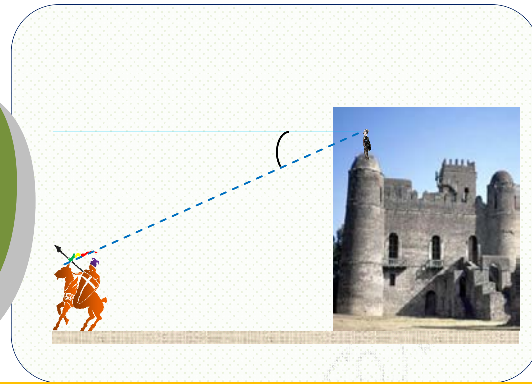
Unit 9



FURTHER ON TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about reciprocal functions.*
- *sketch graphs of some trigonometric functions.*
- *apply trigonometric functions to solve related problems.*

Main Contents:

INTRODUCTION

TRIGONOMETRY IS THE BRANCH OF MATHEMATICS THAT STUDIES THE RELATIONSHIP BETW AND SIDES OF A TRIANGLE. THE VALUES OF THE BASIC TRIGONOMETRIC FUNCTIONS ARE T THE LENGTHS OF THE SIDES OF RIGHT-ANGLED TRIANGLES.

ALTHOUGH "TRIGONOMETRY" ORIGINATED AS THE STUDY OF ANGLES AND LENGTHS IN TRIANGLES, IT HAS MUCH MORE WIDESPREAD APPLICATIONS.

ONE OF THE EARLIEST KNOWN USES OF TRIGONOMETRY IS AN EGYPTIAN TABLE THAT SH RELATIONSHIP BETWEEN THE TIME OF DAY AND THE LENGTH OF THE SHADOW CAST BY STICK THE EGYPTIANS KNEW THAT THIS SHADOW WAS LONGER IN THE MORNING, DECREAS MINIMUM AT NOON, AND INCREASED THEREAFTER UNTIL SUN-DOWN. THE RULE THAT GIVE DAY AS A FUNCTION OF SHADOW LENGTH IS A FORERUNNER OF THE TANGENT AND C FUNCTIONS (TRIGONOMETRIC FUNCTIONS) YOU STUDY IN THIS UNIT.

9.1 THE FUNCTIONS *y* = sec x, *y* = cosec x AND *y* = cot x

YOU HAVE LEARNT THAT THE THREE FUNDAMENTAL TRIGONOMETRIC FUNCTIONS OF THE A ARE DEFINED AS FOLLOWS.

Name of Function	Abbreviation	Value at
SINE	SIN	$\text{SIN} = \dfrac{opp}{hyp}$
COSINE	COS	$\text{COS} = \dfrac{adj}{hyp}$
TANGENT	TAN	$\text{TAN} = \dfrac{opp}{adj}$

CONSIDERING THE STANDARD RIGHT-ANGLED TRIANGLE AND LOOKING AT THE RATIOS THESE BASIC TRIGONOMETRIC FUNCTIONS REPRESENT IN RELATION TO ANGLE A, YOU OBTAIN:

$$\text{SIN}\,A = \frac{opposite}{hypotenuse} = \frac{a}{c}$$

$$\text{COS}\,A = \frac{adjacent}{hypotenuse} = \frac{b}{c}$$

$$\text{TAN}\,A = \frac{opposite}{adjacent} = \frac{a}{b}$$

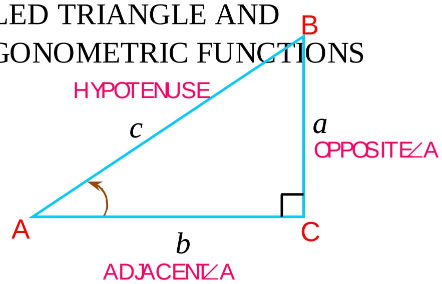
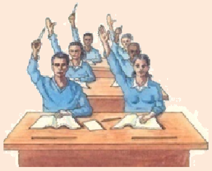


Figure 9.1

ACTIVITY 9.1



1 GIVEN THE TRIANGLE IN FIGURE 9.2 BELOW, FIND

A SIN A **B** SIN B **C** COS B **D** TAN B

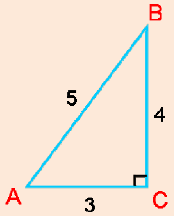


Figure 9.2

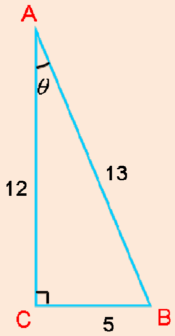


Figure 9.3

2 GIVEN THE TRIANGLE IN FIGURE 9.3 ABOVE, EVALUATE

A $\frac{1}{\text{SIN}}$ **B** $\frac{1}{\text{COS}}$ **C** $\frac{1}{\text{TAN}}$

THERE ARE ACTUALLY SIX TRIGONOMETRIC FUNCTIONS. THE RECIPROCALS OF THE RATIOS TH SINE, COSINE AND TANGENT FUNCTIONS ARE USED TO DEFINE THE REMAINING THREE TRIG FUNCTIONS. THESE RECIPROCAL FUNCTIONS OF AN ACUTE ANGLE ARE DEFINED AS FOLLOWS.

