Unit 6

Leonardo da Vinci obtained the "Mona Lisa" smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes. The outline of the top of the head is the arc of another circle exactly twice as large as the first.

PLANE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *know more theorems special to triangles.*
- *know basic theorems specific to quadrilaterals.*
- *know theorems about circles and angles inside, on and outside a circle.*
- *solve geometrical problems involving quadrilaterals, circles and regular polygons.*

Main Contents

INTRODUCTION

WHY DO YOU STUDY GEOMETRY?

- GEOMETRY TEACHES YOU HOW TO THINK CLEARLY. OF ALL THE SUBJECTS TAU SCHOOL LEVEL, GEOMETRY IS ONE OF THE LESSONS THAT GIVES THE BEST TRAININ AND ACCURATE METHODS OF THINKING.
- THE STUDY OF GEOMETRY HAS A PRACTICAL VALUE. IF SOMEONE WANTS TO B DESIGNER, A CARPENTER, A TINSMITH, A LAWYER OR A DENTIST, THE FACTS AND IN GEOMETRY ARE OF GREAT VALUE.

  Abraham Lincoln BORROWED A GEOMETRY TEXT AND LEARNED THE PROOFS OF MOS PLANE GEOMETRY THEOREMS SO THAT HE COULD MAKE BETTER ARGUMENTS IN C

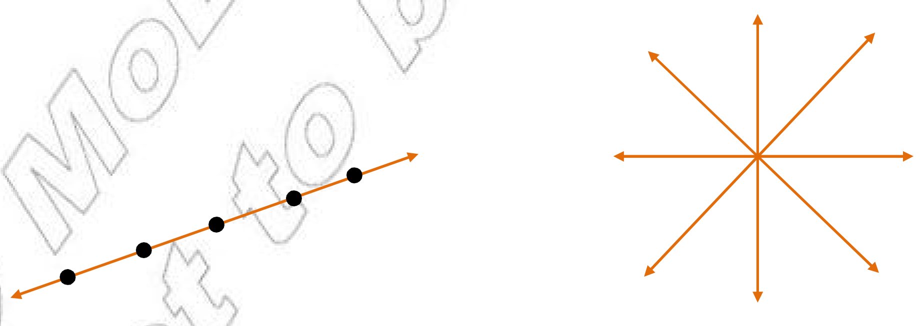
  Leonardo da Vinci OBTAINED THE "MONA LISA" SMILE BY TILTING THE LIPS SO THA ENDS LIE ON A CIRCLE WHICH TOUCHES THE OUTER CORNERS OF THE EYES. THE O TOP OF THE HEAD IS THE ARC OF ANOTHER CIRCLE EXACTLY TWICE AS LARGE AS T SAME ARTIST'S "LAST SUPPER", THE VISIBLE PART OF CHRIST CONFORMS TO THE S EQUILATERAL TRIANGLE.

PLANE GEOMETRY (SOMETIMES CALLED EUCLIDEAN GEOMETRY) IS A BRANCH OF DEALING WITH THE PROPERTIES OF FLAT SURFACES AND PLANE FIGURES, SUCH QUADRILATERALS OR CIRCLES.

6.1 THEOREMS ON TRIANGLES

IN PREVIOUS GRADES, YOU HAVE LEARNT THAT A TRIANGLE IS A POLYGON WITH THR THE SIMPLEST TYPE OF POLYGON.

THREE OR MORE POINTS THAT LIE ON ONE LINE ARE CALLED ***collinear points***. THREE OR MORE LINES THAT PASS THROUGH ONE POINT ARE CALLED ***concurrent lines***

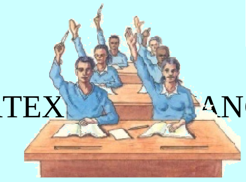


COLLINEAR POINTS CONCURRENT LINES

Figure 6.1

© MoE Ethiopia Not to be republished

ACTIVITY 6.1



1 WHAT DO YOU CALL A LINE SEGMENT JOINING A VERTEX [illegible] ANGLE TO THE MID-POINT OF THE OPPOSITE SIDE?

2 HOW MANY MEDIANS DOES A TRIANGLE HAVE?

3 DRAW TRIANGLE *ABC* WITH $\angle C = 90^{O}$, AC = 8 CM AND CB = 6 CM. DRAW THE MEDIAN FROM *A* TO $\overline{BC}$. HOW LONG IS THIS MEDIAN? CHECK YOUR RESULT USING PYTHAGORAS THEOREM.

4 DRAW A TRIANGLE. CONSTRUCT ALL THE THREE MEDIANS. ARE THEY CONCURREN THINK THAT THIS IS TRUE FOR ALL TRIANGLES? TEST THIS BY DRAWING MORE TRI

5 IS IT POSSIBLE FOR THE MEDIANS OF A TRIANGLE TO MEET OUTSIDE THE TRIANGLE

THEOREMS ABOUT COLLINEAR POINTS AND CONCURRENT LINES ARE CALLED ***incidence theorems***. SOME SUCH THEOREMS ARE STATED BELOW.

RECALL THAT A LINE THAT DIVIDES AN ANGLE INTO TWO CONGRUENT ANGLES IS C BISECTOR OF THE ANGLE.

A LINE THAT DIVIDES A LINE SEGMENT INTO TWO CONGRUENT LINE SEGMENTS IS CALL ***bisector*** OF THE LINE SEGMENT. WHEN A BISECTOR OF A LINE SEGMENT FORMS RIGHT ANGLE WITH THE LINE SEGMENT, THEN IT IS CALLED THE ***perpendicular bisector*** OF THE LINE SEGMENT.

Median of a triangle

A ***median*** OF A TRIANGLE IS A LINE SEGMENT DRAWN FROM ANY VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

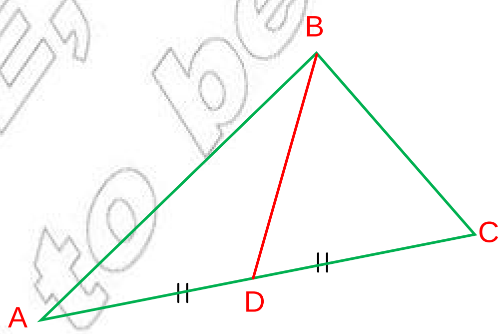


Figure 6.2

$\overline{BD}$ IS A MEDIAN OF TRIANGLE ABC.

© MoE, FDRE ... Not to be republished

ACTIVITY 6.2

COPY △ABC IN FIGURE 6.3.

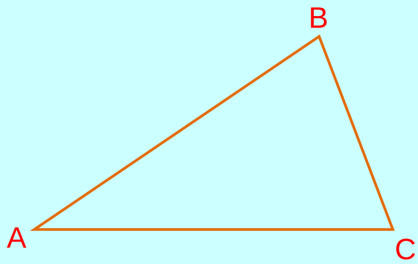


Figure 6.3

1. CONSTRUCT ALL THE MEDIANS OF △ABC CAREFULLY.
2. I MARK THE MID-POINT OF $\overline{BC}$ AS E.
   II MARK THE MID-POINT OF $\overline{AC}$ AS F.
   III MARK THE MID-POINT OF $\overline{AB}$ AS D
3. DID THE MEDIANS INTERSECT AT A POINT?
   IF YOUR ANSWER IS YES, MARK THE POINT O.
4. MEASURE EACH OF THE FOLLOWING SEGMENTS AND DETERMINE THE INDICATED R

I	A	$\overline{AO}$	B	$\overline{OE}$	$AO : OE$
II	A	$\overline{CO}$	B	$\overline{OD}$	$CO : OD$
III	A	$\overline{BO}$	B	$\overline{OF}$	$BO : OF$

5. HOW DO YOU RELATE THE RATIOS OBTAINED IN QUESTION 4 ABOVE?

THE ABOVE ACTIVITY HELPS YOU TO OBSERVE THE FOLLOWING THEOREM

Theorem 6.1

The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

Proof:-

SUPPOSE $\overline{AE}$ AND $\overline{DC}$ ARE MEDIANS OF △ABC THAT ARE INTERSECTING AT POINT O. (*See* FIGURE 6.4).

	Statement		Reason
1	IN ΔABC, $\overline{AE}$ AND $\overline{DC}$ ARE MEDIANS INTERSECT POINT O.	1	GIVEN
2	DRAW $\overline{DE}$	2	CONSTRUCTION
3	DRAW $\overline{EG}$ PARALLEL TO $\overline{DC}$ WITH G ON THE EXTENSION $\overline{AC}$	3	CONSTRUCTION
4	DRAW $\overline{EF}$ PARALLEL TO $\overline{AB}$ WITH F ON $\overline{AC}$	4	CONSTRUCTION
5	DRAW $\overline{FH}$ PARALLEL TO $\overline{DC}$ WITH H ON $\overline{AB}$	5	CONSTRUCTION
6	DRAW LINE ℓ PARALLEL TO $\overline{DC}$ PASING THROUGH A.	6	CONSTRUCTION
7	$AFED$ AND $CGED$ ARE PARALLELOGRAMS WIT SIDE $\overline{DE}$	7	STEPS 3 AND 4
8	THEREFORE, $AF = DE = CG$	8	STEP 7
9	$DE = \frac{1}{2} AC = AF$	9	$\Delta ABC \sim \Delta DBE$ FROM STEP 1
10	$AF = FC = CG$	10	STEPS 8 AND 9
11	$\overline{AG}$ IS TRISECTED BY PARALLEL LINES ℓ, $\overleftrightarrow{HF}$, $\overleftrightarrow{DC}$ AND $\overleftrightarrow{EG}$	11	STEPS 3, 5 AND 10
12	$\overline{AE}$ IS TRISECTED BY ℓ, $\overline{HF}$, $\overline{DC}$ AND $\overline{EG}$	12	STEP 11 AND PROPERTY OF PARALLEL LINES

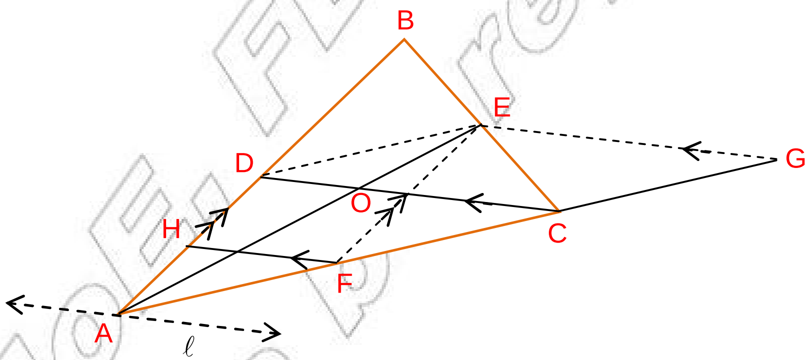


Figure 6.4

THEREFORE, $OE = \frac{1}{3} AE$, $AO = \frac{2}{3} AE$.

YOU HAVE PROVED THAT THE MEDIANS $\overline{DC}$ AND $\overline{AE}$ MEET AT POINT O SUCH THAT $AO = \frac{2}{3} AE$.

YOUR NEXT TASK IS TO PROVE THAT THE MEDIANS $\overline{AE}$ AND $\overline{BF}$ INTERSECT AT THE SAME POINT O. WITH THE SAME ARGUMENT USED ABOVE, LET O' BE THE POINT OF INTERSECTION OF $\overline{AE}$ AND $\overline{BF}$ WHOSE DISTANCE FROM A IS $\frac{2}{3} AE$ THAT IS $AO' = \frac{2}{3} AE$

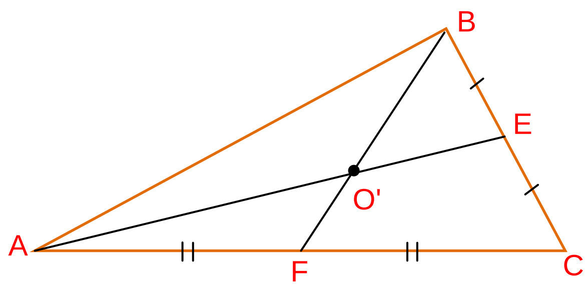


Figure 6.5

IT FOLLOWS THAT $AO = AO'$ AND HENCE $O = O'$ AS O AND O' ARE ON $\overline{AE}$.
THEREFORE, ALL THE THREE MEDIANS OF $\triangle ABC$ ARE CONCURRENT AT A SINGLE POINT O LOCATED $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSI

EXAMPLE 1 IN FIGURE 6.6, $\overline{AN}$, $\overline{CM}$ AND $\overline{BL}$ ARE MEDIANS OF $\triangle ABC$. IF $AN = 12$ CM, $OM = 5$ CM AND $BO = 6$ CM, FIND BL, ON AND OL.

SOLUTION:

BY THEOREM 6.1,

$$BO = \frac{2}{3}\ BL \text{ AND } AO = \frac{2}{3}\ AN$$

SUBSTITUTING $6 = \frac{2}{3}\ BL$ AND $AO = \frac{2}{3} \times 12$

SO $BL = 9$ CM AND $AO = 8$ CM.

SINCE $BL = BO + OL$,

$OL = BL - BO = 9 - 6 = 3$ CM.

NOW, $AN = AO + ON$ GIVES

$ON = AN - AO = 12 - 8 = 4$ CM

$\therefore\ BL = 9$ CM, $OL = 3$ CM AND $ON = 4$ CM

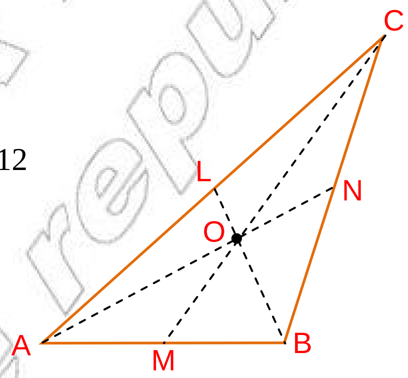


Figure 6.6

Note: THE POINT OF INTERSECTION OF THE MEDIANS OF A TRIANGLE IS CALLED THE ***centroid*** OF THE TRIANGLE.

Altitude of a triangle

THE ***altitude*** OF A TRIANGLE IS A LINE SEGMENT DRAWN FROM A VERTEX PERPENDICULAR TO THE OPPOSITE SIDE, OR TO THE OPPOSITE SIDE PRODUCED.

THE ***altitudes*** THROUGH B AND A FOR THE TRIANGLES ARE SHOWN IN FIGURE 6.7.

© Not FDR Ethiopia Not to be republished

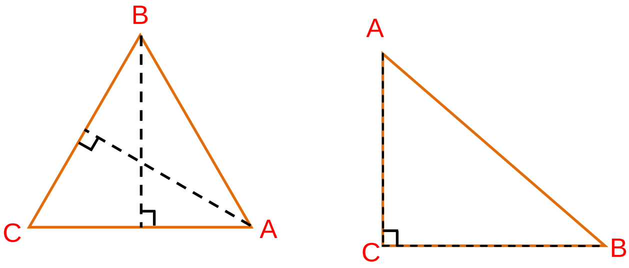


Figure 6.7

ACTIVITY 6.3

1 WHAT IS MEANT BY AN ANGLE BISECTOR?

2 ANY SIDE OF A TRIANGLE MAY BE DESIGNATED AS A BASE. HOW MANY BASES MAY A TRIANGLE HAVE?

3 HOW MANY ALTITUDES CAN A TRIANGLE HAVE?

4 BY DRAWING THE FOLLOWING TYPES OF TRIANGLES WITH THEIR RESPECTIVE ALTITUDES, DETERMINE WHETHER THE ALTITUDES INTERSECT INSIDE OR OUTSIDE THE TRIANGLE:

 A AN ACUTE-ANGLED TRIANGLE; B AN OBTUSE-ANGLED TRIANGLE;

 C A RIGHT-ANGLED TRIANGLE.

