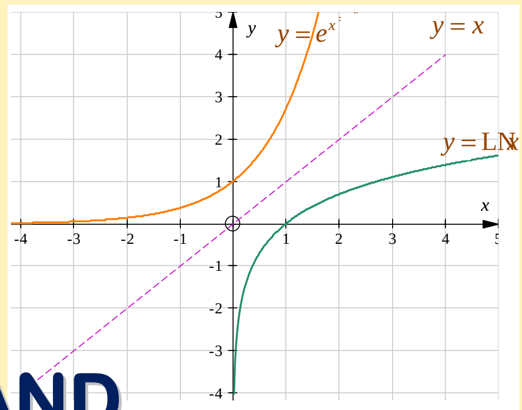
Unit 2

EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Unit Outcomes:

After completing this unit, you should be able to:

- *understand the laws of exponents for real exponents.*
- *know specific facts about logarithms.*
- *know basic concepts about exponential and logarithmic functions.*
- *solve mathematical problems involving exponents and logarithms.*

Main Contents

2.1 Exponents and logarithms

2.2 Exponential functions and their graphs

2.3 Logarithmic functions and their graphs

2.4 Equations involving exponents and logarithms

2.5 Applications of exponential and logarithmic functions

Key Terms
Summary
Review Exercises

INTRODUCTION

EXPONENTIAL AND LOGARITHMIC FUNCTIONS COME INTO PLAY WHEN A VARIABLE IS AN EXPONENT, FOR EXAMPLE, IN AN EXPRESSION SUCH AS 2^x. SUCH EXPRESSIONS ARISE IN MANY APPLICATIONS AND ARE POWERFUL MATHEMATICAL TOOLS FOR SOLVING REAL LIFE P ANALYZING GROWTH OF POPULATIONS OF PEOPLE, ANIMALS, AND BACTERIA; DECAY SUBSTANCES; GROWTH OF MONEY AT COMPOUND INTEREST; ABSORPTION OF LIGHT THROUGH AIR, WATER OR GLASS, ETC.

IN THIS UNIT, YOU WILL STUDY THE VARIOUS PROPERTIES OF EXPONENTIAL AND FUNCTIONS AND LEARN HOW THEY CAN BE USED IN SOLVING REAL LIFE PROBLEMS.

2.1 EXPONENTS AND LOGARITHMS

2.1.1 Exponents

WHILE SOLVING MATHEMATICAL PROBLEMS, THERE ARE OCCASIONS, YOU NEED TO NUMBER BY ITSELF. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64.$$

MATHEMATICIANS USE THE IDEA OF ***exponents*** TO REPRESENT A PRODUCT INVOLVING THE SAME FACTOR. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6.$$

EXPONENTS ARE FREQUENTLY USED IN MANY AREAS OF PHYSICS, ENGINEERING, FINA ETC., TO REPRESENT SITUATIONS WHERE QUANTITIES INCREASE OR DECREASE OVER TIM

OPENING PROBLEM

ETHIOPIA HAS A POPULATION OF AROUND 80 MILLION PEOPLE AND IT IS ESTIMATED TH POPULATION GROWS EVERY YEAR AT AN AVERAGE GROWTH RATE OF 2.3%. IF THE POPULA CONTINUES AT THE SAME RATE,

A WHAT WILL BE THE POPULATION AFTER

I 10 YEARS? **II** 20 YEARS?

B HOW MANY YEARS WILL IT TAKE FOR THE POPULATION TO DOUBLE?

C WHAT WILL THE GRAPH OF THE NUMBER OF PEOPLE PLOTTED AGAINST TIME L

IT IS HOPED THAT AFTER STUDYING THE CONCEPTS DISCUSSED IN THIS CHAPTER, YOU W SOLVE PROBLEMS LIKE THE ONE GIVEN ABOVE.

Exponent notation

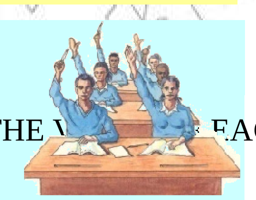
THE PRODUCT $2 \times 2 \times 2 \times 2 \times 2 \times 2$ IS WRITTEN AS 2^6 (READ *"two to the power of six"*).
SIMILARLY, $3^4 = 3 \times 3 \times 3 \times 3$ AND $4^5 = 4 \times 4 \times 4 \times 4 \times 4$.

IF n IS A POSITIVE INTEGER, THEN a^n IS THE PRODUCT OF n FACTORS OF a.

I.E. $a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ FACTORS}}$

IN a^n, a IS CALLED THE **base**, n IS CALLED THE **exponent** AND a^n IS THE n^{TH} **power** OF a.

ACTIVITY 2.1



1 IDENTIFY THE BASE AND THE EXPONENT AND FIND THE VALUE OF EACH OF THE FOLLOWING POWERS:

A 4^3 **B** $(-2)^8$ **C** $\left(\frac{2}{7}\right)^4$ **D** $-(-1)^{23}$ **E** $(5t)^4$

2 FIND THE VALUES OF THE FOLLOWING POWERS:

A $(-1)^1$ **B** $(-1)^2$ **C** $(-1)^3$ **D** $(-1)^4$ **E** $(-1)^5$
F $(-1)^6$ **G** $(-2)^1$ **H** $(-2)^2$ **I** $(-2)^3$ **J** $(-2)^4$
K $(-2)^5$ **L** $(-2)^6$

3 WHICH ONES GIVE YOU A NEGATIVE VALUE: A NEGATIVE BASE RAISED TO AN ODD EXPONENT OR A NEGATIVE BASE RAISED TO AN EVEN EXPONENT?

EXAMPLE 1 EVALUATE:

A $(-3)^4$ **B** -3^4 **C** $(-3)^5$ **D** $-(-3)^5$

SOLUTION:

A $(-3)^4 = -3 \times -3 \times -3 \times -3 = 81$

B $-3^4 = -1 \times 3^4 = -1 \times 3 \times 3 \times 3 \times 3 = -81$

C $(-3)^5 = -3 \times -3 \times -3 \times -3 \times -3 = -243$

D $-(-3)^5 = -1 \times (-3)^5 = -1 \times -243 = 243$

REMEMBER THAT, IN $(-3)^4$ THE BASE IS -3 BUT IN -3^4 ONLY 3 IS THE BASE.

WHAT IS THE BASE IN $(-4t)^3$? THE BASE IS $-4t$ AND $(-4t)^3 = (-4t) \times (-4t) \times (-4t) = -64t^3$

TO WHAT BASE DOES THE EXPONENT 3 REFER IN $-4t^3$? $-4t^3 = -4 \times t \times t \times t$. THEREFORE THE EXPONENT 3 IN $-4t^3$ REFERS TO THE BASE t ONLY.

Laws of exponents

THE FOLLOWING GROUP WORK WILL HELP YOU RECALL THE LAWS OF EXPONENTS DISCUSSE GRADE 9:

Group Work 2.1

1 SIMPLIFY EACH OF THE FOLLOWING:

A $2^3 \times 2^5$	**B** $4^3 \times 4^8$	**C** $\frac{2^7}{2^3}$
D $\frac{2^{-5}}{2^{-9}}$	**E** $(2 \times 3)^3$	**F** $5^{-2} \times 3^{-2}$
G $(3^2)^5$	**H** $\left(\frac{2}{3}\right)^3$	**I** $a^c \times a^d$

2 WHICH LAW OF EXPONENTS DID YOU APPLY TO SIMPLIFY EACH OF THE ABOVE EXPRESSIONS? (DISCUSS WITH YOUR FRIENDS).

IF THE BASES a AND b ARE NON-ZERO REAL NUMBERS AND THE EXPONENTS m AND n ARE INTEGERS, THEN,

1	$a^m \times a^n = a^{m+n}$	TO MULTIPLY POWERS OF THE SAME BASE, KEEP THE BASE AND ADD THE EXPONENTS.
2	$\frac{a^m}{a^n} = a^{m-n}$	TO DIVIDE POWERS OF THE SAME BASE, KEEP THE BASE AND SUBTRACT THE EXPONENTS.
3	$(a^m)^n = a^{m \times n} = a^{mn}$	TO TAKE A POWER OF A POWER, KEEP THE BASE AND MULTIPLY THE EXPONENTS.
4	$(a \times b)^n = a^n \times b^n$	THE POWER OF A PRODUCT IS THE PRODUCT OF THE POWERS.
5	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	THE POWER OF A QUOTIENT IS THE QUOTIENT OF THE POWERS.

EXAMPLE 2 SIMPLIFY EACH OF THE FOLLOWING:

A $(4t)^2 \times (4t)^7$	**B** $r^8 \times r^{-3}$	**C** $\frac{10^3}{10^5}$
D $(x^2)^m$	**E** 16×4^{3t}	**F** $\left(\frac{2y}{25}\right)^2$

SOLUTION:

A $(4t)^2 \times (4t)^7 = (4t)^{2+7} = (4t)^9$

B $r^8 \times r^{-3} = r^{8+(-3)} = r^5$

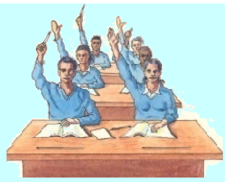
C $\frac{10^3}{10^5} = 10^{3-5} = 10^{-2}$

D $(x^2)^m = x^{2 \times m} = x^{2m}$

E $16 \times 4^{3t} = 2^4 \times (2^2)^{3t} = 2^4 \times 2^{6t} = 2^{4+6t}$

F $\left(\frac{2y}{25}\right)^2 = \frac{2^2 \times y^2}{25^2} = \frac{4y^2}{625}$

ACTIVITY 2.2



1 EVALUATE EACH OF THE FOLLOWING USING THE LAW $\frac{a^m}{a^n} = a^{m-n}$

A $\frac{2^3}{2^3}$; IS 2^0 EQUAL TO 1? WHY? **B** $\frac{10^5}{10^5}$; IS 10^0 EQUAL TO 1? WHY?

C $\frac{(-8)^3}{(-8)^3}$; IS $(-8)^0$ EQUAL TO 1? WHY?

2 FROM YOUR ANSWERS, CAN YOU SUGGEST WHAT ANY NON-ZERO NUMBER RAISED TO ZERO IS?

ANY NON-ZERO NUMBER RAISED TO ZERO IS ONE.

THAT IS, $a^0 = 1$, IF $a \neq 0$

EXAMPLE 3

A $8^0 = 1$ **B** $(-100)^0 = 1$ **C** $\left(\frac{3}{5}\right)^0 = 1$

D $\left(\sqrt{23}\right)^0 = 1$ **E** $(0.153)^0 = 1$

Group Work 2.2

OBSERVE THE FOLLOWING:

- $\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2} = \frac{1}{2^3}$
- IF WE USE THE RULE $\frac{a^m}{a^n} = a^{m-n}$, $\frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$

A USING THE ABOVE TWO STEPS TRY TO SIMPLIFY $\frac{3^5}{3^7}$.

B DISCUSS THE RELATIONSHIP BETWEEN:

I $\frac{1}{2^3}$ AND 2^{-3} **II** $\frac{1}{3^2}$ AND 3^{-2}

C WHAT CAN YOU CONCLUDE ABOUT a^{-n} AND $\frac{1}{a^n}$?

FOR $a \neq 0$ AND $n > 0$

$$a^{-n} = \frac{1}{a^n}$$

ANY NON-ZERO NUMBER RAISED TO A NEGATIVE EXPONENT IS THE RECIPROCAL OF THE SAME POWER WITH POSITIVE EXPONENT.

EXAMPLE 4 SIMPLIFY AND WRITE YOUR ANSWER AS A NON-NEGATIVE EXPONENT.

A 2^{-3} **B** $\frac{2^4}{2^9}$ **C** $\left(\frac{3}{2}\right)^{-3}$

SOLUTION:

A $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B $\frac{2^4}{2^9} = 2^{(4-9)} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

C $\left(\frac{3}{2}\right)^{-3} = \frac{1}{\left(\frac{3}{2}\right)^3} = \frac{1}{\left(\frac{3^3}{2^3}\right)} = 1 \times \frac{2^3}{3^3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

IN EXAMPLE 4C ABOVE YOU HAVE SEEN THAT $\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3$. USE THIS TECHNIQUE TO SIMPLIFY THE FOLLOWING:

EXAMPLE 5

A $\left(\frac{4}{5}\right)^{-1}$ **B** $\left(\frac{2}{5}\right)^{-4}$ **C** $\left(\frac{3}{10}\right)^{-2}$

SOLUTION:

A $\left(\frac{4}{5}\right)^{-1} = \frac{5}{4}$ **B** $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}$ **C** $\left(\frac{3}{10}\right)^{-2} = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Note: FOR $a \neq 0,\ a^{-1} = \frac{1}{a}$

THE ABOVE EXAMPLES LEAD YOU TO THE FOLLOWING FACT:

IF a AND b ARE NON-ZERO REAL NUMBERS THEN IT IS ALWAYS TRUE THAT FOR $n > 0$,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Exercise 2.1

1 USE THE LAWS OF EXPONENTS TO SIMPLIFY EACH OF THE FOLLOWING EXPONENTIAL EXPRESSIONS:

A $t^2 \times t$ **B** $t^3 \times t \times t^5$ **C** $r \times r^4 \times r^5 \times r$ **D** $a^3 \times a \times a^{-5}$

E $\frac{7^6}{7^4}$ **F** $\frac{(-3y)^2}{(-3y)^5}$ **G** $\frac{(2x)^7}{(2x)^8}$ **H** $b^{2x} \div b$

I $(5^5)^{2n}$ **J** $(b^y)^x$ **K** $(7^3)^{-2}$ **L** $(a^{3x})^2$

2 WRITE EACH OF THE FOLLOWING WITH A PRIME NUMBER AS THE BASE:

A 81 **B** $\frac{16^{2x+3}}{16^{2x-3}}$ **C** $\frac{49^x}{7^y}$ **D** $64^a \times 4^a$

3 REMOVE THE BRACKETS FROM EACH OF THE FOLLOWING EXPRESSIONS:

A $(xyz)^2$ **B** $(2ab^2)^5$ **C** $\left(\frac{9}{3}\right)^2$ **D** $\left(-\frac{2}{2n}\right)^6$

4 SIMPLIFY AND GIVE YOUR ANSWERS IN SIMPLEST RATIONAL FORM:

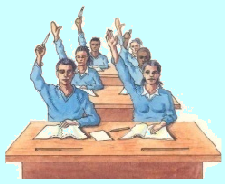
A $\left(\frac{3}{2}\right)^0$ **B** $\left(\frac{8}{3}\right)^{-2}$ **C** $\left(\frac{1}{4^{-3}}\right)^{-1}$ **D** $(-2)^{-5}$ **E** $(3x^2)^{-3}$

Rational exponents

SO FAR WE HAVE CONSIDERED EXPRESSIONS WITH INTEGRAL EXPONENTS. YOU ALL KNOW WHAT 3^5, 2^{-3} AND 7^0 MEAN. BUT WHAT DO EXPRESSIONS SUCH AS $5^{\frac{1}{2}}$ AND $6^{\frac{2}{3}}$ MEAN?

WE NOW EXTEND THE LAWS OF EXPONENTS TO RATIONAL NUMBERS.

ACTIVITY 2.3



USING THE LAW $a^m \times a^n = a^{m+n}$, DO THE FOLLOWING:

1 **A** SIMPLIFY

I $6^{\frac{1}{2}} \times 6^{\frac{1}{2}}$ **II** $\sqrt{6}\times\sqrt{6}$

B COMPARE THE RESULT IN **I** WITH THE RESULT IN **II**. WHAT DO YOU NOTICE?

2 **A** SIMPLIFY

I $6^{\frac{1}{3}} \times 6^{\frac{1}{3}} \times 6^{\frac{1}{3}}$ **II** $\sqrt[3]{6}\times\sqrt[3]{6}\times\sqrt[3]{6}$

B COMPARE THE RESULT IN **I** WITH THE RESULT IN **II**. WHAT DO YOU NOTICE?

3 **A** SIMPLIFY

I $2^{\frac{1}{4}}\times 2^{\frac{1}{4}}\times 2^{\frac{1}{4}}\times 2^{\frac{1}{4}}$ **II** $\sqrt[4]{2}\times\sqrt[4]{2}\times\sqrt[4]{2}\times\sqrt[4]{2}$.

B COMPARE THE RESULT IN **I** WITH THE RESULT IN **II**. WHAT DO YOU NOTICE?

4 IN GENERAL, WHAT DO YOU THINK IS TRUE ABOUT $a^{\frac{1}{n}}$ AND $\sqrt[n]{a}$?

IF $a \geq 0$ AND n IS AN INTEGER WITH $n > 1$, $a^{\frac{1}{n}} = \sqrt[n]{a}$. THIS ALSO HOLDS WHEN $a < 0$ AND n IS ODD. (READ $\sqrt[n]{a}$ AS "THE n^{TH} ROOT OF a".)

EXAMPLE 6 EXPRESS EACH OF THE FOLLOWING IN THE FORM $a^{\frac{1}{n}}$:

A $\sqrt[4]{3}$ **B** $\sqrt[5]{64}$ **C** $\frac{1}{\sqrt{9}}$ **D** $\frac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}}$

SOLUTION:

A $\sqrt[4]{3} = 3^{\frac{1}{4}}$ **B** $\sqrt[5]{64} = 64^{\frac{1}{5}}$ **C** $\dfrac{1}{\sqrt{9}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{(3^2)^{\frac{1}{2}}} = \dfrac{1}{3} = 3^{-1}$

D $\dfrac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}} = \dfrac{\left(32^{\frac{1}{3}}\right)^2}{(2^2)^{\frac{5}{3}}} = \dfrac{32^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \dfrac{(2^5)^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \dfrac{2^{\frac{10}{3}}}{2^{\frac{10}{3}}} = 2^{\left(\frac{10}{3} - \frac{10}{3}\right)} = 2^0 = 1$

WHAT IS THE RESULT OF $6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}}$?

$6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = 6^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 6^{\frac{6}{3}} = 6^2$

ALSO $6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = \left(6^{\frac{2}{3}}\right)^3 = 6^2$ .... *using the law* $(a^m)^n = a^{m \times n}$

THEREFORE, $6^{\frac{2}{3}} = (6^2)^{\frac{1}{3}} = \sqrt[3]{6^2}$

IN GENERAL, IF $a > 0$ AND m, n ARE INTEGERS WITH $n > 1$, THEN $a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

EXAMPLE 7 EXPRESS IN THE FORM $a^{\frac{m}{n}}$, WITH a BEING A PRIME NUMBER.