Name of Function	Abbreviation	Value at
COSECANT	CSC	$\text{CSC} = \frac{hyp}{opp}$
SECANT	SEC	$\text{SEC} = \frac{hyp}{adj}$
COTANGENT	COT	$\text{COT} = \frac{adj}{opp}$

THE RELATIONSHIP OF THESE TRIGONOMETRIC FUNCTIONS IN A STANDARD RIGHT ANGLED TRIANGLE IS SHOWN BELOW.

$$\text{CSC}\,A = \frac{c}{a} = \frac{1}{\text{SIN}\,A}$$

$$\text{SEC}\,A = \frac{c}{b} = \frac{1}{\text{COS}\,A}$$

$$\text{COT}\,A = \frac{b}{a} = \frac{1}{\text{TAN}\,A}$$

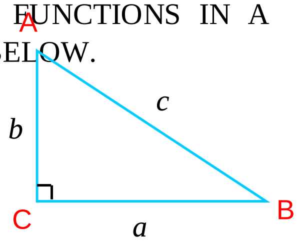


Figure 9.4

Example 1 GIVEN THE TRIANGLE BELOW, FIND:

A COT A **B** CSC B

C SEC A **D** CSC A

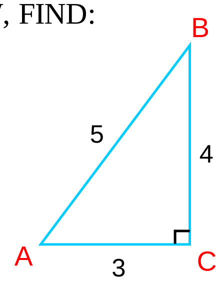


Figure 9.5

Solution

A $\cot A = \frac{3}{4}$ **B** $\csc B = \frac{5}{3}$

C $\sec A = \frac{5}{3}$ **D** $\csc A = \frac{5}{4}$

Graphs of $y = \csc x$, $y = \sec x$ and $y = \cot x$

IN GRADE 10, YOU STUDIED THE GRAPHS OF THE SINE, COSINE AND TANGENT FUNCTIONS. TOPIC YOU WILL STUDY THE GRAPHS OF THE REMAINING THREE TRIGONOMETRIC FUNCTION

Group Work 9.1

1 DETERMINE THE DOMAIN, RANGE AND PERIOD FO THREE TRIGONOMETRIC FUNCTIONS AND DRAW TH

A $y = \sin x$ **B** $y = \cos x$ **C** $y = \tan x$

2 BASED ON YOUR KNOWLEDGE OF TRIGONOMETRIC FUNCTIONS, FILL IN THE FOLLOWING

	0	$\frac{}{6}$	$\frac{}{4}$	$\frac{}{2}$		$\frac{3}{2}$	2
CSC							
SEC							
COT							

3 DETERMINE THE DOMAIN OF

A $y = \csc x$ **B** $y = \sec x$ **C** $y = \cot x$

4 YOU KNOW THAT $-1 \le \sin x \le 1$ FOR ALL $x \in \mathbb{R}$. IN SHORT, $|\sin x| \le 1$; WHAT CAN YOU SAY ABOUT $\frac{1}{|\sin x|}$?

5 YOU ALSO KNOW THAT $\frac{1}{\sin(x + 2)} = \frac{1}{\sin x} = \csc x$.

ARE CSC x, SEC x AND COT x PERIODIC? IF YOUR ANSWER IS YES, DETERMINE THEIR PERIODS.

6 DISCUSS THE SYMMETRIC PROPERTIES OF SECANT, COSECANT AND COTANGENT FUNCTI

FROM GROUP WORK 9.1, YOU SHOULD HAVE DETERMINED THE DOMAIN, RANGE AND PERIOD OF T COSECANT, SECANT AND COTANGENT FUNCTIONS AS FOLLOWS.

1 IF $f(x) = \text{CSC}\, x$, THEN $D_f = \{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

RANGE $= (-\infty, -1] \cup [1, \infty)$

PERIOD, $P = 2\pi$

2 IF $f(x) = \text{SEC}\, x$, THEN $D_f = \left\{x \in \mathbb{R} : x \neq \frac{(2k+1)\pi}{2}; k \in \mathbb{Z}\right\}$

RANGE $= (-\infty, -1] \cup [1, \infty)$

PERIOD, $P = 2\pi$

3 IF $f(x) = \text{COT}\, x$, THEN $D_f = \{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

RANGE $= \mathbb{R}$

PERIOD, $P = \pi$

YOU NOW WANT TO DRAW THE GRAPH OF

$f(x) = \text{CSC}\, x$

THE DOMAIN OF COSECANT FUNCTION IS RESTRICTED, IN ORDER TO HAVE NO DIVISION BY Z
TAKING THE RECIPROCALS OF NON-ZERO ORDINATES ON THE GRAPH OF THE SINE FUNCTION
IN FIGURE 9.6, YOU OBTAIN THE GRAPH OF $f(x) = \text{CSC}\, x$.

