Vectors

Unit 1

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1.3 Some applications of vectors (page 10)	• Define the term equilibrium. • Explain the importance of forming a triangle of three vectors. • Carry out some experiments to investigate vectors.

Whenever you take a measurement of an object you are recording a physical property of that object. Further physical properties can then be calculated using these measurements. All physical quantities are either scalar or vector quantities. This unit looks at vectors in detail, including examples of vectors, how to add them up and why they are used. Vectors are crucial in a wide range of applications, from landing on the Moon to crossing rivers and to keeping bridges standing up!

1.1 Representation of vectors

By the end of this section you should be able to:

- Define the term vector.
- Give some examples of vector quantities.
- Represent vectors both analytically and graphically.

What are vectors?

If you were asked for directions to your house, simply saying '6 km away' would not be very helpful. Instead you need to provide more information. Along with the distance a direction is also required. Saying '6 km due North from here' provides much

KEY WORDS

scalar *a quantity specified only by its magnitude*

magnitude *size*

vector *a quantity specified by its magnitude and direction*

Think about this...

Magnitude is a scientific way of describing the size of a quantity. For example, a velocity of 50 m/s North has a magnitude of 50 m/s. Scalars are quantities that have a magnitude only.

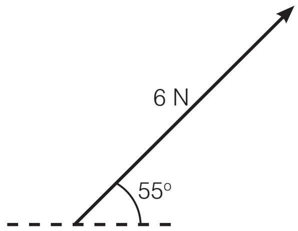


Figure 1.1 *An arrow representing a force of 6 N at about 55° to the horizontal.*

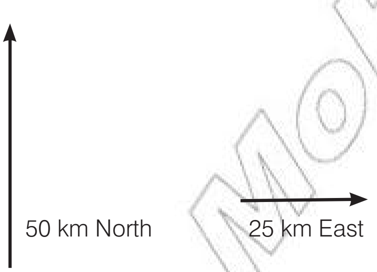


Figure 1.2 *Two different displacement vectors, represented by arrows.*

DID YOU KNOW?

Vectors, as they are understood today, first appeared in a publication called *Vector Analysis* by the American J. W. Gibbs in 1881.

more information. You have provided a **magnitude** (6 km) and a **direction** (North). Quantities that have both a size and a direction are referred to as **vectors**.

Vectors are incredibly useful tools in both mathematics and physics.

- **A vector quantity has both magnitude and direction.**

The alternative, a **scalar** quantity, just has magnitude (size) and there is no direction associated with it. For example, it would be silly to say a chemical energy of 600 J North! Energy is an example of a scalar quantity.

All vector quantities have a direction associated with them. For example, a force of 6 N to the left, or a displacement of 45 km South.

Table 1.1 *Some examples of vector and scalar quantities*

Vector quantities	Scalar quantities
Forces (including weight)	Distance
Displacement	Speed
Velocity	Mass
Acceleration	Energy
Momentum	Temperature

How can we represent vectors?

As all vectors have a direction, we must include one when writing them down. For example, a displacement of 13 km would not be enough information. We must write 13 km South West.

We usually represent vectors using arrows. The length of this arrow represents the size of the quantity and the way it is pointing represents its direction.

Notice in Figure 1.2 that the 50 km vector is twice the size of the 25 km vector.

We often represent vector quantities in equations using bold type or with an arrow above the quantity. For example, to represent force we might write $\mathbf{F}$ or $\vec{F}$. So an important equation like $F = ma$ would be written as $\mathbf{F} = m\mathbf{a}$ or $\vec{F} = m\vec{a}$ as both force and acceleration are vector quantities.

Vectors and scalars should not be confused with SI units.

The International System of Units (SI) defines seven basic units of measurement. These may be seen in Table 1.2 at the top of the next page and all have very exact definitions. For example, the second is defined as the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom!

All other SI units are derived from combining one or more these units. For example, the newton is the SI derived unit of force, 1 N is equivalent to 1 kg m/s^2.

Table 1.2 Some quantities, their units and whether they are vectors

Quantity	SI Unit	Vector or Scalar
Mass	Kilogram (kg)	Scalar
Length	Metre (m)	Sometime scalar (distance) sometime vector (displacement)
Time	Second (s)	Scalar
Temperature	Kelvin (K)	Scalar
Amount	Mole (mol)	Scalar
Electric current	Ampere (A)	Vector
Luminous intensity	Candela (cd)	Scalar

Summary

In this section you have learnt that:

- All physical quantities are either vectors or scalars.
- Vector quantities have both a magnitude and a direction.
- Vectors must include a direction.
- Arrows are used to represent vectors.

