Motion in 2D

Unit 1

Contents

Section	Learning competencies
1.1 Projectile motion (page 2)	• Define the term projectile (and provide several examples). • Explain the difference between 1D and 2D motion. • Correctly use the terms angle of elevation and angle of depression, and explain the importance of the angle when it comes to launching projectiles. • Explain the effect gravity has on the motion of an object. • Describe what happens to the horizontal and vertical velocities of a projectile and the important characteristics of its flight. • Demonstrate how to use the equations for uniform acceleration and to apply these to projectile motion. • Define the term centre of mass. • Conduct simple experiments to determine the centre of mass of 2D objects. • List the characteristics of uniform circular motion. • Describe the relationships between radius, mass, forces and velocity for an object following a circular path.
1.2 Rotational kinematics (page 24)	• Define the terms angular and tangential displacement, and angular and tangential velocity. • Express angles in terms of revolutions, radians and degrees. • Define the term angular acceleration, and list its key characteristics. • Identify the SI unit of angular velocity and angular acceleration. • Explain the relationships between angular displacement, tangential displacement, angular velocity, tangential velocity and angular acceleration. • Demonstrate how to use the equations of constant angular acceleration and compare them with equations of constant acceleration.
1.3 Rotational dynamics (page 30)	• Define the moment of inertia of a point mass. • Define rotational kinetic energy of a body. • Solve simple problems relating to moment of inertia and rotational kinetic energy. • Define the term torque. • Identify the SI unit of torque, N m, which is not the same as the joule. • Express torque in terms of moment of inertia and angular acceleration. • Derive an expression for the work done by the torque. • Use the formula $W = \tau\theta$ to solve problems related to work done by torque.

Contents

Section	Learning competencies
1.3 Rotational dynamics (page 30) (*continued*)	• Define the angular momentum of a particle of mass m and write its SI unit. • State the law of conservation of angular momentum. • Solve problems using the law of conservation of angular momentum. • State the first and second conditions of equilibrium. • Solve problems related to conditions of equilibrium. • Define the term centre of mass (centre of gravity) of a solid body. • Determine the centre of gravity using a plumb-line method. • Define the terms stable, unstable and neutral equilibrium.
1.4 Newton's law of universal gravitation (page 40)	• State Newton's law of universal gravitation. • Determine the magnitude of the force of attraction between two masses separated by a distance r. • Calculate the value of g at any distance above the surface of the Earth. • State Kepler's law of planetary motion. • Use Kepler's law of planetary motion to determine the period of any planet. • Differentiate between orbital and escape velocity of a satellite. • Determine the period of a satellite around a planet. • Calculate the orbital and escape velocity of a satellite. • Describe the period, position and function of a geostationary satellite.

You may have already studied motion in one dimension (1D), including the equation of uniform acceleration. This unit takes motion further, looking into motion in two dimensions. This includes projectiles and circular motion.

When you catch a ball your brain is completing a series of complex calculations relating to the path followed by the ball and the time it takes to reach you. This is hard-wired into our brains from the days when we used to hunt and most of our food lived in trees. A detailed understanding of two-dimensional (2D) motion is essential for physicists, as it enables them to complete the calculations required to design objects from complex rockets to roads around cities.

1.1 Projectile motion

By the end of this section you should be able to:

- Define the term projectile (and provide several examples).
- Explain the difference between 1D and 2D motion.
- Correctly use the terms angle of elevation and angle of depression, and explain the importance of the angle when it comes to launching projectiles.

- Explain the effect gravity has on the motion of an object.
- Describe what happens to the horizontal and vertical velocities of a projectile and the important characteristics of its flight.
- Demonstrate how to use the equations for uniform acceleration and to apply these to projectile motion.
- Define the term centre of mass.
- Conduct simple experiments to determine the centre of mass of 2D objects.
- List the characteristics of uniform circular motion.
- Describe the relationships between radius, mass, forces and velocity for an object following a circular path.

What are projectiles?

A **projectile** is any object moving through the air without an engine or other motive force. This means is it not restricted to cannonballs or bullets. When you throw a stone, toss a cricket ball, or kick a soccer ball, they are classed as projectiles as they fly through the air.

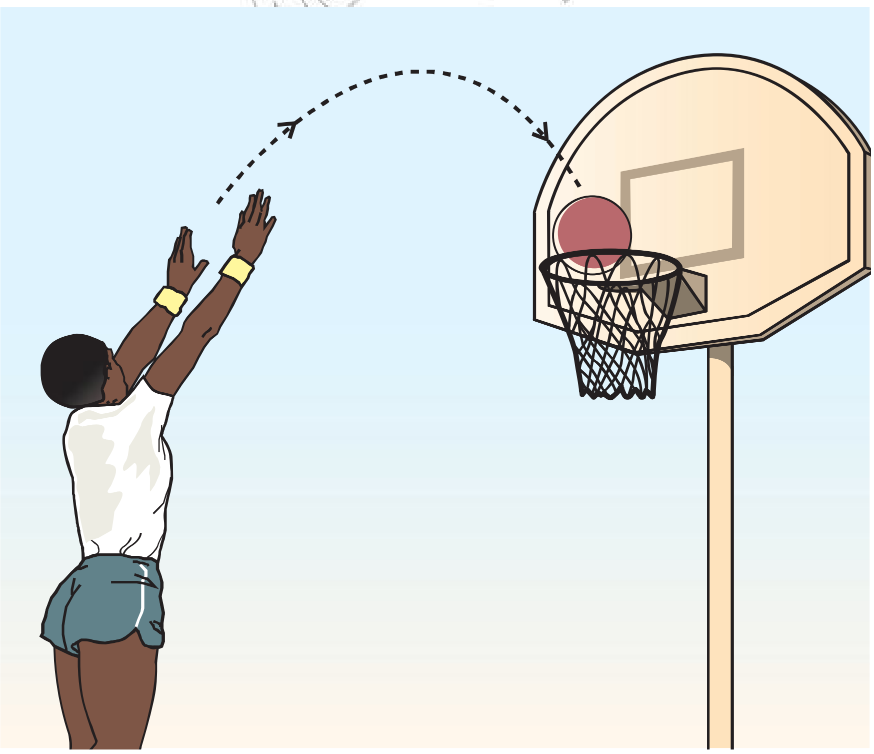


Figure 1.1 *Typical example of a projectile.*

Projectile motion is more complex than 1D motion. It involves motion in two directions, the vertical and the horizontal.

For simplicity we look at the horizontal **velocity** and vertical velocity as separate components of the velocity. These can be treated individually.

KEY WORDS

projectile *any object propelled through space by the exertion of a force which ceases after launch*

velocity *the rate of change of position of a body*

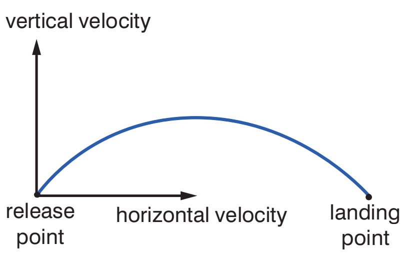


Figure 1.2 *Another projectile.*

KEY WORDS

resolving *splitting a vector into vertical and horizontal components. These components have the same effect as the original vector.*

Activity 1.1: Naming some projectiles

Spend 30 seconds listing as many projectiles as you can think of. Compare your list with a partner.

Resolving velocity

Resolving means splitting one vector into two component vectors (usually one horizontal and one vertical). These components have the same effect as the original vector.

An example can be seen below. The 60 m/s velocity can be resolved into two component vectors that have the same effect when combined:

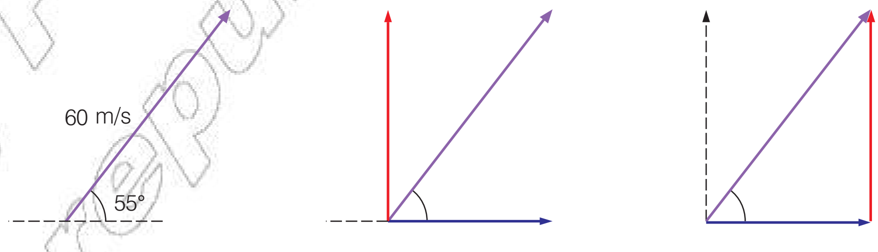


Figure 1.3 *Velocity components are shown in red and blue.*

This allows us to calculate the path of the projectile including its maximum vertical displacement (maximum height), maximum horizontal displacement (range) and flight time.

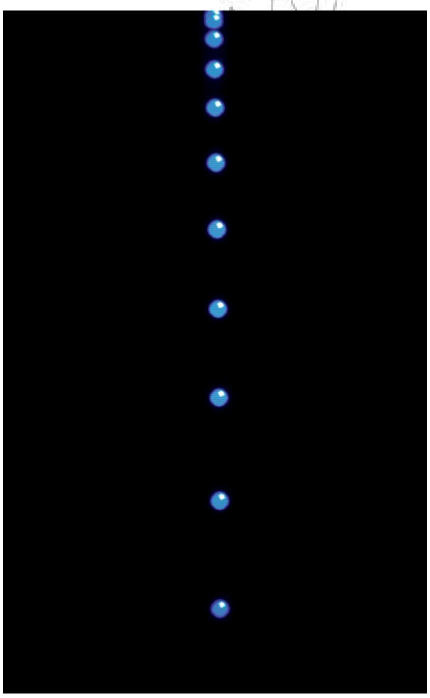


Figure 1.4 *Accelerating under gravity.*

Horizontal motion

If we ignore air resistance, then there are no horizontal forces acting on the projectile as it flies through the air. This means there is no acceleration, so the velocity stays the same horizontally. This is a perfectly valid assumption for many projectiles.

We can apply the following equation:

displacement = *average velocity* × *time taken*

This becomes:

horizontal displacement = *horizontal velocity* × *flight time*

So in order to determine the range of the projectile we would use the total flight time. For example, to find the horizontal displacement after 5.0 s we would use a time of 5.0 seconds.

total horizontal displacement = *horizontal velocity* × *total flight time*

horizontal displacement after 5.0 s = *velocity* × *5.0 s*

Discussion activity

Under what circumstances might it be inappropriate to assume the horizontal velocity of a projectile remains constant?

Vertical motion

The vertical velocity does change. This is because the projectile accelerates under gravity as it moves.

Figure 1.4 shows a ball falling through the air. The images were taken at regular time intervals and you can see the displacement between each shot increases. This is because the ball is moving faster and faster.

Ignoring air resistance once again, gravity causes all objects to accelerate at 9.81 m/s^2.

Figure 1.5 shows two motion graphs for the falling ball. Looking at the displacement–time graph it is evident the gradient is increasing. This is because the object is moving faster as it falls.

This can be seen on the velocity–time graph. A constant gradient indicates a constant acceleration (in this case, 9.81 m/s^2).

The vertical motion of the projectile is an example of uniformly accelerated motion. This means we can use the equations for uniform acceleration:

1. $v = u + at$
2. $s = \frac{1}{2}(u + v)\,t$
3. $s = ut + \frac{1}{2}at^2$
4. $v^2 = u^2 + 2as$
5. $s = vt - \frac{1}{2}at^2$

where:

s = displacement

v = final velocity

u = initial velocity

a = acceleration (in this case, 9.81 m/s^2)

t = time

These can be used to determine the time it takes for a projectile to hit the ground.

Worked example 1.1

Find the time taken for a ball dropped from a height of 6.0 metres:

s (m)	u (m/s)	v (m/s)	a (m/s^2)	t (s)
6.0	0.0 (as dropped)	unknown	9.81	?

We don't know the final velocity so we must use equation 3 (because there is no v in this equation).

DID YOU KNOW?

Due to the shape of the Earth and its uneven density, the value of the acceleration due to gravity varies slightly. At the Equator it is 9.78 m/s^2 whereas at the North Pole it is 9.83 m/s^2. 9.81 m/s^2 is usually referred to as Standard Gravity.

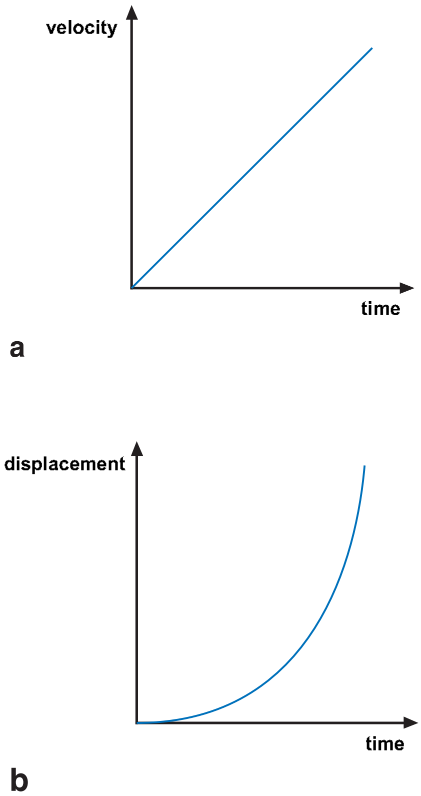


Figure 1.5 *Graphs showing the vertical velocity and displacement as the ball falls.*

DID YOU KNOW?

If we ignore air resistance, the mass of an object does not affect the rate at which it accelerates. Galileo Galilei was the first to realise this back in the 17th century. At the end of the last Apollo 15 moon walk, Commander David Scott tested this theory. He dropped a hammer and a feather at the same time. As the surface of the moon is a vacuum, there was no air resistance and the feather fell at the same rate as the hammer.

$s = ut + \frac{1}{2}at^2$

$ut = 0$ as the ball was dropped so the equation becomes $s = \frac{1}{2}at^2$

This can be rearranged to $t = \sqrt{\frac{2s}{a}}$

$t = \sqrt{\frac{2 \times 6.0 \text{ m}}{9.81 \text{ m/s}^2}}$

$t = 1.1$ s (to two significant figures)

Activity 1.2: Dropping a ball

Drop a ball from several different heights and time how long it takes to hit the ground. Record your data carefully and take repeat measurements for each height.

Using the equation, calculate the time it actually takes to hit the ground. Compare the actual times with your readings and comment on your findings.