THE GRAPH OF COSECANT FUNCTION HAS VERTICAL ASYMPTOTES AT THE POINT WHERE TH
THE SINE FUNCTION CROSSES THE x-AXIS.

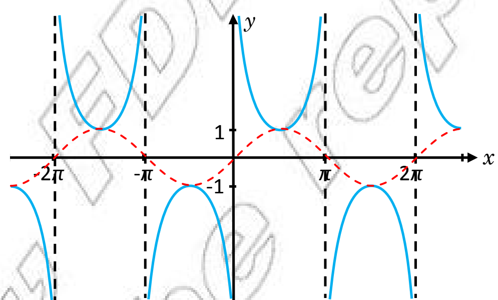


Figure 9.6 *Graph of y = csc x*

APPLYING THE SAME TECHNIQUES AS FOR THE COSECANT FUNCTION, WE CAN DRAW THE C
SECANT AND COTANGENT FUNCTIONS AS FOLLOWS.

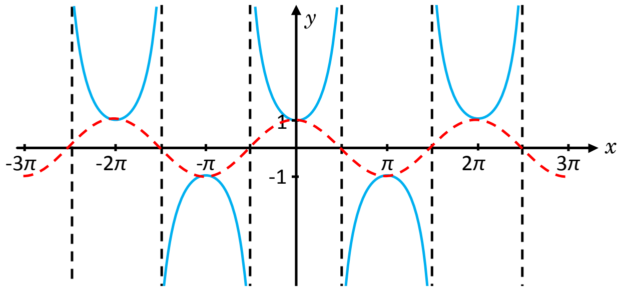


Figure 9.7 *Graph of y = sec x*

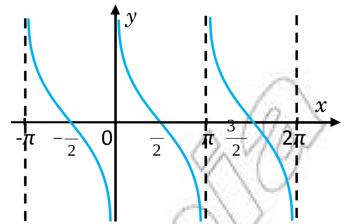


Figure 9.8 *Graph of y = cot x*

Exercise 9.1

1 DETERMINE EACH OF THE FOLLOWING VALUES WITHOUT THE USE OF TABLES OR CALCU

A $\text{SEC}\left(\frac{}{4}\right)$ **B** $\text{CSC}\left(-\frac{}{2}\right)$ **C** $\text{COT}\left(\frac{-3}{4}\right)$

D $\text{SEC}\left(\frac{}{3}\right)$ **E** $\text{CSC}\left(-\frac{}{6}\right)$ **F** $\text{COT}\left(\frac{5}{6}\right)$

G $\text{SEC}\left(\frac{2}{3}\right)$ **H** $\text{CSC}\left(\frac{7}{3}\right)$ **I** $\text{COT}\left(\frac{7\pi}{6}\right)$

J $\text{COT}(-\ )$ **K** $\text{SEC}\left(\frac{5}{2}\right)$ **L** $\text{CSC}\ (3\ )$

2 DETERMINE THE LARGEST INTERVAL $I \subseteq [0, 2\ ]$ ON WHICH

A $f(x) = \text{CSC}\, x$ IS INCREASING. **B** $f(x) = \text{SEC}\, x$ IS INCREASING.

C $f(x) = \text{COT}\, x$ IS INCREASING.

3 SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS.

A $\text{SEC}\, x\ \text{SIN}\, x$ **B** $\text{TAN}\, x\ \text{CSC}\, x$ **C** $1+\frac{\text{TAN}\, x}{\text{COS}\, x}$

D $\text{CSC}\left(x+\frac{}{2}\right)$ **E** $\text{SEC}\left(x-\frac{}{2}\right)$ **F** $\text{TAN}\left(x+\frac{}{2}\right)$

4 FIND THE RANGE OF $y = -3\text{SEC}\, x$.

5 PROVE EACH OF THE FOLLOWING TRIGONOMETRIC IDENTITIES.

A $\text{SEC}^2 x - \text{TAN}^2 x = 1$ **B** $\text{CSC}^2 x - \text{COT}^2 x = 1$

9.2 INVERSE OF TRIGONOMETRIC FUNCTIONS

YOU NOW NEED TO DEFINE INVERSES OF THE TRIGONOMETRIC FUNCTIONS, STARTING WIT REVIEW OF THE GENERAL CONCEPT OF INVERSE FUNCTIONS. YOU FIRST RESTATE A FEW FACTS ABOUT INVERSE FUNCTIONS.

Facts about inverse functions

FOR f A ONE-TO-ONE FUNCTION AND f^{-1} ITS INVERSE:

1 IF (a, b) IS AN ELEMENT OF f, THEN (b, a) IS AN ELEMENT OF f^{-1}, AND CONVERSELY.

2 RANGE OF f = DOMAIN OF f^{-1}

3 DOMAIN OF f = RANGE OF f^{-1}

THE GRAPH OF f^{-1} IS OBTAINED BY REFLECTING THE GRAPH OF f IN THE LINE $y = x$.