Discussion activity

Come up with a list of at least 15 physical properties. Discuss these with your partner and decide if they are scalar or vector quantities. Combine your pairs to form groups of six. Discuss any quantities you are unsure of.

Review questions

1. Give four examples of vector quantities.
2. Explain how vectors differ from scalars. Give some examples.
3. Draw, to scale, three different sized forces acting in different directions. Label them with their size and direction.
4. Abebe wants to lift a 10 N object from the ground. What is the minimum force he needs to exert (include both the magnitude and direction).

1.2 Addition and subtraction of vectors

By the end of this section you should be able to:

- Define the term resultant vector.
- Add two vectors together (including vectors in the same direction, opposite directions and at right angles to each other).
- Determine the angle of a resultant vector.
- Use Pythagoras's theorem to determine the size of the resultant vector.
- Resolve a vector into horizontal and vertical components.
- Find the direction and resultant of two or more vectors using the component method.

Activity 1.1: Representing vectors using arrows

Choosing your own scales draw arrows to represent three vectors:

- 400 km North East
- 32 m/s at an angle of 60° to the horizontal
- A force with a size and direction of your choosing. Include the scale and then pass this to your partner to determine the size and direction of the force.

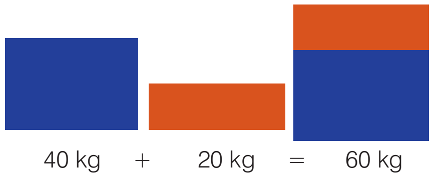


Figure 1.3 *An example of adding scalars*

KEY WORDS

parallel *running in the same direction at the same constant distance apart as another line or surface*

perpendicular *forming an angle of 90 degrees with another line or surface*

parallelogram method

method of resolving *multiple vectors*

resultant *the result of combining multiple vectors*

Think about this...

How fast would you have to run on this bus to have a resultant velocity of 0 km/h?

Activity 1.2: Resultants

Calculate the resultant displacement for the following:

- 10 km left, 20 km right, 30 km left
- 150 km North, 50 km North, 250 km South
- 7 km East, 14 km West, 13 km West.

How do we combine vectors?

Scalars are simple to add. For example, when a mass of 40 kg is added to a mass of 20 kg the total mass is 60 kg.

However, vectors are a bit more complex. When adding two 4 N forces it is possible to get a total of 8 N or 0 N or even 5.7 N!

When you add two or more vectors together the overall vector is called the **resultant**.

Combining parallel vectors

The directions of vectors are really important. If you want to add two **parallel** vectors, for example forces of 6 N and 3 N, you could get 9 N or 3N – as shown below.

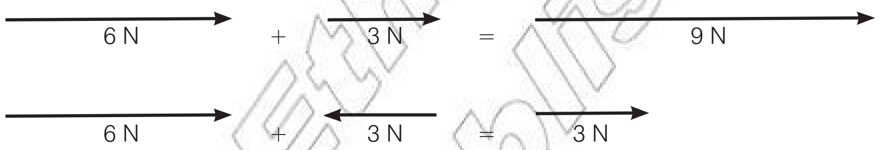


Figure 1.4 *Parallel vector additional (along a line). The resultant is 9 N if they are in the same direction but 3 N if the forces are in opposite directions.*

You could think of the 6 N as positive and the 3 N to the left as negative (–3 N) as it is in the opposite direction. So:

6 N + –3 N = 3 N

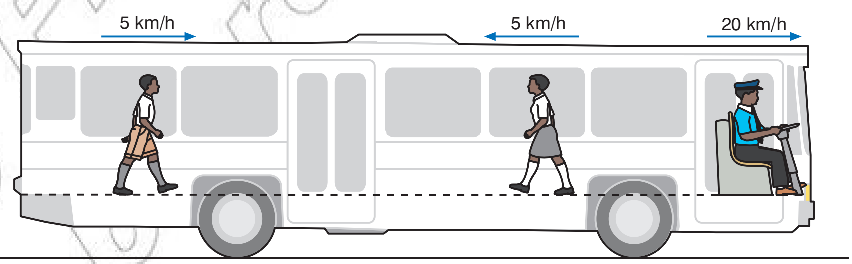


Figure 1.5 *Velocity vectors on a bus*

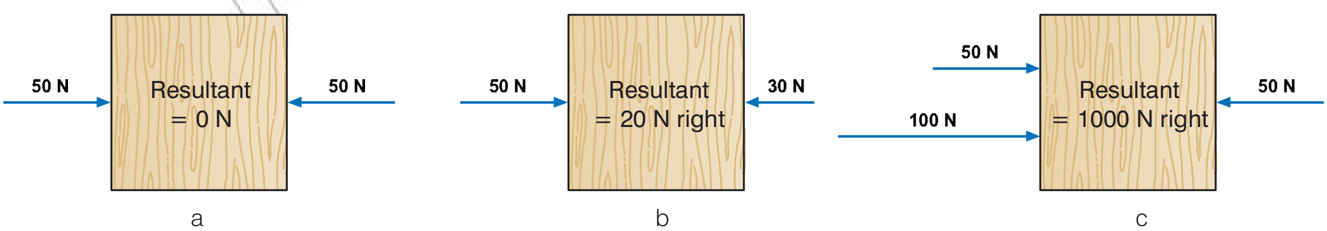
This applies to all vectors. A real world example, this time dealing with velocities, can be seen in Figure 1.5. If the bus is moving North at 20 km/h and you get up and walk towards the front of the bus at 5 km/h your resultant velocity is given by:

v = 20 km/h + 5 km/h = 25 km/h North

If you then turn around and walk to the back of the bus your velocity would be 15 km/h North.

Some other examples involving forces can be seen below.

Figure 1.6 *Adding different parallel vectors (along a line)*



If you have several parallel vectors, the resultant may be found by adding all the vectors in the same direction and subtracting those going in the opposite direction. This can be seen in Figure 1.6c.

Combining perpendicular vectors

But what if the vectors to be added are not parallel?

For example, think about a swimmer swimming from one river bank to another. He swims across the river **perpendicular** to the river bank at 2.0 m/s. However, the river is flowing parallel to the river bank at 1.0 m/s. How can you find his resultant velocity?

One method is referred to as the **parallelogram method.** This involves drawing the two vectors with the same starting point. The two vectors must be drawn to a scale and are made to be the sides of the parallelogram. The resultant will be the diagonal of the parallelogram.

Worked example

1. Choose a scale of 5 cm to represent 1 m/s.
2. Draw the vectors to represent the different velocities of the man starting at the same point.
3. Complete the parallelogram (which in the case of perpendicular vectors is always a rectangle).
4. Draw the resultant vector diagonally across the parallelogram, from A to C (this represents the resultant velocity of the swimmer).
5. Measure the length of AC and convert into m/s. It should be around 11.25 cm long, and this is equivalent to 2.25 m/s (using 1 m/s is 5 cm). The angle from the river bank should be measured as around 64°.

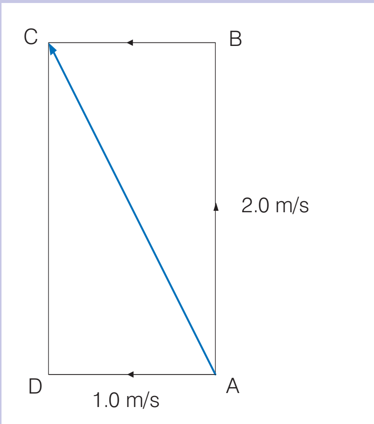


Figure 1.8 *The resultant velocity.*

Pythagoras's theorem

The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

Discussion activity

What are the advantages of the parallelogram method over using mathematics to solve vector problems?

Discussion activity

What is your total displacement during the school day? You begin the day by getting out of bed, and end it by returning to bed.

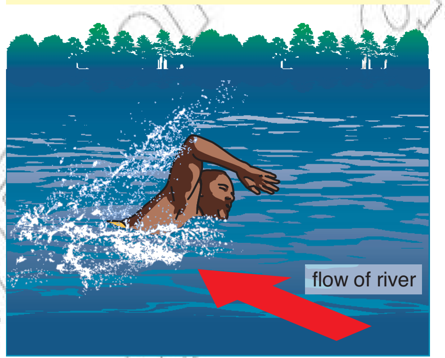


Figure 1.7 *Obang going across Baro River*

Activity 1.3: Using the parallelogram method

Using the parallelogram method, determine the resultant vector in each case:

- 10 km left, 20 km up
- 150 km North, 50 km West
- 7 km East, 14 km North

$$a^2 = b^2 + c^2$$

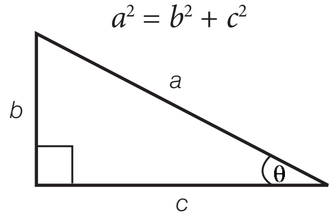


Figure 1.9 *A right-angled triangle*

DID YOU KNOW?

Pythagoras (or to give him his full name, Pythagoras of Samos) was born in ancient Greece around 570 BC. That's over 2500 years ago!

Think about this...

The opposite side is so called because it is opposite the angle.