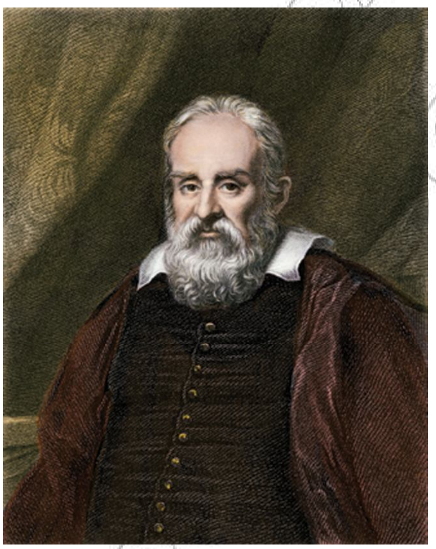


Figure 1.6 Galileo Galilei.

We can also work out the final vertical velocity.

Worked example 1.2

Find the final v ertical velocity for a ball dropped from 6 m. Looking back at the table we now have:

s (m)	*u* (m/s)	*v* (m/s)	*a* (m/s²)	*t* (s)
6.0	0.0 (as dropped)	unknown	9.81	1.1

We could use equations 1, 2, 4, or 5 to determine *v*. However, equation 4 does not require you to calculate time, so this is preferable.

$v^2 = u^2 + 2as$

$v = \sqrt{(u^2 + 2as)}$

$v = \sqrt{(0^2 + 2 \times 9.81 \text{ m/s}^2 \times 6.0 \text{ m})}$

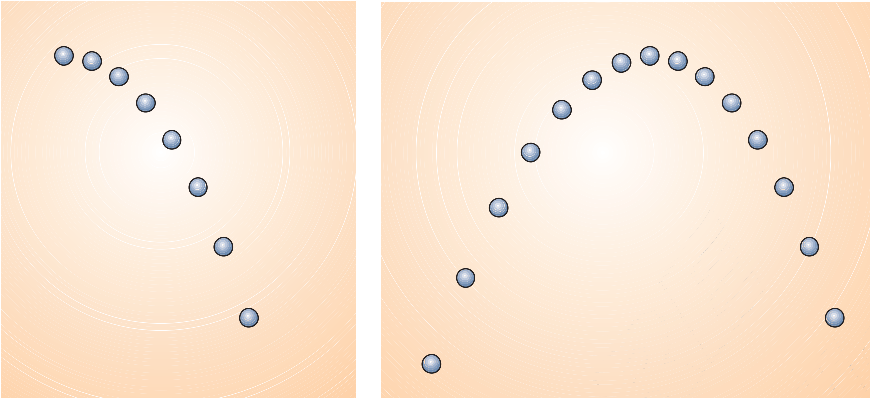
$v = 11$ m/s (to two significant figures)

KEY WORDS

parabola *the curved path a projectile takes through the air*

How does this affect the motion of projectiles?

When a projectile moves through the air, it follows a path caused by the combination of its horizontal and vertical velocities. This path is curved; it forms a special type of curve called a **parabola**.



a b

Figure 1.7 Diagrams of paths of balls.

Figure 1.9 shows a real example of this type of motion. Here one child throws a ball to another. A camera took photos at regular time intervals. If you look carefully, you can see the ball's vertical velocity increases as it falls and decreases as it rises.

Notice:

- Horizontally, the ball moves at a steady speed. The images of each ball are equally spaced horizontally.
- Vertically, the ball accelerates downwards due to gravity. This means the images become further and further apart.

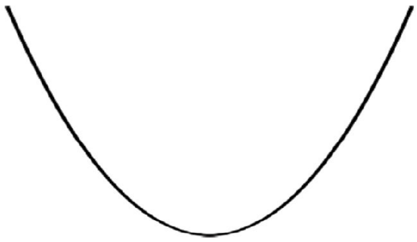


Figure 1.8 A parabola.

Figure 1.9 A multi-flash image showing the motion of a projectile (a ball).

Activity 1.3: The path of a ball

Next time you are outside, get two friends to throw a ball to each other. Stand at the side (some distance away) and watch the path of the ball carefully. Try asking your friends to throw the ball at different angles. What do you notice?

Horizontal and vertical velocities

This special shape is caused by the relationship between the horizontal and vertical velocities.

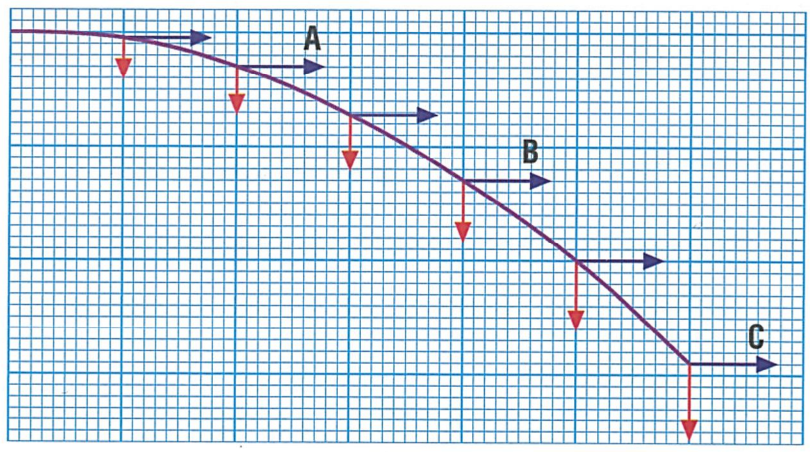


Figure 1.10 *Horizontal and vertical velocities.*

If you look carefully at Figure 1.10 you can see the horizontal velocity remains the same but the vertical velocity increases. This causes the ball to follow a parabolic path.

The same is true for a projectile thrown at an angle.

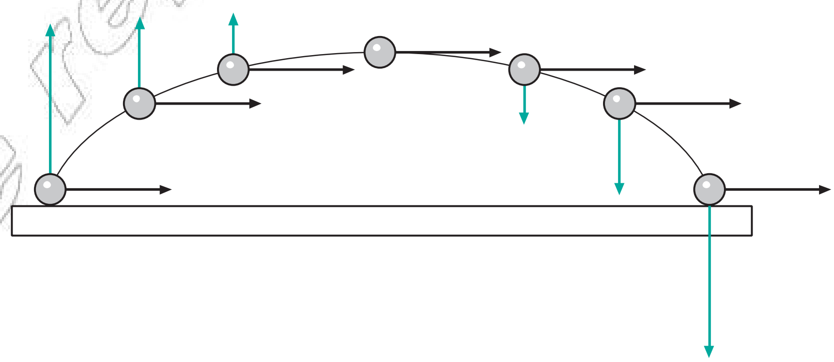


Figure 1.11 *Horizontal and vertical velocities for a projectile thrown at an angle.*

Figure 1.11 shows the velocities for a ball thrown at an angle. Again the horizontal velocity remains constant. The vertical velocity first decreases (on the way up), reaches zero (at the top of the flight), and then increases on the way down. The vertical velocity at any point is given by the equation $v = u + at$.

You need to think carefully about the directions of the vertical velocities and the acceleration. If we use a as 9.81 m/s^2, then the initial vertical velocity must be a negative number as it is in the opposite direction. We have effectively decided that downwards is the positive direction.

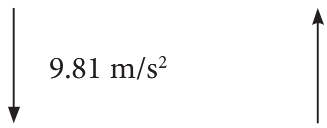


Figure 1.12 *The importance of direction when dealing with vectors.*

This works the other way around, too. It does not matter which way is positive and which way is negative, but you must not mix them up!

Worked example 1.3

An arrow is fired vertically with an initial velocity of 35 m/s. Find its velocity after:

a) 3 s

b) 6 s

Using the table layout seen earlier we get:

s (m)	*u* (m/s)	*v* (m/s)	*a* (m/s^2)	*t* (s)
Unknown	−35	?	9.81	a) 3 b) 6

Notice we have entered −35 m/s for the initial velocity. We are therefore setting the downwards direction as positive.

a)

$v = u + at$

$v = -35\ m/s + (9.81 m/s^2 \times 3\ s)$

$v = -5.6\ m/s$ (to 2 significant figures)

Notice the velocity is still negative as it is still travelling upwards.

b)

$v = u + at$

$v = -35\ m/s + (9.81 m/s^2 \times 6\ s)$

$v = 24\ m/s$ (to 2 significant figures)

Notice the velocity is now positive. This must mean the arrow has changed direction and is heading back down.

Activity 1.4: Heights of arrows

Calculate the height of the arrow in each case for the worked example.

Horizontal projection

Projectiles may be initially travelling horizontally. This might include a ball kicked off a wall, a bullet fired from a horizontal gun, or a parcel dropped from the underside of an aircraft flying horizontally.

The object will follow the path shown in Figure 1.13. It is interesting to note that the time it takes to hit the floor only depends on the original height of the object.

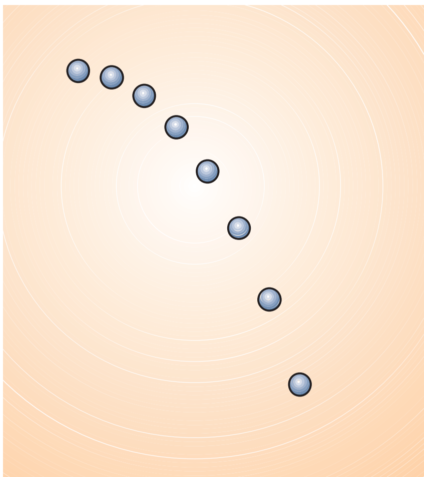


Figure 1.13 Horizontal projection.

The flight time is given by the equation $s = ut + \frac{1}{2}at^2$. Looking at this vertically, $ut = 0$ as the ball initially has no vertical velocity. So the equation becomes $s = \frac{1}{2}at^2$ and the time it takes to hit the floor is given by:

$$t = \sqrt{\frac{2s}{a}}$$

The acceleration is constant (9.81 m/s^2) and so the only factor affecting the time to hit the floor is the drop height.

From this we can draw some counterintuitive conclusions: imagine someone holding a rifle in their right hand and a rifle bullet in their left hand. The rifle is perfectly horizontal. The rifle is then fired and the bullet is dropped at exactly the same time. Which bullet will hit the ground first?

It would be tempting to say the bullet from the rifle takes longer to hit the ground than the one dropped from the hand. However, this would be wrong! They both hit the ground at the same time; the only difference is the bullet from the rifle hits the ground several metres away whereas the dropped bullet hits the ground by the person's feet.

Figure 1.14 shows a time lapse of two balls: one dropped, the other fired horizontally. Both were released simultaneously. You can clearly see they stay at exactly the same height as they fall.

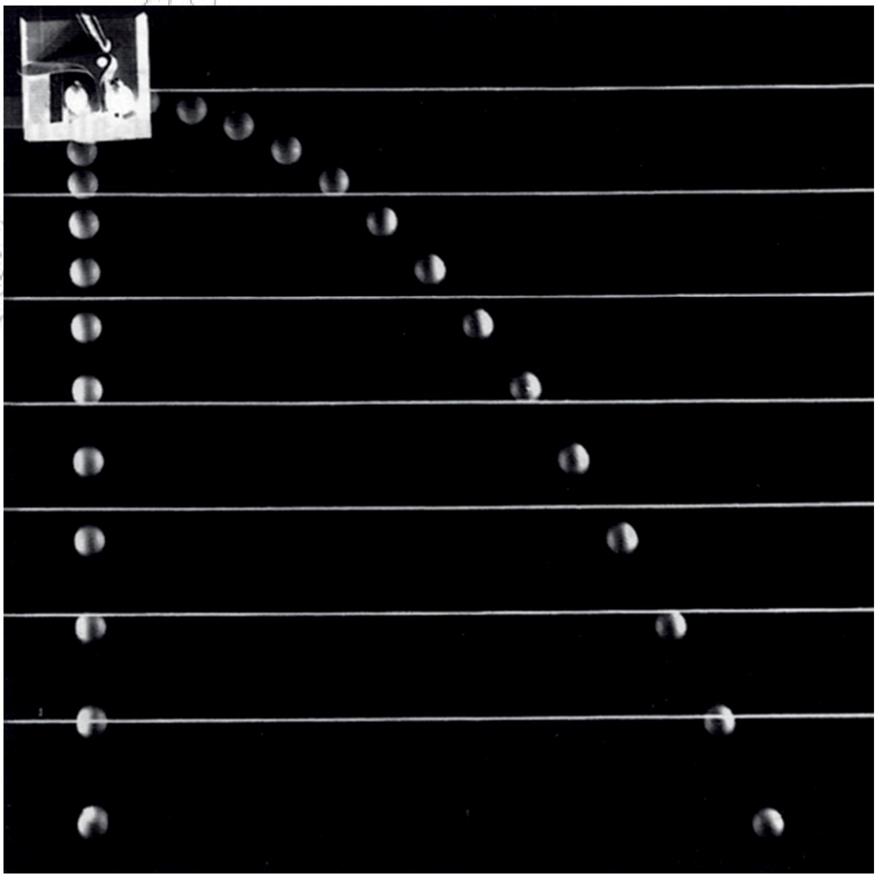


Figure 1.14 *Time lapse of two balls falling.*

Activity 1.5: Test with ruler and two coins

You can test this using a ruler and two coins. Carefully place the ruler on the edge of a desk. Place one coin on the end of the ruler overhanging the desk. Place the other next to the ruler as shown. The aim is to flick the ruler so that the first coin falls vertically where as the second coin gets pushed off the desk horizontally.

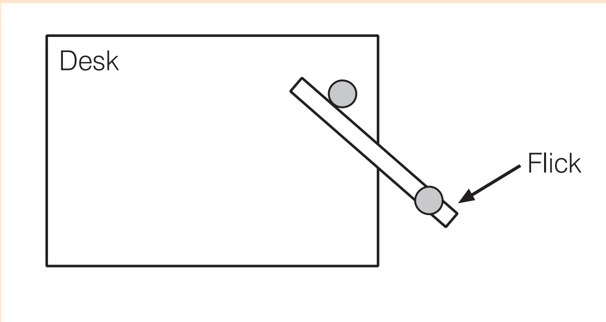


Figure 1.16 *Ruler and coin experiment.*

You have to flick it quite hard.

When you do so, listen for the clink as the coins hit the floor. You will find the two clinks come at once; both coins hit the floor at the same time.

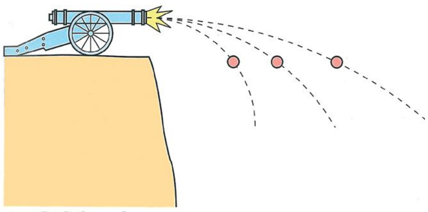


Figure 1.15 *Cannonballs fired horizontally with different velocities.*

Activity 1.6: Rolling a ball down a track

A more complex experiment involves rolling a ball or marble down a track. As you vary the release height, you vary the horizontal velocity of the marble. You could time the time it takes to hit the ground for different release heights.