A $\sqrt[5]{64}$ **B** $\sqrt[3]{16}$ **C** $\sqrt[8]{27}$

SOLUTION:

A $\sqrt[5]{64} = 64^{\frac{1}{5}} = \left(2^6\right)^{\frac{1}{5}} = 2^{\frac{6}{5}}$ **B** $\sqrt[3]{16} = 16^{\frac{1}{3}} = \left(2^4\right)^{\frac{1}{3}} = 2^{\frac{4}{3}}$

C $\sqrt[8]{27} = 27^{\frac{1}{8}} = \left(3^3\right)^{\frac{1}{8}} = 3^{\frac{3}{8}}$

REMEMBER THAT $\sqrt[n]{a}$ IS NOT A REAL NUMBER IF a IS NEGATIVE AND n IS AN EVEN NATURAL NUMBER. HOWEVER, $\sqrt[n]{a}$ IS A REAL NUMBER IF a IS NEGATIVE AND n IS AN ODD NATURAL NUMBER.

FOR EXAMPLE, $\sqrt{-4}$, $\sqrt[4]{-5}$, $\sqrt[6]{-9}$, $\sqrt[8]{-8}$, ETC, ARE NOT REAL NUMBERS, WHEREAS, $\sqrt[3]{-27}$, $\sqrt[5]{-32}$, $\sqrt[3]{-8}$, $\sqrt[9]{-81}$, ETC, ARE REAL NUMBERS.

EXAMPLE 8 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt[3]{-27}$ **B** $\sqrt[7]{-128}$ **C** $\dfrac{\sqrt[5]{-32}}{\sqrt[3]{-64}}$

SOLUTION:

A $\sqrt[3]{-27} = \sqrt[3]{(-3)\times(-3)\times(-3)} = -3$

B $\sqrt[7]{-128} = \sqrt[7]{(-2)^7} = \left(-2^7\right)^{\frac{1}{7}} = -2$

C $\dfrac{\sqrt[5]{-32}}{\sqrt[3]{-64}} = \dfrac{\sqrt[5]{(-2^5)}}{\sqrt[3]{(-4)^3}} = \dfrac{-2}{-4} = \dfrac{1}{2}$

WE CONCLUDE OUR DISCUSSION OF RATIONAL EXPONENTS BY THE FOLLOWING REMAR

ALL RULES FOR INTEGRAL EXPONENTS DISCUSSED EARLIER ALSO HOLD TRUE FOR RATI

Irrational exponents

NOW CONSIDER EXPRESSIONS WITH IRRATIONAL EXPONENTS, SUCH AS $2^{\sqrt{5}}$, $3^{\sqrt{5}}$, 5^{π}, …

EXAMPLE 9 WHICH NUMBER IS THE LARGEST: $2^{\sqrt{5}}$, 4 OR 3?

SOLUTION: THE ANSWER WILL NOT BE SIMPLE BECAUSE WE DO NOT KNOW THE EXACT VALUE OF $2^{\sqrt{5}}$.

TO APPROXIMATE THE NUMBER $2^{\sqrt{5}}$, LET US CONSIDER THE FOLLOWING TABLE FOR 2^x

x	−4	−3	−2	−1	0	1	2	3	4	5
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

FROM THE TABLE WE SEE THAT FOR ANY VALUES OF x_1 AND x_2, IF $x_1 < x_2$, THEN $2^{x_1} < 2^{x_2}$.

THEREFORE, SINCE $2.2 < \sqrt{5} < 2.3$, WE HAVE $2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$.

LET US NOW TAKE CLOSER APPROXIMATIONS TO $2^{\sqrt{5}}$ BY USING A CALCULATOR.

$2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$

$2^{2.23} < 2^{\sqrt{5}} < 2^{2.24}$

$2^{2.236} < 2^{\sqrt{5}} < 2^{2.237}$

$2^{2.2360} < 2^{\sqrt{5}} < 2^{2.2361}$

$2^{2.23606} < 2^{\sqrt{5}} < 2^{2.23607}$

AS WE CAN SEE FROM THE ABOVE LIST, THE NUMBERS $2^{2.2}, 2^{2.23}, 2^{2.236}, \ldots$ APPROACH TO $2^{\sqrt{5}}$. SIMILARLY, THE NUMBERS $2^{2.3}, 2^{2.24}, 2^{2.237}, \ldots$ ALSO APPROACH TO THE SAME NUMBER $2^{\sqrt{5}}$.

SO $2^{\sqrt{5}}$ IS BOUNDED BY TERMS OF CONVERGING RATIONAL APPROXIMATIONS. USING A WE FIND THAT $2^{\sqrt{5}} \approx 4.7111$, TO FOUR DECIMAL PLACES; HENCE $2^{\sqrt{5}}$ IS A NUMBER BETWEEN 4.7 AND 4.8. SO THE LARGEST OF THE NUMBERS 3, 4 AND $2^{\sqrt{5}}$ MUST BE $2^{\sqrt{5}}$.

EXAMPLE 10 GIVE AN APPROXIMATION TO 3^{π}.

SOLUTION: RECALL THAT $\pi \approx 3.1415926$. A CALCULATOR GIVES THE ROUNDED VALUES:

$3^{3.1} \approx 30.1353$
$3^{3.14} \approx 31.4891$
$3^{3.141} \approx 31.5237$
$2^{3.1415} \approx 31.5411$
$3^{3.14159} \approx 31.5442$
$3^{3.141592} \approx 31.5443$
$3^{3.1415926} \approx 31.5443$

HENCE $3^{\pi} \approx 31.5443$, ROUNDED TO FOUR DECIMAL PLACES. A TEN-PLACE CALCULATOR AC
APPROXIMATES 3^{π} BY $3^{3.141592654} \approx 31.5442807002$.

THE ABOVE TWO EXAMPLES SUGGEST THE FOLLOWING:

IF x IS AN IRRATIONAL NUMBER AND $a > 0$, THEN a^x IS THE REAL NUMBER BETWEEN a^{x_1} AND a^{x_2} FOR ALL POSSIBLE CHOICES OF RATIONAL NUMBERS x_1 AND x_2 SUCH THAT $x_1 < x < x_2$.

THE ABOVE STATEMENT ABOUT IRRATIONAL EXPONENTS SUGGESTS THAT THE EXPRESSION a^x IS DEFINED NOT ONLY FOR INTEGRAL AND RATIONAL EXPONENTS BUT ALSO FOR IRRATION

EXAMPLE 11 SIMPLIFY EACH OF THE FOLLOWING:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}}$ **B** $\dfrac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}}$ **C** $\dfrac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}}$.

SOLUTION:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}} = 4^{\sqrt{3}} \times 4^{2\sqrt{3}} = 4^{\sqrt{3}+2\sqrt{3}} = 4^{3\sqrt{3}} = (4^3)^{\sqrt{3}} = 64^{\sqrt{3}}$

B $\dfrac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}} = \dfrac{2^{\sqrt{5}+2\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{2^{3\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{\left(2^3\right)^{\sqrt{5}}}{8^{\sqrt{5}}} = \dfrac{8^{\sqrt{5}}}{8^{\sqrt{5}}} = 1$

C $\dfrac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^0 \times 3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \dfrac{3^{3\sqrt{2}}}{3^{2\sqrt{2}}} = 3^{\left(3\sqrt{2}-2\sqrt{2}\right)} = 3^{\sqrt{2}}$

THE LAWS OF EXPONENTS DISCUSSED EARLIER FOR INTEGRAL AND RATIONAL EXPONEN HOLD TRUE FOR IRRATIONAL EXPONENTS.

IN GENERAL, IF a AND b ARE POSITIVE NUMBERS AND r AND s ARE REAL NUMBERS, THEN

1 $a^r \times a^s = a^{r+s}$ **2** $\dfrac{a^r}{a^s} = a^{r-s}$ **3** $(a^r)^s = a^{rs}$

4 $(a \times b)^s = a^s \times b^s$ **5** $\left(\dfrac{a}{b}\right)^s = \dfrac{a^s}{b^s}$

Group Work 2.3

DISCUSS IN GROUPS AND ANSWER EACH OF THE FOLLO

1 **A** $24 > 23$; IS $24^2 > 23^2$?

B $81 > 16$; IS $81^{\frac{1}{4}} > 16^{\frac{1}{4}}$?

C $20 > 10$; IS $20^{-2} > 10^{-2}$?

D $\frac{1}{100} < \frac{1}{10}$; IS $\left(\frac{1}{100}\right)^2 < \left(\frac{1}{10}\right)^2$?

E $\frac{1}{100} < \frac{1}{10}$; IS $\left(\frac{1}{100}\right)^{-2} < \left(\frac{1}{10}\right)^{-2}$?

2 **A** LET $a > b > 1$.

IS $a^x > b^x$, FOR $x > 0$?

IS $a^x > b^x$, FOR $x < 0$?

B LET $0 < a < b < 1$.

IS $a^x < b^x$, FOR $x > 0$?

IS $a^x < b^x$, FOR $x < 0$?

Exercise 2.2

SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS USING ONE OR MORE OF THE LAWS O

A $a^2 \times a \times a^3$ **B** $(2^{-3} + 3^{-2})^{-1}$ **C** $\left(\sqrt[3]{343}\right)^{-2}$

D $(2a^{-3} \times b^2)^{-2}$ **E** $\frac{(3a)^4}{(3a)^3}$ **F** $\left(\frac{a^2}{b}\right)^3$

G $\left(\frac{a^3}{b^5}\right)^{-2}$ **H** $\frac{(n^2)^4 \times (n^3)^{-2}}{n^{-1}}$ **I** $\left(\frac{m^{-3}m^3}{n^{-2}}\right)^{-2}$

J $\left(\frac{m^{-\frac{2}{3}}}{n^{-\frac{1}{2}}}\right)^{-6}$ **K** $\left(\frac{a^{-\frac{1}{3}}b^{\frac{1}{2}}}{a^{-\frac{1}{4}}b^{\frac{1}{3}}}\right)^6$ **L** $\frac{\left(3^{\sqrt{2}}\right)^2 \times 9^{-\sqrt{3}}}{3^{-\sqrt{12}}}$

M $\left(2^{\sqrt{3}}\right)^2 \div \left(4^{\sqrt{3}}\right)^{-2}$ **N** $\left(\frac{2^{\sqrt{5}} \times 2^{-\sqrt{5}}}{\sqrt{2}}\right)^2$

O $\frac{2^{\sqrt{2}} \times 2^{-\sqrt{2}} \times 32^{\sqrt{2}}}{4^{\sqrt{8}}}$ **P** $\sqrt[6]{64a^6b^{-2}}$

2.1.2 Logarithms

Logarithms CAN BE THOUGHT OF AS ***"the inverse"*** OF EXPONENTS.

FOR EXAMPLE, WE KNOW THAT THE FOLLOWING EXPONENTIAL EQUATION IS TRUE: $3^2 = 9$

IN THIS CASE, THE ***base*** IS 3 AND THE ***exponent*** IS 2. WE WRITE THIS EQUATION IN ***logarithm form*** (WITH IDENTICAL MEANING) AS $\log_3 9 = 2$

WE READ THIS AS "THE LOGARITHM OF 9 TO THE BASE 3 IS 2".

HISTORICAL NOTE:

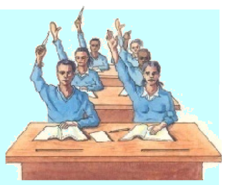
LOGARITHMS ***were developed in the 17th century by the Scottish mathematician, John Napier (1550-1617). They were clever methods of reducing long multiplications into much simpler additions and reducing divisions into subtractions. While he was young, Napier had to help his father, who was a tax collector. John got sick of multiplying and dividing large numbers all day and devised logarithms to make his life easier!***

SINCE $2^4 = 16$, WE CAN SAY THAT $\log_2 16 = 4$.

AS $10^3 = 1000$, $3 = \log_{10} 1000$.

THE FOLLOWING ACTIVITY WILL HELP YOU LEARN HOW TO CONVERT EXPONENTIAL STATEMENTS INTO LOGARITHMIC STATEMENTS AND VICE VERSA.

ACTIVITY 2.4



COMPLETE THE FOLLOWING TABLE:

Exponential statement	Logarithmic statement
$2^3 = 8$	$\log_2 8 = 3$
$2^5 = 32$	
$2^6 = 64$	
	$\log_{10} 100 = 2$
$2^x = y$	

IN GENERAL,

FOR A FIXED ***positive*** NUMBER $b \neq 1$, AND FOR EACH $a > 0$:

$b^c = a$, IF AND ONLY IF $c = \log_b a$.

OBSERVE FROM THE ABOVE NOTE THAT EVERY LOGARITHMIC STATEMENT CAN BE TRA
EXPONENTIAL STATEMENT AND VICE VERSA.

Note: THE VALUE OF $\log_b a$ IS THE ANSWER TO THE QUESTION: " TO WHAT POWER MUST b BE RAISED TO PRODUCE a?"

EXAMPLE 1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

A $3^4 = 81$ **B** $4^3 = 64$ **C** $8^{\frac{1}{3}} = 2$

SOLUTION:

A FROM $3^4 = 81$, WE DEDUCE THAT $\log_3 81 = 4$

B FROM $4^3 = 64$, WE HAVE $\log_4 64 = 3$

C SINCE $8^{\frac{1}{3}} = 2$, $\log_8 2 = \frac{1}{3}$

EXAMPLE 2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

A $\log_{12} 144 = 2$ **B** $\log_4\left(\frac{1}{64}\right) = -3$ **C** $\log_{10}\sqrt{10} = \frac{1}{2}$

SOLUTION:

A FROM $\log_{12} 144 = 2$, WE DEDUCE THAT $12^2 = 144$.

B $\log_4 \frac{1}{64} = -3$ IS THE SAME AS SAYING $4^{-3} = \frac{1}{64}$.

C $\log_{10}\sqrt{10} = \frac{1}{2}$ CAN BE WRITTEN IN EXPONENTIAL FORM AS $10^{\frac{1}{2}} = \sqrt{10}$.

EXAMPLE 3 FIND:

A $\log_2 64$ **B** $\log_3 \frac{1}{9}$ **C** $\log_{1000} 1$

SOLUTION:

A TO FIND $\log_2 64$, YOU ASK "*what power must* 2 *be raised to get* 64?"

AS $2^6 = 64$, $\log_2 64 = 6$ OR FROM THE EXPONENTIAL EQUATIONS DISCUSSED IN GRADE 9, YOU CAN FORM THE EQUATION $2^x = 64$.

SOLVING THIS GIVES $2^x = 2^6 \Rightarrow x = 6$.

... remember that $b^x = b^y$, if and only if $x = y$, for $b > 0$, $b \neq 1$.

B TO FIND $\log_3 \frac{1}{9}$, WE ASK "*to what power must* 3 *be raised to get* $\frac{1}{9}$?"

AS $3^{-2} = \frac{1}{9}$, $\log_3 \frac{1}{9} = -2$ OR $3^x = \frac{1}{9} \Rightarrow 3^x = 3^{-2} \Rightarrow x = -2$.

C TO FIND $\log_{1000} 10$, WE ASK "*to what* POWER *must 1000 be raised to get 10?*"

AS $1000^{\frac{1}{3}} = 10$, $\log_{1000} 10 = \frac{1}{3}$ OR $1000^x = 10 \Rightarrow 10^{3x} = 10^1 \Rightarrow 3x = 1$

$\Rightarrow x = \frac{1}{3}$.

Exercise 2.3

1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

A $100^2 = 10000$ **B** $2^{-5} = \frac{1}{32}$ **C** $125^{\frac{1}{3}} = 5$ **D** $8^{\frac{-2}{3}} = \frac{1}{4}$

2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

A $\log_{10} 10000 = 4$ **B** $\log_7 \sqrt{49} = 1$

C $\log_{10} 0.1 = -1$ **D** $\log_2 \frac{1}{4} = -$

3 FIND:

A $\log_2$ **B** $\log_9 8$

C $\log_{100} 1000$ **D** $\log_{49}$

Laws of logarithms

THE FOLLOWING GROUP WORK WILL HELP YOU OBSERVE DIFFERENT LAWS WHEN USING LOGARITHMS:

Group Work 2.4

1 FIND:

A $\log_2 8 + \log_2 2$; COMPARE THE RESULT WITH $\log_2(8 \times 2)$

B $\log_{10} 100 + \log_{10} 1000$; COMPARE THE RESULT WITH $\log_{10}(100 \times 1000)$

C $\log_3 + \log_3 \left(\frac{1}{27}\right)$; COMPARE THE RESULT WITH $\log_3(9 \times \frac{1}{27})$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b x + \log_b y$?

2 FIND:

A $\log_2 8 - \log_2 2$; COMPARE THE RESULT WITH $\log_2 \left(\frac{8}{2}\right)$

B $\log_{10} 100 - \log_{10} 1000$; COMPARE THE RESULT WITH $\log_{10} \left(\frac{100}{1000}\right)$.

C $\log_3 - \log_3 \frac{1}{27}$; COMPARE THE RESULT WITH $\log_3 \left(9 \div \frac{1}{27}\right)$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b x - \log_b y$?

3 FIND:

A $3\log_2$; COMPARE THE RESULT WITH $\log_2(2^3)$

B $2\log_{10}100$; COMPARE THE RESULT WITH $\log_{10}(100^2)$

C $\frac{1}{2}\log_2 1$; COMPARE THE RESULT WITH $\log_2 16^{\left(\frac{1}{2}\right)}$.

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $k\log_b x$?

4 FIND:

A $\log_3 3$ **B** $\log_8$ **C** $\log_{100} 100$ **D** $\log_{\frac{1}{3}}\frac{1}{3}$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b b$, $b > 0$ AND $b \neq 1$?

5 FIND:

A $\log_3 1$ **B** $\log_4 1$ **C** $\log_{\frac{1}{3}} 1$ **D** $\log_{1000} 1$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR $\log_b 1$, $b > 0$ AND $b \neq 1$?

