
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	**53°**
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	**52°**
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	**51°**
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	**50°**
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	**49°**
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	**48°**
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	**47°**
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	**46°**
45°	0.7071	0.7071	1 .0000	1.000	1.414	1.414	**45°**
	cos	**sin**	**cot**	**tan**	**csc**	**sec**	← ↑

Sample Lesson Plan

Lesson 1

Subject: Mathematics

Main topic: Solving quadratic equations

Objective: At the end of the lesson students will be able to:

- Describe Viete's theorem from roots of a quadratic equation.
- Relate coefficients of a quadratic equation and the roots

Main Content	Teacher Activity	Time (minutes)	Student Activity
Viete's theorem (for addition of roots)	Revise previous lesson on general quadratic solution, and mention the objective of the lesson	4	rehearse previous lesson and state expectations
	For motivating students, give an activity of adding the two roots $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ so that $r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-b}{a}$ and encourage students to relate the roots with the coefficients b and a of $ax^2 + bx + c = 0$, $a \neq 0$ Eg. If x_1 and x_2 are roots of the equation $x^2 + 5x + 6 = 0$, then the sum of the roots $x_1 + x_2$ is ____?	6	Discuss in pairs and share ideas with another group members
	Write the statement of Viete's theorem on the board and encourage students to discuss on the statement and required restrictions	11	Read individually then discuss and explain each point to each other.
	Give some more examples and summarize the lesson	11	Solve some examples and try to summarize the lesson by them selves
	Give some more exercises, organize reflective essay and peer evaluation. Direct the lesson for next session	8	write self assessing reflective essay

Reference Materials

These days search for a reference is at forefront with authentic supply of electronic references. However, with the assumption that there will be limitations in some parts to over utilize ICT, some hard copy reference materials are listed here that can help develop better learning and teaching of mathematics and these units. These books are selected assuming that they are available in many schools. For those who have access to the internet, e-resources are offered as a supplement to those hard copies, if not essentially preferred. You can also access additional reference materials that are available in your school library. These are simply guides to help you use them as references. However, they are not the only to be prescribed. You can also use the web sites given here for reference and demonstration.

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Federal Democratic Republic of Ethiopia

Ministry of Education

Mathematics Syllabus

Grade 9

2009

General Introduction

Mathematics learning at this cycle, should contribute towards educating students to be ready to take part in constructing the future society. At this level everything has to be done to develop personalities characterized by a scientific view of life, high moral qualities and readiness to take part in social activities. Each student should acquire a solid, applicable and extendable mathematical knowledge and develop the appropriate mathematical skills either to pursue with his/her study of preparatory school (Grades 11 and 12) mathematics or join the technical and vocational trainings after which he/she is able to participate in activities of shaping a new society. By including historical facts and real life applications from different fields of social life (agriculture, industry, trade, investment, etc) in word problems, students shall recognize that mathematics is playing an important role in the development of the country.

At this cycle, students should gain solid knowledge of fundamental mathematical notions, theorems, rules and procedures and develop reliable competencies in using this knowledge for solving problems independently.

It is important to identify and realize problems that cause challenging situations to the students and support them in formulating and solving the problems. Formulating and solving problems must be part of a methodical strategy. The task of the teacher is to facilitate in selecting and arranging the order of the problems, as well as helping and motivating students to solve the problems by themselves in a planned and organized way.

Stabilization must have a central place within mathematics learning. It begins with motivation and orientation, by selecting appropriate problems that were already discussed. Concepts that have not been mastered up to now have to be stabilized. A precondition for dealing with new content is always to ensure the necessary level of ability for solving problems. In mathematics learning as a whole, special emphasis has to be put on committing essential facts, notions, definitions, theorems and formulae to the students' memory as well as enabling students to reproduce and interpret what they have learnt in their own words. The main instruments used for stabilization in mathematics learning are activities and exercises.

General Objectives

Objectives of mathematics learning in the first cycle of the secondary education (Grades 9 and 10)

At this cycle students acquire and develop solid mathematics knowledge, skills and attitudes that significantly contribute to the creation of citizens who are conscious of the social, economic, political and cultural realities of Ethiopia and that can actively and effectively participate in the ongoing process of development of the country. To this end, the following are the objectives of mathematics learning at this cycle. Students will be able to:

- appreciate the power, elegance and structure of mathematics.
- use mathematics in their environment and social needs.
- understand the essential contribution of mathematics to Engineering, Science, Economics, Agriculture, etc.
- mathematical knowledge and skills to enable them pursue with their further education or future vocational trainings.
- gain satisfaction and enjoyment from learning and applying mathematics.
- develop their cognitive, creative and appreciative potential by relating mathematics with societal need.

Allotment of Periods

for Units and Sub-units of Mathematics

Grade 9

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
Unit 1: **The number System**	1.1 Revision on the set of rational numbers 1.1.1 Natural numbers, integers, prime numbers and composite number 1.1.2 Common factors and common multiples 1.2 The real number system 1.2.1 Representation of rational numbers by decimals 1.2.2 Irrational numbers 1.2.3 Real numbers 1.2.4 Exponents and radicals 1.2.5 The four operations on real numbers 1.2.6 Limits of accuracy 1.2.7 Standard form (Scientific notation) 1.2.8 Rationalization 1.2.9 Euclid's division algorithm	3 30	33
Unit 2: **Solutions of Equations**	2.1 Equations involving exponents and radicals 2.2 Systems of linear equations in two variables 2.3 Equations involving absolute value 2.4 Quadratic equations	3 8 3 8	22
Unit 3: **Further on Sets**	3.1 Ways to describe sets 3.2 The notion of sets 3.3 Operations on sets	2 4 9	15
Unit 4: **Relations and Functions**	4.1 Relations 4.2 Functions 4.3 Graphs of functions	7 6 9	22
Unit 5: **Geometry and measurement**	5.1 Regular polygons 5.1.1 Measures of each interior angle and each exterior angle of a regular polygon 5.1.2 Properties of regular polygons 5.2 Further on congruency and similarity 5.2.1 Congruency of triangles	5 13	36

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
	5.2.2 Definition of similar figures 5.2.3 Theorems on similarity of triangles 5.2.4 Theorems on similar plane figures 5.2.5 Construction of similar plane figures 5.2.6 Real life problems 5.3 Further on trigonometry 5.3.1 Further on trigonometry 5.3.2 Trigonometric ratios to solve right angled triangles 5.3.3 Trigonometric values of angles from table 5.3.4 Problems on height and distance 5.4 Circles 5.4.1 Symmetrical properties of circles 5.4.2 Angle properties of circles 5.4.3 Arc length, perimeters and areas of segments and sectors 5.5 Measurement 5.5.1 Area of triangles and parallelograms 5.5.2 Further on surface areas and volumes of cylinders and prisms	7 5 6	
Unit 6: Statistics and Probability	6.1 Statistical data 6.1.1 Collection and tabulation of statistical data 6.1.2 Distribution and histogram 6.1.3 Measures of location 6.1.4 Measures of dispersion for un grouped data 6.2 Probability 6.2.1 Probability of an event	15 12	27
Unit 7: Vectors in two dimensions	7.1 Vectors and scalars 7.2 Representation of a vector 7.3 Addition of vectors and multiplication of a vector by a scalar 7.4 Position vector	2 2 6 2	12

Introduction

The curriculum guide for grade 9 is a continuation of the syllabi of mathematics of the preceding grades and is based on the knowledge acquired and competencies developed by students in their mathematics study of the earlier grades. Mathematics learning in grade 9 has to be performed in such a way that students' interest in the subject is stimulated. This can be done by connecting the lesson in the classroom with real life and theory with practice by using students' experience gained from their environment and other subjects. Interesting problems concerning the broad application of mathematics in agriculture, industrial arts, trade, production, investment, the other sciences, etc. should be used.