5 DRAW THE PERPENDICULAR BISECTORS OF THE SIDES OF THE FOLLOWING TRIANGLES, AND NOTE WHERE THE PERPENDICULAR BISECTORS INTERSECT.

 A AN ACUTE-ANGLED TRIANGLE; B AN OBTUSE-ANGLED TRIANGLE;

 C A RIGHT-ANGLED TRIANGLE.

6 DRAW ANY $\triangle ABC$. CONSTRUCT THE PERPENDICULAR BISECTORS OF THE SIDES $\overline{AB}$ AND $\overline{CB}$. LABEL THEIR INTERSECTION AS POINT O.

 A WHY IS POINT O EQUIDISTANT FROM A AND B?

 B WHY IS POINT O EQUIDISTANT FROM B AND C?

 C DO YOU THINK THAT THE PERPENDICULAR BISECTOR OF THE SIDE $\overline{AC}$ PASSES THROUGH THE POINT O? (WHY?)

ACTIVITY 6.3 CAN HELP YOU TO STATE THE FOLLOWING THEOREM:

Theorem 6.2

The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

LET ΔABC BE GIVEN AND CONSTRUCT PERPENDICULAR BISECTORS ON ANY TWO OF THE

PERPENDICULAR BISECTORS OF $\overline{AB}$ AND $\overline{AC}$ ARE SHOWN IN FIGURE 6.8A. THESE PERPENDICULAR BISECTORS INTERSECT AT A POINT O, THEY CANNOT BE PARALLEL. (WHY?)

USING A RULER, FIND THE LENGTHS AO, BO AND CO. OBSERVE THAT THE INTERSECTION POINT O IS EQUIDISTANT FROM EACH VERTEX OF THE TRIANGLE.

NOTE THAT THE PERPENDICULAR BISECTOR OF THE REMAINING SIDE $\overline{BC}$ MUST ALSO PASS THROUGH THE POINT O. THEREFORE, THE POINT OF INTERSECTION OF THE THREE PERPENDICULAR B EQUIDISTANT FROM THE THREE VERTICES OF ΔABC.

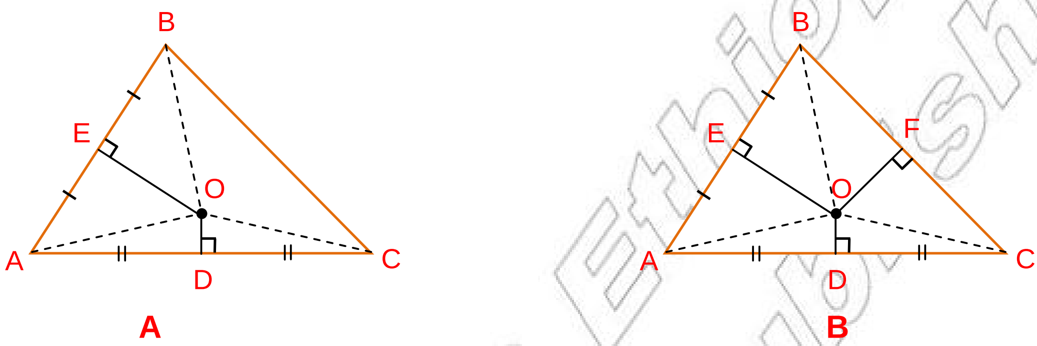


Figure 6.8

LET US TRY TO PROVE THIS RESULT.

WITH O THE POINT WHERE THE PERPENDICULAR BISECTORS OF $\overline{AB}$ AND $\overline{AC}$ MEET, AS SHOWN IN FIGURE 6.8B, $\Delta AOD \equiv \Delta COD$ BY SAS AND HENCE $\overline{AO} \equiv \overline{CO}$.

SIMILARLY, $\Delta AOE \equiv \Delta BOE$ BY SAS AND HENCE $\overline{AO} \equiv \overline{BO}$.

THUS, $\overline{AO} \equiv \overline{BO} \equiv \overline{CO}$. IT FOLLOWS THAT O IS EQUIDISTANT FROM THE VERTICES OF ΔABC.

NEXT, LET F BE THE FOOT OF THE PERPENDICULAR FROM O TO $\overline{BC}$. THEN, $\overline{OF}$ IS THE PERPENDICULAR BISECTOR OF $\overline{BC}$ BECAUSE ΔBOC IS AN ISOSCELES TRIANGLE.

THEREFORE, THE PERPENDICULAR BISECTORS OF THE SIDES OF ΔABC ARE CONCURRENT.

Note: THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF A TRIANGLE IS CALLED ***circumcentre*** OF THE TRIANGLE.

Theorem 6.3

The altitudes of a triangle are concurrent.

TO SHOW THAT THE THREE ALTITUDES OF ΔABC MEET AT A SINGLE POINT, CONSTRUCT $\Delta A'B'C'$ (SHOWN IN FIGURE 6.9) SO THAT THE THREE SIDES OF $\Delta A'B'C'$ ARE PARALLEL RESPECTIVELY TO THE THREE SIDES OF ΔABC:

LET $\overline{EA}$, $\overline{BF}$ AND $\overline{CD}$ BE THE ALTITUDES OF ΔABC.

THE QUADRILATERALS $ABA'C$, $ABCB'$ AND $AC'BC$ ARE PARALLELOGRAMS. (WHY?)

SINCE $ABA'C$ IS A PARALLELOGRAM, $AC = BA'$. (WHY?) AGAIN, SINCE $ACBC'$ IS A PARALLELOGRAM, $AC = BC'$. THEREFORE, $BC' = BA'$ (WHY?) AND $\overline{BF}$ BISECTS $\overline{A'C'}$.

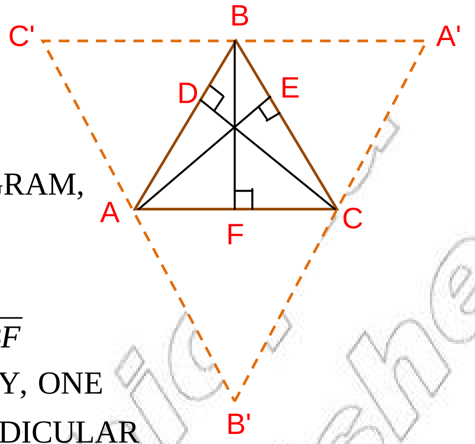


Figure 6.9

ACCORDINGLY, $\overline{BF}$ IS PERPENDICULAR TO $\overline{AC}$ AND SO $\overline{BF}$ IS THE PERPENDICULAR BISECTOR OF $\overline{A'C'}$. SIMILARLY, ONE CAN SHOW THAT $\overline{CD}$ AND $\overline{AE}$ ARE PERPENDICULAR BISECTORS OF $\overline{A'B'}$ AND $\overline{B'C'}$ RESPECTIVELY.

THEREFORE, THE ALTITUDES OF ΔABC ARE THE SAME AS THE PERPENDICULAR BISECTORS OF SIDES OF $\Delta A'B'C'$. SINCE THE PERPENDICULAR BISECTORS OF ANY TRIANGLE ARE CON (THEOREM 6.1), IT IS THEREFORE, TRUE THAT THE ALTITUDES ARE CONCU

Note: THE POINT OF INTERSECTION OF THE ALTITUDES IS CALLED THE ***orthocentre*** OF THE TRIANGLE.

Angle bisector of a triangle

Theorem 6.4

The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

TO SHOW THAT THE ANGLE BISECTORS OF ΔABC MEET AT A SINGLE POINT, DRAW THE BISECTORS O $\angle A$ AND $\angle C$, INTERSECTING EACH OTHER AT O (FIGURE 6.10).

CONSTRUCT THE PERPENDICULARS $\overline{OA'}$, $\overline{OB'}$ AND $\overline{OC'}$.

DO THESE SEGMENTS HAVE THE SAME LENGTH? SHOW THAT $\Delta OBB' \equiv \Delta OBA'$ AND CONCLUDE THAT

$\angle OBB' \equiv \angle OBA'$.

THEREFORE, THE BISECTOR OF $\angle B$ ALSO PASSES THROUGH THE POINT O.

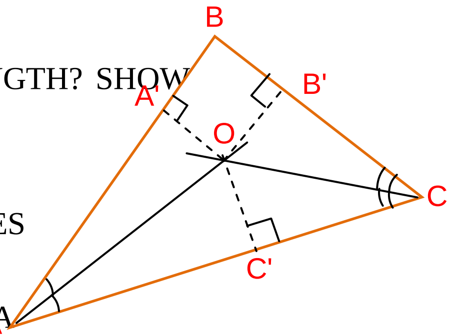


Figure 6.10

THEREFORE, THE ANGLE BISECTORS OF ΔABC MEET AT A SINGLE POINT. ALSO THEIR POINT OF INTERSECTION IS EQUIDISTANT FROM THE THREE SIDES OF ΔABC.

Note: THE POINT OF INTERSECTION OF THE BISECTORS OF THE ANGLES OF A TRIANGLE IS CALLED THE ***incentre*** OF THE TRIANGLE.

EXAMPLE 2 IN A RIGHT ANGLE TRIANGLE ABC, $\angle C$ IS A RIGHT ANGLE, AB = 8 CM AND CA = 6 CM. FIND THE LENGTH OF $\overline{CO}$ WHERE O IS THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF AC AND BC.

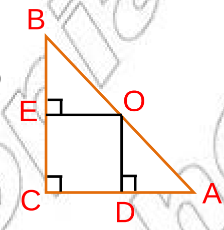


Figure 6.11

SOLUTION: THE PERPENDICULAR BISECTOR OF $\overline{CA}$ IS PARALLEL TO $\overline{CB}$.

HENCE, O IS ON $\overline{AB}$.

THEREFORE, AO = 4. (BY THEOREM 6.2, AO = BO)

BY THEOREM 6.2, O IS EQUIDISTANT FROM A, B AND C.

THEREFORE, CO = AO = 4 CM.

Group Work 6.1

WORK IN A SMALL GROUP ON ONE OR MORE OF THE FO... STATEMENTS. THERE WILL BE A CLASS DISCUSSION ON T... EACH ONE SHOULD BE ATTEMPTED BY AT LEAST ONE GROUP.

Task: CHECK THAT THE FOLLOWING STATEMENTS HOLD TRUE FOR ANY TRIANGLE BY CARRYING OUT THE CONSTRUCTION CAREFULLY.

Materials required: RULER, PROTRACTOR AND COMPASSES

Method: CONSTRUCTION AND MEASUREMENT

1. THE MEDIANS OF ANY TRIANGLE ARE CONCURRENT.
2. THE MEDIANS OF A TRIANGLE ARE CONCURRENT AT A POINT THAT IS $\frac{2}{3}$ THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSTIE SIDE.
3. THE ALTITUDES OF ANY TRIANGLE ARE CONCURRENT.
4. THE PERPENDICULAR BISECTORS OF THE SIDES OF ANY TRIANGLE ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.
5. THE ANGLE BISECTORS OF ANY TRIANGLE ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM THE SIDES OF THE TRIANGLE.
6. GIVEN ANY TRIANGLE, EXPLAIN HOW YOU CAN FIND THE CENTRE OF THE CIRCLE:
   - A INSCRIBED IN THE TRIANGLE (INCENTRE).
   - B CIRCUMSCRIBED ABOUT THE TRIANGLE (CIRCUMCENTRE).

Altitude theorem

THE ALTITUDE THEOREM IS STATED HERE FOR A RIGHT ANGLED TRIANGLE. IT RELATES THE LE
ALTITUDE TO THE HYPOTENUSE OF A RIGHT ANGLED TRIANGLE, TO THE LENGTHS OF TH
HYPOTENUSE.

Theorem 6.5 Altitude theorem

IN A RIGHT ANGLED TRIANGLE ABC WITH ALTITUDE $\overline{CD}$ TO THE HYPOTENUSE $\overline{AB}$,

$$\frac{AD}{DC} = \frac{CD}{DB}$$

Proof:-

CONSIDER ΔABC AS SHOWN IN FIGURE 6.12. $\Delta ABC \sim \Delta ACD$ . . . AA SIMILARITY

SO, $\angle ABC \equiv \angle ACD$

SIMILARLY, $\Delta ABC \sim \Delta CBD$ . . . AA SIMILARITY

SO, $\angle ABC \equiv \angle CBD$.

IT FOLLOWS THAT $\angle ACD \equiv \angle CBD$.

BY AA SIMILARITY, $\Delta ACD \sim \Delta CBD$.

HENCE, $\frac{AD}{CD} = \frac{CD}{BD}$ … (*)

EQUIVALENTLY, $\frac{AD}{DC} = \frac{CD}{DB}$

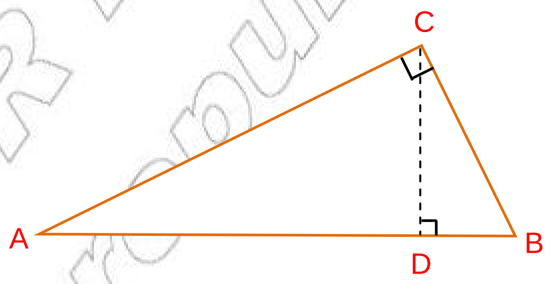


Figure 6.12

THE FOLLOWING ARE SOME FORMS OF THE ALTITUDE THEOREM.

FROM (*), $(CD)^2 = (AD)(BD)$

OR $(AD)(DB) = (CD)(DC)$

THIS CAN BE STATED AS:

THE SQUARE OF THE LENGTH OF THE ALTITUDE IS THE PRODUCT OF THE LEN
SEGMENTS OF THE HYPOTENUSE.

EXAMPLE 3 IN ΔABC, $\overline{CD}$ IS THE ALTITUDE TO THE HYPOTENUSE $\overline{AB}$, $AD = 9$ CM AND $BD = 4$ CM. HOW LONG IS THE ALTITUDE $\overline{CD}$? SEE FIGURE 6.12.

SOLUTION LET $h = CD$. FROM THE ALTITUDE THEOREM, $(CD)^2 = (AD)(BD)$

SUBSTITUTING, $h^2 = 9 \times 4 = 36$ CM^2

SO, $h = 6$ CM.

THE LENGTH OF THE ALTITUDE IS 6 CM.

Menelaus' theorem

Menelaus' theorem WAS KNOWN TO THE ANCIENT GREEKS ALMOST TWO THOUSAND YEARS AGO. IT WAS NAMED IN HONOUR OF THE GREEK MATHEMATICIAN AND ASTRONOMER ***Menelaus*** ***(70 - 140 AD)***.

Theorem 6.6 **Menelaus' theorem**

If points D, E and F on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively of ΔABC (or their extensions) are collinear, then $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$. Conversely, if $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$, then the points D, E and F are collinear.

Note:
1. FOR A LINE SEGMENT AB, WE USE THE CONVENTION $AB = -BA$.
2. IF F IS IN $\overline{AB}$, THEN $\frac{AF}{FB} = r > 0$.

IN FIGURE 6.13, LET D DIVIDE $\overline{BC}$ IN THE RATIO r, E DIVIDES $\overline{CA}$ IN THE RATIO s AND F DIVIDES $\overline{AB}$ IN THE RATIO t,

$$\text{I.E., } r=\frac{BD}{DC},\ s=\frac{CE}{EA} \text{ AND } t=\frac{AF}{FB}.$$

WE SEE FROM THE FIGURE THAT D DIVIDES $\overline{BC}$ AND E DIVIDES $\overline{CA}$ INTERNALLY, BUT F DIVIDES $\overline{AB}$ EXTERNALLY.