THE FOLLOWING ARE LAWS OF LOGARITHMS:

IF b, x AND y ARE POSITIVE NUMBERS AND $b \neq 1$, THEN

I $\log_b xy = \log_b x + \log_b y$ **II** $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

III FOR ANY REAL NUMBER k, $\log_b(x^k) = k\log_b x$

Note: IF $b > 0$ AND $b \neq 1$, THEN

I $\log_b b = 1$ **II** $\log_b 1 = 0$

EXAMPLE 4 USE THE LAWS OF LOGARITHMS TO FIND:

A $\log_2 16 + \log_2$ **B** $\log_4\sqrt{16} - \log_4 4$

C $2\left(\left(\log_{10} 100\right)\right)$ **D** $\log_{10}\sqrt[4]{0.01}$

SOLUTION:

A $\log_2 16 + \log_2 = \log_2(16\times = \log_2 = 6$

… *using the law* $\log_b xy = \log_b x + \log_b y$

B $\log_4\sqrt{16} - \log_4 4 = \log_4\frac{\sqrt{16}}{4} = \log_4\frac{4}{4} = \log_4 = 0$

… *using the law* $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

C $2\left((\log_{10} 100) - \right) = 2(\log_{10}100 - \log_{10}10) = 2\log_{10}\left(\frac{100}{10}\right) = 2\log_{10}10 = 2$

… using the law $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

D $\log_{10}\sqrt[4]{0.01} = \log_{10}(0.01)^{\frac{1}{4}} = \log_{10}\left(\frac{1}{100}\right)^{\frac{1}{4}} = \log_{10}\left(10^{-2}\right)^{\frac{1}{4}} = \log_{10}10^{\frac{-2}{4}}$

$= \frac{-2}{4}\log_{10}10 = \frac{-2}{4} \times 1 = \frac{-2}{4} = \frac{-1}{2}$

… using the law $\log_b\left(x^k\right) = k\log_b x$

Two additional laws of logarithms

IF a, b AND c ARE POSITIVE REAL NUMBERS, $b \neq 1$, $a \neq 1$, THEN

I $\log_a c = \frac{\log_b c}{\log_b a}$ ("CHANGE OF BASE LAW") **II** $b^{\log_b c} = c$

EXAMPLE 5 USING THE ABOVE TWO LAWS FIND

A $\log_{16} 6$

B $\log_3$ (GIVEN THAT $\log_{10}2 = 0.3010$ AND $\log_{10}3 = 0.4771$)

C $10^{\log_{10}}$

SOLUTION

A $\log_{16} 6 = \frac{\log_2 6}{\log_2 1} = \frac{6}{4} = \frac{3}{2}$ OR

$\log_{16} 64 = \frac{\log_{10} 64}{\log_{10} 16} = \frac{\log_{10} 4^3}{\log_{10} 4^2} = \frac{3\log_{10} 4}{2\log_{10} 4} = \frac{3}{2} \times 1 = \frac{3}{2}$

… you can use any base $b > 0$, $b \neq 1$

B $\log_3 = \frac{\log_{10}}{\log_{10}} = \frac{0.3010}{0.4771} = 0.6309$ **C** $10^{\log_{10}} = 7$

Exercise 2.4

1 FIND:

A $\log_{11} 121$ **B** $\log_6 6$ **C** $\log_{10} 100000$ **D** $\log_5 125$

E $\log_3\sqrt{3}$ **F** $\log_9 3$ **G** $\log_{100}\sqrt[5]{100}$ **H** $\log_{\frac{1}{5}} 125$

2 Simplify:

a $\log_2(64 \times 1024)$ **b** $\log_2 \frac{32}{256}$ **c** $\log_2 512^3$

d $\log_{10} 2 \times 10^{-3}$ **e** $\log_2\left(\frac{128\times 64}{512}\right)$ **f** $\log_3 + \log_3 \frac{1}{27}$

g $\log_2 64^2 \div \log_2 1^7$

3 Using the laws $\log_a c = \frac{\log_b c}{\log_b a}$ or $b^{\log_b c} = c$ find:

a $\log_{\left(\frac{1}{3}\right)} 8$ **b** $\log_{\left(\frac{1}{2}\right)}$ **c** $\log_{\frac{1}{3}} \frac{1}{27}$

d $5^{\log_5 3}$ **e** $6^{\log_6 10}$

4 If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, then find:

a $\log_2 \sqrt{3}$ **b** $\log_{\frac{1}{2}} 5$ **c** $\log_{\frac{1}{3}} 0.00$

Logarithms in base 10 (common logarithms)

Our decimal system is based on numbers of the form 10^n. For example,

$10000 = 10^4$ $0.0001 = 10^{-4}$

$1000 = 10^3$ $0.001 = 10^{-3}$

$100 = 10^2$ $0.01 = 10^{-2}$

$10 = 10^1$ $0.1 = 10^{-1}$

$1 = 10^0$

Also numbers like $\sqrt{10}$, $\sqrt{100}$, $10\sqrt{10}$ and $\frac{1}{\sqrt[5]{10}}$ can be written as

$10^{\frac{1}{2}}, 100^{\frac{1}{2}}, 10^1 \times 10^{\frac{1}{2}} = 10^{\frac{3}{2}}$ and $10^{\frac{-1}{5}}$ respectively.

In fact, all positive numbers can be written in the form 10^n. This is the idea introducing the concept of logarithms. The logarithm of a positive number to base 10 is called a ***common logarithm***.

The common logarithm is usually the most convenient one to use for co

involving scientific notations because we use the base 10 number system.

One important usage of common logarithms is in their use in simplifying

computations. Due to the extensive usage of various advanced calculators,

of the usage of logarithms at present is not as it was in the past. However, th

operations like 1.27 that you are able to perform using common logarithms

This is due to the fact that any logarithm to base other than 10 can be e

common logarithm so that one can use the table of common logarithm f

standard books and mathematical tables.

A COMMON LOGARITHM IS USUALLY WRITTEN WITHOUT INDICATING ITS BASE. FOR EXA $\log_{10} x$ IS SIMPLY DENOTED BY LOG x.

SO IF A LOGARITHM IS GIVEN WITH NO BASE, WE TAKE IT TO BE BASE 10.

ACTIVITY 2.5

FIND THE FOLLOWING COMMON LOGARITHMS:

A $\log \sqrt{1}$ **B** $\log 0.000$ **C** $\log 1$ **D** $\log\left(\frac{10}{10^n}\right)$

EXAMPLE 6 FIND THE FOLLOWING COMMON LOGARITHMS:

A $\log 100{,}000$ **B** $\log \sqrt[3]{100}$ **C** $\log 0.001$

SOLUTION

A $\log 100{,}000 = 5$ BECAUSE $10^5 = 100{,}000$ OR $\log 100{,}000 = \log 10^5 = 5 \log 10$

B $\log \sqrt[3]{100} = \frac{2}{3}$ BECAUSE $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{\frac{2}{3}}$ OR

$$\log \sqrt[3]{100} = \log 100^{\frac{1}{3}} = \log (10^2)^{\frac{1}{3}} = \log 10^{\frac{2}{3}} = \frac{2}{3}\log 10 = \frac{2}{3} \times 1$$

C $\log 0.001 = -3$ BECAUSE $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$ OR

$$\log 0.001 = \log \frac{1}{1000} = \log \frac{1}{10^3} = \log 10^{-3} = -3 \log$$

EXAMPLE 7 FIND THE COMMON LOGARITHM OF 526.

SOLUTION: $\log 526 = \log(5.26 \times 10^2) = \log 5.26 + \log 10^2$ … by $\log_b xy = \log_b x + \log_b y$

$= \log 5.26 + 2 \log 10 = \log 5.26 + 2$. NOW WE STILL NEED TO FIND $\log 5.26$.

SINCE $\log 1 = 0$ AND $\log 10 = 1$, WE KNOW THAT $0 < \log 5.26 < 1$.

SO, THE COMMON LOGARITHM OF A NUMBER BETWEEN 1 AND 10 IS A NUMBER BE AND 1. THE SPECIFIC COMMON LOGARITHMIC VALUES FOR NUMBERS BETWEEN 1 A GIVEN IN WHAT IS CALLED A TABLE OF COMMON LOGARITHMS.

A COPY OF THE TABLE IS ATTACHED AT THE END OF THIS BOOK

FROM THE COMMON LOGARITHM TABLE, WE READ THAT $\log 5.26 = 0.7210$

(*It should be noted that this value is only an approximate value.*)

HENCE,

$$\log 526 = \log(5.26 \times 10^2) = \underbrace{\log 5.26}_{\text{Mantissa}} + \underbrace{\log 10^2}_{\text{Characteristic}} = \log 5.26 + 2 = \underbrace{0.7210}_{M} + \underbrace{2}_{C} = 2.7210$$

IF WE WRITE A NUMBER x AS $x = m \times 10^c$, $0 \le m < 10$, THEN THE LOGARITHM OF x CAN BE READ FROM A COMMON LOGARITHM TABLE. THE LOG m IS CALLED THE ***mantissa*** OF THE LOGARITHM OF THE NUMBER x AND c IS CALLED THE ***characteristic*** OF THE LOGARITHM. THEREFORE, THE COMMON LOGARITHM OF A NUMBER IS EQUAL TO ITS ***characteristic*** PLUS ITS ***mantissa***.

EXAMPLE 8 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF EACH OF THE FOLLOWING COMMON LOGARITHMS:

A LOG 0.000415 **B** LOG 239 **C** LOG 0.001

SOLUTION:

A $0.000415 = 4.15 \times 10^{-4}$

THEREFORE, THE CHARACTERISTIC IS –4 AND THE MANTISSA IS LOG 4.15.

B $239 = 2.39 \times 10^{2}$

THEREFORE, THE CHARACTERISTIC IS 2 AND THE MANTISSA IS LOG 2.39.

C $0.001 = 1 \times 10^{-3}$

THEREFORE, THE CHARACTERISTIC IS –3 AND THE MANTISSA IS LOG 1 = 0.

Using the logarithm table

THE LOGARITHM OF ANY TWO DECIMAL PLACE NUMBER BETWEEN 1.00 AND 9.99 CAN BE READ DIRECTLY FROM THE COMMON LOGARITHM TABLE (A PART OF THE TABLE IS GIVEN BELOW FOR REFERENCE).

x	0	1	2	. . .	9
1.0	0.0000	0.0043	0.0086	. . .	0.0374
1.1	0.0414	0.0453	0.0492	. . .	0.0755
1.2	0.0792	0.0828	0.0864	. . .	0.1106
1.3	0.1139	0.1173	0.1206	. . .	0.1430
⋮	⋮	⋮	⋮	⋮	⋮
1.9	0.2788	0.2810	0.2833	. . .	0.2989
2.0	0.3010	0.3032	0.3054	. . .	0.3201
2.1	0.3222	0.3243	0.3263	. . .	0.3404
2.2	0.3424	0.3444	0.3464	. . .	0.3598
⋮	⋮	⋮	⋮	⋮	⋮
9.9	0.9956	0.9961	0.9965	. . .	0.9996

EXAMPLE 9 USE THE TABLE OF LOGARITHMS TO FIND:

A LOG 2.29 **B** LOG 1.21 **C** LOG 1.386 **D** LOG 21,200

SOLUTION:

A READ THE NUMBER AT THE INTERSECTION OF ROW 2.2 AND COLUMN 9.

READING THE NUMBER IN ROW 2.2 UNDER COLUMN 9, WE GET 0.3598.

$\therefore$ LOG 2.29 = 0.3598 .

B READING THE NUMBER AT THE INTERSECTION OF ROW 1.2 AND COLUMN 1, WE GET 0.0828

$\therefore$ LOG 1.21 = 0.0828 .

C 1.386 IS BETWEEN 1.38 AND 1.39 .

SO, ROUND (TO 2 DECIMAL PLACES) LOG 1.386 AS LOG 1.39 . READING IN ROW 1.3 UNDER COLUMN 9, WE GET 0.1430. $\therefore$ LOG 1.386 $\cong$ 0.1430.

D FIRST WRITE 21,200 AS 2.12×10^4

$\therefore$ LOG 21,200 = LOG (2.12×10^4) = LOG 2.12 + LOG 10^4 = LOG 2.12 + 4

= 0.3263 + 4 = 4.3263.

Note: NUMBERS GREATER THAN 10 HAVE LOGARITHMS GREATER THAN 1.

Antilogarithms

SUPPOSE LOG x = 0.6665. WHAT IS THE VALUE OF x?

IN SUCH CASES, WE APPLY WHAT IS CALLED THE *antilogarithm of the logarithm of x*, WRITTEN AS *antilog (LOG x)*. THUS ANTILOG (LOG x) = ANTILOG (0.6665).

WE HAVE TO SEARCH THROUGH THE LOGARITHM TABLE, FOR THE VALUE 0.6665 . WE FIND THIS NUMBER LOCATED WHERE THE ROW WITH HEADING 4.6 MEETS THE COLUMN WITH HEADING 4. THEREFORE LOG 4.64 = 0.6665, AND WE HAVE x = 4.64.

In general, Antilog (log c) = c.

EXAMPLE 10 FIND:

A ANTILOG 0.7348 **B** ANTILOG 0.9335

C ANTILOG 0.8175 **D** ANTILOG 2.4771

SOLUTION:

A THE NUMBER 0.7348 IS FOUND IN THE TABLE WHERE ROW 5.4 AND COLUMN 3 MEET .

$\therefore$ ANTILOG 0.7348 = 5.43 .

B THE NUMBER 0.9335 IS FOUND IN THE TABLE WHERE ROW 8.5 AND COLUMN 8 MEET.

$\therefore$ ANTILOG 0.9335 = 8.58 .

c THE NUMBER 0.8175 DOES NOT APPEAR IN THE TABLE. THE CLOSEST VALUE IS AND 0.8176 = LOG 6.57.

$\therefore$ ANTILOG 0.8175 CAN BE APPROXIMATED BY 6.57 .

d ANTILOG 2.4771 = ANTILOG (0.4771 + 2) = 3×10^2 = 300

(*The antilogarithm of the decimal part* 0.4771 *is found using the table of logarithms and equals* 3. *The antilogarithm of* 2 *is* 10^2 *because* LOG 10^2 = 2.)

EXAMPLE 11 FIND:

a ANTILOG 3.9058 **b** ANTILOG 5.9586. **c** ANTILOG (−1.0150)

SOLUTION:

a ANTILOG 3.9058 = ANTILOG (0. 9058 + 3) = 8.05×10^3 = 8050.

b ANTILOG 5.9586 = ANTILOG (0.9586 + 5) = 9.09×10^5 = 909000.

c ANTILOG(−1.0150) = ANTILOG(2 − 1.0150 − 2) = ANTILOG(0.9850 − 2)

$= 9.66 \times 10^{-2} = 0.0966.$

Note: DO NOT WRITE −1.0150 AS -0.0150; THE ARITHMETIC IS NOT CORRECT!

Computation with logarithms

IN THIS SECTION YOU WILL SEE HOW LOGARITHMS ARE USED IN COMPUTATIONS.

FOR INSTANCE, TO FIND THE PRODUCT OF 32 AND 128 USING LOGARITHM TO THE BASE 2 IT AS FOLLOWS:

LET $x = 32 \times 128$

$\log_2 x = \log_2(32 \times 128)$ …………………….. WHY?

$\log_2 x = \log_2 32 + \log_2 128$ …………………….. WHY?

$\log_2 x = 5 + 7 \Rightarrow \log_2 x = 12$ …………………….. WHY?

$\therefore x = 2^{12}$

IN THE NEXT EXAMPLES YOU WILL SEE HOW COMMON LOGARITHMS ARE USED IN MATH COMPUTATIONS:

REMEMBER THAT ANTILOG (LOG c) = c

In order to compute c you can perform the following two steps:

Step 1 FIND LOG c USING THE LAWS OF LOGARITHMS.

Step 2 FIND THE ANTILOGARITHM OF LOG c.

EXAMPLE 12 COMPUTE $\dfrac{354 \times 605}{8450}$ USING LOGARITHMS.

SOLUTION:

Step 1 LET $x = \dfrac{354 \times 605}{8450}$

$\log x = \log \dfrac{354 \times 605}{8450}$

$\log x = \log (354 \times 605) - \log 8450$

$\log x = \log 354 + \log 605 - \log 8450$

$\log x = (0.5490 + 2 + 0.7818 + 2) - (0.9269 + 3)$

$\log x = 0.4039 + 1$

SO $x = \text{ANTILOG}\ (0.4039 + 1) \Rightarrow x \approx 2.53 \times 10 \approx 25.3$

$\therefore \dfrac{354 \times 605}{8450} \approx 25.3$

EXAMPLE 13 COMPUTE $\sqrt{35}$ USING LOGARITHMS.

SOLUTION: LET $x = \sqrt{35}$

$\log x = \log \sqrt{35} \Rightarrow \log x = \log 35^{\frac{1}{2}} \Rightarrow \log x = \frac{1}{2}[\log 3.5 \times 10]$

$\log x = \frac{1}{2}[0.5441 + 1] \Rightarrow \log x \approx 0.77205$; $\log x \approx 0.7721$

SO $x = \text{ANTILOG}\ (0.7721) \Rightarrow x \approx 5.92$

$\therefore \sqrt{35} \approx 5.92$

EXAMPLE 14 COMPUTE $380^{\frac{1}{3}}$ USING LOGARITHMS.

SOLUTION: LET $x = 380^{\frac{1}{3}}$

$\log x = \log 380^{\frac{1}{3}}$; $\log x = \frac{1}{3}[\log 3.80 \times 10^2]$; $\log x = \frac{1}{3}[0.5798 + 2]$;

$\log x = 0.8599$ SO $x = \text{ANTILOG}\ (0.8599) \Rightarrow x \approx 7.24$ $\therefore 380^{\frac{1}{3}} \approx 7.24$

Group Work 2.5

DISCUSS

1 WHICH BASE IS PREFERABLE FOR MATHEMATICAL COMPUTATIONS? WHY? PRESENT YOUR FINDINGS TO YOUR GROUP.

2 APPROXIMATE $\sqrt{3}$ USING LOGARITHM.

3 USE YOUR RESULT IN 2 TO COMPUTE $10^{\sqrt{3}}$. COMPARE YOUR RESULTS. WHAT DIFFERENCES DO YOU GET?

Exercise 2.5

1 FIND EACH OF THE FOLLOWING COMMON LOGARITHMS:

A $\log(100 \times \sqrt[4]{10})$ **B** $\log \dfrac{100}{\sqrt{10}}$ **C** $\log \dfrac{1}{\sqrt[4]{10}}$ **D** $\log \left(\dfrac{10^m}{10^n}\right)$

2 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF THE LOGARITHM OF EACH OF THE FOLLOWING:

A 0.000402 **B** 203 **C** 5.5 **D** 2190

E $\dfrac{1}{4}$ **F** 8 **G** 23 **H** 35.902

3 USE THE TABLE OF LOGARITHMS TO FIND:

A LOG 3.12 **B** LOG 1.99 **C** LOG 7.2 **D** LOG 5.436

E LOG 0.12 **F** LOG 9.99 **G** LOG 0.00007 **H** LOG 300

4 FIND:

A ANTILOG 0.8998 **B** ANTILOG 0.8 **C** ANTILOG 1.3010

D ANTILOG 0.9953 **E** ANTILOG 5.721 **F** ANTILOG 1.9999

G ANTILOG (−6) **H** ANTILOG(−0.2)

5 COMPUTE USING LOGARITHMS:

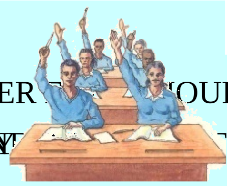
A 6.24×37.5 **B** $\sqrt[9]{125}$ **C** $2^{1.42}$

D $(2.4)^{1.3} \times (0.12)^{4.1}$ **E** $\dfrac{37.9\sqrt{488}}{(1.28)^3}$ **F** $\sqrt[5]{0.0641}$

2.2 THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

IN THIS SECTION YOU WILL DRAW GRAPHS AND INVESTIGATE THE MAJOR PROPERTIES OF FUNCTIONS OF THE FORM $f(x) = 2^x$, $f(x) = 10^x$, $f(x) = 3^{-x}$, $f(x) = (0.5)^x$, ETC.