While planning, the teacher the teacher should always look for hands-on, minds-on and interesting activities that can motivate students to study the subject. Learning has to be facilitated by the teacher in such a way that new subject matter is linked with deepening of the already acquired knowledge and developed abilities and skills.

Objectives

After completing grade 9 mathematics, students should be able to

- deal with and perform the four operations using the set of real numbers.
- solve linear and quadratic equations.
- use basic knowledge about sets to solve related problems.
- develop basic knowledge about relations, functions and their respective graphs.
- know important properties of regular polygons and use the properties to solve related problems.
- use postulates and theorems on congruent and similar figures and solve related real life problems.
- solve real-life problems on height, distance and angle using their knowledge and skills in trigonometry.
- use symmetrical and angle properties of circles to solve related problems.
- calculate are lengths perimeters and areas of segments and sectors
- calculate areas of triangular and parallelogram regions.
- calculate surface areas and volumes of cylinders and prisons.
- collect, tabulate, draw histograms and calculate measures of location and measures of dispersion for ungrouped statistical data.
- calculate probability of an event.
- identify vector and scalar quantities.
- represent vectors pictorially.
- determine the sum of vectors and multiply a given vector by a scalar.
- express any given vector as a position vector.

Unit 1: The Number System *(33 periods)*

Unit outcomes: Students will be able to:

- Know basic concepts and important facts about real numbers
- Justify methods and procedures in computation with real numbers
- Solve mathematical problems involving real numbers.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • identify natural numbers and integers • define prime numbers and composite numbers • determine common factors and common multiples of pairs of numbers.	***1. The Number System*** **1.1 Revision on the set of rational Numbers** *(3 periods)* **1.1.1 Natural numbers, integers, prime numbers and composite numbers** **1.1.2 Common factors and common multiples**	• You can start the lesson by revising the Natural numbers, integers, prime and composite numbers. • Let students revise prime factorization of composite numbers. • Let students revise on finding common factors and common multiples of given numbers. • Discuss with students about the definition of rational numbers together with their important properties	• Turn by turn ask students to tell to the class, what they know about natural numbers, integers, prime numbers, and composite numbers • Asking oral questions • Giving group work exercises' • Giving class activities • Homework and check their work
• show that repeating decimals are also rational numbers	**1.2 The real number system** *(30 periods)*	• After a brief discussion of terminating decimals, then with active participation of students discuss on the method of converting repeating decimals to fraction	• Assignment • quiz III
• identify irrational numbers	**1.2.1 Representation of rational numbers by decimals.**	• You can start the learning by discussing the necessity of irrational numbers as an extension of the number system. For example you may ask students to solve equations of the form $x^2 = 3$	• Ask students to define rational number and to give their own examples of rational numbers.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• locate some irrational numbers on a number line.	**1.2.2 Irrational numbers**	• After considering several examples of irrational numbers (expressed in either decimal form or in radical form), encourage students to describe the nature of irrational numbers and then state the definition of irrational number.	• Ask students to give their own examples of irrational number and let them justify their answers.
• define real numbers. • describe the correspondence between real numbers and points on a numbers line.	**1.2.3 Real numbers**	• Assist students to come to the definition of real numbers in their own words • i.e. real number is the union of rational and irrational numbers and introduce the notation $\Re$ of the set of all real numbers. • Assist students to locate real number say $\sqrt{2}$ or $\sqrt{2}+3$ on the number line and describe the correspondence between real numbers and points on the number line.	• Ask students to give their own examples of real numbers • Give exercise problems on locating a given real number, say $\sqrt{3}$ or $\sqrt{3}+2$ on the number line.
• Realize the relationship between a power with fractional exponent and a radical form. • Convert powers with fractional exponent to radical form and vice-versa	**1.2.4 Exponents and radicals**	• Discuss the relationship between power with fractional exponent form i.e. in which the exponent is $1/_n$ when n is a natural number and radical form. • Assist students to convert from one form to the other form and encourage them to come to the rule which is stated as $a^{1/n} = {}^n\sqrt{a}$ where $a > 0$ and $n \in \mathbf{N}$	• Ask students to describe what is meant by $a^{1/n} = {}^n\sqrt{a}$ for different values of n and a e.g. $9^{½} = \sqrt{9}$, $8^{1/3} = \sqrt[3]{8}$, $4\sqrt{725}$,..

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• perform any one of the four operation on the set of real numbers • use the laws of exponents to simplify expression.	**1.2.5 The four operations on real numbers.**	• You may start the lesson by asking students to perform the four operation on rational numbers; • Assist students to perform the same operations on real numbers. i.e. which involves radicals. • Let students generalize the commutative and associative properties of addition and multiplication on the set of real numbers • Discuss with students the distributive properties of multiplication over addition and discuss also the existence of the additive and multiplicative identities and inverses for every real number (except 0 which has no multiplicative inverse) • Let students state the laws of exponents after considering several examples i.e. (1) $a^n \times a^m = a^{n+m}$ (2) $\frac{a^n}{a^m} = a^{n-m}$, (n > m) (3) $(a^n)^m = a^{n \times m} = (a^m)^n$ (4) $(a \times b)^n = a^n \times b^n$	• Give exercise problems on computing with real numbers (i.e., to find the sum, difference, product and quotient of real numbers. • Ask students to identify important properties of addition and multiplication of real numbers, i.e. for a, b, c ∈ $\mathcal{R}$ • a+b=b+a and a× b= b×a • (a+b)+c=a+(b+c) and (a×b) × c = a × (b×c) • a×(b+c) = (a×b) + (a×c) • Ask students questions like the following and let them justify their answer by giving their own examples. (1) if a is rational number and b is an irrational numbers, then - What type of number is a + b? - What type of number is a × b? (2) if both a and b are irrational numbers, then - What type of number is a + b? - What type of number is a × b?