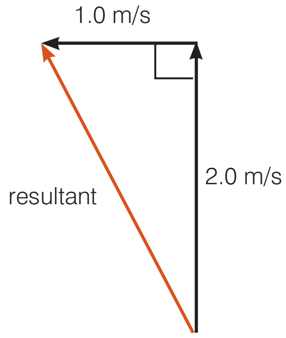


Figure 1.10 The swimmer's velocity vectors shown as a right-angled triangle.

An alternative to the parallelogram method involves using **Pythagoras's theorem** to determine the size of the resultant vector. Trigonometry can then be used to find its direction. This gives a much more precise answer.

Looking again at the swimmer example, a quick sketch of the vectors can be seen in Figure 1.10.

Because the vectors are perpendicular, they form a right-angled triangle. The resultant is the **hypotenuse**, so using Pythagoras's theorem we get:

$a^2 = b^2 + c^2$ *State principle or equation to be used (Pythagoras's theorem)*

$\text{resultant}^2 = 1.0^2 + 2.0^2$ *Substitute in known values*

$\text{resultant}^2 = 5.0$ *Solve for resultant²*

$\text{resultant} = \sqrt{5.0}$ *Rearrange for resultant (take square root) and solve*

resultant = 2.24 m/s (to 3 sig fig) *Clearly state the answer with unit*

This method may be used for any two perpendicular vectors. However, we are missing the direction – all vectors must include a direction.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

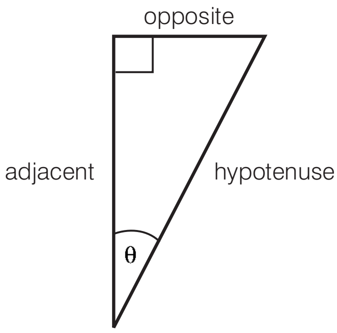


Figure 1.11 Terms used in trigonometry

Trigonometry

Looking back at our simple diagram.

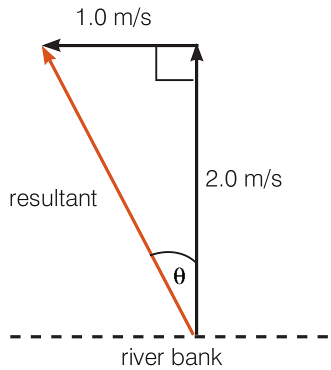


Figure 1.12 The swimmer's velocity vectors shown as a right-angled triangle including the river bank.

Using trigonometry, we can determine angle θ. As we have the side **opposite** the angle (1.0 m/s) and the side adjacent to the angle (2.0 m/s) we should use:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$ *State principle or equation to be used (trigonometry)*

KEY WORDS

hypotenuse *the side of a right-angled triangle opposite the right angle*

opposite *the side of a right-angled triangle opposite the angle being calculated*

Pythagoras's theorem *theorem for calculating the angles and length of the sides of a right-angled triangle*

right angle *an angle of 90 degrees*

$\tan\theta = \frac{1.0}{2.0}$ *Substitute in known values*

$\tan\theta = 0.5$ *Solve for tan θ*

$\theta = \tan^{-1} 0.5$ *Rearrange equation to make θ the subject and solve*

$\theta = 26.6°$ *Clearly state the answer with unit*

This means the angle between the resultant velocity and the river bank is given by 90° – 26.6° = 63.4°.

Both methods give nearly identical answers; the mathematical method offers more precise values.

Table 1.3 *Comparing mathematical and diagrammatic methods for finding resultants*

	Parallelogram method	Mathematical method
Size	2.25 m/s	2.24 m/s
Direction	64°	63.4°

If you have more than two perpendicular vectors you add up the parallel ones first leaving you with two perpendicular vectors from which you can determine the resultant.

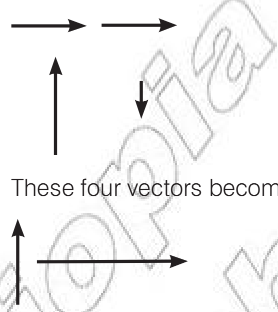


Figure 1.13 *Combining more than two vectors*

Non-parallel and non-perpendicular vectors

So we can now add parallel vectors and perpendicular vectors, but what if the two vectors to be added are not parallel or perpendicular? An example of two forces can be seen below.

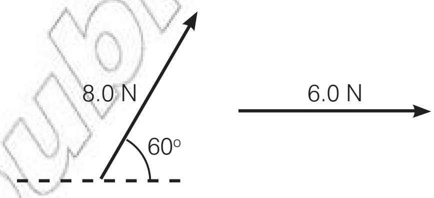


Figure 1.14 *Non-parallel and non-perpendicular vectors (in this case forces)*

We could use the parallelogram method, as before. This can be seen below, but notice that as the vectors are not perpendicular the parallelogram is not a rectangle.