ACTIVITY 2.6



SUPPOSE AN AMOEBA CELL DIVIDES ITSELF INTO TWO AFTER ... HOUR.

A CALCULATE THE NUMBER OF CELLS CREATED BY ... TWO, THREE, FOUR, FIVE ... t HOURS.

B COMPLETE THE FOLLOWING TABLE.

Time in hour (t)	0	1	2	3	4	5	. . .	t
Number of cells created (y)	1							

C WRITE A FORMULA TO CALCULATE THE NUMBER OF CELLS CREATED AFTER t HOURS.

THE FUNCTION $f(x) = b^x$, $b > 0$ AND $b \neq 1$ DEFINES AN EXPONENTIAL FUNCTION.

THE FOLLOWING FUNCTIONS ARE ALL EXPONENTIAL:

A $f(x) = 2^x$ **B** $g(x) = \left(\frac{3}{2}\right)^x$ **C** $h(x) = 3^x$ **D** $k(x) = 10^x$

E $f(x) = \left(\frac{1}{10}\right)^x$ **F** $g(x) = \left(\frac{1}{3}\right)^x$ **G** $h(x) = \left(\frac{1}{2}\right)^x$ **H** $k(x) = \left(\frac{2}{3}\right)^x$

2.2.1 Graphs of Exponential Functions

LET US NOW CONSIDER THE GRAPHS OF SOME OF THE ABOVE EXPONENTIAL FUNCTIONS CAN EXPLORE SOME OF THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF $f(x) = 2^x$.

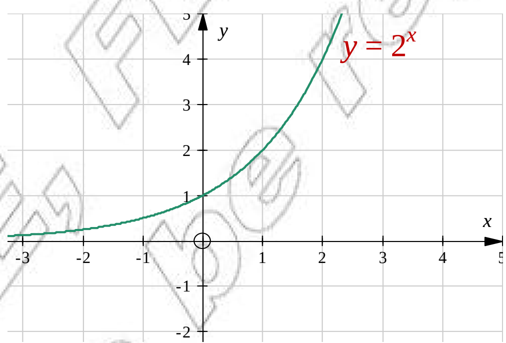
SOLUTION: EVALUATE $y = 2^x$ FOR SOME INTEGRAL VALUES OF x AND PREPARE A TABLE OF VALUES.

FOR EXAMPLE: $f(-3) = 2^{-3} = \frac{1}{8}$; $f(-2) = 2^{-2} = \frac{1}{4}$; $f(-1) = 2^{-1} = \frac{1}{2}$;

$f(0) = 2^0 = 1$; $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

NOW PLOT THESE POINTS ON THE CO-ORDINATE SYSTEM AND JOIN THEM BY A SMOO OBTAIN THE GRAPH OF $f(x) = 2^x$.



Graph of $f(x) = 2^x$

Figure 2.1

ACTIVITY 2.7

1 WHAT IS THE DOMAIN OF THE FUNCTION $f(x) = 2^x$?

2 FOR WHAT VALUES OF x IS 2^x NEGATIVE?

3 CAN 2^x EVER BE 0?

4 WHAT IS THE RANGE OF THE FUNCTION $f(x) = 2^x$?

5 WHAT IS THE y-INTERCEPT OF $f(x) = 2^x$?

6 FOR WHICH VALUES OF x IS 2^x GREATER THAN 1?

7 WHAT CAN YOU SAY ABOUT THE VALUE OF 2^x IF $x < 0$?

8 DOES 2^x INCREASE AS x INCREASES?

9 WHAT HAPPENS TO THE GRAPH OF f WHEN WE TAKE LARGER AND LARGER POSITIVE VALUES OF x?

10 WHAT HAPPENS TO THE GRAPH OF f WHEN x IS NEGATIVE AND $|x|$ IS VERY LARGE?

11 DOES THE GRAPH CROSS THE x-AXIS?

12 WHAT IS THE ASYMPTOTE OF THE GRAPH OF $f(x) = 2^x$?

EXAMPLE 2 DRAW THE GRAPH OF $g(x) = \left(\frac{3}{2}\right)^x$

SOLUTION:

x	−3	−2	−1	0	1	2	3
$g(x) = \left(\frac{3}{2}\right)^x$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

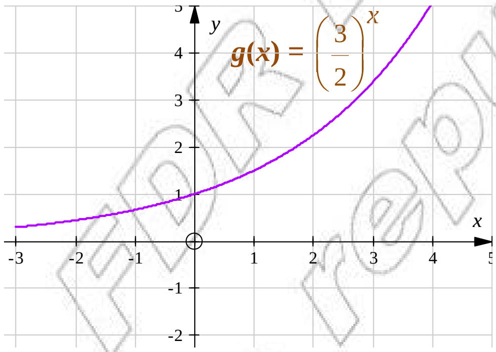


Figure 2.2 Graph of $g(x) = \left(\frac{3}{2}\right)^x$

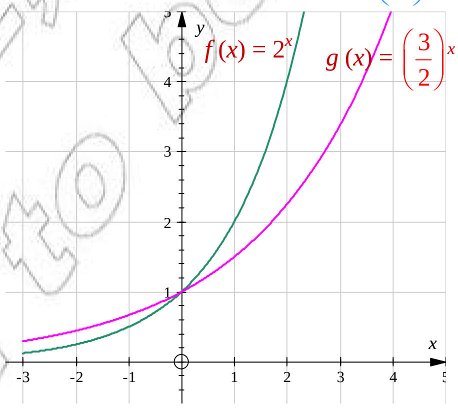


Figure 2.3 Graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{3}{2}\right)^x$ drawn using the same co-ordinate system

IN GENERAL, THE GRAPH OF $f(x) = b^x$, FOR ANY $b > 1$ HAS SIMILAR SHAPE AS THE GRAPHS OF $y = 2^x$ AND $y = \left(\frac{3}{2}\right)^x$.

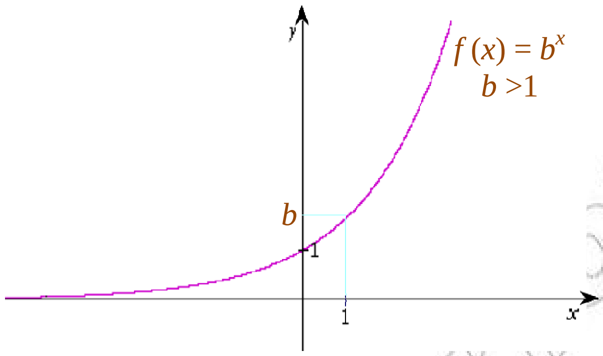


Figure 2.4 Graph of $f(x) = b^x$, for any $b > 1$

Basic properties

THE GRAPH OF $f(x) = b^x$, $b > 1$ *has the following basic properties:*

1 THE DOMAIN IS THE SET OF ALL REAL NUMBERS.

2 THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.

3 THE GRAPH INCLUDES THE POINT (0, 1), I.E. THE y-INTERCEPT IS 1.

4 THE FUNCTION IS INCREASING.

5 THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x > 0$ AND BETWEEN 0 AND 1 FOR $x < 0$.

6 THE GRAPH APPROACHES THE x-AXIS AS AN ASYMPTOTE ON THE LEFT AND INCREASES WITHOUT BOUND ON THE RIGHT.

WE WILL NEXT DISCUSS HOW THE GRAPH OF THE FUNCTION $f(x) = b^x$ LOOKS LIKE WHEN $0 < b < 1$.

EXAMPLE 3 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE AXES. **II** THE SAME COORDINATE AXES.

A $h(x) = \left(\frac{1}{2}\right)^x$ **B** $k(x) = \left(\frac{2}{3}\right)^x$

SOLUTION: AS BEFORE, CALCULATE THE VALUES OF THE GIVEN FUNCTION FOR SOME INTEGRAL VALUES OF x AS SHOWN IN THE TABLES BELOW. THEN PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CUR GET THE GRAPHS AS INDICATED BELOW.

A

x	-3	-2	-1	0	1	2	3
$H(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

B

x	-3	-2	-1	0	1	2	3
$k(x)=\left(\frac{2}{3}\right)^x$	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$

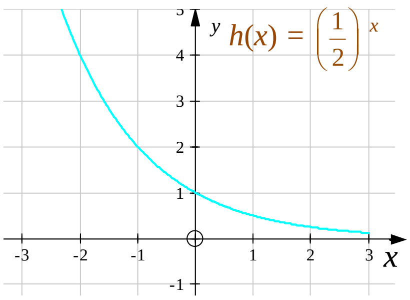


Figure 2.5 Graph of $h(x)=\left(\frac{1}{2}\right)^x$

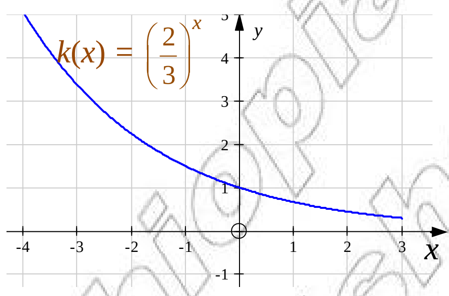


Figure 2.6 Graph of $k(x)=\left(\frac{2}{3}\right)^x$

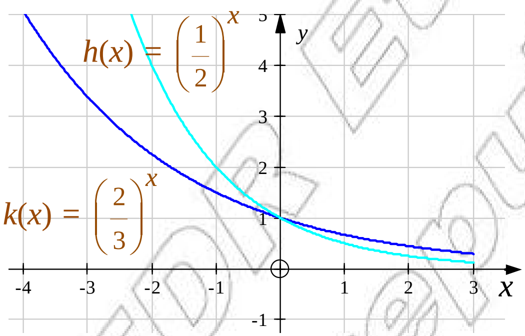


Figure 2.7 Graphs of $h(x)=\left(\frac{1}{2}\right)^x$ and $k(x)=\left(\frac{2}{3}\right)^x$ drawn using the same coordinate axes

THE GRAPH OF $f(x) = b^x$, FOR ANY $0 < b < 1$ HAS SIMILAR SHAPE TO THE GRAPHS OF $y = \left(\frac{1}{2}\right)^x$ AND $y = \left(\frac{2}{3}\right)^x$.

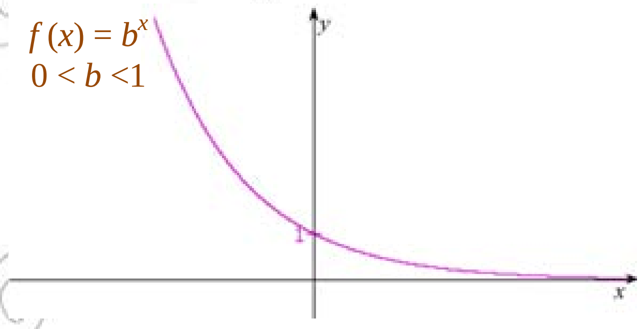


Figure 2.8 Graph of $f(x) = b^x$, for any $0 < b < 1$

Basic properties

THE GRAPH OF $f(x) = b^x$, $0 < b < 1$ *has the following basic properties:*

1. THE DOMAIN IS THE SET OF ALL REAL NUMBERS.
2. THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.
3. THE GRAPH INCLUDES THE POINT (0, 1), I.E. THE y-INTERCEPT IS 1.
4. THE FUNCTION IS DECREASING.
5. THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x < 0$ AND BETWEEN 0 AND 1 FOR $x > 0$.
6. THE GRAPH APPROACHES THE x-AXIS AS AN ASYMPTOTE ON THE RIGHT AND INCREASES WITHOUT BOUND ON THE LEFT.

Exercise 2.6

1. GIVE THREE EXAMPLES OF EXPONENTIAL FUNCTIONS.
2. GIVEN THE GRAPH OF $y = 2^x$ (*see* FIGURE 2.9), WE CAN FIND APPROXIMATE VALUES OF 2^x FOR VARIOUS VALUES OF x. FOR EXAMPLE,

   $2^{1.8} \approx 3.5$ (SEE POINT A).

   $2^{2.3} \approx 5$ (SEE POINT B).

   USE THE GRAPH TO DETERMINE APPROXIMATE VALUES OF

   A $2^{\frac{1}{2}}$ (I.E. $\sqrt{2}$) **B** $2^{0.8}$ **C** $2^{1.5}$ **D** $2^{-1.6}$.

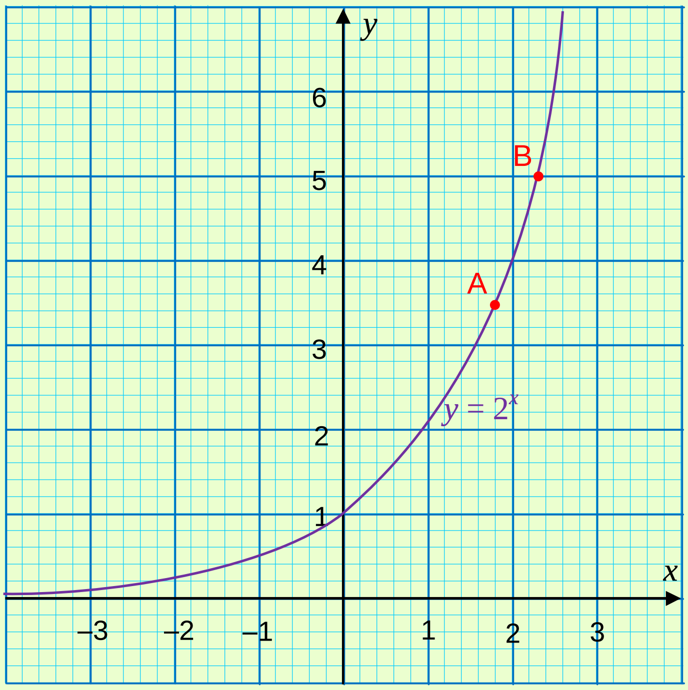


Figure 2.9

3 CONSTRUCT SUITABLE TABLES OF VALUES AND DRAW THE GRAPHS OF

A $h(x) = 3^x$ AND $g(x) = \left(\frac{1}{3}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

B $k(x) = 10^x$ AND $f(x) = \left(\frac{1}{10}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

C $f(x) = 4^x$ AND $g(x) = \left(\frac{1}{4}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

4 REFERRING TO THE FUNCTIONS IN QUESTION 3:

A FIND THE DOMAIN AND THE RANGE OF EACH FUNCTION,

B WHAT IS THE y-INTERCEPT OF EACH FUNCTION?

C WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?

D FIND THE ASYMPTOTE FOR EACH GRAPH

The exponential function with base *e*

UNTIL NOW THE NUMBER π HAS PROBABLY BEEN THE MOST IMPORTANT IRRATIONAL NUMBER YOU HAVE ENCOUNTERED. NEXT, WE WILL INTRODUCE ANOTHER USEFUL IRRATIONAL NUMBER, e, WHICH IS IMPORTANT IN THE FIELD OF MATHEMATICS AND OTHER SCIENCES.

2.2.2 The Number *e*

DO YOU KNOW THAT SOME BANKS CALCULATE INTEREST EVERY MONTH? THIS IS CALLED ***monthly compounding***. OTHER BANKS EVEN ADVERTISE ***continuous*** COMPOUNDING. TO ILLUSTRATE THE IDEA OF *continuous* COMPOUNDING, WE WILL STUDY HOW 1 BIRR GROWS FOR 1 YEAR AT 100 PERCENT ANNUAL INTEREST, USING VARIOUS PERIODS OF COMPOUNDING.

IN THIS CASE, WE USE THE AMOUNT FORMULA $A = P(1 + i)^n$, WHERE THE PRINCIPAL $P = 1$.

TAKING THE ANNUAL RATE $r = 100\% = 1$, $i = \frac{1}{n}$ IF THERE ARE n PERIODS OF COMPOUNDING PER YEAR, THEN THE AMOUNT AFTER 1 YEAR IS GIVEN BY THE FORMULA:

$$A = \left(1 + \frac{1}{n}\right)^n$$

THE FOLLOWING TABLE GIVES THE AMOUNTS (IN BIRR) AFTER 1 YEAR USING VARIOUS PERIODS OF COMPOUNDING.

Number of compounding periods per year	Amount after one year
yearly	$\left(1+\frac{1}{1}\right)^{1}=2$
semi-annually	$\left(1+\frac{1}{2}\right)^{2}=2.25$
quarterly	$\left(1+\frac{1}{4}\right)^{4}=2.44140625$
monthly	$\left(1+\frac{1}{12}\right)^{12}\approx 2.61303529022...$
weekly	$\left(1+\frac{1}{52}\right)^{52}\approx 2.69259695444...$
daily	$\left(1+\frac{1}{365}\right)^{365}\approx 2.71456748202...$
hourly	$\left(1+\frac{1}{8760}\right)^{8760}\approx 2.71812669063...$
every minute	$\left(1+\frac{1}{525600}\right)^{525600}\approx 2.7182792154...$
every second	$\left(1+\frac{1}{31536000}\right)^{31536000}=2.7182817853...$

THE LAST ROW OF THE ABOVE TABLE SHOWS THE EFFECT OF COMPOUNDING APPROXI SECOND. THE IDEA OF CONTINUOUS COMPOUNDING IS THAT THE TABLE IS CONTINUED F LARGER VALUES OF n. AS n CONTINUES TO INCREASE, THE AMOUNT AFTER 1 YEAR TENDS TOWARD THE NUMBER ***2.718281828459...***

THIS IRRATIONAL NUMBER IS REPRESENTED BY THE LETTER e.

e = 2.718281828459...

e IS THE NUMBER THAT $\left(1+\frac{1}{n}\right)^{n}$ APPROACHES AS $n\to\infty$. WHO FIRST DISCOVERED e IS STILL BEING DEBATED. THE NUMBER IS NAMED AFTER THE SWISS MATHEMATICIAN LEO (1707 – 1783), WHO COMPUTED e TO 23 DECIMAL PLACES USING $\left(1+\frac{1}{n}\right)^{n}$.