You must be careful to start timing when the ball leaves the desk.

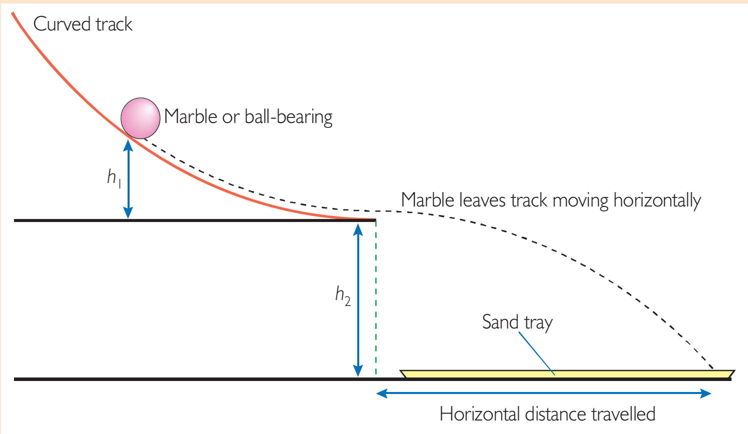


Figure 1.17 *A marble rolling down a track.*

Worked example 1.4

A football is kicked off a wall at an initial horizontal velocity of 12 m/s. The wall is 2.1 m high. Find the time taken for the ball to hit the floor and the range of the ball.

s (m)	**u** (m/s)	**v** (m/s)	**a** (m/s²)	**t** (s)
2.1	12	?	9.81	?

$t = \sqrt{\frac{2s}{a}}$

$t = \sqrt{\frac{2 \times 2.1 \text{ m}}{9.81 \text{ m/s}^2}}$

$t = 0.65$ s

horizontal displacement = horizontal velocity × flight time

horizontal displacement = 12 m/s × 0.65 s

horizontal displacement = 7.8 m

As discussed earlier, the range of any projectile is given by:

horizontal displacement = horizontal velocity (v_h) × flight time

The flight time is given by: $t = \sqrt{\frac{2s}{a}}$

Combining these we get:

horizontal displacement = $v_h \times \sqrt{\frac{2s}{a}}$

horizontal displacement = $v_h \times \sqrt{\frac{2 \text{ vertical height}}{\text{acceleration}}}$

This only applies to projectiles initially travelling horizontally.

Activity 1.7: Trajectory of a ball

Use the equations above to complete the table below. Use this data and a piece of graph paper to carefully plot the trajectory of a ball thrown horizontally with a velocity of 2.0 m/s.

Time (s)	**Vertical displacement** *y* **(m)**	**Horizontal displacement** *x* **(m)**
1.0		
2.0		
3.0		
4.0		
5.0		

Activity 1.8: Range of a ball rolled down a track

Using the same equipment as for Activity 1.6, roll the marble down the track. This time, measure the range of the marble when released from several different heights up the ramp. Be sure to repeat your readings for each height.

Plot a graph of your findings and use your data to calculate the initial velocity in each case.

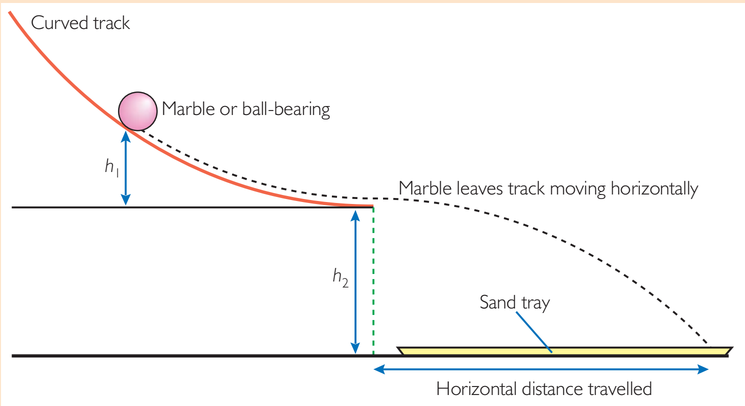


***Figure 1.18** A marble rolling down a track.*

Projectiles at angles

Projectiles may also be travelling at angles to the horizontal. In this case their initial vertical velocity is not zero. In order to find the initial horizontal and initial vertical velocity, the velocity must be resolved into horizontal and vertical components.

For example, a ball kicked with a velocity of 9.0 m/s at an angle of 50° to the horizontal:

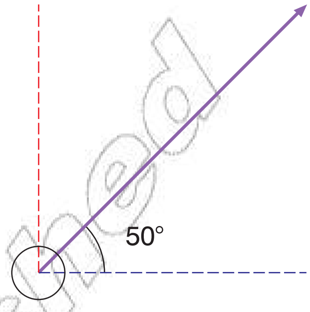


Figure 1.19 Ball kicked at an angle.

To find the components, we use trigonometry.

- initial vertical component = 9.0 m/s × sin 50°
- initial vertical component = 6.9 m/s
- initial horizontal component = 9.0 m/s × cos 50°
- initial horizontal component = 5.8 m/s (remember this will not change throughout the flight of the projectile).

If the angle of the projectile is above the horizontal, then this is referred to as an **angle of elevation** (Figure 1.20a). However, if the angle of the projectile is below the horizontal then this is referred to as an **angle of depression** (Figure 1.20b).

KEY WORDS

angle of elevation *the angle of a projectile's trajectory above the horizontal*

angle of depression *the angle of a projectile's trajectory below the horizontal*

Using the initial vertical velocity, the initial horizontal velocity and the equations of uniform acceleration we can then determine the range, flight time and maximum height of the projectile.

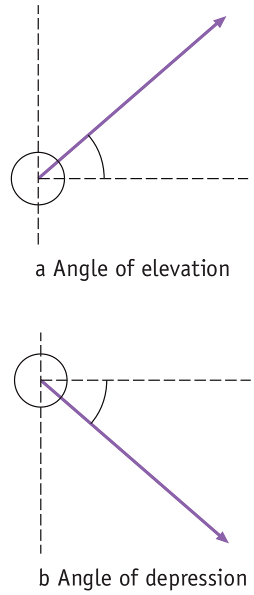


Figure 1.20 Angle of elevation and angle of depression.

Maximum height

At the maximum height the vertical velocity of the projectile will be zero. Using the example in Figure 1.19 (9.0 m/s at 50°) we can use the equations of uniform acceleration as follows:

s (m)	u (m/s)	v (m/s)	a (m/s²)	t (s)
??	$u \sin \theta$ in this case: −6.9	At max height: 0	9.81	unknown

Again, take care to ensure you consider the directions of the velocities; in this case, downwards is positive.

To find the maximum height we use $v^2 = u^2 + 2as$ (as we don't know t):

$$v^2 = u^2 + 2as$$

$$s = \frac{v^2 - u^2}{2a}$$

$$s = \frac{(0^2 - (-6.9^2\ \text{m/s}^2)}{2 \times 9.81\ \text{m/s}^2}$$

$$= -2.4\ \text{m.}$$

The height is of course 2.4 m, the negative just indicates that it is in the opposite direction to the acceleration.

We can then work out the time taken to reach maximum height using $v = u + at$.

$v = u + at$

$t = \frac{v - u}{a}$

$t = \frac{0 - -6.9 \text{ m/s}}{9.81 \text{ m/s}^2}$

$t = 0.70$ s

Projectile range

If the projectile is launched from the ground, the total range will be given by:

- Total horizontal displacement = horizontal velocity × total flight time

The horizontal velocity remains the same throughout the flight and is given by $u \cos \theta$ (in the previous example, 5.8 m/s). All we need to find the range is total flight time.

This can be done in a number of ways; perhaps the simplest is to use $v = u + at$. The final vertical velocity will be the same magnitude as the initial vertical velocity, just in the opposite direction (look at Figure 1.11 to check).

Using the same example as before we get:

s (m)	*u* (m/s)	*v* (m/s)	*a* (m/s²)	*t* (s)
??	$u \sin \theta$ in this case: −6.9	6.9	9.81	unknown

Again, take care to ensure you consider the directions of the velocities; in this case downwards is positive. Notice the final velocity is the same magnitude as the initial velocity, just the opposite direction (hence positive and not negative).

We can then work out the time taken to reach maximum height using $v = u + at$.

$v = u + at$

$t = \frac{v - u}{a}$

$t = \frac{6.9 \text{ m/s} - -6.9 \text{m/s}}{9.81 \text{ m/s}^2}$

$t = 1.4$ s

The total flight time is just double the time to reach maximum height!

So the range of the football is:

Total horizontal displacement = horizontal velocity × total flight time

Total horizontal displacement = 5.8 m/s × 1.4 s

Total horizontal displacement = 8.1 m

Alternatively, you can use algebra to combine the equation above, the horizontal and vertical components of the velocity, and the equations of uniform motion. This gives an expression for the range as:

$$\text{range} = \frac{u \cos\theta \times 2u \sin\theta}{a}$$

or

$$\text{range} = \frac{2u^2 \sin\theta \times \cos\theta}{a} = \frac{u^2 \sin 2\theta}{a}$$

This is usually called the **range equation**.

Activity 1.9

In a small group, discuss why

$$\frac{2u^2 \sin\theta \times \cos\theta}{a} = \frac{u^2 \sin 2\theta}{a}$$

Worked example 1.5

Find the range of a projectile launched at an angle of 50° with an initial velocity of 30 m/s.

range (m)	u (m/s)	a (m/s²)	θ (°)	2θ (°)	sin 2θ
?	30	9.81	50°	100°	0.9848

Use $\text{range} = \dfrac{u^2 \sin 2\theta}{a}$

$$\text{range} = \frac{30 \times 30 \times 0.9848}{9.81}$$

$$= 90.35 \text{ m}$$

KEY WORDS

range equation *an algebraic expression to calculate the range of a projectile*

Flight time

You can also derive an equation for the total flight time for a projectile fired at an angle. This is just a version of $s = vt - \frac{1}{2}at^2$ (equation 5 in our list of equations for constant acceleration).

At the end of the flight the vertical displacement will be 0 m, as the object will be back on the ground. The final vertical velocity will be given by $u \sin\theta$, as the object ends up with the same vertical velocity as it started, just in the opposite direction. So:

$$s = vt - \tfrac{1}{2}at^2$$

$$0 = (u \sin\theta)\, t - \tfrac{1}{2}at^2$$

Rearranging this we get:

$\frac{1}{2}\ a\ t^2 = (u \sin \theta)\ t$

$\frac{1}{2}\ a\ t = u \sin \theta$

$a\ t = 2\ u \sin \theta$

$t = \dfrac{2u \sin \theta}{a}$

Worked example 1.6

Find the flight time for a cannonball launched with a velocity of 60 m/s at an angle of 30°.

t (s)	*u* (m/s)	θ (°)	sin θ (°)	*a* (m/s²)
?	60	30	0.5	9.81

$t = \dfrac{2u \sin \theta}{a}$

$t = \dfrac{2 \times 60 \sin 30}{9.81}$

$t = 6.1$ s

Its range would be:

$\text{Range} = \dfrac{2u^2 \sin \theta \times \cos \theta}{a}$

$\text{Range} = \dfrac{2 \times 60^2 \sin 30 \times \cos 30}{9.81}$

Range = 320 m

Remember that all of these equations ignore air resistance. The equations become exceptionally complex when this is factored in, especially since the air resistance changes as the velocity changes.

Maximum range

The angle of a projectile affects its maximum range.

Figure 1.21 shows different paths of a projectile fired at different angles. The maximum range is achieved when the angle is 45°. This can be explained be referring back to the range equation:

$\text{Range} = \dfrac{2u^2 \sin \theta \times \cos \theta}{a}$

The maximum value of $\sin \theta \times \cos \theta$ is 0.5 and this happens when θ is 45°. This gives us:

$\text{maximum range} = \dfrac{2u^2\ 0.5}{a}$

$\text{maximum range} = \dfrac{u^2}{a}$

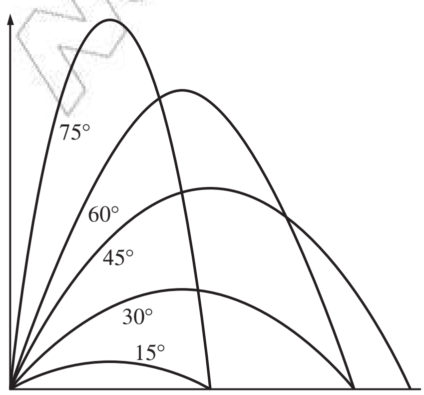


***Figure 1.21** An illustration of the effect of angle on range.*

To determine the range of projectiles launched above the ground you need to use algebra to derive a new equation. The range is given by:

$$\text{Range} = \frac{(u\cos\theta) \times (u\sin\theta + \sqrt{((u\sin\theta)^2 + 2ah)}}{a}$$

where h is the height above the ground. Even this equation ignores air resistance!

Activity 1.10: The range of a cannonball

Use the range equation to determine the range of a cannonball fired with a velocity of 50 m/s when fired at a series of different angles. Use 15°, 30°, 45°, 60° and 75°. Plot a graph of range against angle and comment on your findings.

Worked example 1.7

Use this equation to determine the range of the ball in Figure 1.22.

u (m/s)	θ (°)	cos θ (°)	sin θ	a (m/s²)	h (m)	range (m)
36	39	0.7771	0.6293	9.81	1.6	?