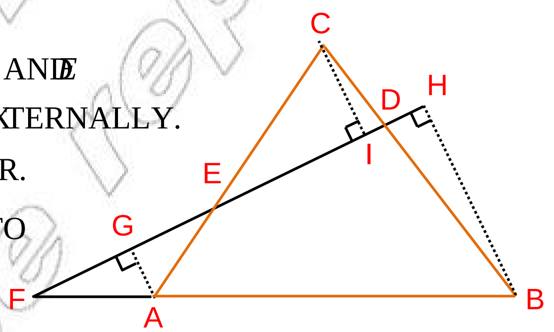


Figure 6.13

ASSUME THAT D, E AND F ARE COLLINEAR.

DRAW $\overline{AG}$, $\overline{BH}$, $\overline{CI}$ PERPENDICULAR TO $\overleftrightarrow{DF}$.

THEN, $\Delta CEI \sim \Delta AEG$ (WHY?),

$$\text{SO, } \frac{CE}{AE}=\frac{CI}{AG} \Rightarrow -\frac{CE}{EA}=\frac{CI}{AG}.$$

SIMILARLY, $\Delta AFG \sim \Delta BFH$ AND $\Delta BDH \sim \Delta CDI$

$$\text{SO, } \frac{AF}{BF}=\frac{AG}{BH},\ \frac{BD}{CD}=\frac{BH}{CI} \Rightarrow -\frac{AF}{FB}=\frac{AG}{BH},\ -\frac{BD}{DC}=\frac{BH}{CI}.$$

$$\text{HENCE, } rst=\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=\left(\frac{-BH}{CI}\right)\left(\frac{-CI}{AG}\right)\left(\frac{-AG}{BH}\right) = -1$$

$$\text{THEREFORE } \left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=-1$$

© MoE FDRE Not to be republished

IT IS ALSO POSSIBLE FOR ALL THREE OF D, E AND F TO DIVIDE THEIR RESPECTIVE SIDES EXTERNALLY, AS YOU CAN SEE BY DRAWING A FIGURE. IN THIS CASE, r, s, t ARE ALL NEGATIVE. OTHERWISE THE PRECEDING PROOF WILL REMAIN UNCHANGED.

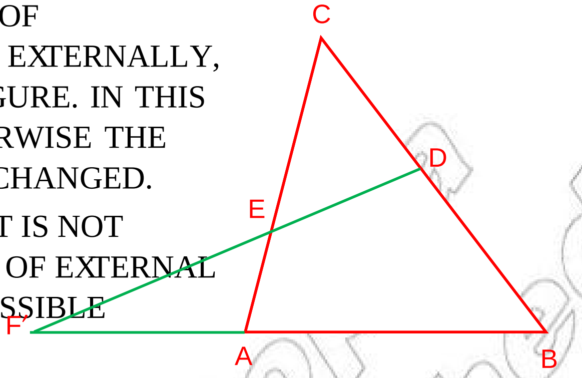


Figure 6.14

THEREFORE, $rst = -1$ IN THIS CASE ALSO. IT IS NOT POSSIBLE TO HAVE AN EVEN NUMBER OF EXTERNAL DIVISIONS, SO $rst = -1$ IN EACH OF THE POSSIBLE CASES.

TO PROVE THE CONVERSE OF MENELAUS' THEOREM, ASSUME THAT $rst = -1$.

EXTEND $\overline{DE}$ UNTIL IT INTERSECTS $\overleftrightarrow{AB}$, SAY AT A POINT F'. LET r' BE THE RATIO IN WHICH F' DIVIDES $\overline{AB}$, THEN $r'st = -1$ (WHY?).

HENCE, $r' = r$ (WHY?)

SINCE F IS THE ONLY POINT THAT DIVIDES $\overline{AB}$ IN THE RATIO r, $F = F'$. THIS IMPLIES THAT D, E AND F ARE COLLINEAR.

Exercise 6.1

1 IN FIGURE 6.15, $\overline{AD} \equiv \overline{DC}$, $\overline{AE} \equiv \overline{EB}$, F IS THE INTERSECTION OF $\overline{CE}$ AND $\overline{BD}$. PROVE THAT $EF = \frac{1}{3}EC$.

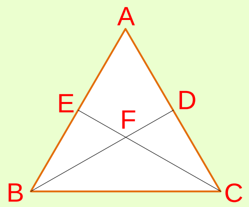


Figure 6.15

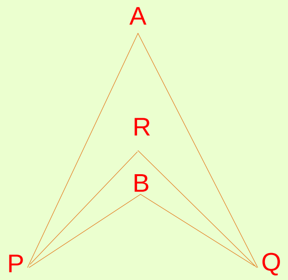


Figure 6.16

2 IN FIGURE 6.16, $\overline{RP}$ AND $\overline{RQ}$ ARE THE BISECTORS OF THE ANGLES QPB AND PQB, RESPECTIVELY. IF $RP \equiv RQ$, PROVE THAT A, R, B LIE ON A STRAIGHT LINE.

Hint: JOIN P AND Q.

3 IF TWO MEDIANS OF A TRIANGLE ARE EQUAL, PROVE THAT THE TRIANGLE FORMED BY A SEGMENT OF EACH MEDIAN AND THE THIRD SIDE IS AN ISOSCELES TRIANGLE.

4 PROVE THAT THE SEGMENT JOINING THE MID-POINTS OF TWO SIDES OF A TRIANGLE IS PARALLEL TO THE THIRD SIDE AND IS HALF AS LONG AS THE THIRD SIDE.

5 **A** LET A (0, 0), B (6, 0) AND C (0, 4) BE VERTICES OF ΔABC.

I FIND THE POINT OF INTERSECTION OF THE MEDIANS OF ΔABC.

ii SHOW THAT THE POINT OBTAINED IN **i** IS $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

b REPEAT 5A FOR ΔDEF WHERE D (0, 0), E (4, 0) AND F (2, 4) ARE THE VERTICES.

6 IN RIGHT ANGLED TRIANGLE *ABC* SHOWN IN FIGURE 6.17, $\overline{CD}$ IS ALTITUDE TO THE HYPOTENUSE $\overline{AB}$. IF AC = 5 UNITS AND AD = 4 UNITS, FIND THE LENGTH OF

a $\overline{BD}$ **b** $\overline{BC}$

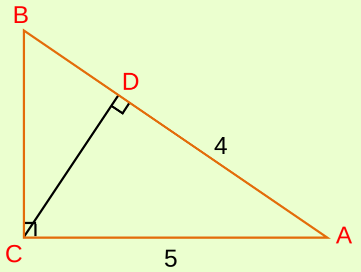


Figure 6.17

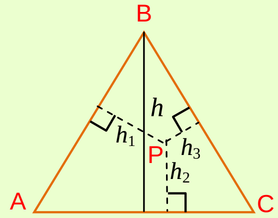


Figure 6.18

7 ***Altitude triangle for equilateral triangle:*** IN FIGURES 6.18, ΔABC IS AN EQUILATERAL TRIANGLE WITH ALTITUDE OF LENGTH *h* AND AN INTERIOR POINT *P*. THE THREE ALTITUDES OF LENGTHS h_1, h_2 AND h_3 ARE DRAWN FROM *P* TO THE SIDES OF THE TRIANGLE. SHOW THAT $h = h_1 + h_2 + h_3$.

Hint: COMPARE THE AREA OF ΔABC WITH THE SUM OF THE AREAS OF ΔAPC, ΔAPB AND ΔBPC.

IN PROBLEMS 8 – 10, THE LETTERS A, B, C, D, E, F, R, S, T HAVE THE MEANING WHICH THEY HAVE IN THE STATEMENT OF MENELAUS' THEOREM.

8 IN FIGURE 6.19, *D* AND *D'* ARE SYMMETRICAL ABOUT THE MID-POINT OF $\overline{BC}$. *E* AND *E'*, *F* AND *F'* ARE ALSO SYMMETRICAL ABOUT THE MID-POINTS OF THEIR CORRESPONDING SIDES. SHOW THAT *D'*, *E'* AND *F'* ARE COLLINEAR IF *D*, *E* AND *F* ARE COLLINEAR.

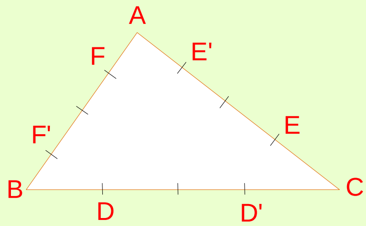


Figure 6.19

9 IN THE PROOF OF THE CONVERSE PART OF MENELAUS' THEOREM, ASSUME THAT $\overleftrightarrow{DE}$ MEETS $\overleftrightarrow{AB}$ AT SOME POINT *F'*.

a PROVE THAT IF $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$, THEN $rst = 1$.

b PROVE THAT IF $rst = -1$, THEN $\overleftrightarrow{DE}$ IS NOT PARALLEL TO $\overleftrightarrow{AB}$.

c PROVE THAT IF $rst = 1$, THEN $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$.

10 IN FIGURE 6.20 BELOW, D DIVIDES $\overline{BC}$ IN THE RATIO r AND D' DIVIDES $\overline{CB}$ IN THE SAME RATIO r. E IS THE MID-POINT OF $\overline{CA}$. D, E, F ARE COLLINEAR AND D', E, F' ARE ALSO COLLINEAR. SHOW THAT $AF = BF'$.

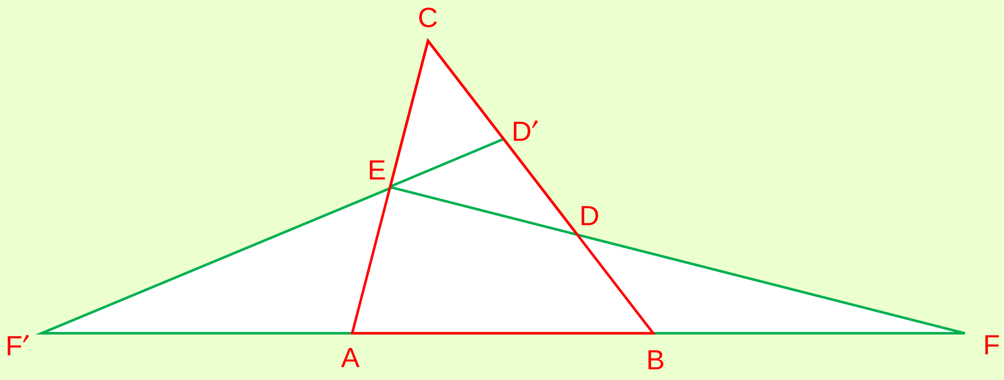


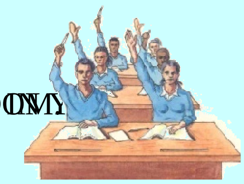
Figure 6.20

6.2 SPECIAL QUADRILATERALS

IN THIS SECTION, WE CONSIDER THE FOLLOWING SPECIAL QUADRILATERALS: ***trapezium***, ***parallelogram***, ***rectangle***, ***rhombus*** AND ***square***.

KEEP IN MIND THE MATHEMATICAL DEFINITIONS OF EACH OF THE ABOVE QUADRILATER

ACTIVITY 6.4



1 DISCUSS PARALLEL LINES BASED ON WHAT YOU SEE IN YOUR CLASSROOM.

2 STATE THE PARALLEL LINES POSTULATE.

3 DISCUSS WHAT IS MEANT BY “EQUIANGULAR QUADRILATERAL” AND “EQUILATERAL QUADRILATERAL”?

4 DEFINE THE FOLLOWING QUADRILATERALS IN YOUR OWN TERMS.

A PARALLELOGRAM **B** RECTANGLE **C** SQUARE

5 WHAT IS AN ALTITUDE OF A PARALLELOGRAM?

6 IN FIGURE 6.21,

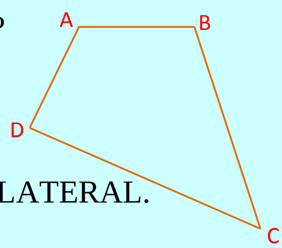


Figure 6.21

I INDICATE A PAIR OF ADJACENT SIDES.

II INDICATE OPPOSITE VERTICES OF THE QUADRILATERAL.

III JOIN TWO OPPOSITE VERTICES.

WHAT DO YOU CALL THIS LINE SEGMENT?

7 WHAT IS A DIAGONAL OF A QUADRILATERAL? HOW MANY DIAGONALS DOES A PARALLELOGRAM OR RECTANGLE HAVE?

Trapezium

Definition 6.1

A ***trapezium*** is a quadrilateral where only two of the sides are parallel.

IN FIGURE 6.22, THE QUADRILATERAL *ABCD* IS A TRAPEZIUM. THE SIDES $\overline{AD}$ AND $\overline{BC}$ ARE NON-PARALLEL SIDES OF THE TRAPEZIUM *ABCD*.

Figure 6.22

NOTE THAT IF THE SIDES $\overline{AD}$ AND $\overline{BC}$ OF TRAPEZIUM *ABCD* ARE CONGRUENT, THEN THE TRAPEZIUM IS CALLED AN ***isosceles trapezium***.

Parallelogram

Definition 6.2

A ***parallelogram*** is a quadrilateral in which both pairs of opposite sides are parallel.

IN FIGURE 6.23, THE QUADRILATERAL *ABCD* IS A PARALLELOGRAM.

$\overline{AB}$ // $\overline{DC}$ AND $\overline{AD}$ // $\overline{BC}$

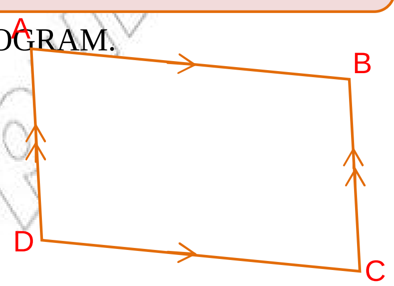


Figure 6.23

ACTIVITY 6.5

1 DRAW A QUADRILATERAL *ABCD*. LET *P*, *Q*, *R* AND *S* BE THE MID-POINTS OF ITS SIDES. CHECK, BY CONSTRUCTION AND MEASUREMENT, THAT *PQRS* IS A PARALLELOGRAM.

2 DRAW A TRAPEZIUM *ABCD* WITH *AB* = 2 CM, *BC* = *DA* = 3 CM AND *DC* = 4 CM.

A INDICATE AND MEASURE THE BASE ANGLES OF TRAPEZIUM *ABCD*.

B DRAW THE DIAGONALS $\overline{DB}$ AND $\overline{AC}$ AND THEN MEASURE THEIR LENGTHS. ALSO, COMPARE THE LENGTHS OF THE TWO DIAGONALS.

3 DRAW A PARALLELOGRAM *ABCD* WITH *AB* = 3 CM AND *BC* = 8 CM.

A MARK POINTS ON $\overline{AB}$ THAT DIVIDE IT INTO THREE CONGRUENT PARTS. THROUGH THESE POINTS, DRAW LINES ACROSS *ABCD* PARALLEL TO $\overline{BC}$. WHY DO THESE LINES DIVIDE *ABCD* INTO THREE SMALLER PARALLELOGRAMS?

B MARK POINTS ON $\overline{BC}$ THAT DIVIDE IT INTO FOUR CONGRUENT SEGMENTS. THROUGH THESE POINTS, DRAW LINES ACROSS ABCD PARALLEL TO $\overline{AB}$. HOW MANY SMALL PARALLELOGRAMS DOES THIS MAKE?

C DRAW THE DIAGONALS OF ALL THE SMALLER PARALLELOGRAMS AND SHOW THAT THESE DIAGONALS ALSO FORM PARALLELOGRAMS.