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
			• Give exercise problems on simplification powers by using the laws of exponents.
• give appropriate upper and lower bounds for a given data to a specified accuracy (e.g. rounding off)	**1.2.6 Limits of accuracy**	• You may start the lesson by revising some important points on the rational number expressed in decimal numerals (i.e. terminating and repeating decimals). • With active participation of students discuss on the idea of "rounding off" and the notion of "significant figures" in a number, in doing so verify the concept using several examples. • For example: 4.5, 4.50 and 4.500 although appearing to represent the same number, but do not. This is because they are written in different degree of accuracy, for instance **4.5** is rounded to one decimal place and therefore any numbers from 4.45 up to but not including 4.55 would be rounded to 4.5. On a number line this would be represented as  Using in equality this can be expressed as $4.45 \le 4.5 < 4.55$ the number **4.45** is called the **lower bound** while **4.55** is known as the **upper bound.**	• Give several exercise problems on rounding off and finding the lower and upper boundaries of number (or measurements). • Ask students to find the sum and difference of numbers to a given number of significant figures.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• By considering examples like: "A carpenter measures the width (w) of a window rounded off to 2.4m to the nearest one decimal place (or 2 significant figures). What are the minimum and maximum values of the width (w) when rounding off?" • Since $2.35 \le w < 2.45$, the number 2.35 is the lower bound while 2.45is the upper bound. • You may consider the effect of operations (addition, subtraction and multiplication) on accuracy. **E.g.** The effect of addition on accuracy The two sides of a triangle are 7.6cm and 5.4cm long. Find their sum. **Ans.** If $\ell_1 = 7.6$cm and $\ell_2 = 5.4$, then $7.55 \le \ell_1 \le 7.65$ and $5.35 \le \ell_2 \le 5.45$ or 7.6 ± 0.05cm and 5.4 ± 0.5. Therefore their sum is $(7.6\pm0.05)+(5.4\pm0.05)$ $=13.0 \pm 0.1$cm and sum lies between 12.9cm and 13.1cm. (***Note****: Care should be taken in the calculations and in the numbers taken, the significant figures not to be more than three (or two decimal places))*	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• express any positive rational number in its standard form.	**1.2.7 Standard form (scientific notation).**	• Assist students to practice writing standard notations of positive rational numbers • Assist students to recognize that this notation is useful in writing very small and very large positive numbers.	• Ask students to write large numbers like the population of Ethiopia in standard form. • Give exercises on expressing large or small numbers by using their standard notation.
• explain the notion of rationalization. • identify a rationalizing factor for a given expression.	**1.2.8 Rationalization**	• With active participating of the students discuss the rules of rationalization with the help of examples, in doing so give emphasis on how to determine the rationalizing factors and also emphasize on rationalizing the denominator as it is commonly used.	• Give exercise problems on rationalizing a given expression.
• use the Euclid's division algorithm to express given quotients of the form $\frac{p}{q}$ where p > q.	**1.2.9 Euclid's division algorithms.**	• Assist students to express, state and generalize the Euclid's division algorithm. i.e. given two numbers **p** and **d** where **p > d** the **p = q.d + r**. Where **q** is the quotient and **r** is the remainder and r ≥ 0, in doing so give emphasis on the nature of the numbers, i.e. all **p, q, d** and **r** are non-negative integers and 0 ≤ r < d	• Give exercise problems on the application of the algorithm e.g. 7 = (2 × 3) + 1.

Unit 2: Solution of Equation *(22periods)*

Unit outcomes: Students will be able to:

- solve problems on equations involving exponents and radicals
- solve systems of simultaneous equations in two variables.
- solve simple equations involving absolute values
- solve quadratic equations.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
students will be able to: • solve equations involving exponents and radicals	***2. Solution of Equation*** **2.1 Equations involving exponents and radicals** *(3 periods)*	• You may start the lesson by the rules for exponents and introduce the fact that for a > 0, $a^x = a^y$ if and only if x = y • Assist students to use the above statement and solve some simple equations involving exponents and radicals.	• Class activities • Group discussions • Giving Assignment • Give exercise problems on equations involving exponents and radicals like find x • $2^x = 8$ • $x^2 = 16$ • $\sqrt{x} = 9$
• solve simultaneous equation • identify the three cases of solutions of simultaneous equations (a unique solution, no solution, infinitely many solutions)	**2.2 Systems of linear equation in to two variables.** *(8 periods)*	• With active participation of the students revise how to find solution for a linear equation, i.e., equation like 2x + 3 = 7 and following this discuss with students how the solution of an equation of the form 2x + 3y = 5 is determined. • Introducing the general form of a system of two linear equations with the help of examples. • Discuss the different methods of finding the solutions of the systems of two linear equations' • Help students them solve the system of simultaneous equations using elimination on substitutions or graphical methods.	• **Homework** • **Quiz** • Give exercise problems on the application of each of the methods for solving system of linear equation.

• solve equations involving absolute value	**2.3 Equations involving absolute value.** *(3 periods)*	• You may start the lesson by asking students to state the definition of absolute values. • Assist students to solve equations involving absolute value such as $\|3x - 2\| = 1$ by using the definition of absolute value.	• Give exercise problems on solving equations involving absolute value of linear expression
• solve quadratic equations by using any one of the three methods. • apply viete's theorem to solve related problems	**2.4 Quadratic equation** *(8 periods)* •Solution of quadratic equation using factorization •solution of quadratic equation using completing the square. •Solutions of quadratic equations using the formula. •The relationship between coefficients of quadratic equations and its roots (Viete's theorem)	• Introduce the general form of a quadratic equation. • Discuss on the different methods of determining the solutions of quadratic equation. • Help students to find the solutions of a quadratic equation by factorization and by completing the square methods. • Help students find the solutions of a quadratic equation using the general quadratic formula. • Let students practice on the application of Viete's theorem through different exercises.	• Give exercise problems on solving quadratic equations (ask the application each method) • Ask students questions about the roots of a given quadratic equations.

Unit 3: Further on Sets *(15 periods)*

Unit outcomes: Students will be able to:

- understand additional facts and principles about sets
- apply rules of operation on sets and find the result
- demonstrate correct usage of venn-diagram in set operations
- apply rules and principles of set theory to practical situations.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • describe sets in different ways	***3. Further on Sets*** **3.1 Ways to describe sets** *(2 periods)*	• Revise important points from previous grade which discussed about sets and emphasise on how the sets are described • Guide the students to write a set whose elements are related by mathematical formula • After revising what the students know about set description , introduce the new way set builder methods of describing sets, by using several examples.	• Ask students to describe a given set in as many ways as possible.
• identify the elements of a given set • explain the notion "empty set" and "universal set"	**3.2 The notion of sets** *(4 periods)*	• Assist students to name some elements of a given set and encourage them to explain whether a given object/number belongs to the set or not and to use the appropriate symbol accordingly. • After describing different sets (using either word description, or set builder method) let the student identify which of these set(s) is/are empty set/s	• Ask students to give examples of empty sets particularly from practical situations (like: the set of dogs that can fly) • Ask the students to explain the difference among each sets: { }, {x} and {0}
• determine the numbers of subsets of a given finite set and list them. • give the power set of a given set		• You may start the lesson by introducing what is meant by "universal set" and explain when and how to use it using Venn-diagram in illustration of relations among sets.	• Group discussions on what they had learnt about sets in earlier grades.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• determine the number of proper subsets of a given finite set and list them. • distinguishes between equal sets and equivalent sets • find equal sets and equivalent sets to a given set		• Assist students to list and count the number of subsets and proper subsets of some given finite sets (i.e. sets with 1, 2, 3, 4 or 5 elements) and encourage the students to derive general formulas to find the numbers of these subsets and proper subsets. • After explaining the meaning of "equal sets" and "equivalent sets" by using several examples, assist the students to determine equal sets and equivalent sets to a given set.	• Ask students to list all the subsets, power sets and proper subsets of a given finite set. • Give exercise problems on finding equal and equivalent sets to a given set.
• determine number of elements in the union of two finite set. • describe the properties of "union" and "intersection" of sets.	**3.3 Operations on sets** *(9 periods)* • "union" and "intersection"	• Let the students be familiar with the notation used for number of elements in a given finite set. • By using some practical examples lead the students to see the relation between the numbers of elements in two finite sets, in their intersection and union. • Let the students apply this relation to find the number of elements in the union of two sets using real life examples/ exercises. • Use several examples and lead your students to conclude that the commutative and associative properties of "union" and "intersection" of sets hold true.	• Ask the students to describe the number of elements using mathematical language • Ask students to find the union and intersection of sets • Ask students to find the number of elements in the union of two sets using both Venn diagrams and formula.
• determine the absolute complement of a given set.	• Complements of a set - De-Morgan's Law	• You can start the lesson by defining the notion "absolute complement" of a given set in terms of the universal set and by using Venn diagram let the students become familiar with the concept and its notation. • Assist students to determine the absolute complement of a given set.	• Ask students to demonstrate De-Morgan's Law using Venn diagram by means of group work approach • Class activities

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• You may use Venn-diagram in your discussion of showing the validity of the De-Morgan's Law and other properties of complements.	
• determine the relative complement of two sets	• Relative complement (difference) of two sets	• Define the notion "Relative complement" or "Difference of two sets" using Venn-diagram. • Let students determine the relative complement of two set and state some of its properties • Lead the students to differentiate between the notions of absolute complement and relative complement.	• Ask the students to explain the difference between relative complement and absolute complements • Ask students to find the relative complement of two given sets.
• determine the symmetric difference of two sets.	• Symmetric difference of two sets	• Start the lesson by defining what is meant by "symmetric difference" of two sets. • Discuss and guide the student to state and explain its properties by using several examples.	• Ask students to describe the symmetric difference of two sets in different ways. i.e. • $A\Delta B = (A \mid B) \cup \Delta (B \mid A)$ • $A\Delta B = ((A \cup B) \mid (A \cap B)$ • Ask students to show the commutative and associative properties of "Δ" by giving specific examples.
• determine the Cartesian product of two sets.	• Cartesian product of two sets.	• Define the notion "Product of two sets" and let the students find out its peculiar nature, i.e., the elements of this set are ordered pairs unlike the sets considered so far. • Let the students explain which properties are true for the product of two sets and which are not true. • Assist the students to see the importance of the Cartesian product of two sets in setting up the coordinate system.	• Ask the students to determine whether an ordered pair/s belong to the product of two given sets or not.

Unit 4: Relations and Functions *(22periods)*

Unit outcomes: Students will be able to:

- know specific facts about relation and function
- understand basic concepts and principles about combination of functions.
- sketch graphs of relations and functions (i.e. of linear and quadratic functions)

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • define the notions "relation", "domain" and "range" • draw graphs of relations • use graphs of relation to determine domain and range	***4. Relations and Functions*** **4.1 Relations** *(7 periods)* • Cartesian product of sets • The Notion "Relation" • Graphs of relations	• You can start the lesson by revising the Cartesian product of two sets. • Assist students to explain the meaning of relation from their daily life. Let them state the formal defining of relation and give examples of relations themselves based on the definition. • You can give some examples of relations and ask students to determine the domain and range • You can start the lesson by discussing with student on how to sketch graphs of relations, like R $= \{(x, y): \quad y < x \}$ R $= \{(x, y) : y > x + 1\}$ etc. and on determining the domain and range from their graphs. • Assist students to draw graphs of relations of the type. R $= \{(x, y) : \quad y \leq x + 1$ and $y \geq 1 - x\}$. and determine the domain and range. • Allow students to practice writing the rule or formula of a relation from its graph.	• Ask students to give their own examples of relation taken from their daily life in mathematical language. • Give exercise problems on algebraic relations, and on their domains and ranges. • Give exercises on determining the domain and range of a relation from its graph. • Give exercises on graphing simple linear inequalities

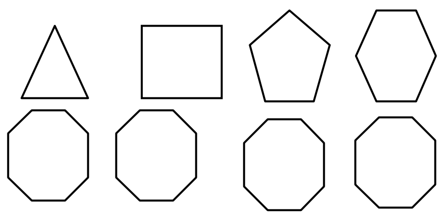
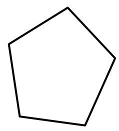
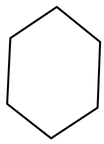
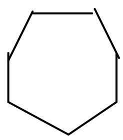
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• define function • determine the domain and range of a given function. • determine the sum difference, produced and quotient of functions.	**4.2 Functions** *(6 periods)*	• You may start the lesson by considering different types of relations. For example, R_1 = {(a, 1) , (b, 2), (c, 3) } R_2 = {(a, 0), (b, 0), c, 0)} R_3 = {(a, 1), (a, 2) , (a, 3) } and • Assist students to observe that types of relations like R_1 and R_2 are function while R_3 is not a function. After considering several examples and discussing them with students state the formal definition of function. • Let students give examples of relations which are functions by themselves. • Assist students to determine the domains and ranges of functions defined by the set of ordered pairs. • You may proceed the lesson by considering functions defined by formulas like.	• Ask students to write down functions written in the form of ordered pairs • Give exercise problems that the students should identify as relations or functions or both and let them give their reason for their answers and let them give the domains and ranges as well. • Ask students to evaluate the value of a given function at given value from its domain.
• Evaluate combination of functions for a given values from their respective domain.	• Combinations of function	f (x) = x + 2 and g(x) = 3 - 3x and guide students to find their sum, difference, product and quotients. Encourage students to determine the relationship between the domains of the component functions and the resulting function. • Assist students to evaluate the sum, difference, product and quotient of functions at a given value of x from the domain.	• Ask students to find combination of simple linear functions, to determine the domain of the resulting function and to find the value of this function at a given value from its domain.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• sketch graphs of linear functions • describe the properties of the graphs of linear functions. • sketch the graphs of a given quadratic function. • describe the properties of the graphs of given quadratic functions • determine the maximum and minimum values of a given quadratic function	**4.3 Graphs of functions** *(9 periods)* • graphs of linear as functions • graphs of quadratic functions	• Define "linear function $y= mx+b$ and quadratic function $y= ax^2 +bx+c \ (a\neq 0)$ " and discuss some basic important properties of each function by using appropriate examples. • You may start the lesson by setting an activity that allows students to construct table of values for given linear and quadratic functions. • Let students sketch the graphs of the given linear and quadratic function whose tables of values are prepared above. • Assist students to describe some of the properties of the graphs of linear and quadratic functions ,the intercepts, the nature of the graphs in relation with the leading coefficients and the coordinates of the vertex of a parabola • Assist students to determine the maximum and minimum values of quadratic function.	• Give exercise problems on sketching the graph of linear and quadratic functions. • Ask students to describe the properties of the graphs of linear and quadratic functions.