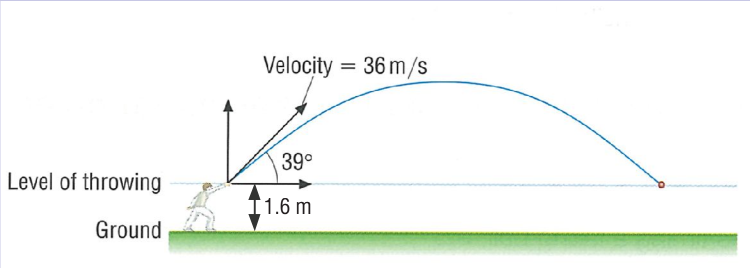


Figure 1.22 Throwing a soccer ball.

$$\text{Use range} = \frac{(u\cos\theta) \times (u\sin\theta) + \sqrt{((u\sin\theta)^2 + 2ah)}}{a}$$

$$= \frac{(36 \times 0.7771) \times (36 \times 0.6293) + \sqrt{((36 \times 0.6293)^2 + 2 \times 9.81 \times 1.6)}}{9.81}$$

$$= \frac{27.9756 \times (22.6548) + \sqrt{((22.6548)^2 + 31.392)}}{9.81}$$

$$= \frac{27.9756 \times (22.6548) + \sqrt{(513.24) + 31.392)}}{9.81}$$

$$= \frac{27.9756 \times (22.6548) + \sqrt{(544.632))}}{9.81}$$

$$= \frac{27.9756 \times (22.6548) + \sqrt{(544.632))}}{9.81}$$

$$= \frac{27.9756 \times (22.6548) + (23.3374)}{9.81}$$

$$= \frac{27.9756 \times 45.9922}{9.81} = \frac{1286.66}{9.81} = 131.16\ \text{m}$$

KEY WORDS

centre of mass *the point at which all the mass of an object may be considered to be concentrated*

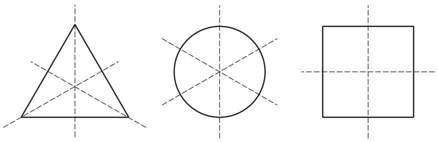


Figure 1.23 *Examples of the centre of mass for difference uniform shapes.*

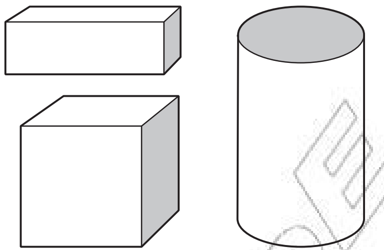


Figure 1.24 *The centre of mass for different uniform 3D objects is in the centre of the object.*

Centre of mass

All objects have a **centre of mass**. This is the point at which all the mass of the object may be considered to be concentrated.

For uniform objects the centre of mass will be at the intersection of all the lines of symmetry; in essence the centre of the object.

This is also true of 3D objects.

Another way to define the centre of mass is: the point through which a single force on a body has no turning effect.

Activity 1.11: Centre of mass of a ruler

Take a ruler and balance it on your finger. When balanced, the centre of mass must be above your finger. Take several other objects and balance them on your finger to find the approximate position of the centre of mass. Try it with a few more irregularly shaped objects.

This idea of balancing an object so there is no net turning effect leads onto the centre of mass theorem. This is a mathematical treatment of the distribution of mass, which is beyond the scope of this course.

The centre of mass theorem

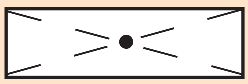
The centre of mass theorem simply states:

- when a force is applied to an object, the object acts as though its mass were a point mass at its centre of mass.

This means that the motion of the centre of mass of a system is identical to the motion of a single particle with the same mass as the system if such a particle were acted on by the same external forces.

Activity 1.12: Motion of the centre of mass

Work in a small group. You need a 2 m plank with four low friction wheels (for example, roller bearings).



- Mark the middle of the plank as shown in the diagram (this is the centre of mass), on the plank and on the floor beneath.
- One person should stand at the middle of the plank and walk four steps.
- Mark the position where the student finishes on the plank and on the floor.
- Record the distance that the centre of mass has moved.
- Use your knowledge of motion to analyse the motion of the centre of mass.

The centre of mass where two or more celestial bodies orbit each other is known as the barycentre. This is the point between the bodies where they balance each other. The Moon does not orbit the exact centre of the Earth, but their masses balance at a point approximately 1710 km below the surface of the Earch on a line between the Earth and the Moon.

Experimental determination of centre of mass

It is quite difficult to accurately determine the centre of mass for a 3D object. Special machines called planimeters are used. However, it is quite simple to determine the mass of a 2D object.

Activity 1.13: Centre of mass of an object

1. Take a piece of thick card and cut it into any shape you like (this example involves using a piece shaped like a jigsaw piece).
2. Make a series of small holes around the edge of the shape.
3. Hang it from one of these holes so the object is free to rotate.
4. Construct a simple plumb line using some wire (or string) and a mass.
5. Hang this from the support so it hangs vertically down.
6. Using a sharp pencil, draw a line to show the position of the plumb line.
7. Repeat this procedure for all the holes you have made, making a series of straight lines on your shape.
8. The lines should all cross; this is the centre of mass of the object. (You can test this by balancing the shape on the sharp pencil.)

The centre of mass of a system does not have to be an object. Take, for example, a cup. The centre of mass will be inside the cup even though there is nothing there but air. A more complex example might be two binary stars; these orbit the centre of mass between the two stars.

Uniform circular motion

Another example of 2D motion is **uniform circular motion**. This does not just refer to objects spinning around in circles, but also to objects following a curved path that is the shape of part of a circle, such as a car going around a bend of constant radius.

- Uniform circular motion specifically refers to following a curved path of constant radius at a steady speed.

In this case, despite the speed remaining constant, the velocity is constantly changing. This is because velocity is a vector quantity, so as the object moves around the circle its direction is changing, and therefore its velocity must be changing. This can be seen in Figure 1.27, where the velocity has changed between points A and B.

DID YOU KNOW?

The terms centre of gravity and centre of mass are often confused. There is a slight technical difference, but this is only apparent if the object is in a non-uniform gravitational field.

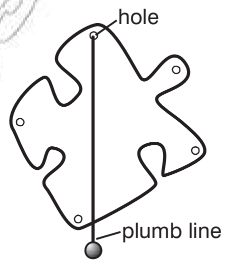


Figure 1.25 *Determining the centre of mass of a 2D object.*

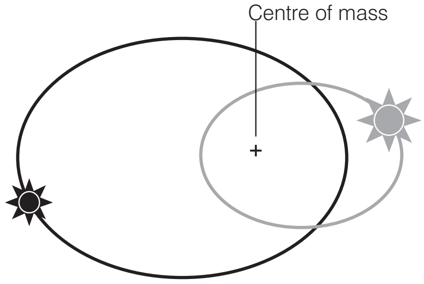


Figure 1.26 *Centre of mass of two binary stars.*

KEY WORD

uniform circular motion *the motion of a body following a curved path of constant radius at constant speed*

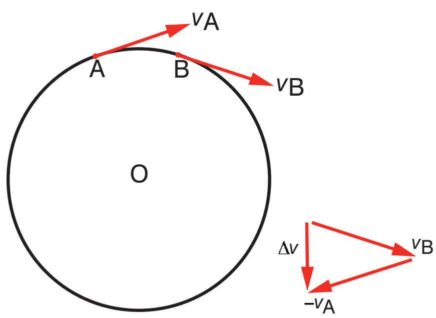


Figure 1.27 *The changing velocity for an object moving in a circular path.*

KEY WORDS

centripetal force *the net force needed to make a body follow a circular path*

As the velocity is changing, the object must be accelerating (acceleration is defined as the rate of change of velocity). According to Newton's second law, acceleration requires a net force and the direction of the acceleration is the same direction as the net force.

Projectile motion is not an example of uniform circular motion. As the path is a parabola, the radius is constantly changing.

Figure 1.28 shows the direction of the net force required to keep an object following a circular path. This force is referred to as the **centripetal force** and it always acts towards the centre of the circle.

What factors affect the size of the centripetal force?

The mass of the object, its velocity, and the radius of the curved path followed by the object affect the centripetal force.

Activity 1.14: A simple pendulum

Make a simple pendulum using a mass and a piece of string. *Carefully* swing this around your head in a horizontal circle, trying to keep the speed constant.

The force required to make the mass move and follow a circular path will come from your hand. You will be able to feel the force required. Experiment by changing the mass, radius and velocity of the object. How does this affect the force required?

To calculate the centripetal force we use the equation below:

$$F = \frac{mv^2}{r}$$

where F = centripetal force, m = mass of the object, v = velocity of the object, and r = radius of the curved path.

Worked example 1.8

The mass of the Earth is 6.0×10^{24} kg. It travels at a steady speed around the Sun at 30 000 m/s at a radius of 1.5×10^9 m. Find the force required to keep it in orbit.

F (N)	m (kg)	v (m/s)	r (m)
?	6×10^{24}	30 000	1.5×10^9

$$F = \frac{mv^2}{r}$$

$$F = \frac{6.0 \times 10^{24}\ \text{kg} \times 30\ 000\ \text{m/s}^2}{1.5 \times 10\ \text{m}}$$

$$F = 3.6 \times 10^{24}\ \text{N}$$

Circular motion examples

There are plenty of examples of circular motion. In each case the centripetal force acts towards the centre of the circular path followed by the object.

Table 1.1 *Examples of centripetal forces.*

Context	Centripetal force	Direction of the force
The Earth orbiting the Sun	Gravitational attraction	Towards the centre of the circular path: towards the centre of the Sun
An electron orbiting an atom	Electrostatic attraction	Towards the centre of the circular path: towards the centre of the nucleus
A ball on a string being whirled around	Tension in the string	Towards the centre of the circular path: towards the centre of the circle
A car going around a bend	Friction	Towards the centre of the circular path: towards the centre of the bend
A bus going over a hump-backed bridge	A component of weight	Towards the centre of the circular path: towards the centre of the bridge

What if the centripetal forces are not large enough?

There is a maximum centripetal force that can be provided. For example, if the tension gets too high, the string will snap. Likewise, if the friction is not high enough the car may skid or slide.

Imagine the maximum frictional force between the road and the tyres of a certain car is 6500 N. The mass of the car is 1200 kg and the bend has a radius of 85 m. Determine the maximum speed at which the car can take the bend.

$$F = \frac{mv^2}{r}$$

$$v^2 = \frac{Fr}{m}$$

$$v = \sqrt{\frac{Fr}{m}}$$

$$v = \sqrt{\frac{6500\ \text{N} \times 85\ \text{m}}{1200\ \text{kg}}}$$

$$v = 21\ \text{m/s}$$

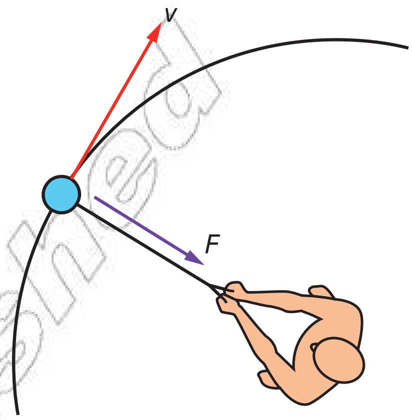


Figure 1.28 *Forces acting on a hammer thrower.*

Worked example 1.9

A hammer thrower swings a hammer around his head at a steady speed of 2.0 m/s at a radius of 1.2 m. He exerts a constant force of 120 N, Calculate the mass of the hammer.

F (N)	*m* (kg)	*v* (m/s)	*r* (m)
120	?	2.0	1.2

$$F = \frac{mv^2}{r}$$

$$Fr = mv^2$$

$$m = \frac{Fr}{v^2}$$

$$m = \frac{120\ \text{N} \times 1.2\ \text{m}}{2\ \text{m/s}^2}$$

$$m = 36\ \text{kg}$$

Discussion activity

Look carefully at the centripetal force equation. What effect does changing each of the variables have on the force required? For example, what would happen to the required force if the mass doubled? What would happen if the velocity doubled?

If the car travels faster than this, friction will not be large enough to provide the required centripetal force. The car will then follow a path of increased radius, and skid towards the edge of the road.

Summary

In this section you have learnt:

- A projectile is any object moving through the air without an engine or other motive force; examples include tennis balls and rifle bullets.
- Projectile motion and uniform circular motion are examples of 2D motion.
- Gravity causes projectiles to follow a parabolic path; this is because the horizontal velocity remains the same but the vertical velocity increases.
- The angle of a projectile affects its flight path and therefore its range and the time in the air.
- The equations for uniform acceleration can be applied to projectile motion.
- Centre of mass is the point at which all the mass of the object may be considered to be concentrated.
- Uniform circular motion is when an object travels at a steady speed around a circular path.

Review questions

1. This question is about a simple model of the physics of the long jump. Figure 1.29 shows a long-jumper at three different stages, A, B and C, during the jump. The horizontal and vertical components of velocity at each position are shown.

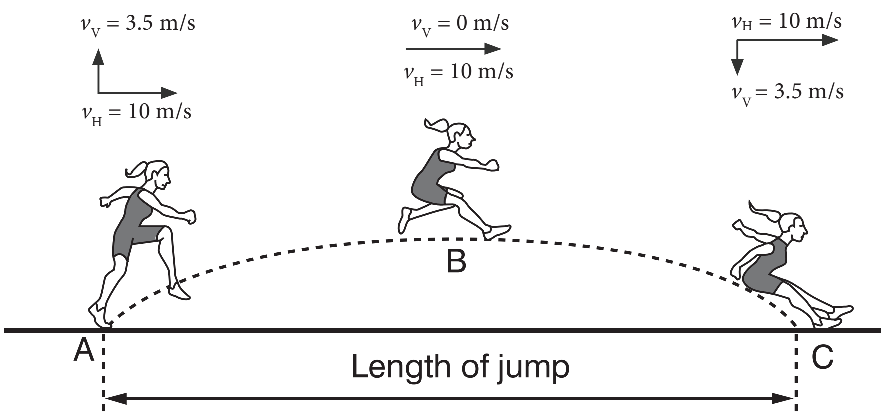


Figure 1.29

a) i) In the model, the horizontal component of velocity v_H is constant at 10 m/s throughout the jump. State the assumption that has been made in the model.

ii) Without calculation, explain why the vertical component of velocity v_H changes from 3.5 m/s at A to 0 m/s at B.

b) i) By considering only the vertical motion, show that it takes about 0.4 s for the jumper to reach maximum height at B after taking off from A. Take $g = 9.8$ m/s^2.

ii) Calculate the length in metres of the jump.

c) Long-jumpers can use this model to help them to improve their performance. Explain why the length of the jump can be increased by:

i) increasing the horizontal component of velocity v_H, keeping v_V the same.

ii) increasing the vertical component of velocity v_V, keeping v_H the same.

2. Define the term centre of mass.

3. Describe in as much detail as you can the motion of a projectile fired at an angle (include descriptions of the vertical and horizontal components of velocity).

4. A football is kicked at a velocity of 15 m/s at an angle of 25° to the horizontal. Calculate:

a) the total flight time

b) the maximum height

c) the range of the ball.

5. A car accidently rolls off a cliff. As it leaves the cliff it has a horizontal velocity of 13 m/s, it hits the ground 60 m from the shoreline. Calculate the height of the cliff.