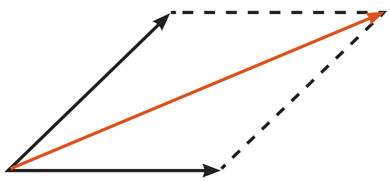


Figure 1.15 *Parallelogram method for non-perpendicular vectors*

The size and the angle of the resultant could then be measured directly. But what if we wanted to find a more precise, mathematical answer?

Draw the two vectors from the same origin. A diagonal passing through their origin describes their resultant.

Trigonometry

hypotenuse × sin θ = opposite

hypotenuse × cos θ = adjacent

Resolving vectors

In order to solve the problem mathematically we need to **resolve** one of the vectors. Resolving means splitting one vector into two component vectors (usually one horizontal and one vertical). These components have the same effect as the original vector. This process is almost the reverse of combining two perpendicular vectors. An example can be seen on the next page in Figure 1.16; the 8.0 N force can be resolved into two component vectors that when combined have the same effect.

KEY WORD

resolve *to split a force or vector into its horizontal and vertical components*

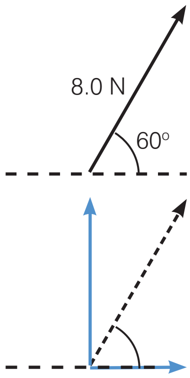


Figure 1.16 *Components shown in blue*

The component vectors form the sides of a right-angled triangle. They make up the opposite and adjacent sides of the triangle. As we know the size of the hypotenuse (in this case 8.0 N) and the angle (in this case 60°) we can then use trigonometry to find their sizes.

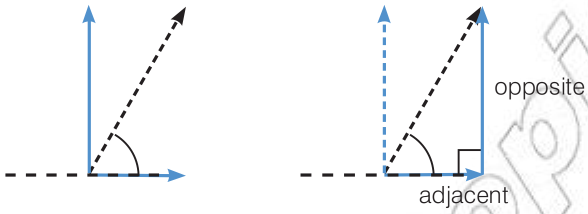


Figure 1.17 *Component vectors as a right-angled triangle*

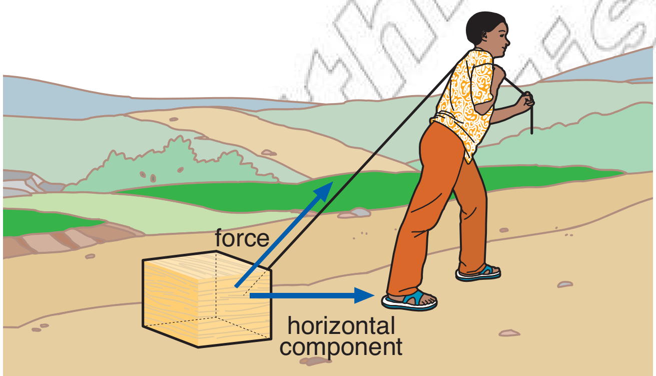


Figure 1.18 *This is an example of where resolving forces may be useful. Although the rope is at an angle it is only the horizontal component that causes the box to move.*

So working through we get:

- hypotenuse × sin θ = opposite
- 8.0 N × sin 60° = 6.9 N, the vertical component
- hypotenuse × cos θ = adjacent
- 8.0 N × cos 60° = 4.0 N, the horizontal component

How is this useful?

We now have three vectors to add together; instead of the 8 N vector we have two components.

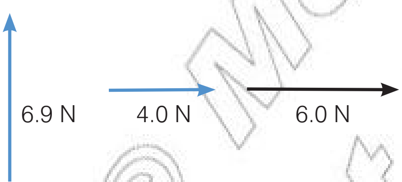


Figure 1.19 *Component vectors to add*

These can then be added to give 10.0 N horizontally and 6.9 N vertically. Using Pythagoras and trigonometry, the size and direction of the resultant can be calculated as before.

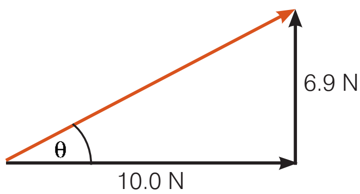


Figure 1.20 *Solution: the resultant is 12.1 N at an angle of 34.6° from the horizontal. Check it yourself!*

Figure 1.21 Vectors are really important to pilots in planning their route.

This technique works for multiple vectors at different angles. For example, adding two velocities (this could be the velocities of an aircraft, one due to the direction it is moving the other due to the wind).

Each of these vectors could then be resolved into horizontal and vertical components. This would give you four vectors to combine.