PROPERTIES OF A PARALLELOGRAM AND TESTS FOR A QUADRILATERAL TO BE A PA STATED IN THE FOLLOWING THEOREM:

Theorem 6.7

A The opposite sides of a parallelogram are congruent.

B The opposite angles of a parallelogram are congruent.

C The diagonals of a parallelogram bisect each other.

D If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

E If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

F If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Proof of A and B:-

Given: PARALLELOGRAM ABCD

To prove: $\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$

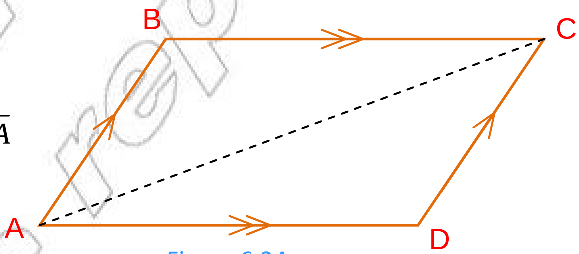


Figure 6.24

	Statement		Reason
1	DRAW DIAGONAL $\overline{AC}$	1	THROUGH TWO POINTS THERE IS EXACTLY ONE STRAIGHT LINE.
2	$\overline{AC} \equiv \overline{CA}$	2	COMMON SIDE.
3	$\angle CAB \equiv \angle ACD$ AND $\angle ACB \equiv \angle CAD$	3	ALTERNATE INTERIOR ANGLES OF PARALLEL LINES.
4	$\Delta ABC \equiv \Delta CDA$	4	ASA POSTULATE.
5	$\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$, AND $\angle ABC \equiv \angle CDA$	5	CORRESPONDING PARTS OF CONGRUENT TRIANGLES

Can you show that $\angle BAD \equiv \angle DCB$?

Proof of C:-

Given: PARALLELOGRAM $ABCD$ WITH DIAGONALS $\overline{AC}$ AND $\overline{BD}$ INTERSECTING AT O.

To prove: $\overline{AO} \equiv \overline{OC}$ AND $\overline{BO} \equiv \overline{DO}$.

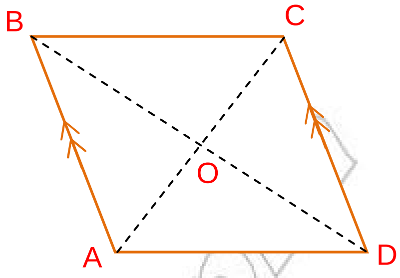


Figure 6.25

	Statement		Reason
1	$\overline{AB} \equiv \overline{CD}$	1	THEOREM 6.7A
2	$\angle CAB \equiv \angle ACD$ AND $\angle ABD \equiv \angle CDB$ HENCE, $\angle OAB \equiv \angle OCD$ AND $\angle ABO \equiv \angle CDO$	2	ALTERNATE INTERIOR ANGLES
3	$\Delta AOB \equiv \Delta COD$	3	ASA POSTULATE
4	$\overline{AO} \equiv \overline{CO}$ AND $\overline{BO} \equiv \overline{DO}$	4	CORRESPONDING PARTS OF CONGRUENT TRIANGLES.

Proof of F:-

Given: A QUADRILATERAL $ABCD$ WITH $\angle A \equiv \angle C$ AND $\angle B \equiv \angle D$.

To prove: $ABCD$ IS A PARALLELOGRAM

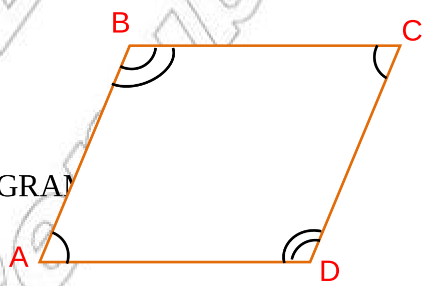


Figure 6.26

	Statement		Reason
1	$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^O$	1	THE SUM OF THE INTERIOR ANGLES OF A QUADRILATERAL IS 360^O
2	$m(\angle A) = m(\angle C)$ AND $m(\angle B) = m(\angle D)$	2	GIVEN
3	$2m(\angle A) + 2m(\angle D) = 360^O$	3	STEPS 1 AND 2
4	$m(\angle A) + m(\angle D) = 180^O$	4	SIMPLIFICATION
5	THEREFORE, $\overline{AB} // \overline{DC}$	5	$\angle A$ AND $\angle D$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL $\overline{AD}$
6	$m(\angle A) + m(\angle B) = 180^O$	6	STEP 2 AND 4.
7	THEREFORE, $\overline{AD} // \overline{BC}$	7	$\angle A$ AND $\angle B$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL $\overline{AB}$.
8	$ABCD$ IS A PARALLELOGRAM	8	DEFINITION OF A PARALLELOGRAM STEPS 5 AND 7.

Rectangle

Definition 6.3

A ***rectangle*** is a parallelogram in which one of its angles is a right angle.

IN FIGURE 6.27, THE PARALLELOGRAM ABCD IS A RECTANGLE WHOSE ∠D IS A RIGHT ANGLE. WHAT IS THE MEASURE OF EACH OF THE OTHER ANGLES OF THE RECTANGLE ABCD?

Figure 6.27

Some properties of a rectangle

I A RECTANGLE HAS ALL PROPERTIES OF A PA

II EACH INTERIOR ANGLE OF A RECTANGLE IS A

III THE DIAGONALS OF A RECTANGLE ARE CONG

Rhombus

Definition 6.4

A ***rhombus*** is a parallelogram which has two congruent adjacent sides.

IN FIGURE 6.28, THE PARALLELOGRAM ABCD IS A RHOMBUS.

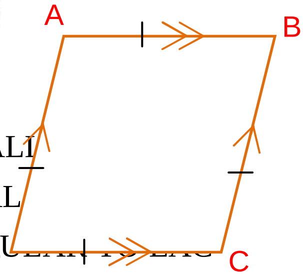


Figure 6.28

Some properties of a rhombus

i A RHOMBUS HAS ALL THE PROPERTIES OF A PARALL

ii A RHOMBUS IS AN EQUILATERAL QUADRILATERAL.

iii THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.

iv THE DIAGONALS OF A RHOMBUS BISECT ITS ANGLES.

Square

Definition 6.5

A ***square*** is a rectangle which has congruent adjacent sides.

IN FIGURE 6.29, THE RECTANGLE ABCD IS A SQUARE.

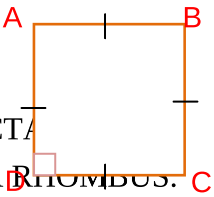


Figure 6.29

Some properties of a square

I A SQUARE HAS THE PROPERTIES OF A RECTA

II A SQUARE HAS ALL THE PROPERTIES OF A RHOMBUS.

Group Work 6.2

1 WHAT ARE SOME SIMILARITIES AND DIFFERENCES BETWEEN A PARALLELOGRAM, A RECTANGLE AND A SQUARE?

2 IF *ABCD* IS A PARALLELOGRAM WITH $AB = 3x + 4$, $BC = 2x + 7$ AND $CD = x + 18$, WHAT TYPE OF PARALLELOGRAM IS *ABCD*?

3 DISCUSS THE RELATIONSHIP AMONG THE FOUR TRIANGLES FORMED BY THE DIAGONALS OF A RHOMBUS.

Theorem 6.8

If the diagonals of a quadrilateral are congruent and are perpendicular bisectors of each other, then the quadrilateral is a square.

Proof:-

Given: $\overline{AC} \equiv \overline{BD}$; $\overline{AC}$ AND $\overline{BD}$ ARE PERPENDICULAR BISECTORS OF EACH OTHER.

To prove: *ABCD* IS A SQUARE.

LET O BE THE POINT OF INTERSECTION OF $\overline{AC}$ AND $\overline{BD}$.

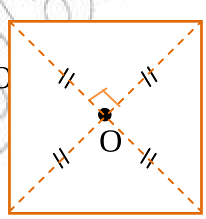


Figure 6.30

	Statement		Reason
1	$\overline{AC} \equiv \overline{BD}$, $\overline{AC}$ AND $\overline{BD}$ ARE PERPENDICULAR BISECTORS OF	1	GIVEN
2	$\overline{AO} \equiv \overline{BO} \equiv \overline{CO} \equiv \overline{DO}$	2	STEP 1
3	$\angle AOB \equiv \angle BOC \equiv \angle COD \equiv \angle DOA$	3	ALL RIGHT ANGLES ARE CONGRUENT
4	$\Delta AOB \equiv \Delta BOC \equiv \Delta COD \equiv \Delta DOA$	4	SAS POSTULATE
5	$\angle CBD \equiv \angle ADB$ AND $\angle DCA \equiv \angle BAC$	5	CORRESPONDING ANGLES OF CONGRUENT TRIANGLES
6	$\overline{BC} // \overline{AD}$ AND $\overline{AB} // \overline{CD}$	6	ALTERNATE INTERIOR ANGLES ARE CONGRUENT
7	*ABCD* IS A PARALLELOGRAM	7	DEFINITION OF A PARALLELOGRAM
8	*ABCD* IS A RECTANGLE	8	DIAGONALS ARE CONGRUENT
9	*ABCD* IS A SQUARE	9	DEFINITION OF A SQUARE, $\overline{AB} \equiv \overline{CD}$ AND STEP 4

Exercise 6.2

1. $ABCD$ IS A PARALLELOGRAM, P IS THE MID-POINT OF $\overline{AB}$ AND Q IS THE MID-POINT OF $\overline{CD}$. PROVE THAT $APCQ$ IS A PARALLELOGRAM.
2. THE MID-POINTS OF THE SIDES OF A RECTANGLE ARE THE VERTICES OF A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.
3. THE MID-POINTS OF THE SIDES OF A PARALLELOGRAM ARE THE VERTICES OF A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.
4. PROVE EACH OF THE FOLLOWING:
   - **A** IF THE DIAGONALS OF A PARALLELOGRAM ARE CONGRUENT, THEN THE PARALLELOGRAM IS A RECTANGLE.
   - **B** IF THE DIAGONALS OF A QUADRILATERAL BISECT EACH OTHER AND ONE ANGLE OF THE QUADRILATERAL IS A RIGHT ANGLE, THEN THE QUADRILATERAL IS A RECTAN
   - **C** IF ALL THE FOUR SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A RHOMBUS.
   - **D** THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.
5. IN EACH OF THE FOLLOWING STATEMENTS, SUFFICIENT CONDITIONS FOR A QUADRILATERAL TO BE A PARALLELOGRAM ARE STATED. PROVE THIS IN EACH CASE.
   - **A** IF THE OPPOSITE SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
   - **B** IF ONE PAIR OF OPPOSITE SIDES OF A QUADRILATERAL IS CONGRUENT AND PARALLEL, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
   - **C** IF THE DIAGONALS OF A QUADRILATERAL BISECT EACH OTHER, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
6. DRAW A PARALLELOGRAM $ABCD$. EXTEND $\overline{AB}$ THROUGH B TO P SO THAT $AB = BP$; EXTEND $\overline{AD}$ THROUGH D TO Q SO THAT $AD = DQ$. PROVE THAT C, P AND Q ALL LIE ON ONE STRAIGHT LINE. (HINT: DRAW $\overline{BD}$)
7. M IS THE MID-POINT OF THE SIDE $\overline{BC}$ OF A PARALLELOGRAM $ABCD$. $\overline{DM}$ AND $\overline{AB}$ PRODUCED MEET AT N. PROVE THAT $\overline{AB} \equiv \overline{BN}$.
8. IF $ABCD$ IS A PARALLELOGRAM AND M AND N THE MID-POINTS OF $\overline{DC}$ AND $\overline{AB}$ RESPECTIVELY, PROVE THAT $\overline{AM} \equiv \overline{CN}$.
9. $ABCD$ IS A PARALLELOGRAM WITH $\overline{AD}$ PRODUCED TO F AND $\overline{CB}$ PRODUCED TO E SUCH THAT $\overline{DF} \equiv \overline{BE}$. PROVE THAT $AECF$ IS A PARALLELOGRAM.

6.3 MORE ON CIRCLES

IN THIS SECTION, YOU ARE GOING TO STUDY CIRCLES AND THE LINES AND ANGLES A
THEM. OF ALL SIMPLE GEOMETRIC FIGURES, A CIRCLE IS PERHAPS THE MOST APPEALIN
EVER CONSIDERED HOW USEFUL A CIRCLE IS? WITHOUT CIRCLES THERE WOULD BE
WAGONS, AUTOMOBILES, STEAMSHIPS, ELECTRICITY OR MANY OTHER MODERN CONVEN

RECALL THAT A ***circle*** IS A PLANE FIGURE, ALL POINTS OF WHICH ARE EQUIDISTANT FROM A GIVEN POINT CALLED THE ***centre*** OF THE CIRCLE.

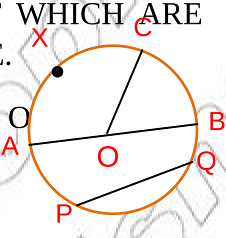


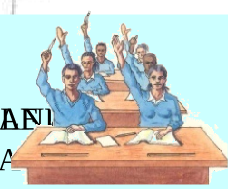
Figure 6.31

AS YOU RECALL FROM GRADE 9, IN FIGURE 6.31, $\overline{PQ}$ IS A CHORD OF THE CIRCLE WITH CENTRE O. $\overline{AB}$ IS A ***chord*** (DIAMETER). $\overset{\frown}{AXC}$ IS AN ***arc*** OF THE CIRCLE.

IF A AND C ARE NOT END-POINTS OF A DIAMETER, $\overset{\frown}{AXC}$ IS A MINOR ARC.

$\angle BOC$ IS A ***central angle***. $\overset{\frown}{AXC}$ OR ARC AXC IS SAID TO ***subtend*** $\angle AOC$ OR $\angle AOC$ ***intercepts*** ARC AXC.

ACTIVITY 6.6



1 DRAW A CIRCLE AND A LINE INTERSECTING THE CIRCLE AT TWO POINTS. DRAW ANOTHER LINE INTERSECTING AT ONE POINT. DRAW A ... OES NOT INTERSECT THE CIRCLE.

2 IF THE LENGTH OF A RADIUS OF A CIRCLE IS r, THEN WHAT IS THE LENGTH OF ITS DIAMETER?

3 REFERRING TO FIGURE 6.32, ANSWER EACH OF THE FOLLOWING QUESTIONS:

A NAME AT LEAST THREE CHORDS, TWO SECANTS AND TWO TANGENTS.

B NAME THREE ANGLES FORMED BY TWO INTERSECTING CHORDS.

C NAME AN ANGLE FORMED BY TWO INTERSECTING TANGENTS.

D NAME AN ANGLE FORMED BY TWO INTERSECTING SECANTS.

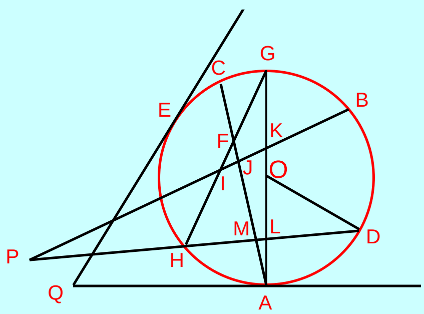


Figure 6.32

4 CONSTRUCT:

A A CENTRAL ANGLE OF 75^{o} IN A CIRCLE. **B** A CENTRAL ANGLE OF 120^{o} IN A CIRCLE.