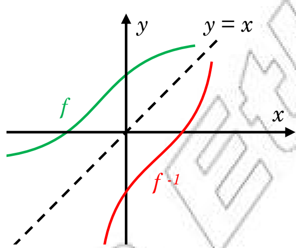
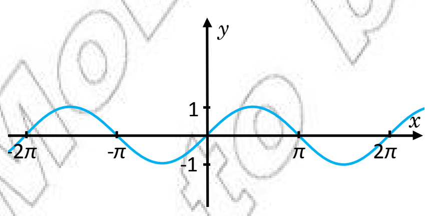
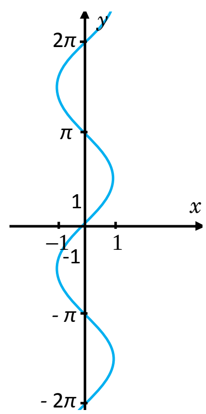


Figure 9.9

YOU KNOW THAT A FUNCTION f IS INVERTIBLE IF IT IS ONE -TO-ONE. ALL TRIGONOMETRIC FUNCTIONS A PERIODIC; HENCE, EACH RANGE VALUE CAN BE ASSOCIATED WITH INFINITELY MANY DOMAI AS A RESULT, NO TRIGONOMETRIC FUNCTION IS ONE-TO-ONE. SO WITHOUT RESTRICTING TH NO TRIGONOMETRIC FUNCTION HAS AN INVERSE FUNCTION AS SHOWN IN FIGURE 9.10 BELOW. TO RESOLVE THIS PROBLEM, YOU RESTRICT THE DOMAIN OF EACH FUNCTION SO THAT IT IS ONE- THE RESTRICTED DOMAIN. THUS, FOR THIS RESTRICTED DOMAIN, THE FUNCTION IS INVERTIBL



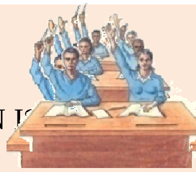
A *Graph of y = sin x*



B *Graph of y = sin^{-1} x on domain = [−1, 1]* and range (−∞, ∞)

Figure 9.10

INVERSE TRIGONOMETRIC FUNCTIONS ARE USED IN MANY APPLICATIONS AND MATHE DEVELOPMENTS AND THEY WILL BE PARTICULARLY USEFUL TO YOU WHEN YOU SOLVE TRI EQUATIONS.



ACTIVITY 9.2

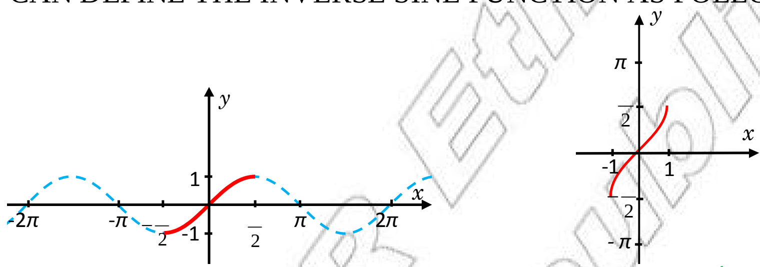
1 FIND SOME INTERVALS ON WHICH THE SINE FUNCTION IS ... NE.

2 DRAW THE GRAPH OF $f(x) = \sin x$ WHEN $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ AND REFLECT IT IN THE LINE $y = x$.

A Inverse sine function

FROM ACTIVITY 9.2, YOU SHOULD HAVE SEEN THAT THE SINE FUNCTION IS INVERTIBLE ON $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

NOW, YOU CAN DEFINE THE INVERSE SINE FUNCTION AS FOLLOWS.



A *Graph of y = sin x* **B** *Graph of $y = \sin^{-1} x$*

Figure 9.11

Definition 9.1 Inverse sine or Arcsine function

THE INVERSE SINE OR ARCSINE FUNCTION, DENOTED BY $\sin^{-1}$ OR arcsin, IS DEFINED BY

$\sin^{-1} x = y$ OR $\arcsin x = y$, IF AND ONLY IF $x = \sin y$ FOR $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

☑ Remark:

1 THE INVERSE SINE FUNCTION IS THE FUNCTION THAT ASSIGNS TO EACH NUMBER x IN $[-1, 1]$ THE UNIQUE NUMBER y IN $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ SUCH THAT $x = \sin y$.