2.2.3 The Natural Exponential Function

FOR ANY REAL NUMBER x, THE FUNCTION $f(x) = e^x$ DEFINES THE EXPONENTIAL FUNCTION WITH BASE e, USUALLY CALLED THE ***natural exponential function***.

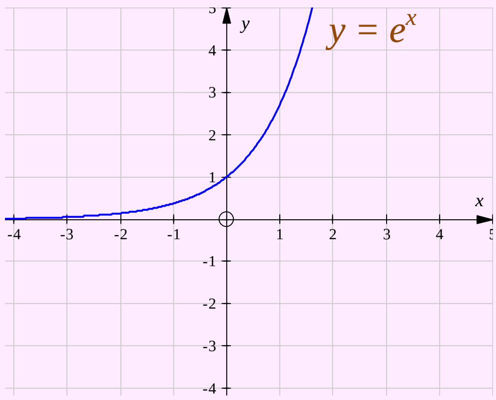


Figure 2.10 The graph OF $y = e^x$.

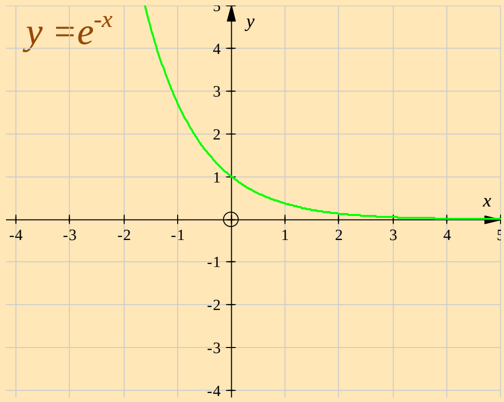


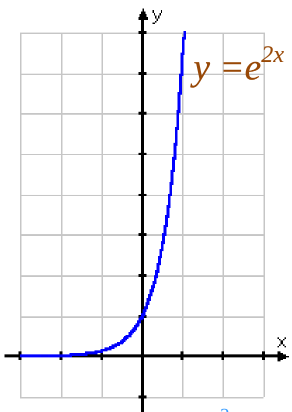
Figure 2.11 The graph of $y = e^{-x}$.

THE DOMAIN OF $y = e^x$ is $\mathbb{R}$.	THE DOMAIN OF $y = e^{-x}$ IS $\mathbb{R}$
THE RANGE IS $(0, \infty)$	THE RANGE IS $(0, \infty)$
$y = e^x$ IS AN INCREASING FUNCTION.	$y = e^{-x}$ IS A DECREASING FUNCTION.
THE GRAPH OF $y = e^x$ INTERSECTS THE y-AXIS AT (0, 1).	THE GRAPH OF $y = e^{-x}$ INTERSECTS THE y-AXIS AT (0, 1).
$e^x > 1$, if $x > 0$	$e^{-x} > 1$, if $x < 0$
$0 < e^x < 1$, if $x < 0$	$0 < e^{-x} < 1$, if $x > 0$

EXAMPLE 1 SKETCH THE GRAPH OF $y = e^{2x}$.

SOLUTION: WE CALCULATE AND PLOT SOME POIN
THE REQUIRED GRAPH, AS SHOWN IN FIGU

x	$y = e^{2x}$
–3	≈ 0.0025
–2	≈ 0.0183
–1	≈ 0.1353
0	= 1
1	≈ 7.7391
2	≈ 54.5981



Graph of $y = e^{2x}$
Figure 2.12

Exercise 2.7

1 SKETCH THE GRAPHS OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = 2^{x-1}$ B $g(x) = 3^{x-2}$ C $k(x) = 3^{2-x}$

2 USE THE e^x KEY ON YOUR CALCULATOR TO EVALUATE EACH OF THE FOLLOWING EXPR 7 DECIMAL PLACES:

A e^3 B $e^{\sqrt{3}}$ C $e^{-7.3011}$ D $e^{\sqrt{5}}$

3 CONSTRUCT TABLES OF VALUES FOR SOME INTEGER VALUES OF x AND THEN GRAPH EACH OF THE FOLLOWING FUNCTIONS:

A $y = -e^x$ B $y = -e^{-x}$ C $y = 10e^{0.2x}$

4 STATE THE DOMAIN AND RANGE OF EACH OF THE FUNCTIONS IN QUESTION 3

2.3 THE LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

FROM SECTION 2.1.2 YOU SHOULD REMEMBER THAT $b^y = x$, IF AND ONLY IF $\log_b x = y$ $(b > 0, b \neq 1$ AND $x > 0)$

HENCE, THE FUNCTION $y = \log_b x$, WHERE $x > 0$, $b > 0$ AND $b \neq 1$ IS CALLED ***a logarithmic function with base b.***

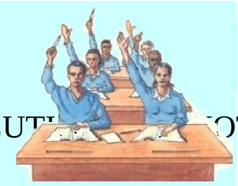
THE FOLLOWING FUNCTIONS ARE ALL LOGARITHMIC:

A $f(x) = \log_2 x$ B $g(x) = \log_{\frac{3}{2}} x$ C $h(x) = \log_3 x$

D $k(x) = \log_{10} x$ E $f(x) = \log_{\frac{1}{10}} x$ F $g(x) = \log_{\frac{1}{3}} x$

G $h(x) = \log_{\frac{1}{2}} x$ H $k(x) = \log_{\frac{2}{3}} x$

ACTIVITY 2.8



THE CONCENTRATION OF HYDROGEN IONS IN A GIVEN SOLUT[illegible]OTED BY $[H^+]$ AND IS MEASURED IN MOLES PER LITER.

FOR EXAMPLE, $[H^+] = 0.0000501$ FOR BEER AND $[H^+] = 0.0004$ FOR WINE.

CHEMISTS DEFINE THE PH OF THE SOLUTION AS THE NUMBER PH $= -\log[H^+]$. THE SOLUTION IS SAID TO BE AN ACID IF PH < 7 AND A BASE IF PH > 7. PURE WATER HAS A PH OF 7, WHICH IT IS NEUTRAL.

A IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?

B WHAT IS THE HYDROGEN ION CONCENTRATION $[H^+]$ OF EGGS IF THE PH OF EGGS IS 7.8?

2.3.1 Graphs of Logarithmic Functions

IN THIS SECTION, WE CONSIDER THE GRAPHS OF SOME LOGARITHMIC FUNCTIONS, SO
EXPLORE THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE SYSTEMS **II** THE SAME COORDINATE SYSTEM.

A $f(x) = \log_2 x$ **B** $g(x) = \log_{\frac{3}{2}} x$.

SOLUTION: THE TABLES BELOW INDICATE SOME VALUES OF $f(x)$ AND $g(x)$. PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY CURVES TO GET THE REQUIRED GRAPHS AS INDICATED IN FIGURES 2.13 AND 2.14.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \log_2 x$	−2	−1	0	1	2

x	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$g(x) = \log_{\frac{3}{2}} x$	−2	−1	0	1	2

A **B**

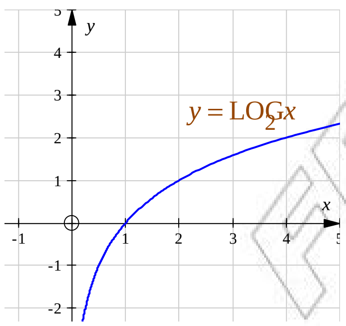


Figure 2.13 Graph of $f(x) = \log_2 x$

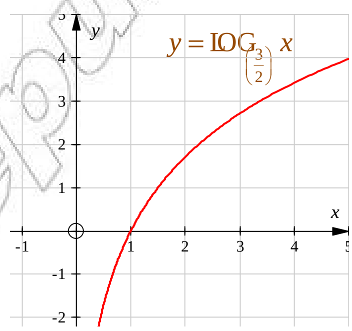


Figure 2.14 Graph of $g(x) = \log_{\left(\frac{3}{2}\right)} x$

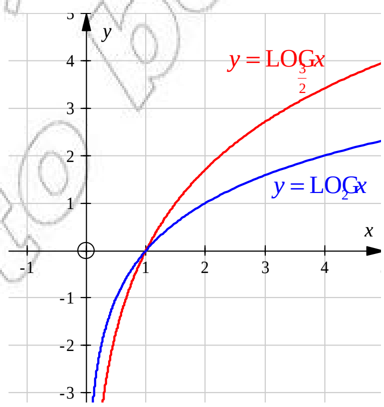
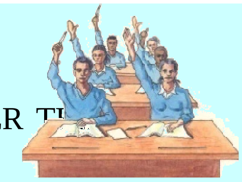


Figure 2.15 Graphs of $y = \log_2 x$ and $y = \log_{\left(\frac{3}{2}\right)} x$ drawn using the same coordinate axes

ACTIVITY 2.9



STUDY THE GRAPHS OF $f(x) = \log_2 x$ AND $g(x) = \log_{\left(\frac{3}{2}\right)} x$ TO ANSWER THE FOLLOWING QUESTIONS:

1. WHAT ARE THE DOMAINS OF f AND g?
2. FOR WHICH VALUES OF x IS $\log_2 x$ NEGATIVE? POSITIVE?
3. FOR WHICH VALUES OF x IS $\log_{\frac{3}{2}} x$ NEGATIVE? POSITIVE?
4. WHAT IS THE RANGE OF f AND g?
5. WHAT IS THE x-INTERCEPT?
6. DOES $\log_2 x$ INCREASE AS x INCREASES? WHAT ABOUT $\log_{\frac{3}{2}} x$?
7. DO THE GRAPHS CROSS THE y-AXIS?
8. WHAT IS THE ASYMPTOTE OF THE GRAPHS?

IN GENERAL, THE GRAPH OF $f(x) = \log_b x$, FOR ANY $b > 1$ LOOKS LIKE THE ONE GIVEN BELOW.

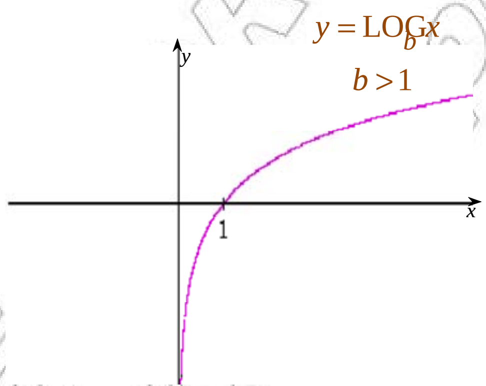


Figure 2.16 Graph of $y = \log_b x$

Basic properties

THE GRAPH OF $y = \log_b x$, $(b > 1)$ HAS THE FOLLOWING PROPERTIES.

1. THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.
2. THE RANGE IS THE SET OF ALL REAL NUMBERS.
3. THE GRAPH INCLUDES THE POINT (1, 0). I.E. THE x-INTERCEPT OF THE GRAPH IS 1.
4. THE FUNCTION INCREASES, AS x INCREASES.
5. THE y-AXIS IS A VERTICAL ASYMPTOTE OF THE GRAPH.
6. THE VALUES OF THE FUNCTION ARE NEGATIVE FOR $x < 1$ AND THEY ARE POSITIVE FOR $x > 1$.

YOU WILL NEXT DISCUSS WHAT THE GRAPH OF THE FUNCTION $y = \log_b x$ LOOKS LIKE WHEN $0 < b < 1$.

EXAMPLE 2 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE SYSTEMS; **II** THE SAME COORDINATE SYSTEM.

A $h(x) = \log_{\frac{1}{2}} x$ **B** $k(x) = \log_{\frac{2}{3}} x$

SOLUTION: CALCULATE THE VALUES OF THE GIVEN FUNCTIONS AS SHOWN IN THE TABLES BELOW. THEN PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CURVES TO GET THE REQUIRED GRAPHS AS INDICATED IN FIGURE 2.17 AND 2.18.

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$h(x) = \log_{\frac{1}{2}} x$	-3	-2	-1	0	1	2	3

x	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$
$k(x) = \log_{\frac{2}{3}} x$	-3	-2	-1	0	1	2	3

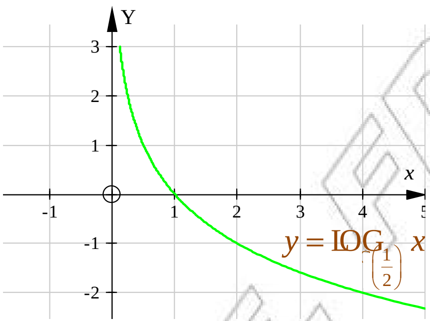


Figure 2.17 Graph of $h(x) = \log_{\left(\frac{1}{2}\right)} x$

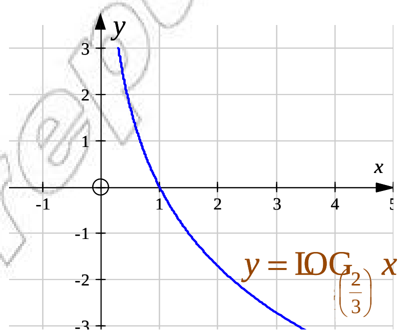


Figure 2.18 Graph of $k(x) = \log_{\left(\frac{2}{3}\right)} x$

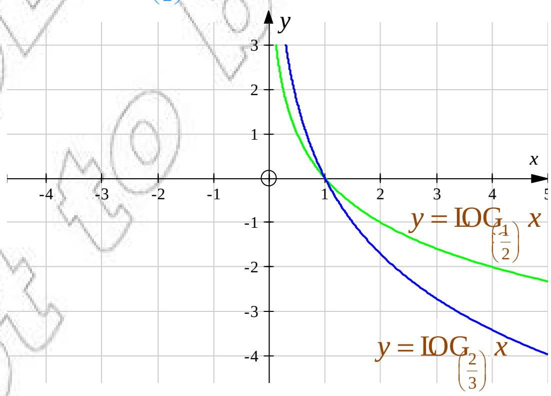


Figure 2.19 Graphs of $y = \log_{\left(\frac{1}{2}\right)} x$ and $y = \log_{\left(\frac{2}{3}\right)} x$ drawn using the same coordinate axes

IN GENERAL, THE GRAPH OF $f(x) = \log_b x$ FOR $0 < b < 1$ LOOKS LIKE THE ONE GIVEN BELOW.

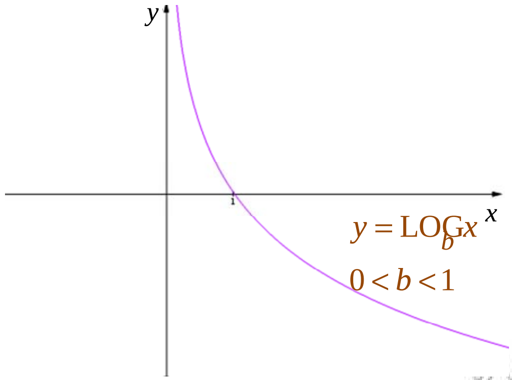


Figure 2.20

Basic properties

THE GRAPH OF $y = \log_b x$, $(0 < b < 1)$ HAS THE FOLLOWING PROPERTIES.

1. THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.
2. THE RANGE IS THE SET OF ALL REAL NUMBERS.
3. THE GRAPH HAS x-INTERCEPT AT (1, 0) I.E. THE x-INTERCEPT IS 1.
4. THE FUNCTION DECREASES AS x INCREASES.
5. THE y-AXIS IS AN ASYMPTOTE OF THE GRAPH.
6. THE VALUES OF THE FUNCTION ARE POSITIVE WHEN $0 < x < 1$ AND THEY ARE NEGATIVE WHEN $x > 1$.

Exercise 2.8

1. DRAW THE GRAPHS OF:
   - A. $h(x) = \log_3 x$ AND $g(x) = \log_{\left(\frac{1}{3}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.
   - B. $k(x) = \log_{10} x$ AND $l(x) = \log_{\left(\frac{1}{10}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.
2. REFERRING TO THE FUNCTIONS IN QUESTION 1,
   - A. WHAT ARE THE DOMAIN AND THE RANGE OF EACH FUNCTION?
   - B. WHAT IS THE x-INTERCEPT OF EACH?
   - C. WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?
   - D. FIND THE ASYMPTOTES OF THE GRAPHS OF THE FUNCTIONS.

2.3.2 The Relationship Between the Functions $y = b^x$ and $y = \log_b x$ $(b > 0,\ b \neq 1)$

CONSIDER THE FOLLOWING TABLES OF VALUES THAT WE CONSTRUCTED IN THE PREVIOUS SECTION FOR $y = 2^x$ AND $y = \log_2 x$.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	-3	-2	-1	0	1	2	3

ACTIVITY 2.10

REFER TO THE TABLES OF VALUES FOR $y = 2^x$ AND $y = \log_2 x$ TO ANSWER THE FOLLOWING QUESTIONS:

1 HOW ARE THE VALUES OF x AND y RELATED IN THE FUNCTIONS $y = 2^x$ AND $y = \log_2 x$?

2 SKETCH THE GRAPHS OF THE TWO FUNCTIONS USING THE SAME CO-ORDINATE SYSTEM.

3 FIND A RELATIONSHIP BETWEEN THE DOMAIN AND THE RANGE OF THE TWO FUNCTIONS.

4 DRAW THE LINE $y = x$ USING THE SAME CO-ORDINATE SYSTEM.

5 HOW ARE THE GRAPHS OF $y = 2^x$ AND $y = \log_2 x$ RELATED?

6 WHAT IS THE SIGNIFICANCE OF THE LINE $y = x$?

EXAMPLE 1 LET US CONSIDER THE FUNCTIONS $y = 10^x$ AND $y = \log_{10} x$.

THE TABLES OF VALUES FOR $y = 10^x$ AND $y = \log_{10} x$ FOR SOME INTEGRAL VALUES OF x ARE GIVEN BELOW:

x	-2	-1	0	1	2
$y = 10^x$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100

x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$y = \log_{10} x$	-2	-1	0	1	2

OBSERVE THAT:

THE VALUES OF x AND y ARE INTERCHANGED IN BOTH FUNCTIONS. THAT IS, THE DOMAIN OF $y = 10^x$ IS THE RANGE OF $y = \log_{10} x$ AND VICE VERSA.