5 HOW LARGE IS A CENTRAL ANGLE THAT IS SUBTENDED BY A CHORD OF LENGTH 3 CM IN A CIRCLE OF RADIUS 3 CM?

6 WHAT IS THE MEASURE OF A SEMI-CIRCLE AS AN ARC?

7 IS THE STATEMENT 'THE MEASURE OF AN ARC IS EQUAL TO THE MEASURE OF THE CORRESPONDING CENTRAL ANGLE' TRUE OR FALSE?

6.3.1 Angles and Arcs Determined by Lines Intersecting Inside and On a Circle

WE NOW EXTEND THE DISCUSSION TO ANGLES WHOSE VERTICES DO NOT NECESSARILY LIE AT THE CENTRE OF THE CIRCLE.

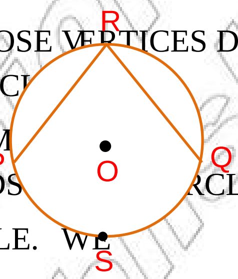


Figure 6.33

IN A CIRCLE, ***an inscribed angle*** IS AN ANGLE WHOSE VERTEX LIES ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE.

IN FIGURE 6.33, ANGLE PRQ IS INSCRIBED IN THE CIRCLE. WE ALSO SAY THAT $\angle PRQ$ IS INSCRIBED IN THE ARC PRQ AND $\angle PRQ$ IS ***subtended*** BY ARC PSQ (OR $\widehat{PSQ}$).

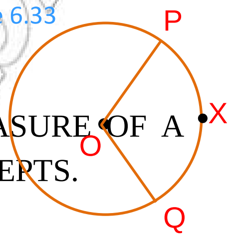


Figure 6.34

MEASURE OF A CENTRAL ANGLE: NOTE THAT THE MEASURE OF A CENTRAL ANGLE IS THE MEASURE OF THE ARC IT INTERCEPTS.

SO, $m(\angle POQ) = m(\widehat{PXQ})$.

Theorem 6.9

THE MEASURE OF AN ANGLE INSCRIBED IN A CIRCLE IS HALF THE MEASURE OF THE ARC SUBTENDING IT.

Proof:-

Given: CIRCLE O WITH $\angle B$ AN INSCRIBED ANGLE INTERCEPTING $\widehat{AC}$.

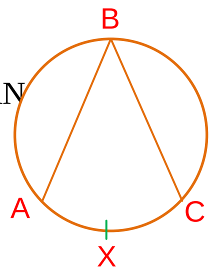


Figure 6.35

To prove: $m(\angle ABC) = \frac{1}{2} m(\widehat{AXC})$, WHERE X IS A POINT AS SHOWN IN FIGURE 6.35.

TO PROVE THEOREM 6.9, WE CONSIDER THREE CASES.

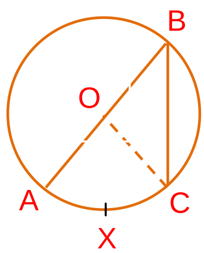


Figure 6.36

Case 1: SUPPOSE THAT ONE SIDE OF $\angle ABC$ IS A DIAMETER OF THE CIRCLE WITH CENTRE O.

	Statement		Reason
1	DRAW RADIUS $\overline{OB}$	1	CONSTRUCTION.
2	$\overline{OC} \equiv \overline{OB}$	2	RADII OF THE SAME CIRCLE.
3	$\angle OBC \equiv \angle OCB$	3	BASE ANGLES OF AN ISOSCELES TRIANGLE.
4	$\angle AOC \equiv \angle OCB + \angle OBC$	4	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO THE SUM OF THE TWO OPPOSITE INTERIOR ANGLES.
5	$m(\angle AOC) = 2m(\angle ABC)$	5	SUBSTITUTION.
6	BUT $m(\angle AOC) = m(\overset{\frown}{AXC})$	6	$\angle AOC$ IS A CENTRAL ANGLE.
7	$2m(\angle ABC) = m(\overset{\frown}{AXC})$	7	SUBSTITUTION.
8	$m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$	8	DIVISION OF BOTH SIDES BY 2.

THEREFORE, $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$

Case 2: SUPPOSE THAT A AND C ARE ON OPPOSITE SIDES OF THE DIAMETER THROUGH B, AS SHOWN IN FIGURE 6.37.

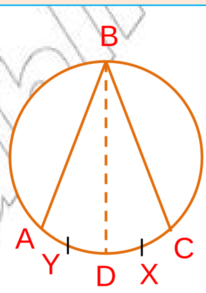


Figure 6.37

	Statement		Reason
1	$m(\angle ABD) = \frac{1}{2} m(\overset{\frown}{AYD})$	1	CASE 1
2	$m(\angle DBC) = \frac{1}{2} m(\overset{\frown}{DXC})$	2	CASE 1
3	$m(\angle ABD) + m(\angle DBC) = \frac{1}{2}(\overset{\frown}{AYD}) + \frac{1}{2} m(\overset{\frown}{DXC})$	3	ADDITION
4	$\therefore m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$	4	SUBSTITUTION

THEREFORE, $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXC})$

Case 3: SUPPOSE THAT A AND C ARE ON THE SAME SIDE OF THE DIAMETER THROUGH B AS SHOWN IN FIGURE 6.38.

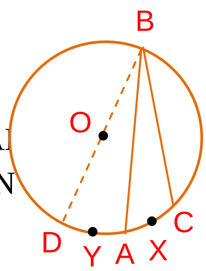


Figure 6.38

	Statement		Reason
1	$m(\angle DBC) = \frac{1}{2}\, m(\overset{\frown}{DAC})$	1	CASE 1
2	$m(\angle DBA) = \frac{1}{2}\, m(\overset{\frown}{DYA})$	2	CASE 1
3	$m(\angle DBC) - m(\angle DBA) = \frac{1}{2}\, m(\overset{\frown}{DAC}) - \frac{1}{2}\, m(\overset{\frown}{DYA})$	3	ADDITION
4	$\therefore\ m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{AXC})$	4	SUBSTITUTION

THEREFORE, $m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{AXC})$ IN ALL CASES AND THE THEOREM HOLDS.

EXAMPLE 1 IN FIGURE 6.39, $m(\overset{\frown}{PXQ}) = 110^{O}$. FIND THE MEASURE OF $\angle PRQ$.

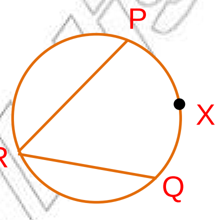


Figure 6.39

SOLUTION: BY THEOREM 6.9, WE HAVE

$$m(\angle PRQ) = \frac{1}{2}\, m(\overset{\frown}{PXQ}) = \frac{1}{2}(110^{O}) = 55^{O}$$

Corollary 6.9.1

An angle inscribed in a semi-circle is a right angle.

Proof:-

IN FIGURE 6.40, $\angle ABC$ IS INSCRIBED IN SEMI-CIRCLE ABC.

$\angle ABC$ IS SUBTENDED BY $\overset{\frown}{ADC}$, WHICH IS A SEMI-CIRCLE.

THE MEASURE OF ARC ADC IS 180^{O} OR RADIANS.

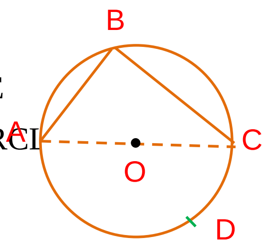


Figure 6.40

BY THEOREM 6.9, $m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{ADC})$

$$= \frac{1}{2}(180^{O}) = 90^{O} \text{ OR } \frac{}{2} \text{ RADIANS.}$$

Corollary 6.9.2

An angle inscribed in an arc less than a semi-circle is obtuse.

Proof:-

$$m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{ADC})$$

BUT $m(\overset{\frown}{ABC})$ < LENGTH OF A SEMI-CIRCLE

$$m(\overset{\frown}{ABC}) < 180^{O}$$

THEREFORE, $m(\overset{\frown}{ADC}) > 180^{O}$

$$m(\angle ABC) = \frac{1}{2}\, m(\overset{\frown}{ADC}) > \frac{1}{2}(180^{O})$$

$m(\angle ABC) > 90^{O}$. SO, $\angle ABC$ IS AN OBTUSE ANGLE.

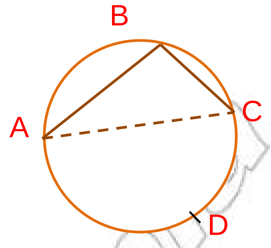


Figure 6.41

Corollary 6.9.3

An angle inscribed in an arc greater than a semi-circle is ***acute***.

Theorem 6.10

Two parallel lines intercept congruent arcs on the same circle.

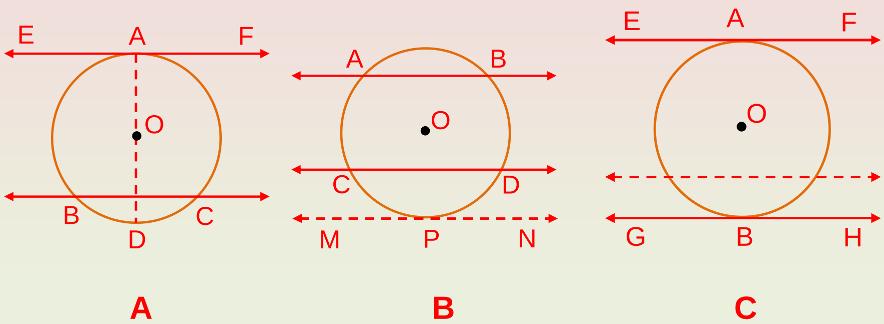


Figure 6.42

Proof:-

TO PROVE THIS FACT, YOU HAVE TO CONSIDER THE THREE POSSIBLE CASES:

A WHEN ONE OF THE PARALLEL LINES $\overleftrightarrow{EF}$ IS A TANGENT LINE AND THE OTHER $\overleftrightarrow{BC}$ IS A SECANT LINE AS SHOWN IN FIGURE 6.42A.

B WHEN BOTH PARALLEL LINES $\overleftrightarrow{AB}$ AND $\overleftrightarrow{CD}$ ARE SECANTS AS SHOWN IN FIGURE 6.42B.

C WHEN BOTH PARALLEL LINES $\overleftrightarrow{EF}$ AND $\overleftrightarrow{GH}$ ARE TANGENTS AS SHOWN IN FIGURE 6.42C.

Case a:

Given: A CIRCLE WITH CENTRE O, $\overleftrightarrow{EF}$ AND $\overleftrightarrow{BC}$ ARE TWO PARALLEL LINES SUCH THAT $\overleftrightarrow{EF}$ IS A TANGENT TO THE CIRCLE AT A AND $\overleftrightarrow{BC}$ IS A SECANT.

To prove: $\overset{\frown}{AB} \equiv \overset{\frown}{AC}$

© Not to be republished

Statement		Reason	
1	DRAW DIAMETER $\overline{AE}$	1	CONSTRUCTION.
2	$\overline{AD} \perp \overline{EF}$ AND $\overline{AD} \perp \overline{BC}$	2	A TANGENT IS PERPENDICULAR TO THE DIAMETER DRAWN TO THE POINT OF TANGENCY AND $\overline{EF} \parallel \overline{BC}$ IS ALSO GIVEN.
3	$\overset{\frown}{BD} \equiv \overset{\frown}{CD}$	3	ANY PERPENDICULAR FROM THE CENTRE OF A CIRCLE TO A CHORD BISECTS THE CHORD AND THE ARC SUBTENDED BY IT.
4	$\overset{\frown}{AB} \equiv \overset{\frown}{AC}$	4	$\overset{\frown}{ABD} \equiv \overset{\frown}{ACD}$ (SEMICIRCLES) AND STEP 3.

PROOFS OF CASE B AND CASE C ARE LEFT AS EXERCISES.

Theorem 6.11

An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

Given: CIRCLE O WITH $\angle ABC$ FORMED BY TANGENT t AND CHORD $\overline{AB}$ AT B, THE POINT OF CONTACT.

To prove: $m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXB})$

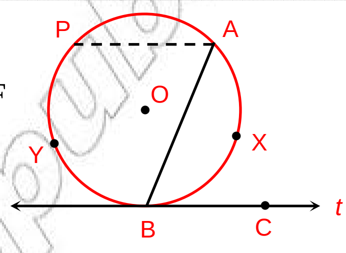


Figure 6.43

Statement		Reason	
1	DRAW $\overline{AP}$ PARALLEL TO t	1	CONSTRUCTION.
2	$\angle PAB \equiv \angle ABC$	2	ALTERNATE INTERIOR ANGLES OF PARALLEL LINES
3	$m(\angle PAB) = \frac{1}{2} m(\overset{\frown}{PYB})$	3	THEOREM 6.9.
4	BUT $\overset{\frown}{PYB} \equiv \overset{\frown}{AXB}$	4	THEOREM 6.10.
5	$\therefore m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AXB})$	5	SUBSTITUTION FROM STEPS 2 – 4.

Theorem 6.12

The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Proof:-

Given: TWO LINES $\overleftrightarrow{AB}$ AND $\overleftrightarrow{CD}$ INTERSECTING AT P INSIDE THE CIRCLE.

To prove: $m(\angle BPD) = \frac{1}{2} m(\overset{\frown}{AXC}) + \frac{1}{2} m(\overset{\frown}{BYD})$.

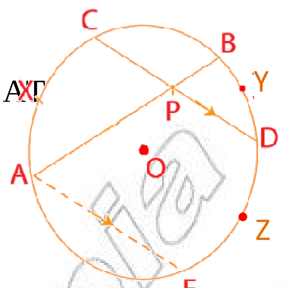


Figure 6.44

Statement		Reason	
1	DRAW A LINE THROUGH A SUCH THAT $\overline{AE} \parallel \overline{CD}$	1	CONSTRUCTION
2	$m(\angle BPD) = m(\angle BAE)$	2	CORRESPONDING ANGLES FORMED BY TWO PARALLEL LINES AND A TRANSVERSAL LINE.
3	$m(\angle BAE) = \frac{1}{2} m(\overset{\frown}{BDE})$	3	THEOREM 6.9
4	$\overset{\frown}{AXC} \equiv \overset{\frown}{DZE}$	4	THEOREM 6.10
5	$\therefore m(\angle BPD) = \frac{1}{2} m(\overset{\frown}{BDE})$ $= \frac{1}{2} m(\overset{\frown}{BYD}) + \frac{1}{2} m(\overset{\frown}{DZE})$	5	THEOREM 6.11
6	$m(\angle BPD) = \frac{1}{2} m(\overset{\frown}{BYD}) + \frac{1}{2} m(\overset{\frown}{AXC})$	6	SUBSTITUTION AND STEP 4.

THEREFORE, $m(\angle BPD) = \frac{1}{2}\left[m(\overset{\frown}{AXC}) + m(\overset{\frown}{BYD})\right]$

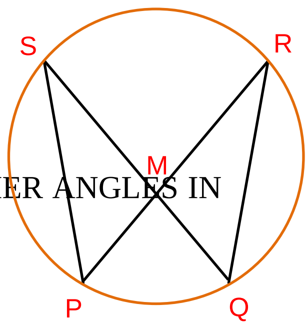


Figure 6.45

EXAMPLE 2 IN FIGURE 6.45, $m(\angle MRQ) = 30^{O}$, AND $m(\angle MQR) = 40^{O}$.

WRITE DOWN THE MEASURE OF ALL THE OTHER ANGLES IN THE TWO TRIANGLES ΔPSM AND ΔQMR. WHAT DO YOU NOTICE ABOUT THE TWO TRIANGLES?

SOLUTION: $m(\angle QMR) = 180^{O} - (30^{O} + 40^{O})$ (WHY?)