6. A car of mass 1200 kg travels at a steady speed of 22 m/s around a bend of radius 50 m. Find the centripetal force required.

7. Find out what is meant by the centre of stability of a ship. Why is it important that the centre of mass of a ship is below the centre of stability?

8. A tennis ball machine fires a ball vertically upwards at time $t = 0$ at 19.6 m/s. Assume that air resistance is negligible and take $g = 9.8$ m/s^2.

a) Calculate the displacement and velocity of the ball at $t = 1.0$ s, 2.0 s, 3.0 s and 4.0 s.

b) Use your answers to part a) to draw

i) a graph of displacement against time for the ball

ii) a graph of velocity against time for the ball.

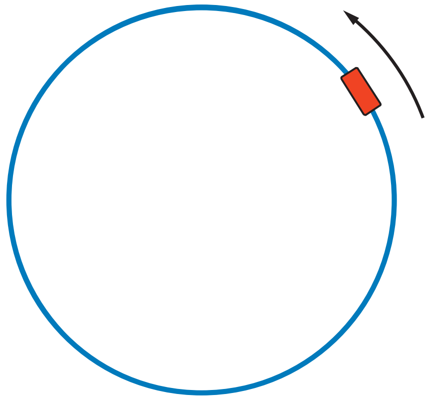


Figure 1.30 *Example of circular motion.*

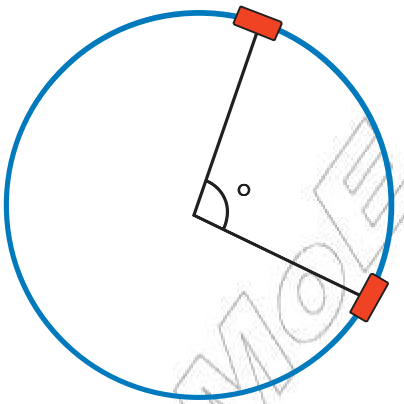


Figure 1.31 *Angular displacement.*

1.2 Rotational kinematics

By the end of this section you should be able to:

- Define the terms angular and tangential displacement, and angular and tangential velocity.
- Express angles in terms of revolutions, radians and degrees.
- Define the term angular acceleration, and list its key characteristics.
- Identify the SI unit of angular velocity and angular acceleration.
- Explain the relationships between angular displacement, tangential displacement, angular velocity, tangential velocity, angular acceleration and tangential velocity.
- Demonstrate how to use the equations of constant angular acceleration and compare them with equations of constant acceleration.

Rotational motion

Whenever objects travel in curved paths their motion can be considered to be rotational. A simple example might be a ball on a piece of string being swung around in a circular path. A more complex example might include the orbit of the planets around the Sun.

Up until this point, whenever we have used the terms displacement, velocity and acceleration this has always applied to a straight line. We now need to consider the motion of objects following curved paths. Let's take an example of an object travelling in a perfect circle.

As discussed earlier, this object is accelerating (it is changing direction, therefore changing velocity, therefore accelerating). However, what if the object was also getting faster? We need to be able to distinguish between its displacement, velocity and acceleration, and whether it is angular or tangential.

Displacement

In one complete revolution an object will travel a distance equal to $2\pi r$ (where r is the radius of the circle). Its tangential displacement will be zero as it is back where it started. If we consider part of this motion, between two points, we can see that the object has an angular displacement equal to θ. This is just the angle the object has subtended.

When describing angular displacement, there are several different units of angular measurement we could use. Perhaps the simplest would be revolutions. Half a circle would be 0.5 revolutions (or 0.5 rev), two complete loops would be 2 revolutions. However, this does not have much scientific merit. A more common unit is the degree.

DID YOU KNOW?

Having 360° in a circle is convenient because it divides easily into a whole a number. It divides by 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360!

Discussion activity

Why are there 360° in a circle? Discuss some possible reasons with a partner.

One revolution is 360°, half a circle would be 180°, and so on.

For scientific and mathematical calculations the **radian** is used. This has a clear mathematical basis and offers a number of advantages (especially when dealing with high level trigonometry). One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle.

This definition means that there must be 2π radians in a circle. This is because the circumference of a circle is given by $2\pi r$ and so the radius fits around a complete circle 2π times. This could be written as 6.28 rad, but it is usually just written as 2π rad. Half a circle would be π rad (or 3.14 rad).

One radian is equal to $\frac{180}{\pi}$ (or $\frac{360}{2\pi}$) degrees, or about 57.3°.

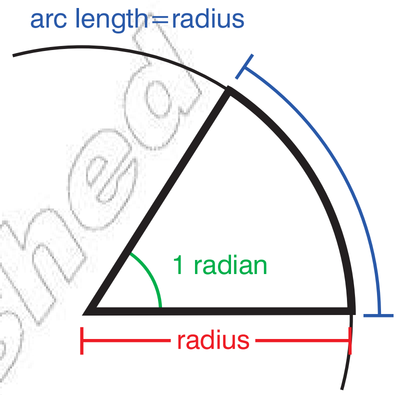


***Figure 1.32** The radian.*

***Table 1.2** Examples of angular measurements*

Revolutions	Degrees	Radians
$\frac{1}{8}$	45	$\frac{\pi}{4}$
$\frac{1}{4}$	90	$\frac{\pi}{2}$
$\frac{1}{2}$	180	π
$\frac{3}{4}$	270	$\frac{3\pi}{2}$
1	360	2π
2	720	4π

Activity 1.15: Calculating angular displacement

Thinking about an analogue clock, calculate the angular displacement in revolutions, radians and degrees for the following:

- A second hand after three minutes
- An hour hand after 20 minutes
- The minute hand as it moves from 9.15 to 12.45.

Velocity

Angular velocity is defined as the rate of change of the angle subtended. This is very similar to the definition of linear velocity (rate of change of displacement). The faster an object rotates, the greater the angle covered per unit of time. This leads us to the relationship:

$$\text{average angular velocity} = \frac{\text{angle covered}}{\text{time taken}}$$

In symbols:

$$\omega = \frac{\theta}{t}$$

ω is the Greek letter omega. As θ is measured in radians and t in seconds, the units of angular velocity are therefore rad/s.

In one complete revolution θ is equal to 2π radians and t becomes T (time for one complete cycle).

So for one complete cycle:

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T}$$

This allows us to work out angular velocities if we know the time taken for one cycle.

KEY WORDS

radian *the angle between two radii of a circle that cut off, on the circumference, an arc equal in length to the radius*

angular velocity *specifies the angular speed of an object and the axis about which the object is rotating*

Activity 1.16: Calulating angular velocity

Other units of angular velocity could be degrees per second or even rpm (revolutions per minute). What would an angular velocity of 3π rad/s be in degrees per second and rpm?

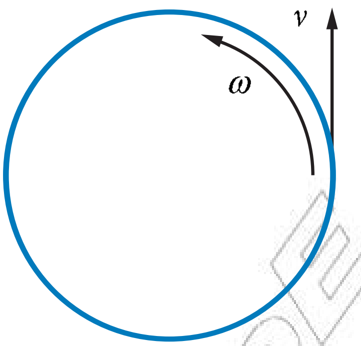


Figure 1.33 *Angular and tangential velocity.*

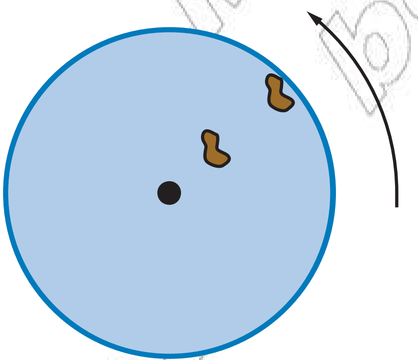


Figure 1.34 *Two stones on a spinning disk.*

KEY WORDS

acceleration *the rate of change of velocity as a function of time*

Worked example 1.10

The Earth takes approximately 365 days to orbit the Sun. Calculate its average angular velocity.

θ	*T*	*t*
2π	?	?

First we must find the time taken for one orbit in seconds

365 days = $365 \times 24 \times 60 \times 60 = 3.2 \times 10^7$ s

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T}$$

$$= \frac{2\pi}{3.2} \times 10^7$$

$$= 2.0 \times 10^{-7} \text{ rad/s}$$

As the object follows a circular path it also has a tangential velocity at any given point (see Figure 1.33).

Angular and tangential velocity may be linked using the equation below:

tangential velocity = radius of path × angular velocity

$v = r\omega$

Using the Earth as an example, its tangential velocity at any one point may be calculated:

$v = r\omega$

$v = 1.5 \times 10^{11} \times 2.0 \times 10^{-7}$

$v = 30\,000$ m/s

Thinking about this equation further leads us to the conclusion that for any given angular velocity the tangential velocity is proportional to the radius.

Imagine two stones on a spinning disk (see Figure 1.34).

Both stones have the same angular velocity as they both cover the same angle in the same time. However, the stone nearer the edge has a higher speed (as it covers a greater distance in the same time) and so a greater tangential velocity.

Another way to think about uniform circular motion is to define it as any motion where both the speed and the angular velocity are constant.

Acceleration

As discussed in the previous section, any object travelling in a circular path is accelerating. This acceleration is called centripetal **acceleration** and it acts towards the centre of the circle.

Centripetal acceleration is given by the equation:

$$\text{Centripetal acceleration} = \frac{\text{tangential velocity}^2}{\text{radius of curvature}}$$

$$a = \frac{v^2}{r}$$

Since $v = r\omega$ this equation could be written as:

$$a = \frac{r^2\omega^2}{r}$$

$$a = r\omega^2$$

Again using the Earth as an example, its centripetal acceleration may be found using the following technique:

$$a = r\omega^2$$

$$a = 1.5 \times 10^{11} \times (2.0 \times 10^{-7})^2$$

$$a = 6.0 \times 10^{-3}\ \text{m/s}^2$$

Activity 1.17: Calculations using planets

Using the data in the table below, calculate the angular velocity, tangential velocity and centripetal acceleration for each of the planets listed.

Planet	Orbital period (days)	Average distance from Sun (m)
Mercury	88	5.8×10^{10}
Venus	225	1.1×10^{11}
Mars	686	2.3×10^{11}
Jupiter	4330	7.8×10^{11}
Neptune	60 000	4.5×10^{12}

If the mass of an object remains constant, then applying Newton's second law of motion gives us:

$F = ma$.

For centripetal acceleration and so centripetal force this equation becomes $F = \frac{mv^2}{r}$.

In the examples we have looked at so far, the object moving in a circular path has been travelling at constant speed. What if the object is also getting faster as it rotates? An example might be a car getting faster as it goes around a bend. In this case its tangential velocity is increasing, and as a result it has a tangential acceleration. The magnitude of this acceleration is given by:

$$\textit{tangential acceleration} = \frac{\textit{change in tangential velocity}}{\textit{time taken}}$$

$$a_T = \frac{\Delta v}{t}$$

In most cases an increase in tangential velocity will lead to an increase of both tangential and centripetal acceleration.

If the radius remains constant, an increase in tangential velocity will also cause an increase in angular velocity. Changes in angular velocity are as common as changes in linear velocity. Just think about a CD spinning up or the wheels of a car as it accelerates.

DID YOU KNOW?

Tangential and angular acceleration are related in the same way as tangential velocity and angular velocity. Instead of $v = r\omega$ we get $a_T = r\alpha$.

Discussion activity

Under what circumstances is it possible to have a constant centripetal acceleration but also be accelerating tangentially?

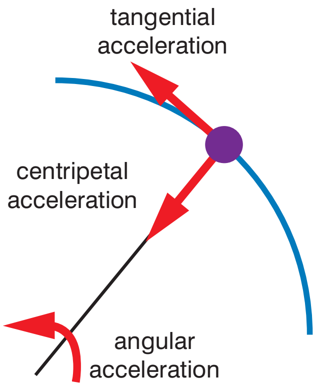


Figure 1.35 Different accelerations in rotational motion.

Angular acceleration is defined in a very similar way to linear acceleration:

$$\textit{angular acceleration} = \frac{\textit{change in angular velocity}}{\textit{time taken}}$$

$$\alpha = \frac{\Delta\omega}{t}$$

α is the Greek letter alpha. As ω is measured in rad/s and t in seconds, the units of angular acceleration are therefore rad/s^2.

This means that for an object travelling with rotational motion there may be three different types of acceleration.

Worked example 1.11

Find the final angular velocity when the initial angular velocity is 2 rad/s, the angular acceleration is 3 rad/s^2 and the time of motion is 4 s.

θ (rad/s)	ω_0 (rad/s)	α (rad/s^2)	t (s)
?	2	3	4

Use $\omega = \omega_0 + \alpha t$

Substitute in given values to get:

$\omega = 2 + 3 \times 4$

$= 14$ rad/s

The equations of constant angular acceleration

You saw on page 5 that there are equations that can be used when there is constant linear acceleration. There is an equivalent set of equations for constant angular acceleration.

Here are the two sets of equations side by side for comparison.

Constant linear acceleration	Constant angular acceleration
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = \frac{1}{2}(u + v)t$	$\theta = \frac{1}{2}(\omega_0 + \omega)t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0{}^2 + 2\alpha\theta$
$s = vt - \frac{1}{2}at^2$	$\theta = \omega t - \frac{1}{2}\alpha t^2$

ω = final angular velocity

ω_0 = initial angular velocity

α = angular acceleration

t = time

θ = angular distance

Worked example 1.12

Find the distance travelled when the final angular velocity is 14 rad/s, the angular acceleration is 3 rad/s^2 and the time of motion is 4 s.

θ (rad)	ω (rad/s)	t (s)	α (rad/s^2)
?	14	4	3

Use $\theta = \omega t - \frac{1}{2}\alpha t^2$

Substitute in given values to get:

$\theta = 14 \times 4 - \frac{1}{2} \times 3 \times 4^2 = 56 - 24 = 32$ rad

Worked example 1.13

Find the distance travelled when the initial angular velocity is 2 rad/s, the final angular velocity is 14 rad/s, and the time of motion is 4 s.