2 DOMAIN OF $\sin^{-1} x$ IS $[-1, 1]$ AND RANGE OF $\sin^{-1} x$ IS $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3 FROM THE DEFINITION, YOU HAVE

$\sin(\sin^{-1} x) = x$ IF $-1 \le x \le 1$ $\sin^{-1}(\sin x) = x$ IF $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

☑ Caution:

$\text{SIN}^{-1}\,x$ IS DIFFERENT FROM $(\text{SIN}\,x)^{-1}$ AND $\text{SIN}\,x^{-1}$;

$(\text{SIN}\,x)^{-1} = \frac{1}{\text{SIN}\,x}$ AND $\text{SIN}\,x^{-1} = \text{SIN}\left(\frac{1}{x}\right)$

Example 1 CALCULATE $\text{SIN}^{-1}\,x$ FOR

A $x = 0$ **B** $x = 1$ **C** $x = \frac{\sqrt{3}}{2}$ **D** $x = -1$

Solution

A $\text{SIN}^{-1}(0) = 0$ SINCE SIN 0 = 0 AND $0 \in \left[-\frac{}{2}, \frac{}{2}\right]$

B $\text{SIN}^{-1}(1) = \frac{}{2}$ SINCE $\text{SIN}\left(\frac{}{2}\right) = 1$ AND $\frac{}{2} \in \left[-\frac{}{2}, \frac{}{2}\right]$

C $\text{SIN}^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{}{3}$ SINCE $\text{SIN}\,\frac{}{3} = \frac{\sqrt{3}}{2}$ AND $\frac{}{3} \in \left[-\frac{}{2}, \frac{}{2}\right]$

D $\text{SIN}^{-1}(-1) = -\frac{}{2}$ SINCE $\text{SIN}\left(-\frac{}{2}\right) = -1$ AND $-\frac{}{2} \in \left[-\frac{}{2}, \frac{}{2}\right]$

Example 2 COMPUTE $\text{COS}\left(\text{SIN}^{-1}\left(\frac{4}{7}\right)\right)$.

Solution LET $= \text{SIN}^{-1}\left(\frac{4}{7}\right)$. THEN, $\text{SIN} = \frac{4}{7}$ AND DRAWING THE REFERENCE TRIANGLE ASSOCIATED WITH, YOU HAVE:

$\text{COS} = \frac{\sqrt{33}}{7}$

WHERE $\sqrt{33}$ IS CALCULATED USING **Pythagoras' theorem**.

THEREFORE, $\text{COS}\left(\text{SIN}^{-1}\left(\frac{4}{7}\right)\right) = \text{COS} = \frac{\sqrt{33}}{7}$

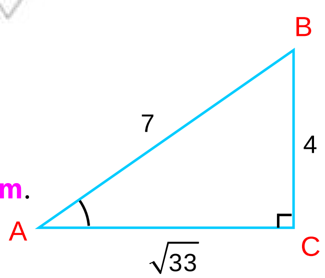


Figure 9.12

Calculator Tips

READ THE USER'S MANUAL FOR YOUR CALCULATOR AND FIND THE VALUE TO 4 SIGNIFICANT DIGITS FOR

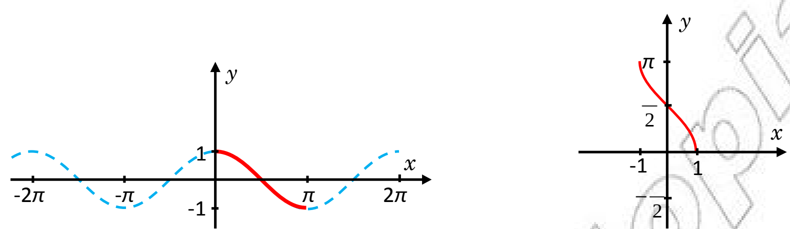
1 ARCSIN (0. 0215)

2 $\text{SIN}^{-1}(-0.137)$

3 $\text{TAN}(\text{SIN}^{-1}(0.9415))$

B Inverse cosine function

YOU KNOW THAT COS x IS NOT ONE-TO-ONE. NOTE, HOWEVER, THAT COS x DECREASES FROM 1 TO –1 IN THE INTERVAL [0,]. THUS IF y = COS x AND x IS RESTRICTED IN THE INTERVAL [0,], THEN FOR EVERY y IN [–1, 1], THERE IS A UNIQUE x SUCH THAT COS x = y.



A *Graph of y = cos x* **B** *Graph of* $y = \cos^{-1} x$

Figure 9.13

USE THIS RESTRICTED COSINE FUNCTION TO DEFINE THE INVERSE COSINE FUNCTION. REFLECTING THE GRAPH OF y = COS x ON [0,] IN THE LINE $y = x$, GIVES THE GRAPH OF $f(x) = \cos^{-1} x$ AS SHOWN IN FIGURES 9.13 AND 9.14.

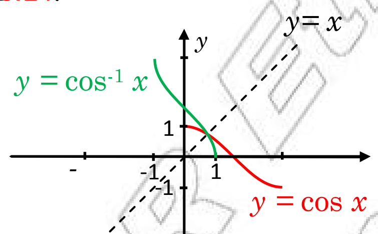


Figure 9.14

> **Definition 9.2**
>
> THE **inverse cosine** OR **arccosine** FUNCTION, DENOTED BY $\cos^{-1}$ OR ARCCOS, IS DEFINED BY
>
> $\cos^{-1} x = y$, IF AND ONLY IF $x = \cos y$, FOR $0 \le y \le$.

☑ Remark:

1 DOMAIN OF $\cos^{-1} x$ IS [–1, 1] AND RANGE OF $\cos^{-1} x$ IS [0,]

2 FROM THE DEFINITION, YOU HAVE

$\cos(\cos^{-1} x) = x$, IF $-1 \le x \le 1$.