$= 180^{O} - 70^{O} = 110^{O}$

$m(\angle RQS) = \frac{1}{2} m(\overset{\frown}{RS})$

THEREFORE, $40^{O} = \frac{1}{2} m(\overset{\frown}{RS})$

$\therefore m(\overset{\frown}{RS}) = 80^{O}$

$$m(\angle PRQ) = \frac{1}{2} m(\overset{\frown}{PQ})$$

HENCE, $30^{O} = \frac{1}{2} m(\overset{\frown}{PQ})$

$\therefore m(\overset{\frown}{PQ}) = 60^{O}$

$$m(\angle PSQ) = \frac{1}{2} m(\overset{\frown}{PQ}) = \frac{1}{2}(60^{O}) = 30^{O}$$

$$m(\angle RPS) = \frac{1}{2} m(\overset{\frown}{RS}) = \frac{1}{2}(80^{O}) = 40^{O}$$

THE TWO TRIANGLES ARE SIMILAR BY AA SIMILARITY.

EXAMPLE 3 AN ANGLE FORMED BY TWO CHORDS INTERSECTING WITHIN A CIRCLE IS 48^{O} AND ONE OF THE INTERCEPTED ARCS MEASURES 42^{O}. FIND THE MEASURES OF THE OTHER INTERCEPTED ARC.

SOLUTION: CONSIDER FIGURE 6.46.

$$m(\angle PRB) = \frac{1}{2} m(\overset{\frown}{PB}) + \frac{1}{2} m(\overset{\frown}{AQ}) \text{ (by THEOREM 6.11)}$$

$$48^{O} = \frac{1}{2}(42^{O}) + \frac{1}{2}(\overset{\frown}{AQ})$$

$$\Rightarrow 96^{O} = 42^{O} + m(\overset{\frown}{AQ})$$

$$\therefore 54^{O} = m(\overset{\frown}{AQ})$$

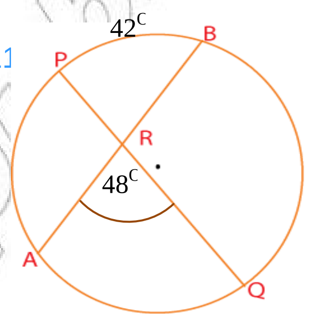


Figure 6.46

Remark: THE FOLLOWING RESULT IS SOMETIMES CALLED THE ***product of rectangle property of a circle***.

IF TWO CHORDS INTERSECT IN A CIRCLE AS SHOWN IN FIGURE 6.47, THEN $(AP)(PB) = (XP)(PY)$.

HINT FOR PROOF:

1	$\angle XAP \equiv \angle BYP$ AND $\angle AXP \equiv \angle YBP$	(WHY?)
2	$\Delta PAX \sim \Delta PYB$	(WHY?)
3	$\frac{AP}{YP} = \frac{PX}{PB}$	(WHY?)
4	$\therefore (AP)(PB) = (YP)(PX)$	(WHY?)

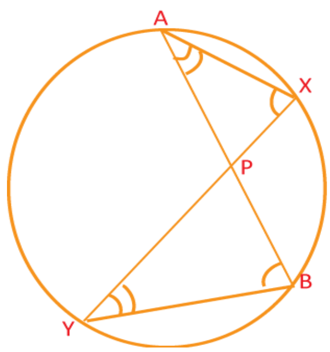


Figure 6.47

Not FDRE Ethiopia be republished

EXAMPLE 4 IN FIGURE 6.48, CALCULATE THE RADIUS OF THE CIRCLE.

SOLUTION: LET THE RADIUS OF THE CIRCLE BE r UNITS LONG.

THEN, $OD = r$ AND $PD = 2r – 2$.

SINCE $(AP)(PB) = (CP)(PD)$, YOU HAVE

$$4 \times 4 = 2(2r – 2)$$
$$16 = 4r – 4$$
$$r = 5$$

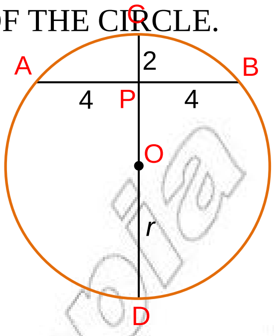


Figure 6.48

Group Work 6.3

1 IN FIGURE 6.49, $\overline{AB}$ AND $\overline{PQ}$ ARE PARALLEL, $m(\angle BOQ) = 70^{O}$ AND O IS THE CENTRE OF THE CIRCLE. WHAT IS THE MEASURE OF $\angle AOP$?

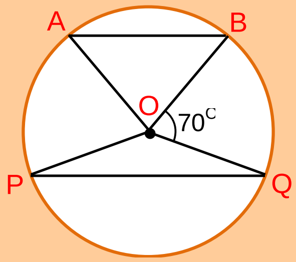


Figure 6.49

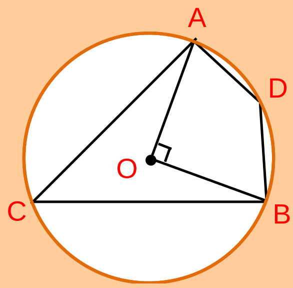


Figure 6.50

2 IN FIGURE 6.49, IF $PO = 5$ UNITS AND $m\left(\overset{\frown}{PQ}\right) = 120^{o}$, FIND THE LENGTH OF $\overline{PQ}$.

3 IN FIGURE 6.50, IF CENTRAL ANGLE $\angle AOB$ IS A RIGHT ANGLE.

A WHAT ARE THE DEGREE MEASURES OF $\angle ACB$ AND $\angle ADB$?

B FIND THE DEGREE MEASURE OF $\angle CBO$, IF $m(\angle CAO) = 20°$.

Exercise 6.3

1 IN FIGURE 6.51, $\overline{AB}$ IS A DIAMETER, O IS THE CENTRE OF THE CIRCLE AND $\overline{OC} \parallel \overline{BD}$. IF $m(\angle ABD) = 60^{O}$, FIND $m(\angle OCD)$.

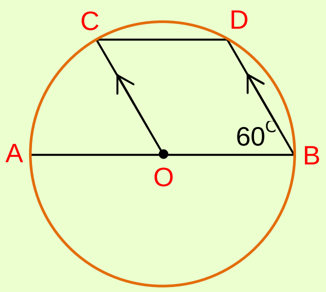


Figure 6.51

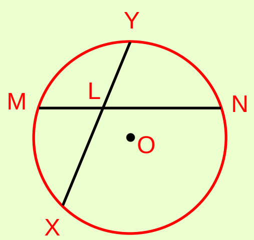


Figure 6.52

2 PROVE THAT, IF AN ANGLE INSCRIBED IN AN ARC OF A CIRCLE IS A RIGHT ANGLE, THEN THE ARC IS A SEMICIRCLE.

3 IN FIGURE 6.52, $\overset{\frown}{MX}$ IS AN ARC OF 28°, AND $\overset{\frown}{YN}$ IS AN ARC OF 50°.

A WHAT IS THE DEGREE MEASURE OF $\angle YLN$?

B IF $ML = 4$ UNITS, $LX = 5$ UNITS AND $LN = 7$ UNITS, FIND YL

4 IN FIGURE 6.52 OF QUESTION 3, WOULD IT BE POSSIBLE FOR $\angle MLX$ TO BE A 30° ANGLE AND FOR THE MEASURE OF $\overset{\frown}{MX}$ TO BE 40°? IF SO, WHAT WOULD BE THE MEASURE OF $\overset{\frown}{YN}$?

5 IN FIGURE 6.53, O IS THE CENTRE OF THE CIRCLE. IF $m(\angle AOB) = 40^{\circ}$ AND $m(\angle COD) = 60^{\circ}$, FIND

A $m(\angle AQB)$ **B** $m(\angle APB)$?

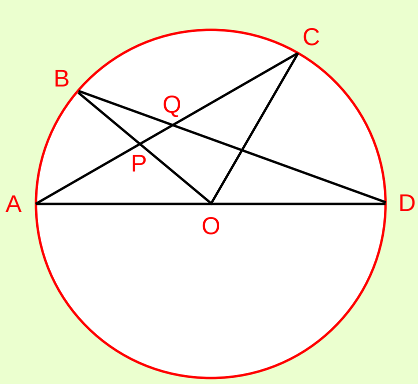


Figure 6.53

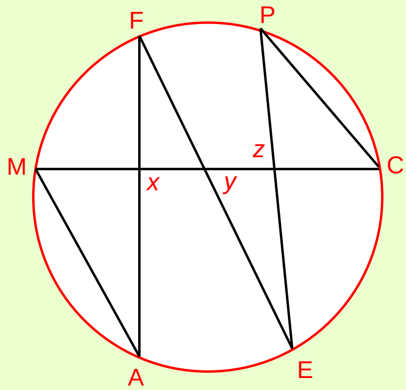


Figure 6.54

6 IN FIGURE 6.54, IF $m(\angle FAM) = 40^{\circ}$ AND $m(\angle CPE) = 50^{\circ}$, WHAT IS THE DEGREE MEASURE OF $\angle EYC$?

7 **A** IN FIGURE 6.55, THE VERTICES OF QUADRILATERAL $ABCD$ LIE ON THE CIRCLE O. SUCH A QUADRILATERAL IS CALLED ***cyclic quadrilateral***.

I WHAT IS THE SUM OF THE MEASURE OF ARCS ABC AND ADC ?

II PROVE THAT OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL ARE SUPPLEMENTARY.

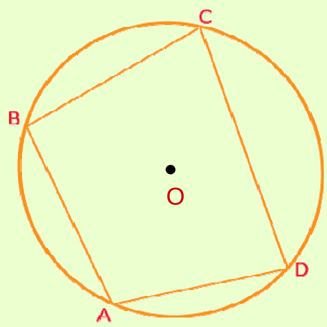


Figure 6.55

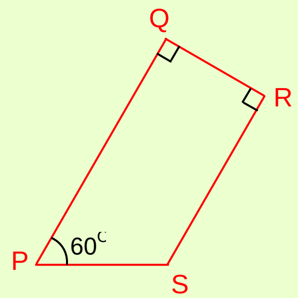


Figure 6.56

B IN FIGURE 6.56, IS THERE A CIRCLE CONTAINING P, Q, R AND S?

8 IN FIGURE 6.57, FIND THE VALUES OF x AND y GIVEN THAT O IS THE CENTRE OF THE CIRCLE AND $m(\angle AOC) = 160^{O}$

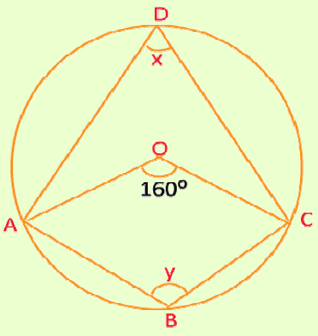


Figure 6.57

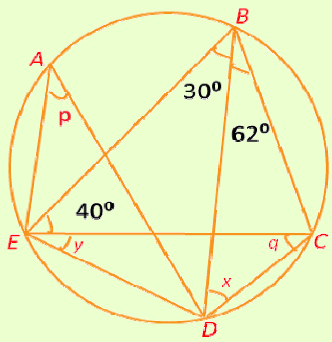


Figure 6.58

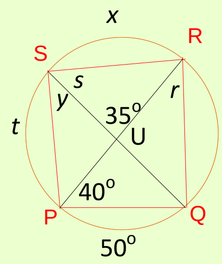


Figure 6.59

9 IN FIGURE 6.58, CALCULATE THE ANGLES MARKED p, q, x AND y.

10 FIND THE VALUES OF THE ANGLES MARKED x, y, s, r AND t AS SHOWN IN FIGURE 6.59.

6.3.2 Angles and Arcs Determined by Lines Intersecting Outside a Circle

WHAT HAPPENS IF TWO SECANT LINES INTERSECT OUTSIDE A CIRCLE? IN FIGURE 6.60, $\overleftrightarrow{AB}$ AND $\overleftrightarrow{XY}$ INTERSECT AT P OUTSIDE THE CIRCLE. THEY INTERCEPT ARCS BY AND AX. DRAW THE CHORD $\overline{AC}$ PARALLEL TO $\overleftrightarrow{XY}$. CAN YOU SEE THAT THE MEASURE OF $\angle APX$ IS HALF THE DIFFERENCE BETWEEN THE MEASURES OF BY AND AX? CAN YOU PROVE IT?

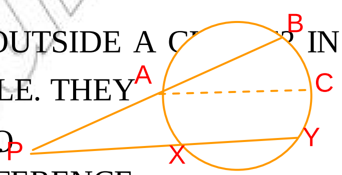
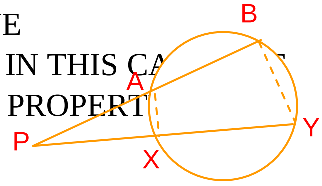


Figure 6.60

THIS IS STATED IN THEOREM 6.13.

> **Theorem 6.13**
>
> The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept.

THE PRODUCT PROPERTY $(PA)(PB) = (PX)(PY)$ IS ALSO TRUE WHEN TWO CHORDS INTERSECT OUTSIDE A CIRCLE. IN THIS CASE, THE PROOF IS SIMILAR TO THE PROOF OF THE PRODUCT PROPERTY IN SECTION 6.3.1



DRAW $\overline{AX}$ AND $\overline{BY}$. TWO SIMILAR TRIANGLES ARE FORMED.

Figure 6.61

BY CONSIDERING CORRESPONDING SIDES, WE SEE THAT

$(PA)(PB) = (PX)(PY)$.

Can you point out the similar triangles, in FIGURE 6.61 and put in the other details?

Theorem 6.14

The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

Proof:-

Given: SECANT PBA AND TANGENT $\overline{PD}$ INTERSECTING AT P.

To prove: $m(\angle P) = \frac{1}{2}[m(\overset{\frown}{AXD}) - m(\overset{\frown}{BD})]$

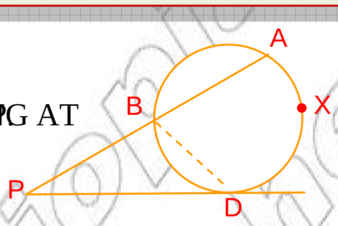


Figure 6.62

	Statement		Reason
1	DRAW $\overline{BD}$	1	CONSTRUCTION.
2	$\angle ABD \equiv \angle BDP + \angle DPA$	2	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO THE SUM OF THE TWO OPPOSITE INTERIOR ANGLES OF A TRIANGLE.
3	$\angle ABD - \angle BDP \equiv \angle DPA \equiv \angle P$	3	SUBTRACTION.
4	$m(\angle ABD) = \frac{1}{2} m(\overset{\frown}{AXD})$ AND $m(\angle PDB) = \frac{1}{2} m(\overset{\frown}{BD})$	4	THEOREM 6.9 AND THEOREM 6.11.
5	$m(\angle ABD) - m(\angle BDP) = \frac{1}{2} m(\overset{\frown}{AXD}) - \frac{1}{2} m(\overset{\frown}{BD})$	5	SUBSTITUTION.
6	$\therefore m(\angle P) = \frac{1}{2} m(\overset{\frown}{AXD}) - \frac{1}{2} m(\overset{\frown}{BD})$	6	SUBSTITUTION.

Theorem 6.15

If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by

$$(PA)^2 = (PB)(PC).$$

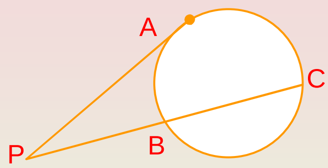


Figure 6.63

Proof:-

Given: A CIRCLE WITH SECANT $\overline{PC}$ AND TANGENT $\overline{PA}$ AS IN FIGURE 6.64

To prove: $(PA)^2 = (PB)(PC)$

DRAW $\overline{AB}$ AND $\overline{CA}$. THEN $\Delta PCA \sim \Delta PAB$ (SHOW!)