θ (rad)	ω_0 (rad/s)	ω (rad/s)	t (s)
?	2	14	4

Use $\theta = ½(\omega_0 + \omega)t$

Substitute in given values to get:

$\theta = ½(2 + 14) \times 4 = ½ \times 16 \times 4 = 32$ rad

Summary

- Angular displacement is the distance an object travels on a circular path and is often measured in radians. Tangential displacement is the distance an object moves in a straight line. Thus, an object that moves round a complete circle will have an angular displacement of 2π radians but a tangential displacement of 0 m (see Figure 1.36).
- Angular velocity is the angular displacement in a given unit of time. It is often measured in radians per second. Tangential velocity is the linear distance moved in a given unit of time and is often measured in metres per second.
- You can express angles in terms of revolutions, radians and degrees. For example, 1 revolution is the same as 2π radians or 360° (see Figure 1.37).
- Angular acceleration is the change in angular velocity per unit time. It is angular acceleration that contributes to the centripetal force which enables objects to move in circular paths. It is often measured in rad/s^2.
- Tangential displacement and angular displacement are related by the equation:

  tangential displacement = radius of path × angular displacement

- Tangential velocity and angular velocity are related by the equation

  tangential velocity = radius of path × angular velocity

- Angular acceleration and angular velocity are related by the equation:

  $$\text{angular acceleration} = \frac{\text{angular velocity}}{\text{time}}$$

- The equations of motion with constant angular acceleration are related to the equations of motion with constant linear acceleration as shown in the table on page 28.

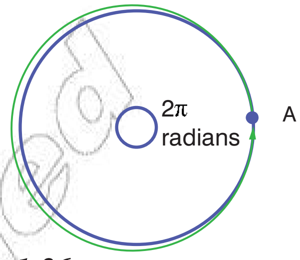


Figure 1.36

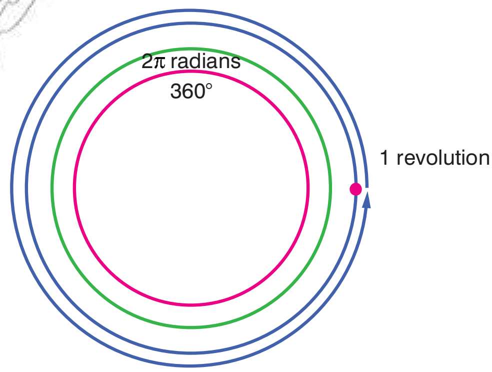


Figure 1.37

Review questions

1. a) Define angular displacement and gives its units.
   b) Define tangential displacement and give its units.
   c) Explain why an object that moves round a complete circle will have an angular displacement of 2π radians but a tangential displacement of 0 m.
2. a) Define angular velocity and give its units.
   b) Define tangential velocity and give its units.
   c) Find the tangential velocity of an object moving in a path of radius 2 m with an angular velocity of 3 rad/s.

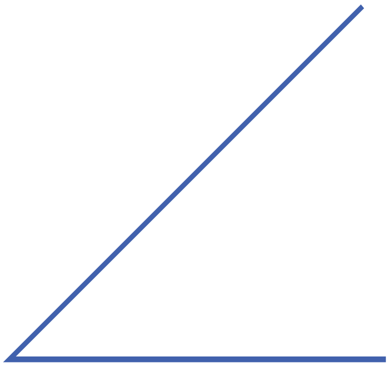


Figure 1.38

3. a) Give the three ways of expressing angles.
   b) Express the angle shown in Figure 1.38 in three different ways.
4. a) Define angular acceleration and give its units.
   b) How does angular acceleration enable objects to move in circular paths?
5. a) State the equations of motion with constant angular acceleration.
   b) Find the final angular velocity when the initial angular velocity is 5 rad/s, the angular acceleration is 2 rad/s and the time of motion is 10 s.
   c) Find the distance travelled when the final angular velocity is 20 rad/s, the angular acceleration is 2 rad/s and the time of motion is 5 s.
6. A satellite is in orbit 35600 km above the surface of the Earth. Its angular velocity is 7.27×10^{-5} rad/s. What is the velocity of the satellite? (The radius of the Earth is 6400 km.)
7. Astronauts in training are subjected to extreme acceleration forces by the centripetal forces in a giant centrifuge. The radius of the centrifuge is approximately 5 m.
   a) Calculate the approximate centripetal force on an astronaut of mass 80 kg if the centrifuge rotates once every 2 s.
   b) Approximately how many times greater than the astronauts weight is this force?

1.3 Rotational dynamics

By the end of this section you should be able to:

- Define the moment of inertia of a point mass.
- Define rotational kinetic energy of a body.
- Solve simple problems relating to moment of inertia and rotational kinetic energy.
- Define the term torque.
- Identify the SI unit of torque, which is N m, which is not the same as a joule.
- Express torque in terms of moment of inertia and angular acceleration.
- Derive an expression for the work done by the torque.

- Use the formula $W = \tau\theta$ to solve problems related to work done by torque.
- Define the angular momentum of a particle of mass m and write its SI unit.
- State the law of conservation of angular momentum.
- Solve problems using the law of conservation of angular momentum.
- State the first and second conditions of equilibrium.
- Solve problems related to conditions of equilibrium.
- Define the term centre of mass (centre of gravity) of a solid body.
- Determine the centre of gravity using a plumb-line method.
- Define the terms: stable, unstable and neutral equilibrium.

The moment of inertia of a point mass

The **moment of inertia** of a body is a measure of the manner in which the mass of that body is distributed in relation to the axis about which that body is rotating. It is dependent on the:

- mass of the body
- size of the body
- shape of the body
- which axis is being considered.

A spinning flywheel possesses **kinetic energy**, but how much? The expression KE = $\frac{1}{2}mv^2$ applies here, but the difficulty is that different parts of the flywheel are moving at different speeds – the regions further from the axis of rotation are going faster.

With linear motion, the kinetic energy is determined solely by the mass of the body and its speed. With the flywheel, the mass and the angular velocity are important, but there is now a third factor – how that mass is distributed in relation to the axis.

Consider two wheels each of mass M, but one is made in the form of a uniform disc whereas the other consists of a ring of the same radius R fixed to the axle by very light spokes rather like a bicycle wheel (Figure 1.39).

Both are spinning with the same angular velocity, ω. We can easily calculate the kinetic energy stored in the second flywheel because all its mass is at the rim.

You know that for a point mass going steadily in a circle of radius r at a linear speed v, that speed is related to its angular speed by:

$\omega = \frac{v}{r}$ so $v = r\omega$

KEY WORDS

moment of inertia *a measure of the distribution of the mass of a body in relation to its axis of rotation*

kinetic energy *the work needed to accelerate a body of a given mass from rest to its current velocity*

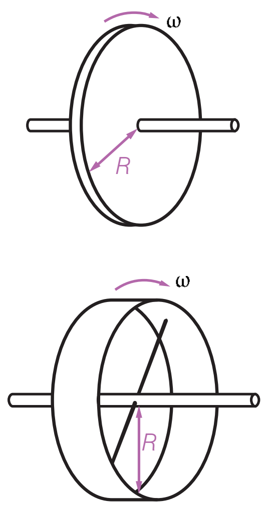


Figure 1.39 *Two wheels of the same mass but different construction.*

This flywheel has an angular speed ω, and since the whole of its mass is going in a circle of radius R it is all travelling at a linear speed $v = R\omega$. The kinetic energy, $\frac{1}{2}mv^2$, here will total $\frac{1}{2}MR^2\omega^2$.

What can we say about the first flywheel? Its total kinetic energy will be less, because most of its mass is moving at a speed slower than v. How can we go further than that?

A way forward is to think of the wheel as consisting of a large number of separate particles. There is no need to relate them to the individual atoms of the metal – we are just imagining it to be made up of a huge number of very small bits.

One of these bits, of mass m and distance r out from the axis, will have its share of the kinetic energy given by $\frac{1}{2}mv^2$ which can be expressed as $\frac{1}{2}mr^2\omega^2$ (since $v = r\omega$).

Activity 1.18: Rotational inertia

Place two different sized rolls of adding machine tape or other rolled paper on a dowel. Attach heavy clips to the rolls and hold them so that they cannot unwind. Release the rolls at the same time and note which unrolls most rapidly. Which roll has the greatest rotational inertia?

Rotational kinetic energy of a body

The total kinetic energy of the whole flywheel is just the sum of every particle in it. Those particles have different speeds v, but every one has the same angular velocity ω.

Adding all those kinetic energies, and denoting each particle with a subscript 1, 2, 3... and on for ever, we get:

Total kinetic energy $= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + ...$

We can rewrite this as:

$\frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2)\omega^2 + ...$

The expression in the bracket is the rotational equivalent of mass in the expression of linear kinetic energy ($\frac{1}{2}mv^2$), and we call it the body's moment of inertia, I. Its units will be kg m^2.

We can simplify it if we replace all those separate values of r^2 by the *average* value of r^2 for *all* the particles in that body. It then becomes the total mass M of the body (which can be measured with an ordinary balance) multiplied by the average value of r^2 for all the particles (which can be calculated by a geometrical exercise for various shapes of body).

The moment of inertia of a body is its rotational inertia. It is not a constant for that body, because it depends on the axis chosen for it to rotate around. It may be defined for a given body about a given axis as the sum of mr^2 for every particle in that body (where m is the mass of the particle and r is its distance from the axis.

For a rotating body, its kinetic energy (KE) = $\frac{1}{2} I \omega^2$

Worked example 1.14

Find the moment of inertia of a point mass of 0.001 kg at a perpendicular distance of 2 m from its axis of rotation.

I (kg m²)	M (kg)	R (m)
?	0.001	2

Note that the perpendicular distance from a given axis of rotation is specified. This is the value you need for R in the equation.

Use $I = MR^2$

Substitute given values: $I = 0.001 \times 2^2 = 0.004$ kg m²

Worked example 1.15

Find the moment of inertia of a disc of radius 0.25 m and mass 0.5 kg.

I (kg m²)	M (kg)	R (m)
?	0.5	0.25

Use $I = \frac{1}{2}MR^2$

Substitute in given values: $I = \frac{1}{2} \times 0.5 \times 0.25^2 = 0.015625$ kg m²

Worked example 1.16

Find the moment of inertia of a sphere of mass 0.5 kg and radius 0.15 m.

I (kg m²)	M (kg)	R (m)
?	0.5	0.55

Use $I = \frac{2}{5}MR^2$

Substitute in given values: $I = \frac{2}{5} \times 0.5 \times 0.15^2 = 0.0045$ kg m²

Worked example 1.17

Find the kinetic energy of a rotating body with moment of inertia 0.004 kg m² and angular velocity of 0.5 rad/s.

KE (J)	I (kg m²)	ω (rad/s)
?	0.004	0.55

Use $KE = \frac{1}{2}I\omega^2$

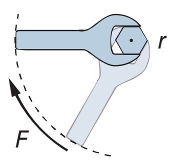
Substitute in given values: $KE = \frac{1}{2} \times 0.004 \times 0.5^2 = 0.0005$ J

Torque

A **torque** is a turning effect. It is the total moment acting on that body about the axis of rotation, and is measured by multiplying the force by its perpendicular distance from the axis.

torque = force × perpendicular distance

$$\tau = F \times r_{\text{perpendicular}}$$



The linear equivalent of a torque is a force.

The SI unit of torque is N m. This is not the same as a joule!

Activity 1.19: Comparing linear and rotational kinetic energy

Make sure that you understand why the expression for kinetic energy of a rotating body is $\frac{1}{2}I\omega^2$ by comparing it with the expression for kinetic energy of a body moving with linear motion.

Some moments of inertia

A point mass, M: $I = MR^2$

A disc of mass M and radius R (as with the first flywheel): $I = \frac{1}{2}MR^2$

A sphere of mass M and radius R rotating on its axis: $I = \frac{2}{5}MR^2$

KEY WORDS

torque *the tendency of a force to rotate an object about its axis of rotation*

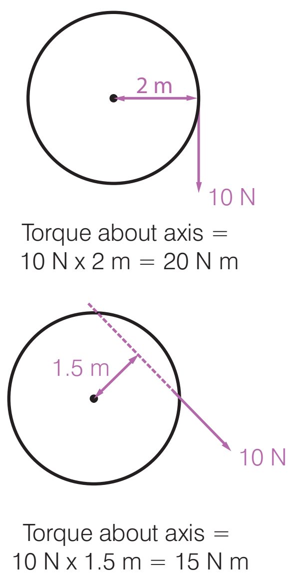


***Figure 1.40** The value of the torque depends on the distance from the axis.*

Activity 1.20: Distance in the torque definition

In a small group, discuss why distance is part of the definition of a torque.

Worked example 1.18

Find the work done when a torque of 5 N m moves through an angle of π radians.

W (J)	τ (N m)	θ (rad)
?	5	π

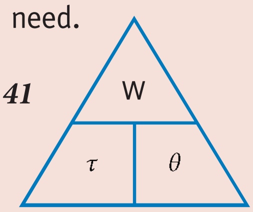
Use $W = \tau\,\theta$

Substitute given values

$W = 5\pi = 5\pi$ J

Remember that you can use this triangle to rearrange the formula. Cover up the term you are trying to find and the triangle will give you the form you need.

Figure 1.41



Worked example 1.19

Find the torque when the work done to move through an angle of π radians is 10 J.

W (J)	τ (N m)	θ (rad)
10	?	π

Use $W = \tau\,\theta$

Rearrange it so that torque is on the left-hand side.

$$\tau = \frac{W}{\theta}$$

Substitute given values

$$\tau = \frac{10}{\pi} \text{ N m}$$

Torque in terms of moment of inertia and angular acceleration

Since the linear equivalent of a torque, a force, can be found using Newton's second law:

force = mass × acceleration

you can see that, using the same law,

torque = moment of inertia × angular acceleration

If we give torque the symbol τ, then in symbols this equation is written as:

$\tau = I\alpha$

Work done by the torque

In linear terms the work done by a force is Fs, the force (F) multiplied by the distance (s) it moves in that direction. Likewise when a torque turns a body, the work it does = $\tau\,\theta$, the torque multiplied by the angle it turns through.

Angular momentum of a particle and its SI unit

The linear momentum of a body of mass m moving with a velocity v is defined to be mv. Likewise the **angular momentum** of a body of moment of inertia I rotating at an angular velocity ω is defined to be $I\omega$. Both linear momentum and angular momentum are vector quantities – they have magnitude and direction. Its units are N m s.

The law of conservation of angular momentum

The effect of an unbalanced force on a body is to cause its momentum to change. By Newton's second law of motion the momentum changes at a rate that is proportional to the magnitude of that force, and this leads to $F = ma$.

Similarly, the effect of an unbalanced torque on a body that can rotate is to cause its angular momentum to change, at a rate which is proportional to the magnitude of the torque. This leads to:

$\tau = I\alpha$ (the moment of inertia multiplied by the angular acceleration).