$\cos^{-1}(\cos x) = x$, IF $0 \le x \le$.

Example 3 CALCULATE $y = \cos^{-1} x$ FOR

A $x = 0$ **B** $x = 1$ **C** $x = \frac{\sqrt{3}}{2}$ **D** $x = -1$

Solution

A $\cos^{-1}(0) = \frac{}{2}$ SINCE $\cos \frac{}{2} = 0$ AND $\frac{}{2} \in [0,]$

B $\cos^{-1}(1) = 0$ SINCE COS 0 = 1 AND 0 ∈ [0,]

C $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{}{6}$ SINCE $\cos\frac{}{6} = \frac{\sqrt{3}}{2}$ AND $\frac{}{6} \in [0,$

D $\cos^{-1}(-1) =$ SINCE COS = –1 AND ∈ [0,]

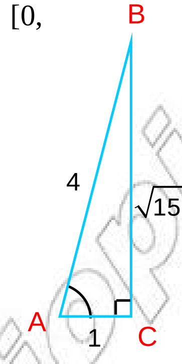


Figure 9.15

Example 4 COMPUTE $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$

Solution LET $= \cos^{-1}\left(\frac{1}{4}\right)$, SO THAT $\cos = \frac{1}{4}$.

THE OPPOSITE SIDE IS $\sqrt{4^2 - 1^2} = \sqrt{15}$

THUS, $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \tan = \frac{\sqrt{15}}{1} = \sqrt{\;}$

Example 5 SHOW THAT $\cos^{-1}(-x) = - \cos^{-1} x$

Solution LET $y = - \cos^{-1} x$ THEN, $\cos^{-1} x = - y$

$\Rightarrow x = \cos(- y) \Rightarrow x = - \cos y \Rightarrow -x = \cos y$

$\Rightarrow \cos^{-1}(-x) = y = - \cos^{-1} x$

Calculator Tips

FIND TO 4 SIGNIFICANT DIGITS

1. ARCCOS (0.5214)
2. $\cos^{-1}(-0.0103)$
3. SEC (ARCCOS (0.04235))

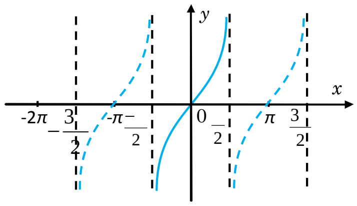
Example 6 COMPUTE $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = - \frac{}{4} = \frac{3}{4}$

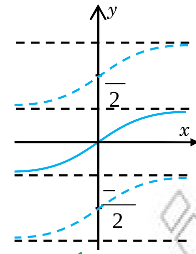
C Inverse tangent function

THE FUNCTION TAN x IS NOT ONE-TO-ONE ON ITS DOMAIN AS IT CAN BE SEEN FROM ITS GRAPH.

TO GET A UNIQUE x FOR A GIVEN y, YOU RESTRICT x TO THE INTERVAL $\left(-\frac{}{2}, \frac{}{2}\right)$.



A *Graph of $y = \tan x$*



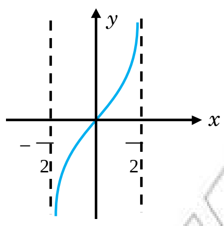
B *Graph of $y = \tan^{-1} x$ and*

range = $\{y \in \mathbb{R}: y \neq (2n-1)\frac{\pi}{2};\ n \in \mathbb{N}\}$.

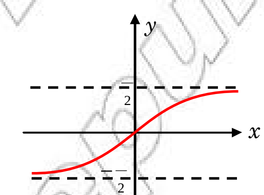
Figure 9.16

> **Definition 9.3**
>
> THE **inverse tangent function** IS A FUNCTION DENOTED BY $\tan^{-1} x$ OR $\arctan x$ THAT ASSIGNS TO EACH REAL NUMBER x THE UNIQUE NUMBER y IN $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ SUCH THAT $x = \tan y$.

REFLECTING THE GRAPH OF $y = \tan x$ ON THE LINE $y = x$ GIVES THE GRAPH OF $f(x) = \tan^{-1} x$ AS SHOWN IN THE FIGURES 9.16 AND 9.17.



A *Graph of $y = \tan x$*



B *Graph of $y = \tan^{-1} x$*

Figure 9.17

> ☑ **Remark:**
>
> **1** DOMAIN OF $\tan^{-1} x$ IS $(-\infty, \infty)$ AND RANGE OF $\tan^{-1} x$ IS $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
>
> YOU STRESS THAT $\frac{\pi}{2}$ IS NOT IN THE RANGE OF $\tan^{-1} x$ BECAUSE $\tan \frac{\pi}{2}$ IS NOT DEFINED.
>
> **2** FROM THE ABOVE DEFINITION, YOU HAVE,
>
> $\tan(\tan^{-1} x) = x$ FOR ALL REAL x
>
> $\tan^{-1}(\tan x) = x$, IF $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 7 COMPUTE (IN RADIANS).