HENCE, $\frac{PC}{PA} = \frac{PA}{PB}$ AND $(PA)^2 = (PB)(PC)$

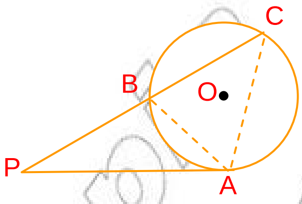


Figure 6.64

EXAMPLE 5 IN FIGURE 6.65, FROM P SECANTS $\overline{PA}$ AND $\overline{PC}$ ARE DRAWN SO THAT $m(\angle APC) = 30^{O}$; CHORDS $\overline{AB}$ AND $\overline{CD}$ INTERSECT AT F SUCH THAT $m(\angle AFC) = 85^{O}$. FIND THE MEASURE OF ARC AC, MEASURE OF ARC BD AND MEASURE OF $\angle ABC$.

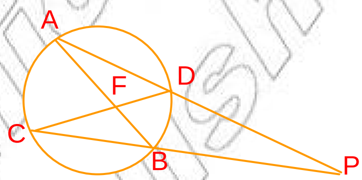


Figure 6.65

SOLUTION: LET $m(\overset{\frown}{AC}) = x$ AND $m(\overset{\frown}{DB}) = y$

SINCE $m(\angle AFC) = \frac{1}{2} m(\overset{\frown}{AC}) + \frac{1}{2} m(\overset{\frown}{BD})$

$$85^{O} = \frac{1}{2}(x + y)$$

$$x + y = 170^{O} \; .\,.\,.\,.\,.\,.\,.\,.\,. \; (1)$$

AGAIN AS $m(\angle APC) = \frac{1}{2} m(\overset{\frown}{AC}) - \frac{1}{2} m(\overset{\frown}{BD})$

$$30^{O} = \frac{1}{2}(x - y)$$

$$x - y = 60^{O} \; .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \; (2)$$

SOLVING *equation 1* AND *equation 2* SIMULTANEOUSLY, WE GET

$$\begin{cases} x + y = 170^{O} \\ x - y = 60^{O} \end{cases}$$

$$2x = 230^{O}$$

$$x = 115^{O}$$

SUBSTITUTING x FOR IN EQUATION 2,

$$115^{O} - y = 60^{O}$$

$$y = 55^{O}$$

THEREFORE, $m(\overset{\frown}{AC}) = 115^{O}$ AND $m(\overset{\frown}{DB}) = 55^{O}$.

$$m(\angle ABC) = \frac{1}{2} m(\overset{\frown}{AC}) = \frac{1}{2}(115^{O}) = 57.5^{O}.$$

© MoE FDR Ethiopia Not to be republished

Group Work 6.4

1. $\overline{AB}$ IS A DIAMETER OF A CIRCLE CENTRE O. C IS A POINT ON THE CIRCUMFERENCE. D IS A POINT ON $\overline{AC}$ SUCH THAT $\overline{OD}$ BISECTS $\angle AOC$. PROVE THAT $\overline{OD}$ IS PARALLEL TO $\overline{BC}$.

2. IN FIGURE 6.66, SUPPOSE LINES $\overleftrightarrow{PA}$ AND $\overleftrightarrow{PX}$ ARE TANGENTS TO A CIRCLE. PROVE THAT
$m(\angle APX) = \frac{1}{2}$ (MEASURE OF MAJOR ARC AX) $- \frac{1}{2}$ (MEASURE OF MINOR ARC AX)

OR $m(\angle P) = \frac{1}{2} m(\overset{\frown}{ACX}) - \frac{1}{2} m(\overset{\frown}{ABX})$

Hint: DRAW A LINE THROUGH A PARALLEL TO $\overline{PX}$

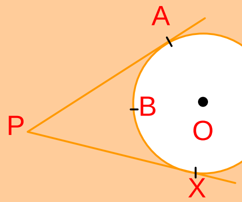


Figure 6.66

3. SUPPOSE A GEOSTATIONARY SATELLITE S ORBITS 35,800 KM ABOVE EARTH, ROTATING SO THAT IT APPEARS TO HOVER DIRECTLY OVER THE EQUATOR. USE FIGURE 6.67 TO DETERMINE THE MEASURE OF THE ARC ON THE EQUATOR VISIBLE TO THIS GEOSTATIONARY SATELLITE.

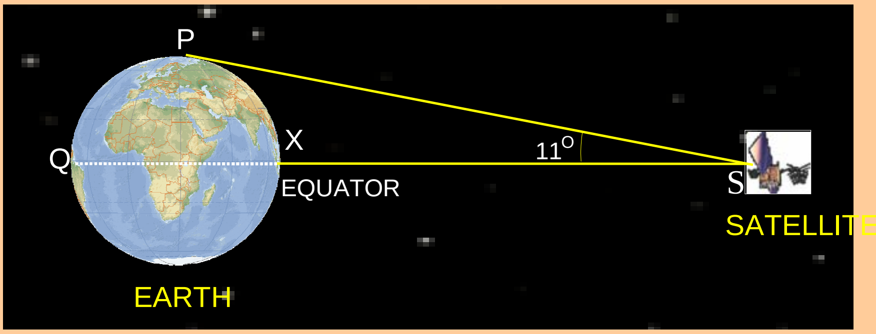


Figure 6.67

Exercise 6.4

1. IF THE MEASURE OF ARC AQ IS 30^{o} AND THE MEASURE OF ARC BR IS 60^{o}, WHAT IS THE MEASURE OF $\angle P$? REFER TO FIGURE 6.68.

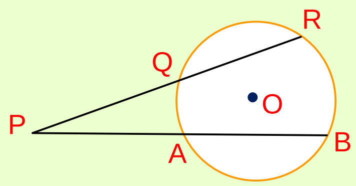


Figure 6.68

2 In Figure 6.69, $\overleftrightarrow{AP}$ is a tangent to the circle. Prove that $\angle CAP \equiv \angle ABC$.

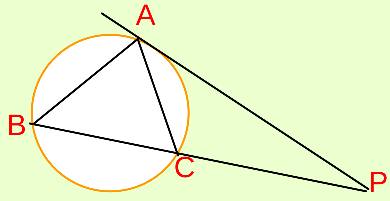


Figure 6.69

3 In Figure 6.70, $\overline{CD}$ is a diameter and $\overline{AB}$ is is bisected by $\overline{CD}$ at P. A square with side $\overline{AP}$ and a rectangle with sides $\overline{CP}$ and $\overline{PD}$ are drawn. Prove that the areas of the square and the rectangle are equal.

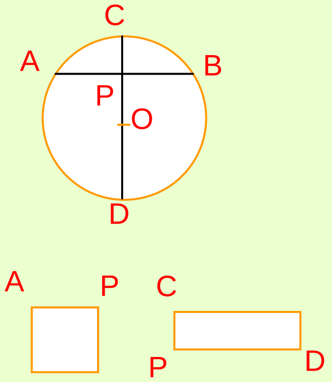


Figure 6.70

4 In Figure 6.71, $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EG}$ are tangents to the circle with centre O at B, D and F respectively. Prove that $CB + EF = CE$.

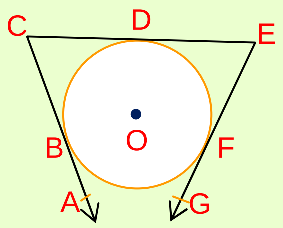


Figure 6.71

5 Use the circle in Figure 6.72 with tangent $\overline{PT}$, secants $\overline{PE}$, $\overline{PC}$ and chord $\overline{BD}$ to find the lengths of $\overline{GB}$, $\overline{EF}$ and $\overline{PT}$, if CG = 4 units, GA = 6 units, DG = 3 units, PF = 9 units and PB = 8 units.

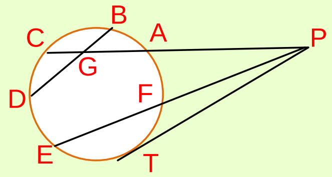


Figure 6.72

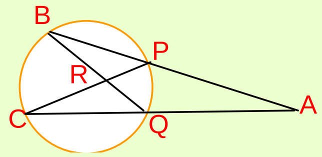


Figure 6.73

6 IN FIGURE 6.73, $m(\angle BPC) = 48^{O}$, $m(\angle BRC) = 68^{O}$ AND $m(\angle BCR) = 62^{O}$. CALCULATE THE MEASURES OF ANGLES OF $\triangle ABC$.

7 THE DIAGONALS $\overline{AC}$ AND $\overline{BD}$ OF THE PARALLELOGRAM $ABCD$ ARE OF LENGTHS 20 CM AND 12 CM RESPECTIVELY. IF THE CIRCLE BCD CUTS $\overline{CA}$ AT F, FIND THE LENGTH OF $\overline{AF}$.

8 IN FIGURE 6.74, BP = 6 CM, DC = 10 CM AND CP = 8 CM. CALCULATE THE LENGTHS OF THE CHORD $\overline{AB}$ AND THE TANGENT $\overline{PT}$.

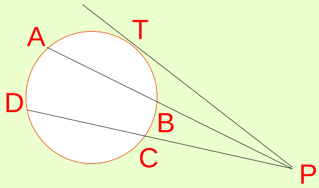


Figure 6.74

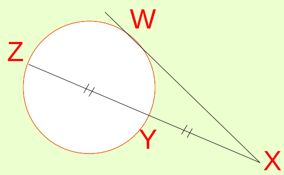


Figure 6.75

9 IN FIGURE 6.75, Y IS THE MID-POINT OF $\overline{XZ}$ AND $\overleftrightarrow{WX}$ IS TANGENT TO THE CIRCLE. FIND WX IN TERMS OF XY. EXPLAIN YOUR REASONING.

6.4 REGULAR POLYGONS

A POLYGON WHOSE VERTICES ARE ON A CIRCLE IS SAID TO BE ***inscribed*** IN THE CIRCLE. THE CIRCLE IS ***circumscribed*** ABOUT THE POLYGON.

IN FIGURE 6.76, THE POLYGON $ABCDE$ IS INSCRIBED IN THE CIRCLE OR THE CIRCLE IS CIRCUMSCRIBED ABOUT THE POLYGON.

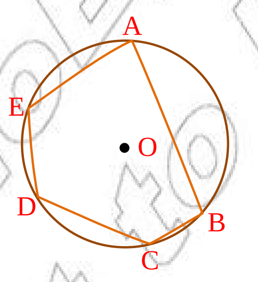


Figure 6.76

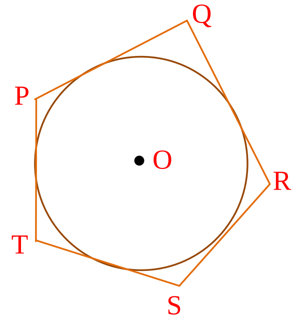


Figure 6.77

A POLYGON WHOSE SIDES ARE TANGENT TO A CIRCLE IS CIRCUMSCRIBED ABOUT THE CIRCLE. IN FIGURE 6.77, THE PENTAGON $PQRST$ IS CIRCUMSCRIBED ABOUT THE CIRCLE. THE CIRCLE IS INSCRIBED IN THE PENTAGON.

ACTIVITY 6.7

1 WHAT IS A REGULAR POLYGON? GIVE EXAMPLES.

2 DRAW THREE CIRCLES OF RADIUS 5 CM. CIRCUMSCRIBE A QUADRILATERAL ABOUT THE FIRST CIRCLE, A TRIANGLE ABOUT THE SECOND, AND A 7-SIDED POLYGON ABOUT THE THIRD.

3 CIRCUMSCRIBE A CIRCLE ABOUT A SQUARE.

4 DRAW A CIRCLE SUCH THAT THREE OF THE SIDES OF A RECTANGLE ARE TANGENT TO IT. GIVE REASONS WHY A CIRCLE CANNOT BE INSCRIBED IN THE RECTANGLE OF UNEQUAL S

5 SHOW THAT A CIRCLE CAN ALWAYS BE CIRCUMSCRIBED ABOUT A QUADRILATERAL IF TWO OPPOSITE ANGLES ARE RIGHT ANGLES.

6 SHOW THAT, IF A CIRCLE CAN BE CIRCUMSCRIBED ABOUT A PARALLELOGRAM, THEN THE PARALLELOGRAM IS A RECTANGLE.

7 WHAT IS THE MEASURE OF AN ANGLE BETWEEN THE BISECTORS OF TWO ADJACENT ANGLES IN A REGULAR POLYGON OF n SIDES, 3, 5, 10,

8 WHAT IS THE MEASURE OF AN ANGLE BETWEEN THE PERPENDICULAR BISECTORS OF TWO ADJACENT SIDES OF A REGULAR POLYGON OF n SIDES? 3, 7, 10,

9 DRAW A SQUARE WITH SIDE 5 CM. DRAW THE INSCRIBED AND CIRCUMSCRIBING CIRCLES.

6.4.1 Perimeter of a Regular Polygon

YOU HAVE STUDIED HOW TO FIND THE LENGTH OF A SIDE (S) AND PERIMETER (P) OF A RE POLYGON WITH RADIUS "r" AND THE NUMBER OF SIDES "n" IN GRADE 9. THE FOLLOWING EXAMPLE IS GIVEN TO REFRESH YOUR MEMORY.

EXAMPLE 1 THE PERIMETER OF A REGULAR POLYGON WITH 9 SIDES IS GIVEN BY:

$$P = 9 \times 2r \text{ SIN} \frac{180^{\circ}}{9} = 9d \text{ SIN} \frac{180^{\circ}}{9}, \text{ WHERE } d = 2r \text{ IS DIAMETER}$$

$$= 9d \text{ SIN } 20^{\circ} \approx 3.0782d$$

EXAMPLE 2 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR QUADRILATERAL WITH RADIUS 5 UNITS.

SOLUTION:

$$s = 2r \text{ SIN} \frac{180^{\circ}}{n} \qquad P = 2nr \text{ SIN} \frac{180^{\circ}}{n}$$

$$s = 2 \times 5 \text{ SIN} \frac{180^{\circ}}{4} = 10 \text{ SIN } 45^{\circ} \qquad P = 2 \times 4 \times 5 \text{ SIN} \frac{180^{\circ}}{4} = 40 \text{ SIN } 45^{\circ}$$

$$= 10 \times \frac{\sqrt{2}}{2} \qquad = 40 \times \frac{\sqrt{2}}{2}$$

$$\therefore s = 5\sqrt{2} \text{ UNITS.} \qquad \therefore P = 20\sqrt{2} \text{ UNITS.}$$

6.4.2 Area of a Regular Polygon

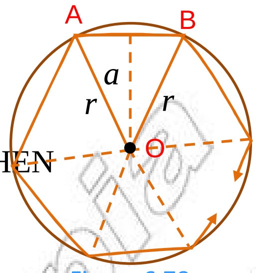


Figure 6.78

DRAW A CIRCLE WITH CENTRE O AND RADIUS r. INSCRIBE IN IT A REGULAR POLYGON WITH n SIDES AS SHOWN IN FIGURE 6.78.

JOIN O TO EACH VERTEX. THE POLYGONAL REGION IS THEN DIVIDED INTO n TRIANGLES. ΔAOB IS ONE OF THEM.

$\angle AOB$ HAS DEGREE MEASURE $\frac{360^{\circ}}{n}$.