Just as linear momentum is conserved in the absence of a force, so is the angular momentum in the absence of a torque. The conservation of angular momentum says:

- if no resultant torque is acting, the angular momentum of a body cannot change.

This principle is demonstrated readily by a spinning skater, as shown in Figure 1.42. An ice skater holds her arms outstretched to either side and starts to spin round as fast as she can. She then folds her arms and her rate of rotation rises even further. By bringing more of her mass in closer to the axis of rotation, she has reduced her moment of inertia. Since no external torque is acting, $I\omega$ has to stay constant. Here I is made less, so ω has to increase.

Activity 1.21: Experience the conservation of angular momentum

If you have access to a chair that can swivel round freely, you can do this for yourself.

Hold a large mass in each hand and extend your arms fully to either side. Sit in the chair and get your friends to turn the chair round as fast as they can. Tell them to stop turning, and bring the masses close in to your body. Your moment of inertia has fallen so you should spin faster. Your angular momentum is conserved.

Your rotational kinetic energy is $\frac{1}{2} I\omega^2$. This can be written as $\frac{1}{2}(I\omega)\omega$. As you bring your arms in, the $(I\omega)$ term stays constant. You are spinning faster, though, so that extra factor of ω means you have gained in rotational kinetic energy.

Where has it come from? If you have tried this, you may well have felt the answer. As those two masses were pulled into a tighter circle, you had to provide the increasing centripetal force to achieve this. You had to exert a force, and had to move that force through a distance. In other words, you did some work.

Therefore you had to release some of the chemical energy in your food – and that is where the extra kinetic energy came from.

Figure 1.42 Spinning skater.

Worked example 1.20

No external force acts on a skater. Her moment of inertia is initially 60 kg m^2. Her angular velocity is 0.2 rad s^{-1} at the beginning of a spin. She brings in her arms and her angular velocity increases so that her moment of inertia decreases to 50 kg m^2. Find her final angular velocity.

initial angular momentum = final angular momentum

I (kg m^2)	ω (rad/s)
60	0.2

I (kg m^2)	ω (rad/s)
50	?

angular momentum at end of spin = angular momentum at beginning of spin

$60 \times 0.2 = 50 \times$ final angular velocity

final angular velocity $= \dfrac{60 \times 0.2}{50}$

0.24 rad/s

Activity 1.22: Applications of the principle of conservation of angular momentum

In a small group, make a list of applications of the principle of conservation of angular momentum. To start you off, think about other sports where spinning is required.

Activity 1.23: Comparing linear and rotational motion

In a small group, make a poster to compare linear and rotational motion. You should include all the information you have learnt in this unit, and rotational equivalents of Newton's first and second laws.

KEY WORDS

equilibrium *a stable situation in which forces cancel one another out*

This table summarises the relationships between linear and angular quantities.

Linear	Angular
$P = mv$	$L = I\omega$
$F = ma$	$\tau = I\alpha$
$F = \frac{\Delta p}{\Delta t}$	$\tau = \frac{\Delta L}{\Delta t}$
$\Delta p = 0$	$\Delta L = 0$
$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$

The conditions of equilibrium

There are two conditions that must be satisfied if a body is to be in equilibrium.

1 The forces acting on it must sum to zero.

2 The turning effects of the forces must sum to zero.

Worked example 1.21

Explain why these systems are in equilibrium.

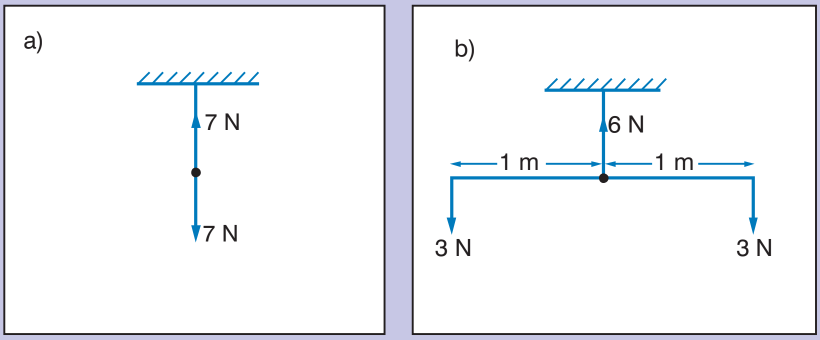


Figure 1.43

In a) there is no turning effect so condition 2 above is satisfied. The forces are 7 N upwards and 7 N downwards so the forces acting sum to zero, satisfying condition 1 above. So a) is in equilibrium.

In b) there would be a turning effect but one is 3 N m anticlockwise and the other is 3 N m clockwise to they sum to zero so condition 2 above is satisfied. The forces are 6 N upwards and a total of 6 N downwards so these sum to zero so condition 1 above is satisfied. So b) is in equilibrium.

The centre of mass (centre of gravity) of a solid body

The **centre of mass** of a solid body is the point at which the body's whole mass can be considered to be concentrated for the purpose of calculations. In a solid body, the position of its centre of mass is fixed in relation to the object (but not necessarily in contact with it). The centre of mass is often called the centre of gravity but this is only true in a system where the gravitational forces are uniform, such as on Earth.

Activity 1.24: Determine the centre of gravity using a plumb-line method

Repeat Activity 1.13 (see page 19) but this time, rather than using a piece of thick card, use an object made from modelling clay.

Stable, unstable and neutral equilibrium

An object is in **stable equilibrium** if, when it is slightly displaced, it returns to its original position.

An object is in **unstable equilibrium** if, when it is slightly displaced, it moves further away from its original position.

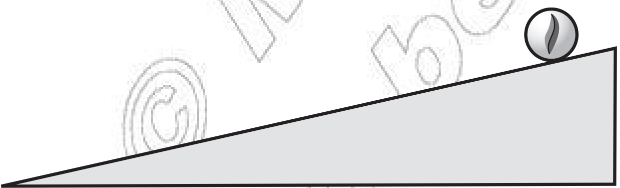


Figure 1.45 *If this marble is displaced it will run down the ramp and not return to its original position.*

An object is in **neutral equilibrium** if, when it is slightly displaced, the system does not necessarily return to its original position but neither does it move further away. For example, if you kick a football along the ground it will roll a little way and then stop at another spot. The kick changes its position but not its stability.

Figure 1.46 *This is neutral equilibrium.*

DID YOU KNOW?

Engineers try to design a sports car's centre of mass as low as possible to make the car handle better. When high jumpers perform a 'Fosbury Flop', they bend their body in such a way that it is possible for the jumper to clear the bar while his or her centre of mass does not.

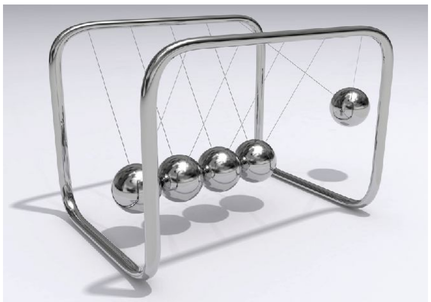


Figure 1.44 *These spheres will return to their original position when they are displaced.*

KEY WORDS

stable equilibrium *the tendency of an object, if it is displaced, to return to its original position*

unstable equilibrium *the tendency of an object, if it is displaced, to keep moving and not to return to its original position*

neutral equilibrium *the tendency of an object, if it is displaced, to neither return to its original position nor to move farther from it*

DID YOU KNOW?

The concept of centre of mass was first introduced by the ancient Greek mathematician, physicist, and engineer Archimedes. He showed that the torque exerted on a lever by weights resting at various points along the lever is the same as what it would be if all of the weights were moved to a single point — their centre of mass. In work on floating bodies he demonstrated that the orientation of a floating object is the one that makes its centre of mass as low as possible.

DID YOU KNOW?

The centre of mass on an aircraft significantly affects the stability of the aircraft. To ensure the aircraft is safe to fly, the centre of mass must fall within specified limits.

Summary

- The moment of inertia of a point mass, M, is $I = MR^2$.
- The moment of inertia for a disc of mass M and radius R is: $I = \frac{1}{2} MR^2$.
- The moment of inertia for a sphere of mass M and radius R rotating on its axis is: $I = \frac{2}{5} MR^2$.
- The rotational kinetic energy of a body is $\frac{1}{2}I\omega^2$.
- You can use the above formulae for simple problems relating to moment of inertia and rotational kinetic energy.
- A torque is a turning effect. It is the total moment acting on that body about the axis of rotation, and is measured by multiplying the force by its perpendicular distance from the axis. $\tau = F \times r_{perp}$. Its unit is N m, which is not the same as a joule.
- torque = moment of inertia × angular acceleration
- If we give torque the symbol τ, then in symbols this equation is written as:

  $\tau = I\alpha$
- When a torque turns a body the work it does = $\tau\,\theta$, the torque multiplied by the angle it turns. You can use this formula $W = \tau\theta$ to solve problems related to work done by torque.
- The angular momentum of a body of moment of inertia I rotating at an angular velocity ω is defined to be $I\omega$. Its units are N m s.
- The conservation of angular momentum says: if no resultant torque is acting, the angular momentum of a body cannot change.
- You can use this law to solve problems.
- There are two conditions that must be satisfied if a body is to be in equilibrium.

  1 The forces acting on it must sum to zero.

  2 The turning effects of the forces must sum to zero.
- You can solve problems using the conditions of equilibrium.
- The centre of mass of a solid body is the point at which the body's whole mass can be considered to be concentrated for the purpose of calculations. You can determine the centre of gravity using a plumb-line method.
- An object is in stable equilibrium if, when it is slightly displaced, it returns to its original position.
- An object is in unstable equilibrium if, when it is slightly displaced, it moves further away from its original position.

- An object is in neutral equilibrium if, when it is slightly displaced, the system does not necessarily return to its original position but neither does it move further away.

Review questions

1. Define the moment of inertia of a point mass.
2. Define rotational kinetic energy of a body.
3. a) Find the moment of inertia of a point mass of 0.005 g at a perpendicular distance of 3 m from its axis of rotation.
   b) Find the moment of inertia of a sphere of mass 0.3 kg and radius 0.6 m.
   c) Find the kinetic energy of a rotating body with moment of inertia 0.003 kg m^2 and angular velocity of 0.6 rad s^{-1}.
4. Define the term torque and identify its SI unit.
5. Express torque in terms of moment of inertia and angular acceleration.
6. Derive an expression for the work done by the torque.
7. a) Find the work done when a torque of 3.5 N m moves through an angle of $\frac{1}{2}\pi$ radians.
   b) Find the torque when the work done to move through an angle of $\frac{1}{4}\pi$ radians is 3 J.
8. Define the angular momentum of a particle of mass *m* and write its SI unit.
9. State the law of conservation of angular momentum.
10. Give examples of uses of the law of conservation of angular momentum.
11. State the first and second conditions of equilibrium.
12. Explain why the system in Figure 1.47 is in equilibrium?

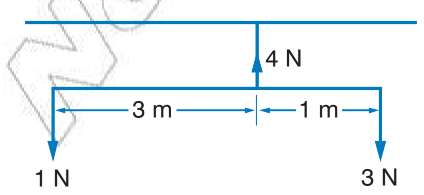


Figure 1.47

13. Define the term centre of mass (centre of gravity) of a solid body.
14. Explain how you can determine the centre of gravity using a plumb-line method.
15. Define the terms stable, unstable and neutral equilibrium.

1.4 Newton's law of universal gravitation

By the end of this section you should be able to:

- State Newton's law of universal gravitation.
- Determine the magnitude of the force of attraction between two masses separated by a distance *r*.
- Calculate the value of *g* at any distance above the surface of the Earth.
- State Kepler's laws of planetary motion.
- Use Kepler's laws of planetary motion to determine the period of any planet.
- Differentiate between the orbital and escape velocity of a satellite.
- Determine the period of a satellite around a planet.
- Calculate the orbital and escape velocity of a satellite.
- Describe the period, position and function of a geostationary satellite.

Newton's law of universal gravitation

To simplify matters we shall consider that the planets go round the Sun in circular orbits, which is nearly the case. There is no force opposing their motion, which is why they just keep going.

There has to be a force pulling them towards the Sun, though. Otherwise they would keep moving in a straight line. If the planet has a mass m, is travelling at a speed v and follows an orbit of radius r, the magnitude of the force straight towards the Sun has to be $\frac{mv^2}{r}$.

There is nothing but space between the planet and the Sun. What is producing a force that size? The answer is gravitation. This is a force of attraction that exists between any two lumps of matter, but which becomes noticeable only if at least one of them is of astronomical size.

Based on observations of the path of the Moon, Newton proposed a formula to describe how this force must behave. In doing so, he imagined that the laws of nature that applied to objects on Earth also applied to heavenly bodies. At the time this was a very daring idea.

The law has since been confirmed by incredibly sensitive experiments carried out between a pair of masses on Earth, and is known as Newton's universal law of gravitation.

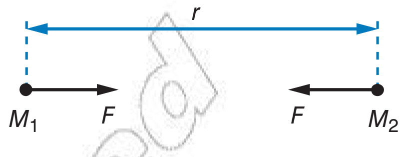


Figure 1.48 *A gravitational force attracting two masses.*

If two masses M_1 and M_2 are a distance r apart, Newton claimed that the gravitational force F between them was proportional to each of the masses, and decreased as they moved apart by an inverse square relationship – move three times as far apart and the force drops to one ninth, for example.

Putting this together, we get:

$$F = \frac{GM_1M_2}{r^2}$$

G is a constant for all matter everywhere, and is called the gravitational constant. You should be able to see that it will have units of N m^2 kg^{-2}.

Its value has been measured to be 6.67×10^{-11} N m^2 kg^{-2}. There is no need to try to remember that, but its small size does illustrate the weakness of the force.

Worked example 1.22

Find the gravitational force between the Earth and the Moon. The mass of the Moon is 7.35×10^{22} kg, the mass of the Earth is 5.98×10^{24} kg and the distance between them is approximately 3×10^6 km.