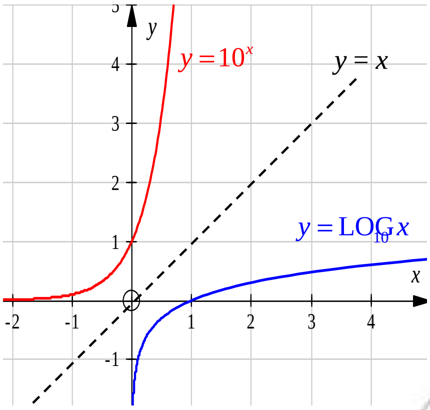


Figure 2.21

$y = 10^x$ IS OBTAINED BY REFLECTING $y = \log_{10} x$ ALONG THE LINE $y = x$.

IN SUCH CASES WE SAY ONE OF THE FUNCTIONS IS THE INVERSE OF THE OTHER.

IN GENERAL, THE RELATION BETWEEN THE FUNCTIONS $y = b^x$ AND $y = \log_b x$ ($b > 1$) IS SHOWN BELOW:

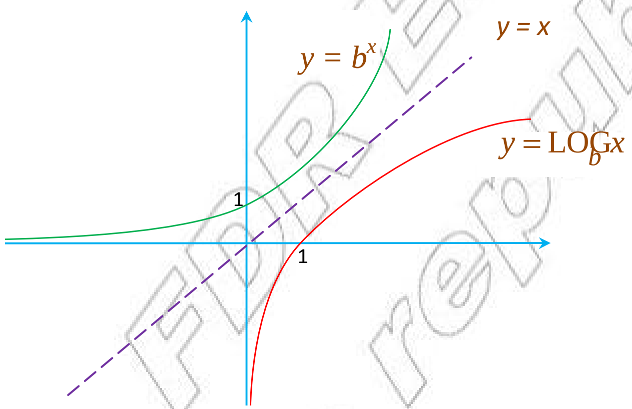


Figure 2.22

FROM THE GRAPHS ABOVE, WE OBSERVE THE FOLLOWING RELATIONSHIPS:

1 THE DOMAIN OF $y = b^x$ IS THE SET OF ALL REAL NUMBERS, WHICH IS THE SAME AS THE R
OF $y = \log_b x$.

2 THE RANGE OF $y = b^x$ IS THE SET OF ALL POSITIVE REAL NUMBERS, WHICH IS THE SAME A
DOMAIN OF $y = \log_b x$.

3 THE x-AXIS IS THE ASYMPTOTE OF $y = b^x$, WHEREAS THE y-AXIS IS THE ASYMPTOTE OF $y = \log_b x$.

4 $y = b^x$ HAS A y-INTERCEPT AT (0, 1) WHEREAS $y = \log_b x$ HAS AN x-INTERCEPT AT (1, 0).

DOMAIN OF $y = b^x$ IS EQUAL TO THE RANGE OF $y = \log_b x$.

RANGE OF $y = b^x$ IS EQUAL THE DOMAIN OF $y = \log_b x$.

THE FUNCTIONS $f(x) = b^x$ AND $g(x) = \log_b x$ ($b > 1$) ARE INVERSES OF EACH OTHER.

2.3.3 The Natural Logarithm

IF WE START WITH NATURAL EXPONENTIAL FUNCTION $y = e^x$ AND INTERCHANGE x AND y, WE OBTAIN $x = e^y$ WHICH IS THE SAME AS $y = \log_e x$.

$y = \log_e x$ IS THE MIRROR IMAGE OF $y = e^x$ ALONG THE LINE $y = x$.

Notation: $\ln x$ IS USED TO REPRESENT $\log_e x$.
$\ln x$ IS CALLED THE NATURAL LOGARITHM OF x.

THE GRAPHS OF $y = e^x$, $y = \ln x$ AND THE LINE $y = x$ ARE SHOWN BELOW:

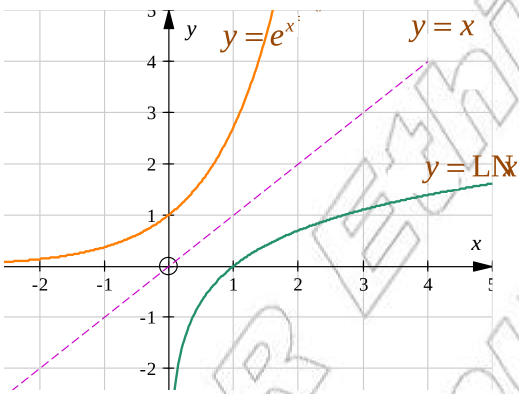


Figure 2.23

EXAMPLE 1 FIND:

A $\ln 1$ **B** $\ln e$ **C** $\ln e^2$ **D** $\ln \sqrt{e}$ **E** $\ln \frac{1}{e}$

SOLUTION:

A $\ln 1 = 0$ BECAUSE $e^0 = 1$

B $\ln e = 1$ BECAUSE $e^1 = e$

C $\ln e^2 = 2\ln e = 2 \times 1 = 2$

D $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2} \ln e = \frac{1}{2}$

E $\ln \frac{1}{e} = \ln e^{-1} = -1 \ln e = -1$

Note: IN GENERAL, $\ln e^x = x$.

Exercise 2.9

1 SKETCH THE GRAPHS OF:

A $f(x) = 4^x$, $g(x) = \log_4 x$ AND $y = x$ USING THE SAME COORDINATE SYSTEM.

B $h(x) = \left(\frac{1}{4}\right)^x$ AND $k(x) = \log_{\left(\frac{1}{4}\right)} x$ USING THE SAME COORDINATE SYSTEM.

C HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS f AND g GIVEN IN QUESTION 1A?

D HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS h AND k GIVEN IN QUESTION 1B?

2 FIND:

A $\ln \sqrt[3]{e}$ **B** $\ln \frac{1}{e^2}$ **C** $\ln e^{3x}$ **D** $e^{\ln 3}$

3 SIMPLIFY:

A $\ln e^{8}$ **B** $\ln(e \times e)$ **C** $\ln(e^x \times e^y)$ **D** $\ln\left(\frac{e^x}{e^y}\right)$

2.4 EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

AN EXPONENTIAL EQUATION IS AN EQUATION WITH THE UNKNOWN IN THE EXPONENT.

EXAMPLES OF EXPONENTIAL EQUATIONS ARE:

$4^x = 8$ $4^x - 2^{x+1} - 8 = 0$

$2^{3x-2} = 5$ $9^{x^2+4x} = 3^{3x+7}$

A LOGARITHMIC EQUATION IS AN EQUATION THAT INVOLVES THE LOGARITHM OF AN UNI

EXAMPLES OF LOGARITHMIC EQUATIONS ARE:

$4\log x = -5$ $\log_x(x - \;) = 6$

$\log(x+3) + \log x = 1$ $\log_2 x + \log_4(x + \;) = 2$

2.4.1 Solving Exponential Equations

PROPERTIES OF EXPONENTS DISCUSSED IN THE PREVIOUS SECTIONS PLAY A MAJOR RO EXPONENTIAL EQUATIONS. READ CAREFULLY THROUGH THE PROPERTIES BELOW, TO MEMORY!

IF a AND b ARE POSITIVE NUMBERS, $a \neq 1$, $b \neq 1$ AND m AND n ARE REAL NUMBERS, THEN

1 $a^m \times a^n = a^{m+n}$ **2** $(a^m)^n = a^{mn}$

3 $(a \times b)^n = a^n \times b^n$ **4** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

5 $\frac{a^m}{a^n} = a^{m-n}$ **6** $a^{-n} = \frac{1}{a^n}$ AND $\frac{1}{a^{-n}} = a^n$

7 IT IS ALWAYS TRUE THAT FOR $k > 0$, $\left(\frac{a}{b}\right)^{-k} = \left(\frac{b}{a}\right)^k$

Additional properties:

Property of equality for exponential equations

FOR $b > 0$, $b \neq 1$, x AND y REAL NUMBERS,

1 $b^x = b^y$, IF AND ONLY IF $x = y$

2 $a^x = b^x$, $(x \neq 0)$, IF AND ONLY IF $a = b$

EXAMPLE 1 SOLVE FOR x.

A $3^x = 81$ **B** $2^x = \frac{1}{32}$ **C** $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$ **D** $4^x = \left(\frac{1}{2}\right)^{x-3}$

SOLUTION:

A $3^x = 81 = 3^4$ … *look for a common base*

$\Rightarrow x = 4$ … *property of equality of bases*

B $2^x = \frac{1}{2^5} = 2^{-5}$ … *look for a common base*

$\Rightarrow x = -5$ … *property of equality of bases*

C $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$

$\Rightarrow \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{3}{2}\right)^{2x} = \left(\frac{2}{3}\right)^{-2x}$

$\Rightarrow 2x + 1 = -2x$

$\Rightarrow 2x + 2x = -1$

$\Rightarrow x = -\frac{1}{4}$

D $4^x = \left(\frac{1}{2}\right)^{x-3}$

$\Rightarrow 4^x = (2^{-1})^{x-3} = 2^{(-x+3)}$

$\Rightarrow (2^2)^x = 2^{(-x+3)}$

$\Rightarrow 2^{2x} = 2^{(-x+3)}$

$\Rightarrow 2x = -x + 3 \Rightarrow x = 1$

IF YOU CANNOT EASILY WRITE EACH SIDE OF AN EXPONENTIAL EQUATION USING THE S
CAN SOLVE THE EQUATION BY TAKING LOGARITHMS OF EACH SIDE.

EXAMPLE 2 SOLVE FOR x, BY TAKING THE LOGARITHM OF EACH SIDE:

A $4^x = 10$ **B** $2^{3x-2} = 5$ **C** $2^{2x} = 11$

SOLUTION:

A $4^x = 10$

$\text{LOG}\,4^x = \text{LOG}\,10$ … *taking the logarithm of each side*

$x\,\text{LOG}\,4 = 1$ … *since* LOG10 = 1, AND $\text{LOG}\,a^k = k\,\text{LOG}\,a$

$x = \frac{1}{\text{LOG}\,4} = \frac{1}{0.6021} = 1.6609$

B $2^{3x-2} = 5$

$\Rightarrow \log 2^{(3x-2)} = \log 5$

$\Rightarrow (3x-2)\log 2 = \log 5$

$\Rightarrow 3x-2 = \dfrac{\log 5}{\log 2}$

$\Rightarrow 3x = \dfrac{\log 5}{\log 2} + 2$

$\Rightarrow x = \dfrac{1}{3}\left(\dfrac{\log 5}{\log 2} + 2\right) = 1.4408$

C $2^{2x} = 11$

$\Rightarrow \log 2^{2x} = \log 11$

$\Rightarrow 2x \log 2 = \log 11$

$\Rightarrow 2x = \dfrac{\log 11}{\log 2}$

$\Rightarrow x = \dfrac{1}{2}\left(\dfrac{\log 11}{\log 2}\right) = 1.730$

Exercise 2.10

1 Solve for x

A $5^x = 625$ **B** $2^{3-x} = 16$ **C** $4^{3x-8} = 2^{3x+9}$

D $\dfrac{1}{27} = \left(\dfrac{1}{9}\right)^{2x}$ **E** $3^{-x} = 81$ **F** $2^{x^2-2} = 4$

G $7^{x^2+x} = 49$ **H** $3^{6(x+2)} = 9^{x+2}$ **I** $3\left(\dfrac{27}{8}\right)^{\frac{2}{3}x-1} = 2\left(\dfrac{32}{243}\right)^{2x}$

2 Solve for x by taking the logarithm of each side:

A $2^x = 15$ **B** $10^x = 14.3$ **C** $10^{3x+1} = 92$ **D** $1.05^x = 2$

E $6^{3x} = 5$ **F** $4^{2x} = 61$ **G** $10^{5x-2} = 348$ **H** $2^{-x} = 0.238$

2.4.2 Solving Logarithmic Equations

Properties of logarithms discussed in the previous sections play a major role in solving logarithmic equations. Remember that

If a, b, c, x and y are positive numbers and $a \neq 1$, $b \neq 1$, then

1 $\log_b xy = \log_b x + \log_b y$

2 $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3 For any real number k, $\log_b(x^k) = k\log_b x$

4 $\log_b b = 1$

5 $\log_b 1 = 0$

6 $\log_a x = \dfrac{\log_b x}{\log_b a}$ … *change of base law*

7 $b^{\log_b x} = x$

EXAMPLE 1 SOLVE EACH OF THE FOLLOWING FOR x, CHECKING THAT YOUR SOLUTIONS ARE VALID.

A $\log_2(x-3) = 5$ **B** $\log_4(5x-1) = 3$

C $\log(x+3) + \log x = 1$ **D** $\log_3(x+1) - \log_3(x+3) = 1$

E $\log x^8 + \log(x-20) = 3$

SOLUTION:

A $\log_2(x-3) = 5 \Rightarrow 2^5 = x-3$ … *changing to exponential form*

HENCE, $32 = x - 3$

THEREFORE, $x = 35$

Check!

FROM THE DEFINITION OF LOGARITHMS, WE KNOW THAT $\log_2(x-3)$ IS VALID ONLY WHEN $x - 3 > 0$, I.E. WHEN $x > 3$. SO $\{x \mid x > 3\} = (3, \infty)$ IS KNOWN AS THE ***universe*** FOR $\log_2(x-3)$. SINCE $x = 35$ IS AN ELEMENT OF THE UNIVERSE, $x = 35$ IS THE SOLUTION OF THE GIVEN EQUATION.

A UNIVERSE IS THE LARGEST SET FOR WHICH THE GIVEN EXPRESSION IS DEFINED.

B $\log_4(5x-1)$ IS VALID WHEN $5x - 1 > 0$

SO $x > \frac{1}{5}$. THEREFORE, THE UNIVERSE, $U = \left(\frac{1}{5}, \infty\right)$

$\log_4(5x-1) = 3$

$\Rightarrow 5x - 1 = 4^3$

$\Rightarrow 5x = 64 + 1$

$\Rightarrow x = \frac{65}{5} = 13$. SINCE $13 \in \left(\frac{1}{5}, \infty\right)$, $x = 13$ IS THE SOLUTION.

C REMEMBER THAT $\log(x+3)$ IS VALID FOR $x > -3$ AND $\log x$ IS VALID FOR $x > 0$.

THEREFORE $\log(x+3) + \log x$ IS VALID FOR $x > 0$. SO $U = (0, \infty)$.

NOW $\log(x+3) + \log x = 1$

$\Rightarrow \log x(x+3) = 1$ … *since* $\log x + \log y = \log xy$

$\Rightarrow x(x+3) = 10^1$ … *changing to exponential form*

$\Rightarrow x^2 + 3x - 10 = 0$

$\Rightarrow (x+5)(x-2) = 0$

Thus, $x = -5$ OR $x = 2$

BUT -5 IS NOT AN ELEMENT OF THE UNIVERSE.

SO, THE ONLY SOLUTION IS $x = 2$.

D $\log_3(x+1) - \log_3(x+3)$ IS VALID FOR $x+1>0$ AND $x+3>0$,

I.E. FOR $x > -1$ AND $x > -3$.

THEREFORE THE U = $(-1, \infty)$.

$$\log_3(x+1) - \log_3(x+3) = 1$$

$$\Rightarrow \quad \log_3\left(\frac{x+1}{x+3}\right) = 1 \qquad \ldots \textit{since } \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\Rightarrow \quad \frac{x+1}{x+3} = 3^1$$

$$\Rightarrow \quad x+1 = 3(x+3) = 3x+9$$

THEREFORE $-2x = 8$ AND $x = -4$.

HOWEVER, -4 IS NOT IN THE UNIVERSE. HENCE, THERE IS NO x SATISFYING THE GIVEN EQUATION AND THE SOLUTION SET IS THE EMPTY SET.

E $\log 8x + \log(x-20)$ IS VALID FOR $8x > 0$ AND $x - 20 > 0$; I.E. FOR $x > 0$ AND $x > 20$.

SO U = $(20, \infty)$.

NOW $\log 8x + \log(x-20) = 3$

$$\Rightarrow \log 8x(x-20) = 3 \qquad \ldots \log_b xy = \log_b x + \log_b y$$

$$\Rightarrow 8x(x-20) = 10^3 = 1000$$

$$\Rightarrow 8x^2 - 160x = 1000$$

$$\Rightarrow 8x^2 - 160x - 1000 = 0$$

$$\Rightarrow 8(x^2 - 20x - 125) = 0$$

$$\Rightarrow x^2 - 20x - 125 = 0$$

$$\Rightarrow (x-25)(x+5) = 0$$

SO $x = 25$ OR $x = -5$. BUT $-5 \notin (20, \infty)$

SO THE ONLY SOLUTION IS $x = 25$.

Property of equality for logarithmic equations

IF b, x, AND y ARE POSITIVE NUMBERS, $b \neq 1$, THEN

$\log_b x = \log_b y$, IF AND ONLY IF $x = y$.

FOR INSTANCE, IF $\log_2 x = \log_2 7$, THEN $x = 7$. IF $x = 7$, THEN $\log_2 x = \log_2 7$.

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING FOR x.

A $\log 3x - \log(2 - x) = 0$

B $\log_5(4x - 7) = \log_5(x + 5)$

C $\log(x - 5) + \log(10 - x) = \log(x - 6) + \log(x - 1)$

SOLUTION:

A $\log 3x$ IS VALID WHEN $x > 0$ AND $\log(2 - x)$ IS VALID WHEN $2 - x > 0$ I.E. $x < 2$.

SO U = (0, 2).

NOW $\log 3x - \log(2 - x) = 0$ GIVES

$\log 3x = \log(2 - x)$

HENCE, $3x = 2 - x$ … *property of equality*

$\Rightarrow \quad 3x + x = 2$

SO $x = \frac{1}{2}$ IS THE SOLUTION IN (0, 2).

B $\log_5(4x - 7)$ IS VALID WHEN $x > \frac{7}{4}$ AND $\log_5(x + 5)$ IS VALID WHEN $x > -5$.

SO U = $\left(\frac{7}{4}, \infty\right)$. NEXT $\log_5(4x - 7) = \log_5(x + 5)$ GIVES

$4x - 7 = x + 5 \quad \Rightarrow \; 3x = 12$. SO $x = 4$ IS THE SOLUTION.

C THE TERM $\log(x - 5)$ IS VALID WHEN $x > 5$, THE TERM $\log(10 - x)$ IS VALID WHEN $x < 10$, THE TERM $\log(x - 6)$ IS VALID WHEN $x > 6$, AND THE TERM $\log(x - 1)$ IS VALID WHEN $x > 1$.

IF WE RESTRICT THE UNIVERSE TO THE SET OF ALL REAL NUMBERS x BETWEEN 6 AND 10 OR $6 < x < 10$, EVERY TERM IN THE EQUATION IS VALID.