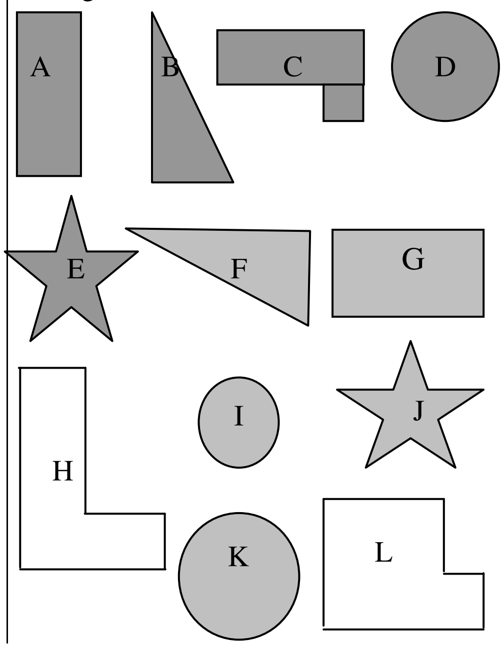
Unit 5: Geometry and Measurement (*36 periods)*

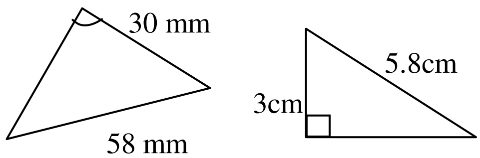
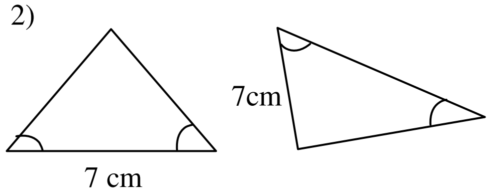
Unit outcomes: Students will be able to:

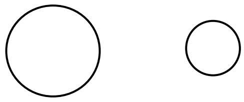
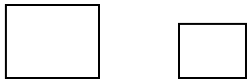
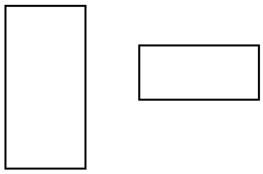
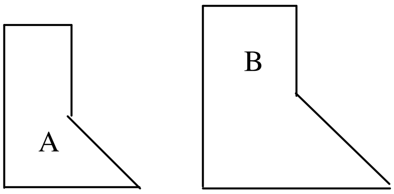
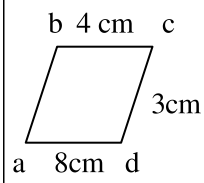
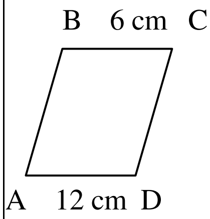
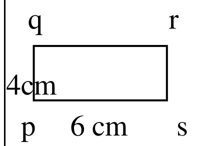
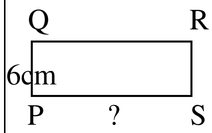
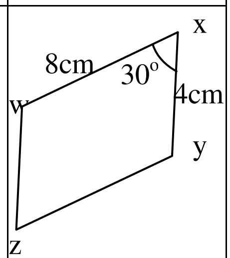
- know basic concepts about regular polygons
- apply postulates and theorems in order to prove congruence and similarity of triangles
- construct similar figures
- apply the concept of trigonometric ratio to solve problems on practical situations
- know specific facts on circles
- solve problems on areas of triangles and parallelograms.

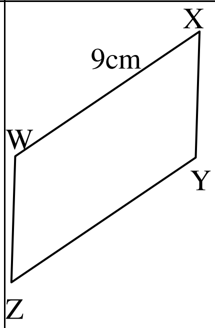
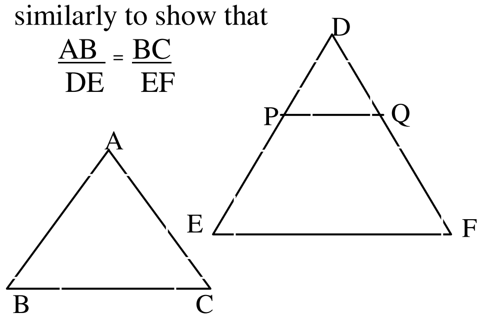
Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • show that the sum of the measures of the interior angles of a triangle is 180^0 • define a regular polygon and related terms	***5. Geometry and Measurement*** **5.1 Regular Polygons** *(5 periods)* **5.1.1 Measures of angles of a regular polygon.**	• It is suggested that the teacher begins this lesson by providing activities to students so that they revise what they have already studied about polygon, then define regular polygon as "a polygon which is equiangular (all angles are congruent) and equilateral (all sides have the same length)" Set of regular polygons  Regular polygons	•"Consider each of the following polygons, and show into how many triangles can it be divided by the diagonals from one vertex to the other vertices?" a  b  c  d 
• find the measure of each interior or exterior angle of a regular polygon.		• **Interior angles:** The interior angles of a polygon are those angles at each vertex on the inside of the polygon. There is one per vertex. For a polygon with n sides, there are n interior angles.	