A $\tan^{-1}(0)$ **B** $\tan^{-1}\left(\sqrt{3}\right)$ **C** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

SOLUTION

A $\tan^{-1}(0) = 0$ BECAUSE $\tan(0) = 0$ AND $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

B $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ BECAUSE $\tan\frac{\pi}{3} = \sqrt{3}$ AND $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

C $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ BECAUSE $\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ AND $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 8 EXPRESS $\tan(\sin^{-1} x)$ IN TERMS OF x.

Solution HERE, YOU CONSIDER THE FOLLOWING CASES.

I SUPPOSE $x = 0$, THEN $\tan(\sin^{-1}(0)) = \tan 0 = 0$.

II SUPPOSE $0 < x < 1$. LET $\theta = \sin^{-1} x$, THEN $\sin\theta = x$ AND $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

LOOK AT THE REFERENCE TRIANGLE GIVEN.

HENCE, $\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$

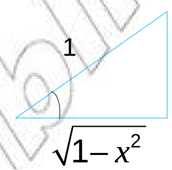


Figure 9.18

III IF $-1 < x < 0$, THEN $\tan(\sin^{-1} x) = -\tan(\sin^{-1}(-x))$

$\Rightarrow \tan(\sin^{-1} x) = -\frac{-x}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

$\therefore \tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$ FOR ALL $x \in \mathbb{R}$

Inverse cotangent, secant, and cosecant functions

HERE, THE DEFINITIONS OF THE INVERSE COTANGENT, SECANT AND COSECANT FUNCTIONS ARE GIVEN. WHEREAS DRAWING THE GRAPHS IS GIVEN AS EXERCISE.

Definition 9.4

I THE **inverse cotangent** FUNCTION $\cot^{-1}$ OR ARCCOT IS DEFINED BY

$y = \cot^{-1} x$, IF AND ONLY IF $x = \cot y$ WHERE $0 < y < \pi$ AND $-\infty < x < \infty$.

II THE **inverse secant function** $\sec^{-1} x$ OR ARCSEC x IS DEFINED BY

$y = \sec^{-1} x$, IF AND ONLY IF $x = \sec y$ WHERE $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$, $|x| \ge 1$.

III THE **inverse cosecant function** $\csc^{-1} x$ OR ARCCSC x IS DEFINED BY

$y = \csc^{-1} x$, IF AND ONLY IF $x = \csc y$ WHERE $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$, $|x| \ge 1$.

Example 9 FIND THE EXACT VALUES OF

A $\cot^{-1}\left(\sqrt{\ }\right)$ **B** $\sec^{-1}(2$ **C** $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution

A $y=\cot^{-1}\left(\sqrt{\ }\right) \Rightarrow \cot y=\sqrt{\ }3$ AND $0<y<$ $\Rightarrow y=\frac{}{6}$

B $\sec^{-1}(2)=\frac{}{3}$ BECAUSE $\sec\left(\frac{}{3}\right)=$ AND $0<\frac{}{3}<\frac{}{2}$

C $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{}{3}$

Exercise 9.2

1 FIND THE EXACT VALUES OF EACH OF THE FOLLOWING EXPRESSIONS WITHOUT USING A CALCULATOR OR TABLES.

A $\sin^{-1}\left(-\frac{1}{2}\right)$ **B** $\cos^{-1}(3$ **C** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

D $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ **E** $\sec^{-1}\left(\sqrt{\ }2\right)$ **F** $\cot^{-1}-($

G $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ **H** $\sin^{-1}\left(\sin\frac{}{4}\right)$ **I** $\sin^{-1}\left(\sin\frac{5}{4}\right)$

J $\arccos\left(\cos\frac{5}{6}\right)$ **K** $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ **L** $\tan\left(\tan^{-1}\sqrt{\ }\right)$

M $\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right)$ **N** $\cos^{-1}\left(\tan\left(\frac{-}{4}\right)\right)$

2 EXPRESS EACH OF THE FOLLOWING EXPRESSIONS IN TERMS OF x.

A $y=\sin(\arctan x)$ **B** $y=\cos(\arcsin x)$ **C** $y=\tan(\arccos x)$

3 PROVE EACH OF THE FOLLOWING IDENTITIES.

A $\tan^{-1}(-x)=-\tan^{-1}x$ **B** $\operatorname{arcsec} x=\arccos\left(\frac{1}{x}\right)$ FOR $|x|\geq 1$

C $\sec^{-1}x=\sin^{-1}\left(\frac{1}{x}\right)$ FOR $|x|\geq 1$

4 SKETCH THE GRAPH OF:

A $y=\operatorname{arccsc} x$ **B** $y=\operatorname{arcsec} x$ **C** $y=\operatorname{arccot} x$

5 LET $y=3+2\arcsin(5x-1)$. EXPRESS x IN TERMS OF y AND DETERMINE THE RANGE OF VALUES OF x AND y.

9.3 GRAPHS OF SOME TRIGONOMETRIC FUNCTIONS

IN THE PREVIOUS SECTION, THE GRAPHS OF y = SIN x AND y = COS x HAVE BEEN DISCUSSED. IN THIS SECTION, YOU WILL CONSIDER GRAPHS OF THE MORE GENERAL FORMS:

$$y = a\ \text{SIN}\ k(x + b) + c \text{ AND } y = a\ \text{COS}\ k(x + b) + c$$

THESE EQUATIONS ARE IMPORTANT IN BOTH MATHEMATICS AND RELATED FIELDS. THEY ARE ANALYSIS OF SOUND WAVES, x-RAYS, ELECTRIC CIRCUITS, VIBRATIONS, SPRING-MASS SYSTEMS, ETC.