THEREFORE (6 ,10) IS THE UNIVERSE.

$\log(x - 5) + \log(10 - x) = \log(x - 6) + \log(x - 1)$

$\Rightarrow \log((x - 5)(10 - x)) = \log((x - 6)(x - 1))$

$\Rightarrow (x - 5)(10 - x) = (x - 6)(x - 1)$

$\Rightarrow -x^2 + 15x - 50 = x^2 - 7x + 6$

$\Rightarrow 15x - 50 = 2x^2 - 7x + 6$ … *adding x^2 to both sides*

$\Rightarrow -50 = 2x^2 - 22x + 6$

$\Rightarrow 0 = 2x^2 - 22x + 56$

$\Rightarrow 0 = x^2 - 11x + 28$ … *dividing both sides by 2*

$\Rightarrow (x - 7)(x - 4) = 0$.

$\Rightarrow x = 7$ OR $x = 4$, BUT ONLY 7 IS IN THE UNIVERSE.

HENCE $x = 7$ IS THE SOLUTION.

Exercise 2.11

1 STATE THE UNIVERSE AND SOLVE EACH OF THE FOLLOWING FOR x:

A $\log_3 (2x-1) = 5$ **B** $\log_{\sqrt{2}} x = -$

C $\log_3 (x^2 - 2x) = 1$ **D** $\log_2 (x^2 + 3x + 2) = 1$

E $\log_2 (1 + \frac{1}{x}) = 3$ **F** $\log_2 (x-1) + \log_2 3 = 3$

G $\log (x^2 - 121) - \log (x + 11) = 1$ **H** $\log_3 (x+4) - \log_3 (x-1)$

I $\log (6x + 5) - \log 3 = \log 2 - \log x$ **J** $\log x - \log 3 = \log 4 - \log (x + 4)$

K $\log_3 (x+1) + \log_3 (x+3) = 1$ **L** $\log_2 2 + \log_2 (x+2) - \log_2 (3x-5) = 3$

M $\log_x (x+6) = 2$

2 APPLY THE PROPERTY OF EQUALITY FOR LOGARITHMIC FUNCTIONS TO SOLVE THE FOLLOWING EQUATIONS (CHECK THAT YOUR SOLUTIONS ARE VALID):

A $\log_3 x + \log_3 = 5$ **B** $\log_3 25 \quad 2\log_3 x =$

C $\log_5 x + \log_5 (x + \ ) = 1 \quad \log_5$ **D** $\log 2^x - \log 16 = 0$

E $\log_4 (3^{6(x+2)}) - \log_4 (9^{x+2}) = 0$ **F** $\log_2 (x^2 - 9) - \log_2 (3 + x) = 2$

2.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

AS MENTIONED AT THE START OF THIS UNIT, EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE USED IN DESCRIBING AND SOLVING A WIDE VARIETY OF REAL-LIFE PROBLEMS. IN THIS SECT DISCUSS SOME OF THEIR APPLICATIONS.

EXAMPLE 1 *Population Growth*

A SUPPOSE THAT YOU ARE OBSERVING THE BEHAVIOUR OF CELL DUPLICATION IN A LABORATORY. IN AN EXPERIMENT, YOU STARTED WITH ONE CELL AND THE C EVERY MINUTE.

I WRITE AN EQUATION TO DETERMINE THE NUMBER OF CELLS AFTER ONE HOUR.

II DETERMINE HOW LONG IT WOULD TAKE FOR THE NUMBER OF CELLS TO R 100,000.

B ETHIOPIA HAS A POPULATION OF AROUND 80 MILLION. IT IS ESTIMATED THAT THE POPULATION GROWS EVERY YEAR AT AN AVERAGE GROWTH RATE THE POPULATION GROWTH CONTINUES AT THE SAME RATE;

I WHAT WILL BE THE POPULATION AFTER

- 10 YEARS?
- 20 YEARS?

II HOW MANY YEARS WILL IT TAKE THE POPULATION TO DOUBLE? (REFER BACK TO THE OPENING PROBLEM)

SOLUTION AND EXPLANATION:

A **I** FIRST RECORD YOUR OBSERVATIONS BY MAKING A TABLE WITH TWO ROWS, ONE FOR THE TIME AND THE OTHER FOR THE NUMBER OF CELLS. THE NUMBE DEPENDS ON THE TIME.

FOR EXAMPLE, AT $t = 0$, THERE IS 1 CELL, AND THE CORRESPONDING POINT IS (0, 1).

AT $t = 1$, THERE ARE 2 CELLS, AND THE CORRESPONDING POINT IS (1, 2).

AT $t = 2$, THERE ARE 4 CELLS, AND THE CORRESPONDING POINT IS (2, 4).

AT $t = 3$, THERE ARE 8 CELLS, AND THE CORRESPONDING POINT IS (3, 8), ETC.

THIS RELATIONSHIP IS SUMMARIZED IN THE FOLLOWING TABLE:

Time (in min.) (t)	0	1	2	3	4	5	6
No. of cells (y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	$64 = 2^6$

THEREFORE, THE FORMULA TO ESTIMATE THE NUMBER OF CELLS AFTER t MINUTES IS GIVEN BY

$$f(t) = 2^t$$

DETERMINE THE NUMBER OF CELLS AFTER ONE HOUR:

CONVERT ONE HOUR TO MINUTES. (1 HR = 60 MIN)

SUBSTITUTE 60 FOR T IN THE EQUATION, $f(t) = 2^t$:

$$f(60) = 2^{60} = 1.15 \times 10^{18} = 1{,}150{,}000{,}000{,}000{,}000{,}000$$

SO THE NUMBER OF CELLS AFTER 1 HOUR WILL BE 1,150,000,000,000,000,000 = 1.15×10^{18}

II IN THIS EXAMPLE, YOU KNOW THE NUMBER OF CELLS AT THE BEGINNING OF THE EXPERIMENT (1) AND AT THE END OF THE EXPERIMENT (100,000), BUT YOU NOT KNOW THE TIME. SUBSTITUTE 100,000 FOR $f(t)$ IN THE EQUATION $f(t) = 2^t$:

$$100{,}000 = 2^t$$

TAKE THE NATURAL LOGARITHM OF BOTH SIDES:

$$\ln(100{,}000) = \ln\left(2^t\right) \Rightarrow \ln(100{,}000) = t\ln(2)$$

DIVIDE BOTH SIDES BY LN(2):

$$t = \frac{\ln(100{,}000)}{\ln(2)} \Rightarrow t = 16.60964 \text{ MINUTES}$$

IT WOULD TAKE ABOUT 16.6 MINUTES, FOR THE NUMBER OF CELLS TO REACH 1

B **I** LET P REPRESENT THE CURRENT POPULATION WHICH IS 80 MILLION = 8.0×10^7;

LET r REPRESENT THE ANNUAL GROWTH RATE WHICH IS 2.3%;

LET t REPRESENT THE TIME IN YEARS FROM NOW.

THE TOTAL POPULATION AFTER ONE YEAR:

$$A_1 = 80 \text{ MILLION} + 2.3\% \,(80 \text{ MILLION}) = 8.0 \times 10^7 + 2.3\% \,(8.0 \times 10^7)$$
$$= 8.0 \times 10^7 (1 + 2.3\%)$$

THE TOTAL POPULATION AFTER TWO YEARS:

$$A_2 = A_1 + 2.3\% (A_1) = A_1(1 + 2.3\%) = 8.0 \times 10^7 (1 + 2.3\%)(1 + 2.3\%)$$
$$= 8.0 \times 10^7 (1 + 2.3\%)^2$$

THE TOTAL POPULATION AFTER THREE YEARS:

$$A_3 = A_2 + 2.3\% (A_2) = A_2 (1 + 2.3\%) = 8.0 \times 10^7 (1 + 2.3\%)^2(1 + 2.3\%)$$
$$= 8.0 \times 10^7(1 + 2.3\%)^3$$

FROM THE ABOVE PATTERN WE CAN GENERALIZE:

THE TOTAL POPULATION AFTER t YEARS IS GIVEN BY THE FORMULA:

$$A_t = P (1 + r)^t$$

SO THE TOTAL POPULATION AFTER 10 YEARS WILL BE

$$A_{10} = 8.0 \times 10^7 (1 + 2.3\%)^{10} = 100{,}426{,}036.81$$

THE TOTAL POPULATION AFTER TWENTY YEARS WILL BE

$$A_{20} = 8.0 \times 10^7 [1 + 2.3\%]^{20} = 126{,}067{,}360.86$$

II WHEN WILL THE TOTAL POPULATION DOUBLE? (I.E. 160 MILLION)

THE TOTAL POPULATION AFTER t YEARS IS:

$$8.0 \times 10^7 [1 + 2.3\%]^T = 160{,}000{,}000$$

$$\Rightarrow [1 + 2.3\%]^t = \frac{160{,}000{,}000}{80{,}000{,}000} = 2 \Rightarrow \text{LOG}(1 + 2.3\%)^t = \text{LOG}2$$

$$\Rightarrow t\,\text{LOG}(1 + 0.023) = 0.3010 \Rightarrow t\,\text{LOG}(1.023) = 0.3010$$

THEREFORE, $t = \dfrac{0.3010}{\text{LOG}(1.023)} \approx \dfrac{0.3010}{0.0099} \approx 30.40$

THEREFORE, THE CURRENT POPULATION IS EXPECTED TO DOUBLE IN ABOU

EXAMPLE 2 *Compound Interest*

IF BIRR 5000 IS INVESTED AT A RATE OF 6% COMPOUNDED QUARTERLY (4 TIMES A YEAR), THEN

A WHAT IS THE AMOUNT AT THE END OF 4 YEARS AND 10 YEARS?

B HOW LONG DOES IT TAKE TO DOUBLE THE INVESTMENT?

SOLUTION: WE USE THE FORMULA $A = p\left(1+\dfrac{r}{n}\right)^{nt}$

HERE $p = 5000$, $r = 6\% = 0.06$

$n = 4$ (COMPOUNDED 4 TIMES)

A TO FIND THE BALANCE AT THE END OF THE 4TH YEAR.

$$A = p\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.06}{4}\right)^{4\times 4} = 5000\,(1 + 0.015)^{16}$$

$= 5000\,(1.015)^{16} \cong 5000\,(1.2690) =$ BIRR 6345

THE BALANCE AT THE END OF THE 10TH YEAR IS

$$A = p\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.06}{4}\right)^{4\times 10} = 5000\,(1 + 0.015)^{40} = 5000\,(1.015)^{40}$$

$\cong 5000\,(1.8140) =$ BIRR 9070

B IF THE INVESTMENT IS TO BE DOUBLED, $2P = 2 \times 5000 = 10{,}000$

$$A = p\left(1+\frac{r}{n}\right)^{nt}$$

$$\Rightarrow \quad 10{,}000 = 5000\left(1+\frac{0.06}{4}\right)^{4t} = 5000\,(1 + 0.015)^{4T}$$

$$\Rightarrow \quad 10{,}000 = 5000\,(1.015)^{4T}$$

$2 = (1.015)^{4T}$ … *dividing both sides by* 5000

$\text{LOG}\,2 = \text{LOG}\,(1.015)^{4T} = 4T\ \text{LOG}\,(1.015)$

$$4t = \frac{\text{LOG}\,2}{\text{LOG}(1.015)} = \frac{0.3010}{0.0065} = 46.30769 \Rightarrow t = \frac{46.30769}{4} \approx 11.58 \text{ YEARS}$$

IT TAKES ABOUT 12 YEARS TO DOUBLE THE INVESTMENT.

EXAMPLE 3 *Chemistry* (REFER BACK TO ACTIVITY 2.8)

THE CONCENTRATION OF HYDROGEN IONS IN A SOLUTION IS GIVEN BY $[H^+]$ AND IS MEASURED IN MOLES PER LITRE. FOR EXAMPLE, $[H^+] = 0.0000501$ FOR BEER AND $[H^+] = 0.0004$ FOR WINE. CHEMISTS DEFINE THE PH OF THE SOLUTION AS THE NUMBER $\text{PH} = -\text{LOG}[H^+]$. THE SOLUTION IS SAID TO BE AN ACID IF PH < 7 AND A BASE IF PH > 7. PURE WATER HAS A PH OF 7, WHICH MEANS IT IS NEUTRAL.

A IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?

B WHAT IS THE HYDROGEN ION CONCENTRATION $[H^+]$ OF EGGS IF THE PH OF EGGS IS 7.8?

SOLUTION:

A (TEST FOR BEER)

$\text{PH} = -\text{LOG}[H^+]$

$\text{PH} = -\text{LOG}[0.0000501] = -\text{LOG}[5.01\times 10^{-5}] = -[\text{LOG}\,5.01 + (-5)] = -[0.6998 + (-5)] = 4.3$

SINCE PH = 4.3 < 7 BEER IS AN ACID.

(TEST FOR WINE)

$$\text{PH} = -\text{LOG}[H^+] = -\text{LOG}[0.0004] = -\text{LOG}[4 \times 10^{-4}] = -[\text{LOG}\,4 + (-4)]$$

$$= -[0.6021+(-4)] \approx 3.4 \Rightarrow \text{PH} = 3.4 < 7.$$

SO WINE IS AN ACID.

B $\text{PH} = -\text{LOG}[H^+] \Rightarrow -\text{LOG}[H^+] = 7.8$

$$\Rightarrow \text{LOG}[H^+] = -7.8 \Rightarrow [H^+] = 10^{-7.8}$$

$$\Rightarrow [H^+] = 1.58 \times 10^{-8}$$

Group Work 2.6

NEWTON'S LAW OF COOLING STATES THAT AN OBJECT ... PROPORTIONAL TO THE DIFFERENCE BETWEEN THE TE... OF THE OBJECT AND THE ROOM TEMPERATURE. THE TEMPERATURE OF THE OBJECT AT A TIME t IS GIVEN BY A FUNCTION

$$f(t) = ce^{rt} + a,$$

WHERE a = ROOM TEMPERATURE

c = INITIAL DIFFERENCE IN TEMPERATURE BETWEEN THE OBJECT AND THE RO

r = CONSTANT DETERMINED BY DATA IN THE PROBLEM

PROBLEM: SUPPOSE YOU MAKE YOURSELF A CUP OF TEA. INITIALLY THE TEA HAS A TEMPERATURE OF 95°C; 5 MINUTES LATER THE TEA HAS COOLED TO 65°C. WHEN WILL THE TEA REACH A DRINKABLE TEMPERATURE OF 40°C?

Hint: ASSUME THAT THE ROOM TEMPERATURE a = 22°C. FIRST SOLVE FOR r AND THEN FIND t APPLYING THE NATURAL LOGARITHM.

Exercise 2.12

1 SUPPOSE YOU ARE OBSERVING THE BEHAVIOUR OF CELL DUPLICATION IN A LABORATORY. IN ONE EXPERIMENT, YOU START WITH ONE CELL AND THE CELL POPULATION IS TRIPLING EV

A WRITE A FORMULA TO DETERMINE THE NUMBER OF CELLS AFTER t MINUTES.

B USE YOUR FORMULA TO CALCULATE THE NUMBER OF CELLS AFTER AN HOUR

C DETERMINE HOW LONG IT WOULD TAKE THE NUMBER OF CELLS TO REACH 100,000.

2 SUPPOSE IN AN EXPERIMENT YOU STARTED WITH 100,000 CELLS AND OBSERVED CELL POPULATION DECREASED BY ONE HALF EVERY MINUTE.

A WRITE A FORMULA FOR THE NUMBER OF CELLS AFTER t MINUTES.

B DETERMINE THE NUMBER OF CELLS AFTER 10 MINUTES.

C DETERMINE HOW LONG IT WOULD TAKE THE CELL POPULATION TO REACH 10.

3 A BIRR 1,000 DEPOSITS IS MADE AT A BANK THAT PAYS 12% INTEREST COMPOU MONTHLY. HOW MUCH WILL BE IN THE ACCOUNT AT THE END OF 10 YEARS?

4 IF YOU START A BIOLOGY EXPERIMENT WITH 5,000,000 CELLS AND 25% OF THE C DYING EVERY MINUTE, HOW LONG WILL IT BE BEFORE THERE ARE FEWER THAN 1,0

5 ***Learning curve:*** IN PSYCHOLOGICAL TESTS, IT IS FOUND THAT STUDENTS CAN MEMORIZE A LIST OF WORDS AFTER t HOURS, ACCORDING TO THE LEARNING CURVE $y = 50 - 50e^{-0.3t}$, WHERE y IS THE NUMBER OF WORDS A STUDENT CAN LEARN DURING THE t^{TH} HOUR OF STUDY. FIND HOW MANY WORDS A STUDENT WOULD BE EXPECTED TO LEARN IN THE NINTH HOUR OF

6 THE ENERGY RELEASED BY THE LARGEST EARTHQUAKE RECORDED, MEASURED IN JOULES, IS ABOUT 100 BILLION TIMES THE ENERGY RELEASED BY A SMALL EARTHQUAKE THAT FELT. IN 1935 THE CALIFORNIA SEISMOLOGIST ***Charles Richter*** DEVISED A LOGARITHMIC SCALE THAT BEARS HIS NAME AND IS STILL WIDELY USED. THE MAGNITUDE M ON THE RICHTER SCALE IS GIVEN AS FOLLOWS:

$$M = \frac{2}{3}\text{LOG}\frac{E}{E_0} \quad \text{RICHTER SCALE}$$

WHERE E IS THE ENERGY RELEASED BY THE EARTHQUAKE MEASURED IN JOULES, AND E_0 IS THE ENERGY RELEASED BY A VERY SMALL REFERENCE EARTH QUAKE WHICH HAS BEEN STANDARDIZED TO BE $E_0 = 10^{4.40}$ JOULES.

QUESTION

AN EARTH QUAKE IN A CERTAIN TOWN X RELEASED APPROXIMATELY 5.96×10^{16} JOULES OF ENERGY. WHAT WAS ITS MAGNITUDE ON THE RICHTER SCALE? GIVE YOUR ANSWER DECIMAL PLACES.

7 ***Physics:*** THE BASIC UNIT OF SOUND MEASUREMENT IS CALLED A BEL, NAMED AFTER THE INVENTOR OF TELEPHONE, ALEXANDER GRAHAM BELL (1847-1922). THE LOUDEST SO HEALTHY PERSON CAN HEAR WITHOUT DAMAGE TO THE EARDRUM HAS AN INTENS (10^{12}) TIMES THAT OF THE SOFTEST OF SOUND A PERSON CAN HEAR. THE RELATIONSHIP OF LOUDNESS OF SOUND L AND INTENSITIES I AND I_0 IS GIVEN BY

$$L = 10\,\text{LOG}\frac{I}{I_0},$$

WHERE L IS MEASURED IN DECIBELS, I_0 IS THE INTENSITY OF THE LEAST AUDIBLE SOUND TH AN AVERAGE HEALTHY PERSON CAN HEAR, WHICH IS TAKEN 10^{-12} WATT PER SQUARE METER, AND I IS THE INTENSITY OF THE SOUND IN QUESTION.