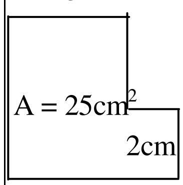
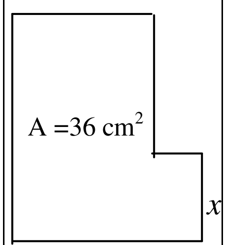
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• Let students discover the theorem "the sum of the measures of the angles of a triangle is 180^0" by cutting the corners of triangle formed from manila paper and fitting the cutouts to form straight angle. • Assist students to arrive at the fact that each interior angle of a regular polygon is given by $\frac{180^o(n-2)}{n}$ by using the angle sum theorem for *n* triangle. • Define exterior angle as "the angle formed on the outside of a polygon between a side and the extended adjacent side. • The students should be aware that the exterior angle and interior angle are measured from the same line, so that they add up to 180^0, so the external angle is just 180^0 - (Measure of interior angle).	• Give exercise problems on calculation of interior angles and exterior angles of a regular polygon such as: a) An equilateral triangle b) regular pentagon c) regular hexagon
• state properties and related terms of regular polygons. • determine the lines of symmetry of regular polygons	**5.1.2 Properties of regular polygon** • Pentagon, hexagon, octagon and decagon	• Under "properties of regular polygon" in radius/apothem, circum radius, in circle, circum circle, diagonals, perimeter, area and symmetry of regular polygons, particularly of pentagon, hexagon, octagon and decagon will be dealt with. • Students should be encouraged to find the rules for finding measures of interior and exterior angles, apothem, perimeter and area of a given regular polygon.	• Students can be asked to construct (draw) regular pentagon, hexagon, octagon and decagon and to state the properties of these regular polygons and to show the lines of symmetry of these polygons.
▪ determine the perimeter of a given regular polygon.		• Assist students to find general formulae for finding perimeters and areas of regular polygons, given the length of sides, radius and/or apothem.	• Students calculate perimeters and areas of the polygons

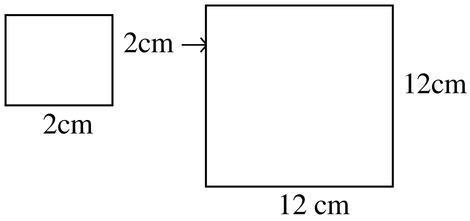
Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
▪ determine the area of a given regular polygon			(Supported by measurement). • Students can be asked to make wall charts on which regular pentagons, hexagons, octagons and decagons and their corresponding lines of symmetries are drawn.
• use the postulates and theorem on congruent triangle in solving related problems.	**5.2 Further on congruency and similarity** (*13 periods)* **5.2.1 Congruency of triangles**	• The lesson can be started by giving revision activity questions like: Which of the following shapes are congruent? 	• Various exercise problems can be given on using the conditions given for triangles to be congruent.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		• Students are encouraged and motivated to revise conditions for triangles to be congruent:- SSS, SAS, AAS and RHS for right - angled triangles. Exercises such as For each of the following pairs of triangles, state whether they are congruent or not. If they are congruent, give reason. 1)  2) 	
• Define similar plane figures and similar solid figures.	**5.2.2 Definition of similar figures**	• The teacher can start this lesson by defining similar figures as: similar figures are identical in shape, but not necessarily in size." • Students can be given different activity problems, and make groups to discuss and come up with answers and reason out.	• Oral questions can be asked demanding students to give examples of similar figures. • Ask students to draw different plane figures and sketch their similar ones.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		Activity: Which of the following pairs are always similar? 1.  Any two circles 2.  Any two squares 3.  Any two rectangles 4.  B is an enlargement of figure A	• Exercise problems such as:- 1.   Figures abcd and ABCD are similar, find the lengths of CD and AB.   The rectangles pqrs and PQRS are similar, What is the length of PS?
		• Students should be encouraged and assisted to come to the conclusion. "For any pair of similar figures, corresponding sides are in the same ratio and corresponding angles are equal"	

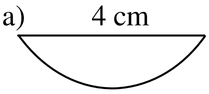
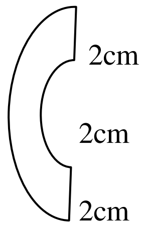
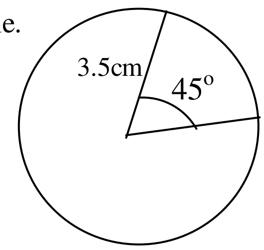
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
			 wxyz and WXYZ are similar figures. a) What is the length of XY? b) What is the size of angle WXY?
• apply the SSS, SAS and AA similarity theorems to prove similarity of triangles	**5.2.3 Theorems on similarity of triangles**.	• The teacher can state these theorems and activities to students so that students verify these theorems E.g. Theorem: If two angles of a triangle are respectively equal to two angles of another triangle then the two triangles are similar. **Activity** Consider 'two triangles ABC and DEF such that $\angle A \equiv \angle D$ and $\angle B \equiv \angle E$ show that Δ ABC is similar to Δ'DEF. • Encourage students to cut DP =AB and DQ = AC and join PQ and show that" $\frac{DP}{DE} = \frac{DQ}{DF}$ so that $\frac{AB}{DE} = \frac{AC}{DF}$ similarly to show that $\frac{AB}{DE} = \frac{BC}{EF}$ 	• Various exercise problems on the application of the similarity theorems can be given and corrected to get feedback.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• discover the relationship between the perimeters of similar plane figures and use this relationship to solve related problems.	**5.2.4 Theorems on similar plane figures** • Ratio of perimeters of similar plane figures. • Ratio of areas of similar plane figures. … .	• Similar activity problems can be given to prove the other theorems as well; for instance assist and encourage them to prove that if a perpendicular is drawn from the vertex of the right angle of a right angled triangle to the hypotenuse, then the triangles on each side of the perpendicular are similar to the given triangle and to each other. • Before students generalize the ratio of perimeters, ratio of areas of similar polygons and .. let them be given various activity problems on finding these ratios from given similar polygons...	• Giving different exercise problems such as:
• discover the relationship between the areas of similar plane figures and use this' relationship to solve related problems. ..			1. The following two shapes are similar. What is the length of *x*?   2. Two similar pyramids have volumes 64 cm^3 and 343 cm^3. What is the ratio of their surface areas?

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• enlarge and reduce plane figures by a given scale factor.	**5.2.5 Construction of similar figures**	• Students can exercise drawing plane figures similar to give ones by multiplying each side by a given scale factor. E.g. Draw a square which has a length six times the given one. 	•Let enough exercise problems be given on construction of similar plane figures. •Let enough exercise problems be given on construction of similar plane figures.
• solve real life problems using the concepts of similarity and congruency.	**5.2.6 Real life problems using congruency and similarity.**		•Different real life exercise problems and activities on the use of similarity and congruence concepts can be given.
• describe radian measure of an angle. • convert radian measure to degree measure and vice versa.	**5.3 Further on Trigonometry** *(7 periods)* **5.3.1 Radian measure of angle** • Conversion between radian and degree measures.	• The teacher can start this lesson by defining the radian measure like: Radian is a central angle subtended in a circle by an arc whose length is equal to the radius of the circle 1 rad. $\therefore$ 1 rad = 57.296^0 1 rad $\approx 57^0$ Thus the **radian measure** of an angle is the ratio of the length of the arc subtending it to the radius of the in which it is the central angle" The circumference of the circle is given by $C = 2\pi r$ Substituting $r = 1$ gives $C = 2\pi (1)$ $C = 2\pi$	• Different exercise problems on the radian measure and on conversion between radian and degree measures can be given Ex 1) Express each of the following in radian a) 45^0 b) 60^0 c) 270^0