F (N)	G (N m^2 kg^{-2})	M_1 (kg)	M_2 (kg)	r (m)
?	6.67×10^{11}	7.35×10^{22}	5.98×10^{24}	3×10^9

Use $F = \frac{GM_1M_2}{r^2}$

Substitute in given values and value for G.

$$F = \frac{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \times 7.35 \times 10^{22} \text{ kg} \times 5.98 \times 10^{24} \text{ kg}}{(3 \times 10^9 \text{ km})^2}$$

$$F = \frac{2.93 \times 10^{37}}{9 \times 10^{18}} = 3.257 \times 10^{18} \text{ N}$$

Finding the value of g at any distance above the surface of the Earth

The value of g varies a little above the surface of the Earth. The law of universal gravitation tells us that the force on a body acted upon by Earth's gravity is given by:

$$F = G\frac{m_1m_2}{r^2} = \left(G\frac{m_1}{r^2}\right)m_2$$

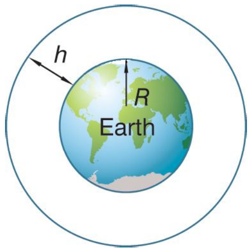


Figure 1.49 *To calculate* g *at a height* h *above the Earth, use* R + h *instead of* r *in the formula.*

where r is the distance between the centre of the Earth and the body, and m_1 is the mass of the Earth and m_2 is the mass of the body.

Newton's second law, $F = ma$, where m is mass and a is acceleration, tells us that:

$$F = m_2 g$$

Comparing the two formulae you can see that:

$$g = G\frac{m_1}{r^2}$$

To calculate the value of g at a distance h above the surface of the Earth, you need to substitute the value of the radius of the Earth R, + h, for r in the above formula (see Figure 1.49).

Worked example 1.23

Find the value of g at a distance of 2 km above the surface of the Earth.

g (m/s²)	G (N m² kg⁻²)	m_1 (kg)	r_2 (m)
?	6.67×10^{-11}	5.98×10^{24}	$6.4 \times 10^6 + 2000$

Use $g = \frac{G\,m_1}{r^2}$

$$g = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.4 \times 10^6 + 2 \times 10^3)^2}$$

$$g = \frac{3.98 \times 10^{14}}{(6\ 402\ 000)^2}$$

$$g = \frac{3.98 \times 10^{14}}{4.099 \times 10^{13}}$$

$$g = 9.71 \text{ m/s}^2$$

Kepler's laws of planetary motion

Kepler's laws are experimental laws that describe the rotation of satellites about their parent body. They apply to all the satellites around the Earth, but Kepler's data were collected in the late 1500s – before the telescope had been invented – by closely observing the paths of the planets as they went around the Sun.

Kepler began with many sheets of figures, and amazingly succeeded in spotting three unexpected patterns to them all. These are now known as **Kepler's laws**:

1 Each planet moves in a path called an ellipse, with the Sun at one focus.

KEY WORDS

Kepler's laws *describe the motion of a planet around the Sun*

2 The line that joins the Sun to the orbiting planet sweeps out equal areas in equal times.

3 The square of the time it takes the planet to go round the Sun (that is, the square of its year) is proportional to the cube of its average distance from the Sun.

This law can be written in equation form as

$$T^2 = \left[\frac{4\pi}{GM}\right] a^3$$

where T is the time it takes the planet to go round the Sun, G is the gravitational constant, M is the mass of the planet and a is the mean distance between the planet and the Sun.

We will look at the first two laws. Figure 1.50 illustrates what they are saying. The two shaded areas are equal.

The second law follows from the conservation of energy. As the planet gets closer to the Sun it speeds up, and as it climbs back to a more distant part of its orbit some of that kinetic energy is transferred into its extra gravitational potential energy.

The planets all have orbits round the Sun that are close to being circles. There are some natural satellites of the Sun that are very different, though – the comets. These are only visible to us as they pass quickly through the part of their orbit which is close to the Sun. Most of the time they are remote from the Sun, in the darkness of space and travelling more slowly.

The path of the Moon as it travels round the Earth is very nearly a circle, though there are times in its orbit when it is closer to us and so appears slightly larger in the sky.

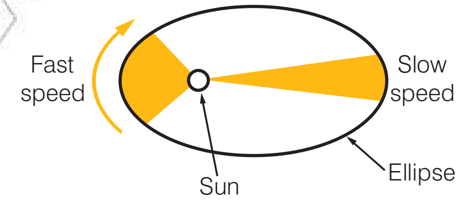


***Figure 1.50** Kepler's laws 1 and 2.*

Halley's Comet

We know that this has appeared to us in the sky every 75–76 years during recorded history, right back to the year 240 BC. It was last visible in 1986 and is next due in 2061. All the planets go round the Sun in the same direction. Halley's Comet goes the opposite way round, and its orbit is tilted at about 18° to that of the planets. This strongly suggests that it did not form when the planets formed, but passed by the Sun at a later time close enough to find itself captured by it.

The period, position and function of a geostationary satellite

The Earth nowadays has a large number of satellites in orbit round it. Most were made by humans, but the biggest one of them all is natural – the Moon.

Since the path of the Moon as it travels around the Earth is very nearly a circle, we can say that it has constant speed and is subject to a centripetal force

$$F = \frac{mv^2}{r}$$

where m is the mass of the Moon, v is its speed and r is its distance from the Earth.

If Earth has mass M, Newton's law of universal gravitation tells us that

$$F = \frac{GMm}{r^2}$$

So $\dfrac{mv^2}{r} = \dfrac{GMm}{r^2}$

$$v^2 = \frac{GMr}{r^2} = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

This equation can be applied to all satellites. This means that the smaller the distance from Earth, the greater the speed of the satellite.

Worked example 1.24

Find the speed of the Moon if the mass of the Earth is 6×10^{24} kg and the distance from the Moon to Earth is 4×10^{8} m.

v (m/s)	G (N m² kg⁻²)	M (kg)	r (m)
?	6.67×10^{-11}	6×10^{24}	4×10^{8}

Use $v = \sqrt{\frac{GM}{r}}$

$$v = \sqrt{\frac{6.67 \times 10^{11} \times 6 \times 10^{24}}{4 \times 10^{8}}}$$

$$v = \sqrt{\frac{4.002 \times 10^{36}}{4 \text{ x } 10^{8}}}$$

$$v = \sqrt{1.0005 \text{ x } 10^{28}}$$

$$v = 1.00025 \text{ x } 10^{14}$$

Geostationary satellites orbits are above the equator and they are going round with the same angular velocity as the Earth. This means that they have the same period as the Earth. They are always in the same spot in the sky and so they are ideal for communications purposes and for satellite navigation systems.

KEY WORDS

geostationary satellite *a satellite that orbits the equator with the same angular velocity as the Earth*

Orbital velocity of a satellite

Other satellites have orbits that are always moving relative to the Earth. Many of these are used to look down on the Earth beneath. Some are military spy satellites, but others observe the Earth – the weather over the whole globe, the temperatures of the oceans and so on. One is even used to provide a platform for a telescope to study the rest of the universe clearly without having to peer through the Earth's atmosphere.

The satellites are all launched from sites as close to the equator as possible, and certainly not from polar regions. The Earth is spinning on its axis: the nearer you are to the equator, the greater the distance you must travel to go round once in a day, and therefore the faster your speed is due to this motion. If you wanted to throw a ball at as great a speed as you could, propelling it forward from a fast-moving car would give you a good start. Satellites launched by humans similarly take advantage of the Earth's motion.

Activity 1.25: Verify Kepler's third law

On page 27, the orbital period of five planets is given. Use this data to verify Kepler's third law.

Activity 1.26: Use of geostationary satellites

Research some uses of geostationary satellites.

A satellite is not always sent into its final orbit in one step. Sometimes it will be launched so it goes into a temporary 'parking orbit'. From there a second carefully controlled rocket can be fired to lift it into its permanent orbit.

All satellites are in orbits that are distant from Earth. There is no mathematical reason why one should not orbit just above the Earth's surface, but the inconvenience of it is not the main reason why nobody does that. The satellites are deliberately positioned so that they are clear of the Earth's atmosphere – this means there is no resistance force to oppose the satellite's horizontal motion, otherwise an engine would be needed to cancel out such a force. Each satellite moves at a particular velocity in its orbit – this is its orbital velocity and is calculated by using the equation $\omega = \frac{2\pi}{T}$ where ω is its angular velocity and T is its period.

Activity 1.27: Calculate the orbital velocity of a satellite

Use $\omega = \frac{2\pi}{T}$ to calculate the orbital velocity of a geostationary satellite.

Escape velocity of a satellite

If a satellite is launched vertically upward at a sufficiently large velocity, it will be able to climb right out of the Earth's potential well and escape completely.

We can calculate this speed. Suppose the satellite has a mass m. At the Earth's surface the potential is $\frac{GM}{R}$ in J kg^{-1},

where M and R are the Earth's mass and radius. Thefore the satellite on the Earth's surface will have a potential energy of $\frac{GMm}{R}$ in joules.

When completely clear of the Earth, its potential energy will be zero. Therefore we must raise its potential energy by $\frac{GMm}{R}$ in joules. If the satellite is launched with a speed of v upwards, it will have an amount of kinetic energy given by $\frac{1}{2}mv^2$. If this kinetic energy is enough to supply what is needed, it can escape.

Thus the minimum escape velocity is given by:

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

whence $v = \sqrt{\frac{2GM}{R}}$

The expression for the escape speed can be made even simpler. If we substitute $M = \frac{gR^2}{G}$ in the above, we get:

escape velocity $v = \sqrt{(2gR)}$

Worked example 1.25

Find the escape velocity for a satellite leaving Earth. The radius of the Earth is 6.4×10^6 m and g is 9.81 N kg^{-1}

v (m/s)	g (N kg^{-1})	R (m)
?	9.81	6.4×10^6

Use $v = \sqrt{(2gR)}$

$v = \sqrt{(2 \times 9.81 \times 6.4 \times 10^6)}$

$v = \sqrt{(125\ 568\ 000)}$

$v = 11\ 205.71$ m/s

$v = 11$ km/s

Summary

- Newton's law of universal gravitation states that if two masses M_1 and M_2 are a distance r apart, the gravitational force F between them is proportional to each of the masses, and decreases as they move apart by an inverse square relationship:

$$F = \frac{GM_1M_2}{r^2}$$

- G is a constant for all matter everywhere, and is called the gravitational constant. You should be able to see that it will have units of N m^2 kg^{-2}.
- You can use Newton's law of universal gravitation to determine the magnitude of the force of attraction between two masses separated by a distance r.
- You can calculate the value of g at any distance above the surface of the Earth using:

$$g = G\frac{m_1}{r^2}$$

- To calculate the value of g at a distance h above the surface of the Earth, substitute the value of the radius of the Earth, $R + h$ for r in the above formula.
- Kepler's laws of planetary motion are:

  1. Each planet moves in a path called an ellipse, with the Sun at one focus.
  2. The line that joins the Sun to the orbiting planet sweeps out equal areas in equal times.
  3. The square of the time it takes the planet to go round the Sun (that is, the square of its year) is proportional to the cube of its average distance from the Sun.

- You can use Kepler's laws of planetary motion to determine the period of any planet.
- You can determine the period of a satellite around a planet using $T = \frac{2\pi}{\omega}$.
- Geostationary satellite orbits are above the equator and they are going round with the same angular velocity as the Earth. This means that they have the same period as the Earth. They are always in the same spot in the sky and so they are ideal for communications purposes and for satellite navigation systems.
- The orbital velocity of a satellite is the speed with which it goes round the Earth (or other planet). The escape velocity of a satellite is the speed it needs to escape the potential well of the Earth.

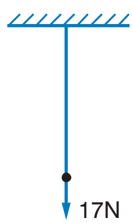
- You can calculate the orbital velocity of a satellite using $\omega = \frac{2\pi}{T}$ and the escape velocity of a satellite using $v = \sqrt{(2gR)}$.

Review questions

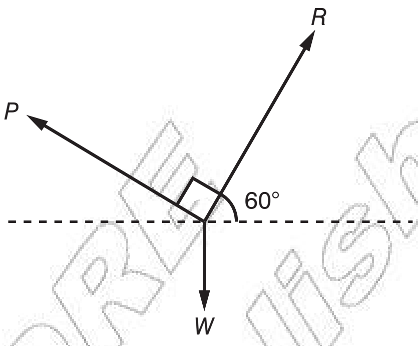
1. State Newton's law of universal gravitation.
2. Determine the magnitude of the force of attraction between Mercury and the Sun. They are approximately 5.8×10^{10} m apart. The mass of Mercury is about 3.3×10^{23} kg and the mass of the Sun is about 2×10^{30} kg. G is 6.67×10^{-11} N m^2 kg^{-2}.
3. Calculate the value of g at 1000 m above the surface of the Earth. The radius of the Earth is 6378.1 km.
4. State Kepler's laws of planetary motion.
5. Describe the period, position and function of a geostationary satellite.
6. Differentiate between orbital and escape velocity of a satellite.

End of unit questions

1. a) Define the term 'projectile'.
   b) Give some examples of projectiles.
2. a) Explain why the angle is important when launching projectiles.
   b) Find the range of a projectile launched at an angle of 45° with an initial velocity of 25 m/s.
   c) Explain why the vertical velocity of a projectile does not change.
3. a) State the centre of mass theorem.
   b) Give some practical applications of centre of mass.
4. Draw a table to compare the equations of motion with constant angular acceleration with the equations of motion with constant linear acceleration.
5. Find the distance travelled when the initial angular velocity is 3 rad/s, the final angular velocity is 25 rad/s , and the time of motion is 10 s.
6. Find the moment of inertia of a disc of radius 0.4 m and mass 0.75 kg.
7. Copy and complete this diagram so that the system is in equilibrium.



8. This system is in equilibrium. Write an expression linking *P*, *R* and *W*.



9. Determine the magnitude of the force of attraction between the Moon and the Earth. They are approximately 4×10^8 m apart. The mass of the Moon is 2×10^{24} kg and the mass of the Earth is 6×10^{24} kg.

10. Geostationary satellites are placed in orbits of radius 4.2×10^4 km. Use this information to deduce *g* at that height.

11. A climber of mass 80 kg is on a steep rock face. The force X that the rock exerts on the climber is at an angle of 50° to the vertical. Y, the other force on the climber, keeps him in equilibrium and is provided by a rope at an angle of 40° to the vertical.
    a) Draw a sketch to show the forces acting on the climber.
    b) From your sketch of the forces, sketch a triangle of forces to show equilibrium.
    c) Use your triangle of forces to find
        i) X, the force the rock face exerts on the climber
        ii) Y, the force provided by the rope.

12. Which of these answers is correct? Justify your answer.

    If the polar ice caps melt completely as a result of rising global temperatures, then
    a) the Earth will rotate faster
    b) the Earth will rotate slower
    c) there will be no change in the angular speed
    d) the duration of a day on the Earth will increase.