RECALL THAT THE FORMULA FOR THE AREA OF A TRIANGLE WITH SIDES a AND b UNITS LONG AND $\angle C$ INCLUDED BETWEEN THESE SIDES IS:

$$A = \frac{1}{2} ab \sin(\angle C)$$

HENCE, AREA OF ΔAOB IS

$$A = \frac{1}{2} r \times r \sin(\angle AOB) = \frac{1}{2} r^2 \sin\frac{360^{\circ}}{n}$$

THEREFORE, THE AREA A OF THE POLYGON IS GIVEN BY

$$A = \frac{1}{2} nr^2 \sin\frac{360^{\circ}}{n} \quad \text{(WHY?)}$$

Theorem 6.16

The area A of a regular polygon with n sides and radius r is

$$A = \frac{1}{2} nr^2 \sin\frac{360^{o}}{n}.$$

THIS FORMULA FOR THE AREA OF A REGULAR POLYGON CAN BE USED TO FIND THE AREA (NUMBER OF SIDES INCREASES, THE AREA OF THE POLYGON BECOMES CLOSER TO THE ARE

ACTIVITY 6.8

SQUARE ABCD IS INSCRIBED IN A CIRCLE OF RA[DIUS] r.

a WHAT IS THE MEASURE OF ANGLE AOB?

b FIND THE AREA OF THE SQUARE ABCD.

c FIND THE AREA OF THE SQUARE IF r = 10 CM.

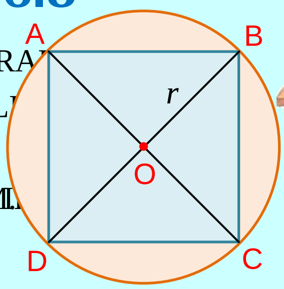


Figure 6.79

EXAMPLE 3 SHOW THAT THE AREA A OF A REGULAR HEXAGON INSCRIBED IN A CIRCLE WITH RADIUS r IS $\frac{3\sqrt{3}}{2} r^2$.

SOLUTION: $A = \frac{1}{2}nr^2 \sin\frac{360^{\circ}}{n} = \frac{1}{2}\times 6\times r^2 \sin\frac{360^{\circ}}{6} = 3r^2 \sin 60^{\circ}$

$$A = 3r^2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}r^2}{2} \text{ SQ UNITS.}$$

Exercise 6.5

1 FIND THE AREA OF A REGULAR NINE-SIDED POLYGON WITH RADIUS 6 UNITS.

2 FIND THE AREA OF A REGULAR TWELVE-SIDED POLYGON WITH RADIUS 3 UNITS.

3 PROVE THAT THE AREA A OF AN EQUILATERAL TRIANGLE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = \frac{3\sqrt{3}r^2}{4}$. USE THIS FORMULA TO FIND THE AREA OF AN EQUILATERAL TRIANGL
IN A CIRCLE WITH RADIUS:

A 2 CM **B** 3 CM **C** $\sqrt{2}$ CM **D** $2\sqrt{3}$ CM.

4 PROVE THAT THE AREA A OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS r IS $A = 2r^2$. USE THIS FORMULA TO FIND THE AREA OF A SQUARE INSCRIBED IN A CIRCLE WITH RAD

A 3 CM **B** 2 CM **C** $\sqrt{3}$ CM **D** 4 CM.

5 SHOW THAT ALL THE DISTANCES FROM THE CENTRE OF A REGULAR POLYGON TO THE SIDES ARE EQUAL.

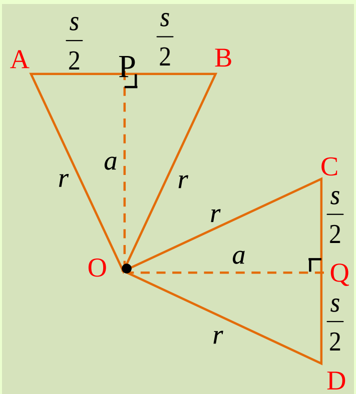


Figure 6.80

6 USE FIGURE 6.80 GIVEN ABOVE TO PROVE THE FORMULA FOR THE APOTHEM a:

$$a = r\cos\frac{180^{\circ}}{n}.$$

7 USE THE FORMULA $a = r\cos\frac{180^{\circ}}{n}$ TO CALCULATE THE APOTHEMS OF THE FOLLOWING REG
POLYGONS INSCRIBED IN A CIRCLE OF RADIUS 12 CM:

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** NONAGON.

8 SHOW THAT A FORMULA FOR THE AREA A OF A REGULAR POLYGON WITH n SIDES, APOTHEM a AND PERIMETER P IS $A = \frac{1}{2}aP$.

USE THIS FORMULA TO CALCULATE THE AREA OF A REGULAR;

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** OCTAGON.

GIVE YOUR ANSWER IN TERMS OF ITS RADIUS.

9 **A** SHOW THAT ANOTHER FORMULA FOR THE AREA A OF A REGULAR POLYGON WITH n SIDES, RADIUS r AND PERIMETER P IS

$$A = \frac{1}{2}Pr\cos\frac{180^{\circ}}{n}.$$

B SHOW THAT THE RATIO OF THE AREAS OF TWO REGULAR n-SIDED POLYGONS IS THE SQUARE OF THE RATIO OF THEIR RADII.

C USE THE FORMULA FOR THE APOTHEM $a = r\cos\frac{180^{o}}{n}$ AND TO SHOW THAT THE RATIO OF THE AREAS OF TWO REGULAR POLYGONS WITH THE SAME NUMBER OF SIDES I OF THE SQUARES OF THE LENGTHS OF CORRESPONDING SIDES.

D CAN YOU PROVE THE RESULT IN **C** ABOVE WITHOUT USING ANY OF THE FORMULAE OF THIS SECTION?

10 A CIRCULAR TIN IS PLACED ON A SQUARE. IF THE SIDE OF THE SQUARE IS CONGRUENT TO THE DIAMETER OF THE TIN, CALCULATE THE PERCENTAGE OF THE SQUARE WHI UNCOVERED. GIVE YOUR ANSWER CORRECT TO 2 DECIMAL PLACES.

Key Terms

altitude	concurrent lines	plane geometry
apothem	Euclidean Eeometry	product property
arc	incentre	quadrilateral
bisector	incircle	rectangle
central angle	inscribed angle	regular polygon
centroid	major arc	rhombus
chord	median	semi-circle
circle	minor arc	square
circumcentre	orthocenter	trapezium
circumcircle	parallelogram	
collinear points	perpendicular	

Summary

1 THE MEDIANS OF A TRIANGLE ARE CONCURRENT AT A POINT $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

2 THE PERPENDICULAR BISECTORS OF THE SIDES OF ANY TRIANGLE ARE CONCURRENT AT A POINT CALLED CIRCUMCENTER WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.

3 THE ALTITUDES OF A TRIANGLE ARE CONCURRENT AT A POINT CALLED THE ORTHOCENTRE OF THE TRIANGLE. IF POINTS D, E AND F ON THE SIDES $\overline{BC}$, $\overline{CA}$ AND $\overline{AB}$ RESPECTIVELY OF $\triangle ABC$ (OR THEIR EXTENSIONS) ARE COLLINEAR, THEN $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$. CONVERSELY, IF $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=-1$, THEN THE POINTS D, E AND F ARE COLLINEAR.

4 A TRAPEZIUM IS A QUADRILATERAL THAT HAS EXACTLY TWO SIDES PARALLEL.

5 A PARALLELOGRAM IS A QUADRILATERAL IN WHICH BOTH PAIRS OF OPPOSITE SIDES ARE PARALLEL.

6 A RECTANGLE IS A PARALLELOGRAM IN WHICH ONE OF ITS ANGLES IS A RIGHT ANGLE.

7 A RHOMBUS IS A PARALLELOGRAM WHICH HAS TWO ADJACENT SIDES CONGRUENT.

8 A SQUARE IS A RECTANGLE WHICH HAS CONGRUENT ADJACENT SIDES.

9 IN A CIRCLE, AN INSCRIBED ANGLE IS AN ANGLE WHOSE VERTEX LIES ON THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE.

10 IN FIGURE 6.81, $m(\angle APB) = \frac{1}{2} m\left(\overset{\frown}{AXB}\right)$

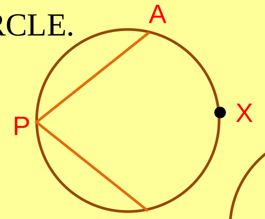


Figure 6.81

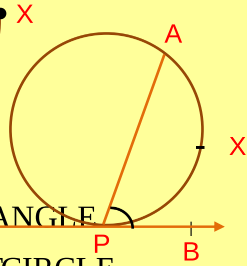


Figure 6.82

11 AN ANGLE INSCRIBED IN A SEMI-CIRCLE IS A RIGHT ANGLE.

12 AN ANGLE INSCRIBED IN AN ARC LESS THAN A SEMI-CIRCLE IS OBTUSE.

13 AN ANGLE INSCRIBED IN AN ARC GREATER THAN A SEMI-CIRCLE IS ACUTE.

14 IN FIGURE 6.82, $m(\angle APB) = \frac{1}{2}\, m\,(\overset{\frown}{AXP})$.

15 IN FIGURE 6.83, $m(\angle BPD) = \frac{1}{2} m(\overset{\frown}{AXC}) + \frac{1}{2} m(\overset{\frown}{BYD})$

AND $(AP)(PB) = (CP)(PD)$

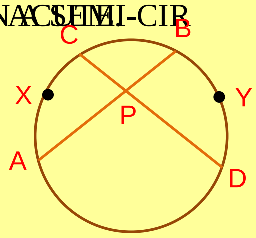


Figure 6.83

16 IN FIGURE 6.84:

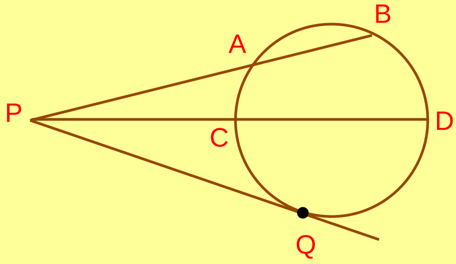


Figure 6.84

A $m(BPD) = \frac{1}{2}m(\overset{\frown}{BD}) - \frac{1}{2}m(\overset{\frown}{AC})$

B $m(DPQ) = \frac{1}{2}m(\overset{\frown}{DQ}) - \frac{1}{2}m(\overset{\frown}{QC})$

C $(PA)(PB) = (PC)(PD)$

D $(PQ)^2 = (PC)(PD)$

17 THE LENGTH OF A SIDE s AND PERIMETER P OF A REGULAR POLYGON OF n SIDES AND RADIUS r ARE:

$$s = 2r \text{ SIN } \frac{180^{O}}{n} \qquad P = 2n\,r \text{ SIN } \frac{180^{O}}{n} \qquad P = ns$$

18 THE AREA A OF A REGULAR POLYGON OF n SIDES AND RADIUS r IS

$$A = \frac{1}{2}nr^2 \text{ SIN } \frac{360^{O}}{n}.$$

Review Exercises on Unit 6

1 THE POINTS E AND F ARE THE MID-POINTS OF SIDES $\overline{AB}$ AND $\overline{AD}$ OF PARALLELOGRAM $ABCD$. PROVE THAT AREA(ΔCEF) = $\frac{1}{2}$ AREA($ABCD$). (*See* FIGURE 6.85)

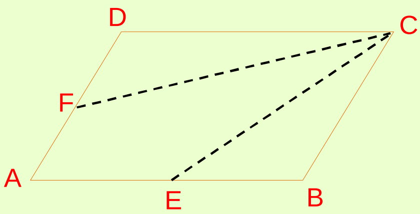


Figure 6.85

2 TWO CHORDS $\overline{AB}$ AND $\overline{CD}$ OF A CIRCLE INTERSECT AT RIGHT ANGLES AT A POINT INS
CIRCLE. IF $m(\angle BAC) = 35^{O}$, FIND THE MEASURES OF $\overset{\frown}{CB}$ AND $\overset{\frown}{AD}$.

3 IN FIGURE 6.86, O IS THE CENTRE OF THE CIRCLE. CALCULATE x AND y.

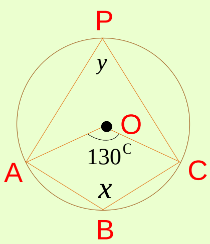


Figure 6.86

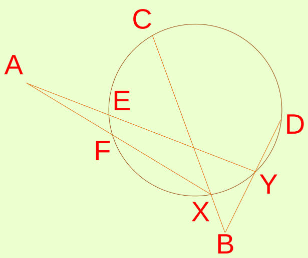


Figure 6.87

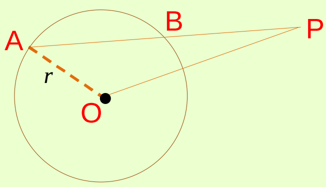


Figure 6.88

4 IN FIGURE 6.87, IF $m(\angle A) = 10^{O}$, $m(\overset{\frown}{EF}) = 15^{\circ}$ AND $m(\overset{\frown}{CD}) = 95^{O}$, FIND $m(\angle B)$.

5 FROM ANY POINT OUTSIDE A CIRCLE O WITH CENTRE AND RADIUS r, A LINE IS DRAWN CUTTING THE CIRCLE AT A AND B. PROVE THAT $(PA)(PB) = (PO)^2 - r^2$, AS SHOWN IN FIGURE 6. 88

6 TWO CHORDS $\overline{AB}$ AND $\overline{CD}$ OF A CIRCLE INTERSECT WHEN PRODUCED AT A POINT P OUTSIDE THE CIRCLE. $\overline{PT}$ IS A TANGENT FROM P TO THE CIRCLE.

PROVE THAT $(PA)(PB) = (PC)(PD) = (PT)^2$.

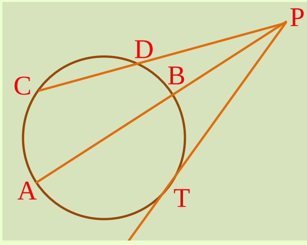


Figure 6.89

7 A CHORD OF A CIRCLE OF RADIUS 6 CM IS 8 CM LONG. FIND THE DISTANCE OF THE CHORD FROM THE CENTRE.

8 $\overline{MN}$ IS A DIAMETER AND $\overline{QR}$ IS A CHORD OF A CIRCLE, SUCH THAT $\overline{MN} \perp \overline{QR}$ AT L (AS SHOWN IN FIGURE 6.90). PROVE THAT $(QL)^2 = (ML).(LN)$.

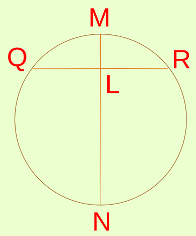


Figure 6.90

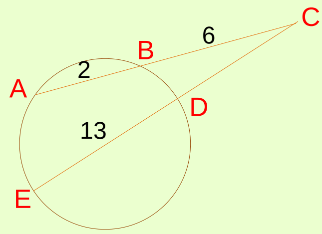


Figure 6.91

9 SECANTS $\overleftrightarrow{CA}$ AND $\overleftrightarrow{CE}$ INTERSECT A CIRCLE AT A, B, D AND E AS SHOWN IN FIGURE 6.91. IF THE LENGTHS OF THE SEGMENTS ARE AS SHOWN, FIND THE LENGTH OF $\overline{CD}$.

10 $\overleftrightarrow{AOB}$, $\overleftrightarrow{COD}$ ARE TWO STRAIGHT LINES SUCH THAT AB = 20 CM, CD = 19 CM, AO = 6 CM, CO = 7 CM. PROVE THAT $ACBD$ IS A CYCLIC QUADRILATERAL.

11 $ABXY$ IS A PARALLELOGRAM OF AREA 18 CM^2, AB = 6 CM, AY = 4 CM AND C IS A POINT ON $\overline{YX}$ OR EXTENDED SUCH THAT BC = 5 CM. FIND:

A THE AREA OF ΔABC **B** THE DISTANCE FROM B TO $\overline{AY}$.

C THE DISTANCE FROM A TO $\overline{CB}$