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
			2) Express each of the angles in degrees a) $\frac{\pi}{6}$ rad b) $\frac{5\pi}{6}$ rad c) $\frac{4\pi}{3}$ rad
• use the trigonometrical ratios to solve right angled triangles.	**5.3.2 Trigonometrical ratios to solve right angled triangle.**	• Assist students to arrive at conversion formula:- a) from degree measure to radian measure b) from radian measure to degree measure. • Let students exercise solving right - angled triangles using the trigonometrical ratios. (Trigonometrical values for 30^0, 45^0, 60^0 should be studied by heart). • Let students summarize trigonometric ratios of 0^0, 30^0, 45^0 60^0 and 90^0 using table.	
• find the trigonemetrical values of angles from trigonometrical table. • find the angle whose trigonometrical value is given (using trigonometrical table.) • determine the trigonometrical values for obtuse angles using trigonometrical table.	**5.3.3 Trigonometrical values of angles from table (sinθ, cos θ, and tan θ for $0^0 \leq \theta \leq 180^0$)**	• Let students revise the definitions of the trigonometrical ratios sine, cosine and tangent for acute angles using right - angled triangle. • Let students study the values of the ratios for 30^0, 45^0, 60^0 and 90^0. • Assist students how to read trigonometrical table to find the values of the ratios and vice versa. • Encourage the students to use the Cartesian coordinate plane the unit circle and trigonometrical table to find the values for the ratios of obtuse angles.	• Let students exercise finding trigonometrical values of different angles, including obtuse angles using the trigonometrical tables.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• discover the symmetrical properties of circles • use the symmetrical properties of circles to solve related problems	**5.4 Circles** *(5 periods)* **5.4.1 Symmetrical properties of circles**	• Assist students to construct circles and find out that: 1) A circle is symmetrical about every diameter, hence any chord AB perpendicular to a diameter is bisected by the diameter. 2) In Equal circles or in the same circle equal chords are equidistant from the center. Chords which are equidistant from the center are equal 3) Tangents from an external point are equal in length.	• Let students do exercise problems on stating the symmetrical properties of circles and solve related exercise problems using these properties.
• state angle properties of circles in their own words. • apply angle properties of circles to solve related problems	**5.4.2 Angle properties of circles**.	• Let students be familiar with the angle properties of circles and re-state the properties in their own words. 1) An angle at the centre of a circle is twice any angle at the circumference subtended by the same arc 2) Every angle at the circumference subtended by the diameter of a circle is a right angle. 3) Angles in the same segment of a circle are equal.	• Different exercise problem are given and students' works are checked to get feedback.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Find arc length, perimeters and areas of segments and sectors	**5.4.3 Arc length, perimeters and areas of segment and sectors**.	• Let the teacher define arc, segment and sector of a circle and students try to explain the concepts in their own words. • Assist students to calculate lengths of arcs, perimeters and areas of segments and sectors. E.g. 1) Find the perimeter of each of the following. a)  b)  2) Find the area of the shaded region of the circle. 	• Various exercise problems on calculations of arc length, sector, segment areas and perimeters can be given.
• At the end of this lesson the students should be able to: • calculate areas of triangles using Heron's formula, whenever the lengths of the three sides only are given. • calculate areas of parallelograms.	**5.5 Measurement** *(6 periods)* **5.5.1 Areas of triangles and parallelograms** • Heron's formula for the area of triangles. • Area of parallelograms.	Let students be familiarized with Heron's formula to calculate areas of triangles whenever only the lengths of the three sides of a triangle are given. If a, b, c are the lengths of the sides of a triangle, and $s = \frac{a+b+c}{2}$, then the area A of ΔABC is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		The following type of questions can be given. E.g. Find the area of a triangle whose sides are of lengths 20cm, 12cm and 16cm, using Heron's formula. Also find its perimeter. $s = \frac{20 + 12 + 16}{2} = 24$ cm $A = \sqrt{24(24-20)(24-12)(24-16)}$ $= \sqrt{24 \times 4 \times 12 \times 8}$ cm^2 $= (24 \times 4)$ $= 96$ cm^2 $P = (20 + 12 + 16)$ cm $= 48$ cm	• Different exercise problems on calculations of areas of triangles and parallelograms can be given. Like: • A triangular field has dimensions 24m, 7m and 25m. • It has a road running around it which is 1m wide. Find the area of the road.
• Calculate the surface areas of cylinders and prisms. • Calculate volumes of cylinders and prisms.	**5.5.2 Further on surface areas and volumes of cylinders and prisms**.	• Let students revise on the definitions of cylinders and prisms. • Let students make models of these solids. • Assist students to formulate and use area formula for these solids. • The formula for the volumes of the solids are given as V = Area of base $\times$ height $V_{cylinder} = (\pi r^2) \times h$ $V_{Prism} = (\ell \times w) \times h$ Let students exercise on calculations of surface areas and volumes of cylinders and prisms	• Different activities on calculation of surface areas and volumes of cylinders and prisms can be given.

Unit 6: Statistics and Probability *(22 periods)*

Unit outcomes: Students will be able to:

- know methods and procedures in collecting and presenting simple statistical data.
- know basic concepts about statistical measures.
- understand facts and basic principles about probability
- solve simple mathematical problems on statistics and probability

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • differentiate primary and secondary data • collect data from their environment • classify and tabulate primary data according to the required criteria.	***6. Statistics and Probability*** **6.1 Statistical Data** *(14periods)* **6.1.1 Collection and tabulation of statistical data**	• You may begin the lesson by discussing the importance and purposes of statistics by raising issues like HIV/AIDS, population growth, Health and Transport, etc. • After introducing the notions "Population" and "Population function" discuss the concept of "Descriptive statistics" and by using several examples let the student differentiate between "primary data" and "secondary data". • Arrange students in groups and let them collect data from their environment (for instance, in their school compound about students achievement in the National Exams, etc.) • After setting a certain criteria let the student either classify or tabulate the data that they collected from their environment (for instance for the data they collected from their school let them present a table for the medium achievers according to their age	• Ask students about some important concepts that they had learnt in the previous grades. • After forming groups among the students let them collect data and present it in tabular form then let them explain and defend their findings (or conclusion) in the class.
• Construct a frequency distribution table for ungrouped data		• Assist students to construct frequency distribution table for various ungrouped data, for instance you may take students' test score as an example.	• Give exercise problems on drawing histograms for a given data.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• construct a histogram for a given data • interpret a given histogram	**6.1.2 Distribution and Histogram**	• After explaining how to draw a histogram, let the students draw a histogram for a certain given frequency distribution. • Guide students to interpret a given histogram i.e., let them explain what they understand from the given histogram.	• Ask students to tell, as much as they can, about the data by observing its histogram.
• determine the Mean, Median and Mode of a given data • describe the purposes and uses of Mean, Median and Mode • identify the properties of the Mean of a given data (population function)	**6.1.3 Measures of Location (Mean, median and Mode(s)**	• You may start the lesson by revising the definitions of Mean, Median and Mode that the students had learnt in grade 7 • Assist students to determine the mean, median and mode for a given data, (the data can be given as a frequency distribution table) • Discuss with your students about the purpose and uses of the Mean, Median and Mode (or why we calculate them). • By using several examples assist students to generalize the properties of "Mean" of a given data (population function) that are. 1) The sum of the deviations from the mean, taken with their proper signs is zero. 2) The mean of the sum or difference of two population functions (of equal numbers of observations) is equal to the sum or difference of the means of the two population functions. 3) The mean of a constant times a population function is equal to the constant times the mean of the population function.	• Give exercise problems on computations of the Mean, Median and Mode of a given data. • Ask students to explain with their own words about the use of these measures of location in interpreting the data (they can also give examples)