QUESTION FIND THE NUMBER OF DECIBELS:

A FROM AN ORDINARY CONVERSATION WITH SOUND INTENSITY $I = 3.2 \times 10^{-6}$ WATT PER SQUARE METER.

B FROM A ROCKMUSIC CONCERT WITH SOUND INTENSITY $I = 5.2 \times 10^{-3}$ WATT PER SQUARE CENTIMETRE.

Key Terms

antilogarithm	***exponential expression***	***logarithmic expression***
base	***exponential function***	***logarithmic function***
characteristics	***logarithm***	***mantissa***
common logarithm	***logarithm of a number***	***natural logarithm***
exponent	***logarithmic equation***	***power***
exponential equation		

Summary

1 IF n IS A POSITIVE INTEGER, THEN a^n IS THE PRODUCT OF n FACTORS OF a.

I.E. $a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ FACTORS}}$

IN a^n, a IS CALLED THE ***base***, n IS CALLED THE ***exponent*** AND a^n IS THE n^{TH} ***power*** OF a.

2 **Laws of Exponents**

FOR a AND b POSITIVE AND r AND s REAL NUMBERS

A $a^r \times a^s = a^{r+s}$

B $\dfrac{a^r}{a^s} = a^{r-s}$

C $(a^r)^s = a^{rs}$

D $(a \times b)^s = a^s \times b^s$

E $\left(\dfrac{a}{b}\right)^s = \dfrac{a^s}{b^s}$

3 ANY NON – ZERO NUMBER RAISED TO ZERO IS 1, I.E. $a^0 = 1$ FOR $(a \neq 0)$

4 FOR $a \neq 0$ AND $n > 0$, $a^{-n} = \dfrac{1}{a^n}$.

5 FOR $a \neq 0$, $b \neq 0$ AND $n > 0$, $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

6 FOR ANY REAL NUMBER a AND ANY INTEGER $n > 1$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

$\sqrt[n]{a} \in \mathbb{R}$ IF $a \in \mathbb{R}$ AND n IS ODD; $\sqrt[n]{a} \notin \mathbb{R}$ IF $a < 0$ AND n IS EVEN.

7 IF $a > 0$ AND m, n ARE INTEGERS WITH $n > 1$, THEN $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

8 IF x IS AN IRRATIONAL NUMBER AND $a > 0$, THEN a^x IS A REAL NUMBER BETWEEN a^{x_1} AND a^{x_2} FOR ALL POSSIBLE CHOICES OF RATIONAL NUMBERS x_1 AND x_2 SUCH THAT $x_1 < x < x_2$.

9 FOR A FIXED POSITIVE NUMBER $b \neq 1$, AND FOR EACH $a \in \mathbb{R}$, $b^c = a$, IF AND ONLY IF $c = \log_b a$. ($c = \log_b a$ IS READ AS "c IS THE LOGARITHM OF a TO THE BASE b")

10 Laws of logarithms

IF b, x AND y ARE POSITIVE NUMBERS AND $b \neq 1$, THEN

A $\log_b xy = \log_b x + \log_b y$

B $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

C FOR ANY REAL NUMBER k, $\log_b x^k = k \log_b x$

D $\log_b b = 1$

E $\log_b 1 = 0$

11 LOGARITHMS TO BASE 10 ARE CALLED ***common logarithms***.

12 THE CHARACTERISTIC OF A COMMON LOGARITHM SHOWS THE LOCATION OF THE DECIMAL POINT. THE MANTISSA IS A POSITIVE DECIMAL LESS THAN 1.

13 IF a, b, c ARE POSITIVE REAL NUMBERS, $a \neq 1$, $b \neq 1$, THEN

A $\log_a c = \dfrac{\log_b c}{\log_b a}$ ("CHANGE OF BASE LAW") **B** $b^{\log_b c} = c$

14 $\log_e x = \ln x$ IS CALLED THE ***natural logarithm*** OF x.

15 THE FUNCTION $f(x) = b^x$, $b > 0$ AND $b \neq 1$ DEFINES AN EXPONENTIAL FUNCTION.

16 THE FUNCTION $f(x) = e^x$ IS CALLED THE ***natural exponential function***.

17 ALL MEMBERS OF THE FAMILY $f(x) = b^x$ ($b > 0$, $b \neq 1$) HAVE GRAPHS WHICH

✓ PASS THROUGH THE POINT (0 , 1)

✓ ARE ABOVE THE x-AXIS FOR ALL VALUES OF x

✓ ARE ASYMPTOTIC TO THE x-AXIS

✓ HAVE DOMAIN THE SET OF ALL REAL NUMBERS.

✓ HAVE RANGE THE SET OF ALL POSITIVE REAL NUMBERS.

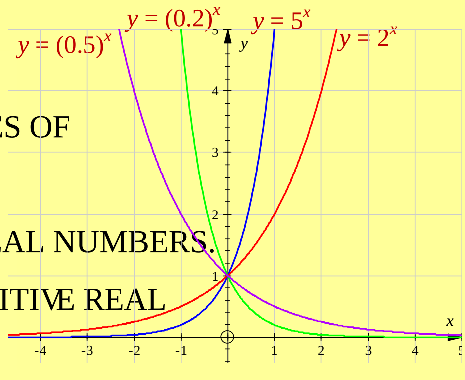


Figure 2.24

18 THE FUNCTION $f(x) = \log_b x$, $b > 0$ AND $b \neq 1$ IS CALLED A LOGARITHMIC FUNCTION WITH BASE b.

19 THE FUNCTION $f(x) = \log_e x = \ln x$ IS CALLED THE ***natural logarithm*** OF x.

20 ALL MEMBERS OF THE FAMILY $y = \log_b x$, $(b > 0,\ b \neq 1)$ HAVE GRAPHS WHICH

- ✓ PASS THROUGH THE POINT (1, 0)
- ✓ ARE ASYMPTOTIC TO THE y-AXIS
- ✓ HAVE DOMAIN THE SET OF ALL POSITIVE REAL NUMBERS
- ✓ HAVE RANGE THE SET OF REAL NUMBERS.

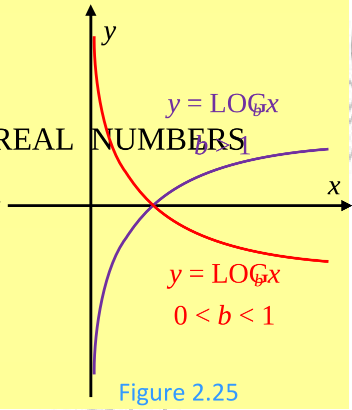


Figure 2.25

Review Exercises on Unit 2

1 WRITE THE SIMPLIFIED FORM OF EACH OF THE FOLLOWING EXPRESSIONS:

A 2^5 **B** -2^5 **C** 2^{-5} **D** -2^{-5}

E $\left(\dfrac{2}{3}\right)^2$ **F** $\left(\dfrac{2}{3}\right)^{-2}$ **G** $\dfrac{2^{-2}}{3^{-2}}$ **H** $\left(-\dfrac{2}{3}\right)^2$

2 USE THE LAWS OF EXPONENTS TO SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS:

A $2^5 \times 2^2$ **B** $\left(6^{\frac{1}{2}}\right)^2$ **C** $\dfrac{64^{\frac{3}{2}}}{8^{\frac{3}{2}}}$ **D** $a^{-3}b^{-3}$

E $(4n^5)^2$ **F** $\left(\dfrac{x}{2y}\right)^2$ **G** $\dfrac{d^{-4}}{d^{-2}}$ **H** $(x^{-3})^2$

I $E^{3x-1}\,E^{4-x}$ **J** $\dfrac{3^x}{3^{1-x}}$ **K** $\dfrac{5^{x-3}}{5^{x-4}}$ **L** $(2^x3^y)^z$

3 CHANGE EACH LOGARITHMIC FORM TO AN EQUIVALENT EXPONENTIAL FORM:

A $\log_3 81 = 4$ **B** $\log_{25} 5 = \dfrac{1}{2}$

C $\log_2 \dfrac{1}{4} = -2$ **D** $\log_{\frac{1}{2}} \dfrac{1}{4} =$

4 FIND x IF:

A $\log_2 x = 5$ **B** $\log_4 16 = x$ **C** $\log_7 7 = x$

D $\log_x 16 = 2$ **E** $\log_8 x = \frac{1}{3}$ **F** $\log_{\frac{1}{3}} 9 = x$

G $\log_{49} \frac{1}{7} = x$ **H** $\log_x 1000 = \frac{3}{2}$

5 USE THE PROPERTIES OF LOGARITHMS TO WRITE EACH OF THE FOLLOWING EXPRESSIONS AS A SINGLE LOGARITHM:

A $\log_{10} 2 + \log_{10} 25$ **B** $\log_5 18 - \log_5 3$

C $3\log_3 5 - 2\log_3 7$ **D** $5\log_a x + 3\log_a y$

E $\log_a x^3 + \log_a \left(\frac{b}{\sqrt[3]{x}}\right)$ **F** $\ln x^3 - \ln \sqrt{x}$

6 USE THE TABLE OF COMMON LOGARITHMS TO FIND:

A LOG 4.21 **B** LOG 0.99 **C** LOG 8.2

D LOG 123 **E** LOG 0.34 **F** LOG 8.88

G LOG 0.00001 **H** LOG 500

7 FIND:

A ANTILOG 0.4183 **B** ANTILOG 0.3507 **C** ANTILOG 0.5428

D ANTILOG 0.8831 **E** ANTILOG 5.9736 **F** ANTILOG 1.7559

G ANTILOG (-10) **H** ANTILOG (-0.3)

8 STUDY THE FOLLOWING GRAPH (FIGURE 2.26) AND ANSWER THE QUESTIONS THAT FOLLOW:

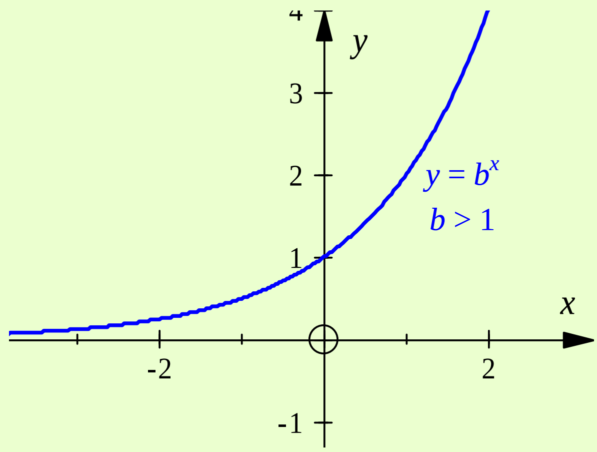


Figure 2.26

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE y-INTERCEPT?

E FOR WHICH VALUES OF x IS b^x GREATER THAN 1?

F WHAT CAN YOU SAY ABOUT THE VALUE OF b^x IF x IS NEGATIVE?

G FOR WHICH VALUES OF x IS b^x LESS THAN ZERO?

9 STUDY THE FOLLOWING GRAPH (FIGURE 2.27) AND ANSWER THE QUESTIONS GIVEN BELOW.

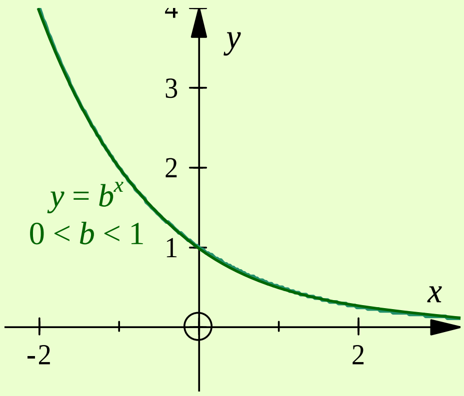


Figure 2.27

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE y-INTERCEPT?

E FOR WHICH VALUES OF x IS $b^x > 1$?

F WHAT IS THE VALUE OF b^x IF x IS POSITIVE?

G FOR WHICH VALUES OF x IS $b^x < 0$?

10 SKETCH THE FOLLOWING PAIRS OF FUNCTIONS USING THE SAME COORDINATE SYSTEM:

A $f(x) = 2^x - 3$ AND $g(x) = 2^x + 3$

B $f(x) = 3^x$ AND $g(x) = 3^x + 2$

C $f(x) = \left(\frac{3}{5}\right)^x$ AND $g(x) = \left(\frac{3}{5}\right)^{x+1}$

D $f(x) = 5^x$ AND $g(x) = \left(\frac{1}{5}\right)^x$

11 STUDY THE FOLLOWING GRAPH (Figure 2.28) AND ANSWER THE QUESTIONS THAT FOLLOW:

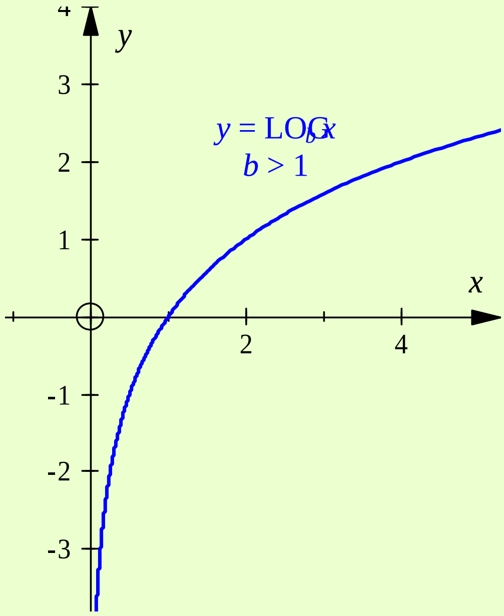


Figure 2.28

A GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.

B WHAT IS THE ASYMPTOTE OF THE GRAPH?

C IS THE FUNCTION INCREASING OR DECREASING?

D WHAT IS THE x-INTERCEPT?

E FOR WHICH VALUES OF x IS $\log_b x > 0$?

F WHEN IS $\log_b x < 0$?

12 SKETCH THE FOLLOWING PAIRS OF FUNCTIONS USING THE SAME COORDINATE SYSTEM:

A $f(x) = \log_3 x$ AND $g(x) = \log_3(x-2)$

B $f(x) = \ln x$ AND $g(x) = \ln(x+2)$

C $f(x) = \log_5 x$ AND $g(x) = \log_{\left(\frac{1}{5}\right)} x$

D $f(x) = 5^x$ AND $g(x) = \log_5 x$

13 STATE THE UNIVERSE FOR EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = \log_5 x$ **B** $g(x) = \log_{\left(\frac{1}{3}\right)}(x+3)$

C $f(x) = \log_3(3-x)$ **D** $g(x) = \log_9(7x-12)$

E $f(x) = \log_2(3-x) + \log_2(3+x)$ **F** $f(x) = \log_2\left(x^2 - 2x\right)$

14 SOLVE EACH OF THE FOLLOWING EXPONENTIAL EQUATIONS:

A $3^x = 27$ **B** $2^{3-x} = 16$ **C** $5^{(4x-5)} = \frac{1}{25}$

D $4^{3x-8} = 2^{3x+9}$ **E** $36^{5x} = 6$ **F** $7^{x^2+x} = 49$

G $2^{6(x+2)} = 4^{x+2}$ **H** $2\left(\frac{243}{32}\right)^{2x} = 3\left(\frac{8}{27}\right)^{\left(\frac{2}{3}x-1\right)}$

15 SOLVE EACH OF THE FOLLOWING FOR x, CHECKING VALIDITY OF SOLUTIONS:

A $\mathrm{LOG}_3\, x = 3$ **B** $\mathrm{LOG}_{10}\, x = \frac{3}{2}$

C $\mathrm{LOG}_x\, e^5 = 5$ **D** $\mathrm{LOG}\, 3x^2 - \mathrm{LOG}\, 9x = 2$

E $\mathrm{LOG}\, x - \mathrm{LOG}\, 3 = \mathrm{LOG}\, 4 - \mathrm{LOG}\,(x+4)$ **F** $\mathrm{LN}\,(x+3) - \mathrm{LN}\, x = 2\,\mathrm{LN}\, 2$

G $\mathrm{LN}\,(2x+1) - \mathrm{LN}\,(x-1) = \mathrm{LN}\, x$ **H** $\mathrm{LOG}\,(x^2-3) = 2\,\mathrm{LOG}\,(x-1)$

I $\mathrm{LOG}\,(4x)^5 = 5$ **J** $\mathrm{LOG}_2\, x + \mathrm{LOG}_2\, x^2 = 15$

K $\mathrm{LOG}_5\,(3+x) - \mathrm{LOG}_5\, x = 2$

16 IF 2000 BIRR IS INVESTED AT 4% INTEREST, COMPOUNDED EVERY YEAR FOR 5 YEARS, WHAT IS THE AMOUNT REALIZED AT THE END OF 5 YEARS?

17 SUPPOSE THAT THE NUMBER OF BACTERIA IN A CERTAIN COLONY GROWS AT THE RATE OF 5% PER DAY. IF THERE ARE 1000 BACTERIA PRESENT INITIALLY, THEN W THE NUMBER OF BACTERIA PRESENT AFTER:

A 1 DAY? **B** 2 DAYS? **C** 3 DAYS? **D** 10 DAYS? **E** n DAYS?

18 THE POPULATION OF COUNTRY A IS 3.25×10^7 AND THAT OF COUNTRY B IS 1.11×10^8. IF THE ANNUAL GROWTH OF POPULATION OF COUNTRIES A AND B ARE 5.2% AN RESPECTIVELY, WHEN WILL COUNTRIES A AND B HAVE THE SAME POPULATION?

19 A CAR PURCHASED FOR 30,000 BIRR DEPRECIATES AT 5% PER ANNUM, THE DEPRECIATION BEING WORKED OUT ON THE VALUE OF THE CAR AT THE BEGINN YEAR. FIND ITS VALUE AFTER 10 YEARS.

Hint: IF V_O IS THE VALUE OF A CERTAIN OBJECT AT A CERTAIN TIME, AND r% IS THE RATE OF DEPRECIATION PER YEAR, THEN THE VALUE V_t AT THE END OF t YEARS IS GIVEN BY: $V_t = V_O\left(1 - \frac{r}{100}\right)^t$, WHERE V_O IS THE INITIAL VALUE.