These could then be added to give two perpendicular vectors. Notice that the horizontal vectors are in different directions and so should be subtracted.

Finally you can use Pythagoras and trigonometry to determine the size and direction of the resultant.

Activity 1.4: Resultant forces

Mathematically determine the resultant force if two forces, A and B, act on an object. Force A is 85 N and is at an angle of 20° to the horizontal. Force B is 125 N and is at an angle of 60° to the horizontal.

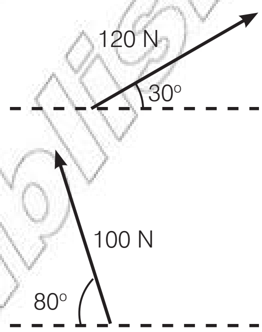


Figure 1.22 *Two vectors at different angles*

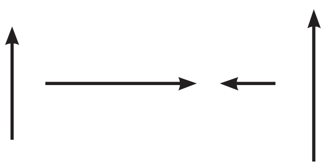


Figure 1.23 *Four components from the two vectors in Figure 1.22*

Summary

In this section you have learnt that:

- The resultant is the sum of two or more vectors.
- When adding vectors their direction is very important.
- The parallelogram method is a quick and easy way to determine the resultant vector.
- To add perpendicular vectors mathematically you use Pythagoras's theorem to find the size of the resultant and trigonometry to determine its direction.
- Resolving a vector means splitting it into two components.
- Resolving vectors enables you to find the result for vectors at different angles.

Review questions

1. Calculate the resultant force in each of the examples below.

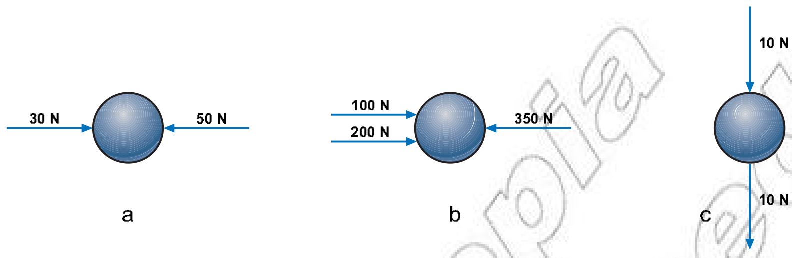


***Figure 1.24** Examples for Question 1*

2. An aircraft is travelling due North with a velocity of 100 m/s. A strong wind blows from the West with a velocity of 25 m/s. Find the resultant velocity, using both the parallelogram method and the mathematical method.
3. Find the resultant force in Figure 1.22.

1.3 Some applications of vectors

By the end of this section you should be able to:

- Define the term equilibrium.
- Explain the importance of forming a triangle of three vectors.
- Carry out some experiments to investigate vectors.

What does equilibrium mean?

As well as their importance in navigation (displacement and velocity vectors), force vectors are incredibly important to all buildings and structures. Often huge forces are involved but in the case of a bridge or building there should be no resultant forces acting. If there was, the bridge would move and perhaps topple over.

When there is no resultant force acting on an object it is said to be in **equilibrium**.

This is easy to imagine in one dimension.

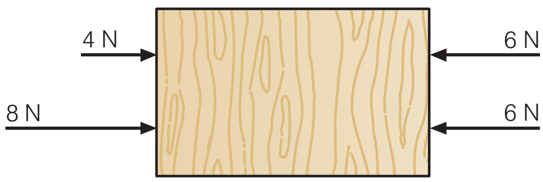


***Figure 1.25** Forces in equilibrium*

The sum of the forces to the left is 12 N. The sum of the forces to the right is 12 N (you could say –12 N). Adding these together gives a

KEY WORD

equilibrium *a state of balance where there are no resulting forces acting on a body* resultant of 0 N. This object is in equilibrium, there is no resultant force acting on it.

In two dimensions this gets a little more difficult. If the vectors are just perpendicular you add up the horizontal forces (those in the *x* direction) and these should give a resultant force of zero. You would then repeat the process for the vertical forces (those in the *y* direction). If all the forces add up to zero then the object is in equilibrium.

DID YOU KNOW?

When in equilibrium, all the horizontal forces (those in the *x* direction) must add up to equal zero. This can be written as:

$\Sigma F_x = 0$

Σ means 'sum of'. So this literally means that the sum of all the forces in the *x* direction is zero.

The same is true for the vertical forces (those in the *y* direction). This can be written as:

$\Sigma F_y = 0$

Scale diagrams

If the forces are not perpendicular then there are two techniques you could use to check if the object is in equilibrium. The first involves drawing a scale diagram.