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• compute the measures of dispersion for ungrouped data (manually and using scientific calculator) • describe the purpose and use of measures of dispersion for ungrouped data.	**6.1.4 Measures of dispersion for ungrouped data**	• You may start the lesson by introducing the notion "Measures of dispersion" and then discuss what is meant by "range" "variance" and "standard Deviation" of a population function (data) • By using several examples, guide students in computation of range, variance and standard deviation. • Discuss with your students about the purposes and uses of the range, variance and standard deviation.	• Give exercise problems on computation of the range, variance and standard deviation for a given data. • Ask students to describe, with their own words, the purpose of the measures of dispersion in understanding /interpreting a given data.
• determine the probability of an event from a repeated experiment.	**6.2 Probability** *(8 periods)* • **..Probability of an event** Experimental approach	• You can start the lesson by revising important point from grade 8 such as experiments, events, impossible out comes, certain out-come; uncertainty, possibility set, probability (expressed as fraction, decimal or percentage). • You may begin the topic "experimental approach" as follows: form several groups among the students and let each group perform simple activities which lead to the concept of probability, for instance, let them take a coin and toss it 5 times 10 times or 15 times,... and record their observations in the following table.	• Either by letting the students to perform experiments or from a record of events obtained from similar activities let the students obtain probability of the events in the performed experiments.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr><td></td><td></td><td>
<table>
<tr><th rowspan="2">Number of times the coin is tossed</th><th colspan="2">Number of times the</th></tr>
<tr><th>Head turns up</th><th>Tail turns up</th></tr>
<tr><td>5</td><td>....</td><td>....</td></tr>
<tr><td>10</td><td>....</td><td>....</td></tr>
<tr><td>15</td><td>....</td><td>....</td></tr>
</table>
After completing the above activity guide them to calculate the values of the following two fractions F_1 and F_2 for each trials
$F_1 = \frac{\text{Number of times a head turns up}}{\text{Total number of times the coin is tossed}}$
$F_2 = \frac{\text{Number of times the tail turns up}}{\text{Total number of times the coin is tossed}}$
• From the results they obtained (for F_1 and F_2) guide the students to observe the situation that, as the number of tosses increase the values of F_1 and F_2 gets closer and closer to ½
• At the end encourage them to state "experimental probability, denoted by P(E), of an event E, in n trials is:
$P(E) = \frac{\text{Number of trials in which the event (E) has occurred}}{\text{Total number of trials}}$
and let them internalize the formula by performing such kind of several experiments (like throwing a die) or from a record of events obtained from similar activities so that they can calculate the experimental probability of an event.
</td><td></td></tr>
<tr><td>• determine the probability of an event.</td><td>• Theoretical approach</td><td>• You may start the lesson by revising the main idea of experimental probability of an event and then explain that, in order to compute probability of an event using the experimental approach, it is necessary that the experiment should be done for a large number of times and this makes it difficult.</td><td></td></tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• Through discussion, let the students see the need for an efficient method and hence, introduce the "Theoretical approach of probability" as a second and effective method which is formulated after performing several experiment to compute probability of an event. In the discussion define related terms like "equally likely outcomes" or "possible outcomes" etc. • Assist students in writing all the possible outcomes of a given experiment and in identifying out comes that are favorable to an event in the given experiment for computing probability of the event. You may consider examples like: e.g. If a fair die is thrown the numbers 1, 2, 3, 4, 5 and 6 are "equally likely" to appear (i.e. 1, 2, 3, 4, 5 and 6 are the possible outcomes) thus we say the probability of showing up any one of the six number is 1/6. • From similar activities encourage the students to suggest the following definition of theoretical probability of an event. "Theoretical probability of an event E, written as P(E), is defined as follows. $P(E) = \frac{\text{Number of outcomes favorable to the event E}}{\text{Total number of possible outcomes}}$ • By giving various types of several exercises let the students familiarize themselves with the calculation and concepts of probability.	

Unit 7: Vectors In Two Dimensions *(12 periods)*

Unit outcomes: Students will be able to:

- know basic concept specific facts about vectors.
- perform operations on vectors

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • differentiate Vectors from scalars quantities. • represent vectors pictorially • explain what is meant by magnitude and direction of a vector.	***7. Vectors in Two Dimensions*** **7.1 Introduction to vectors and scalars.** *(2 periods)* **7.2 Representation of a vector.** *(2 periods)* **7.1.3 The magnitude and direction of a vector**;	• You may start the lesson with an activity which deals with the concepts "vector quantity" and "scalar quantity" and then state the formal definitions of "vector quantities" and "scalar quantities" • Discuss the representation of vectors by arrows through different examples • Guide students to realize that the magnitude of a vector is proportion and represented by the length of the arrow while its direction is given by the angle that is formed by the arrow with horizontal line or vertical (in case of compass direction) lines, in doing so, use several examples. • Assist students to realize and define opposite vector of a given vector.	• From a given list of different quantities. • Ask students to list vectors and scalars quantities separately and check their work. • Ask students questions like a) Can a vector and its opposite vector have the same initial point? b) Can a vector and its opposite vector lie on the same straight line?
• determine the sum of given vectors • determine the difference of two vectors	**7.3 Addition and Subtraction of vectors and multiplication of a vector by a scalar.** *(6 periods)*	• Discuss the laws of addition of vectors. (triangular law and the parallelogram law) • Introduce the concept of scalar multiplication of vectors. $\vec{a} = k\ \vec{y},\quad k \in \mathbb{R}$	• Ask students to determine the sum of some pair of vector.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• multiply a given vector by a given scales.			• Ask students to enlarge or shorten the pictorial representation of a given vector quantity and let them explain the physical interpretation of enlarging or shortening a vector.
• express any given vector as position vector.	**7.4 Position vector of a point.** *(2 periods)*	• Help students practice how to find position vectors of given vectors. Provided the coordinates of its terminal and initial points are given.	• Ask students to determine the coordinate representation of vectors.

MATHEMATICS

TEACHER'S GUIDE

GRADE 9

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