To do this you simply:

- select a scale for your forces
- draw them to scale, one after the other (in any order), lining them up head to tail ensuring the directions are correct.

If you end up where you started then all the forces cancel out and there is no resultant force (Figure 1.26). However, if there is a gap then there must be a resultant force and the object is not in equilibrium (Figure 1.27).

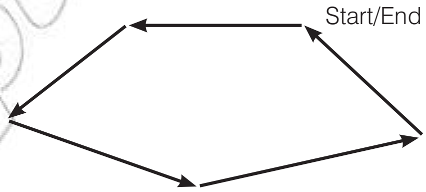


Figure 1.26 *Scale diagram showing no resultant force*

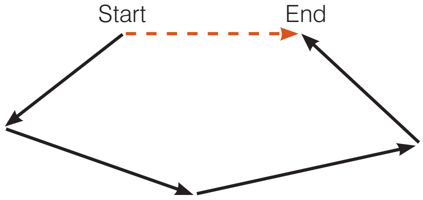


Figure 1.27 *Scale diagram showing a resultant force (the red arrow)*

Triangle of vectors

If there are only three forces acting, then the scale diagram will always be a triangle if the object is in equlibrium.

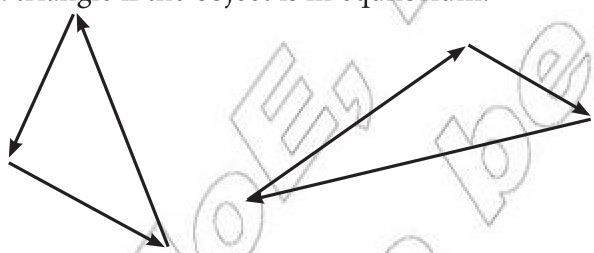


Figure 1.28 *Triangle of vectors*

Proving equilibrium mathematically

If you have several forces you can check they are in equilibrium mathematically.

Take three forces below.

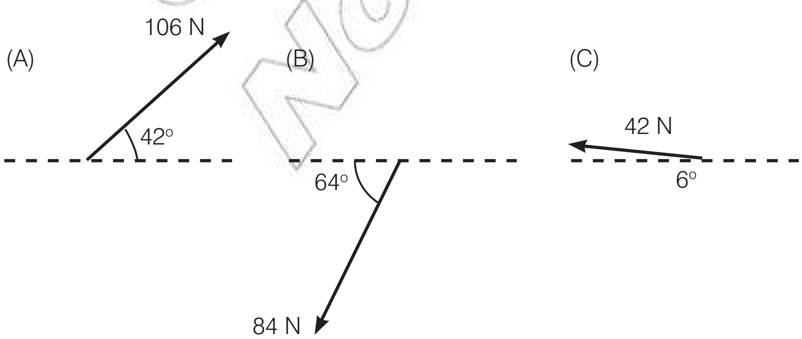


Figure 1.29 *Three forces, A, B and C at different angles*

Each of these forces could then be resolved into horizontal and vertical components. This would give six component vectors – three vertical and three horizontal.

Discussion activity

If you had two forces could you work out the size and direction of a third force required to keep the object in equilibrium?

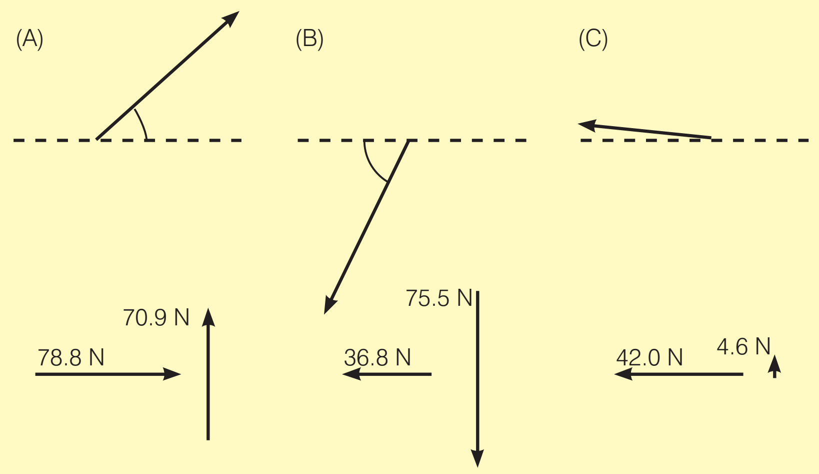


Figure 1.30 *Six components from the three forces in Figure 1.29*

Adding up the vertical vectors:

70.9 N – 75.5 N + 4.6 N = 0 N

Adding up the horizontal vectors:

78.8 N – 36.8 N – 42.0 N = 0 N

There is no resultant force so the object must be in equilibrium. Be careful to ensure you add or subtract the vectors depending on their direction.

You could repeat this technique for any number of forces! If the components don't all cancel out then the object will not be in equilibrium.

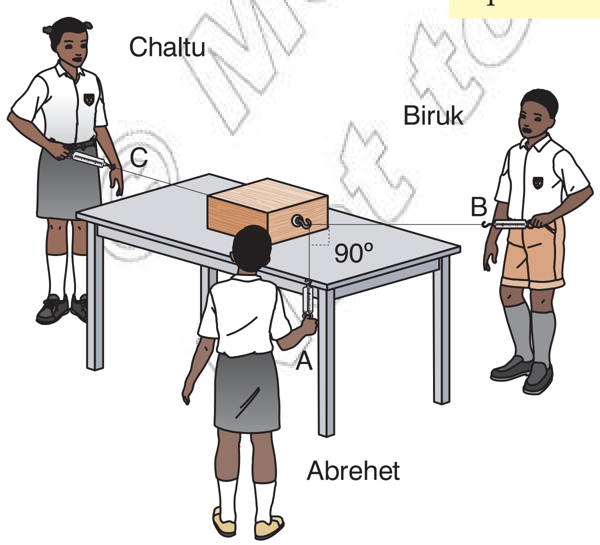


Figure 1.31 *Investigating vectors*

The box pulled by Chaltu, Biruk and Abrehet is in equilibrium. This means that:

The sum of the forces exerted by Abrehet and Biruk is equal to the force exerted by Chaltu

OR

The sum of the forces exerted by Biruk and Chaltu is equal to the force exerted by Abrehet

OR

The sum of the forces exerted by Chaltu and Abrehet is equal to the force exerted by Biruk.

Activity 1.5: Experimentally determining equilibrium

There are a number of experiments you could do to investigate forces in equilibrium. Here is one example.

You are going to pull on a block of wood with two forces. You will find the resultant of the two forces, and then check your findings by vector addition.

- Find a suitable block of wood, and three forcemeters (newtonmeters or spring balances). Place the block on a sheet of plain paper.
- Attach two of the forcemeters (A and B) to one end of the block, as shown in Figure 1.31. Attach the third (C) to the opposite end.
- One person pulls on each forcemeter. A and B should be at an angle of 90° to each other. C is in the opposite direction. Pull the forcemeters so that their effects balance.
- On the paper, record the magnitudes and directions of the three forces.
- Now find the resultant of forces A and B (either by scale drawing or by calculation).
- Because force C balances forces A and B, it must be equal and opposite to the resultant of A and B. Did you find this?
- Repeat the experiment with different forces at a different angle.

You could repeat the experiment without one of the forcemeters. You could then, either by scale diagram or mathematically, determine the size and direction of the unknown force.

Review questions

1. What is meant by the term equilibrium?
2. Give three examples of objects in equilibrium found in the classroom and draw an approximate scale diagram for the object.
3. Three forces are acting on an object (Figure 1.32) which is in equilibrium. Determine force A.

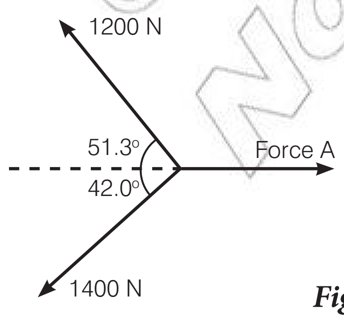


***Figure 1.32** Three forces, acting on a ship.*

Summary

In this section you have learnt that:

- An object is said to be in equilibrium when there are no resultant forces acting on it.
- Scale diagrams can be used to determine whether or not an object is in equilibrium.
- If there are three forces acting on an object in equilibrium then when drawn they form a triangle.
- Using the component method you can mathematically determine if an object is in equilibrium.

End of unit questions

1. Distinguish between a vector and a scalar quantity. Give four examples of each.
2. State which of the following are vectors and which are scalars: distance, mass, time, weight, volume, density, speed, velocity, acceleration, force, temperature and energy.
3. A velocity of magnitude 40 m/s is directed at an angle of 40° East of North. Draw a vector on paper to represent this velocity.
4. A car travels 3 km due North, then 5 km East. Represent these displacements graphically and determine the resultant displacement.
5. Two forces, one of 12 N and another of 24 N, act on a body in such a way that they make an angle of 90° with each other. Find the resultant of the two forces.
6. Two cars A and B are moving along a straight road in the same direction with velocities of 25 km/h and 40 km/h, respectively. Find the velocity of car B relative to car A.
7. Calculate the component of a force of 200 N at a direction of 60° to the force.