Group Work 9.2

1 FOR THE FOLLOWING VALUES OF x, FILL IN A TABLE FOR THE G FUNCTIONS.

x	$\sin x$	$2 \sin x$	$\cos x$	$-3 \cos x$	$\frac{2}{3}\cos x$
0	0	0	1	–3	$\frac{2}{3}$
$\frac{\ }{6}$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{-3\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\ }{4}$					
⋮					
2					

COPY AND COMPLETE THE TABLE FOR

$$x = 0, \frac{\ }{6}, \frac{\ }{4}, \frac{\ }{3}, \frac{\ }{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \ , \frac{7}{6}, \frac{5}{4}, \frac{4}{3}, \frac{7}{4}, \frac{11}{6}, 2$$

2 USING THE ABOVE TABLE, SKETCH THE GRAPHS OF THE FOLLOWING PAIRS OF FUNCTION THE SAME COORDINATE AXES.

A y = SIN AND y = 2 SIN

B y = SIN AND $y = \frac{1}{2}$ SIN

C y = COS AND y = –3 COS

D y = COS AND $y = \frac{2}{3}$ COS

3 FOR EACH OF THE FOLLOWING FUNCTIONS, FIND THE PERIODS AND RANGES

A $y = 2 \sin x$ **B** $y = \frac{1}{2} \sin x$

C $y = -3 \cos x$ **D** $y = \frac{2}{3} \cos x$

4 LET $a \in \mathbb{R}$; EXPRESS THE RANGE OF $y = a \cos x$ IN TERMS OF a. HERE, $|a|$ IS SAID TO BE THE AMPLITUDE OF $a \cos x$. IN GENERAL, IF f IS A PERIODIC FUNCTION, THE AMPLITUDE OF f IS GIVEN BY

$$|a| = \frac{\text{Maximum value of } f - \text{Minimum value of } f}{2}.$$

FIND THE AMPLITUDES OF EACH OF THE FOLLOWING TRIGONOMETRIC FUNCTIONS.

A $f(x) = \sin x$ **B** $g(x) = -\cos x$ **C** $h(x) = 0.25 \sin x$

D $k(x) = 4 \tan x$ **E** $s(x) = -6 \cos x$ **F** $f(x) = |\sin x|$

FROM GROUP WORK 9.2 YOU SHOULD HAVE OBSERVED THAT THE GRAPH OF $y = a \sin x$ CAN BE OBTAINED FROM THE GRAPH OF $y = \sin x$ BY MULTIPLYING EACH y VALUE OF THE GRAPH OF $y = \sin x$ BY a.

- ✓ THE GRAPH OF $y = a \sin x$ STILL CROSSES THE x-AXIS WHERE THE GRAPH OF $y = \sin x$ CROSSES THE x-AXIS, BECAUSE $a \times 0 = 0$.
- ✓ SINCE THE MAXIMUM VALUE OF $\sin x$ IS 1, THE MAXIMUM VALUE OF $a \sin x$ IS $|a| \times 1 = |a|$. THE CONSTANT $|a|$, THE **amplitude** OF THE GRAPH OF $y = a \sin x$, INDICATES THE MAXIMUM DEVIATION OF THE GRAPH OF $y = a \sin x$ FROM THE x-AXIS.
- ✓ THE PERIOD OF $y = a \sin x$ IS ALSO 2, SINCE $a \sin(x + 2) = a \sin x$.

Example 1 DRAW THE GRAPHS OF $y = \sin x$, $y = \frac{1}{2} \sin x$ AND $y = -2 \sin x$, ON THE SAME COORDINATE SYSTEM FOR $0 \le x \le 2$.

Solution THE AMPLITUDES OF $y = \frac{1}{2} \sin x$ AND $y = -2 \sin x$ ARE $\frac{1}{2}$ AND 2, RESPECTIVELY, AND THE AMPLITUDE OF $y = \sin x$ IS 1. THE NEGATIVE SIGN IN $y = -2 \sin x$ REFLECTS THE GRAPH OF $y = 2 \sin x$ ACROSS THE x-AXIS. TOGETHER WITH THE RESULTS FROM GROUP WORK 9.2, THIS GIVES YOU THE GRAPHS OF ALL THE THREE FUNCTIONS AS SHOWN IN FIGURE 9.19A.

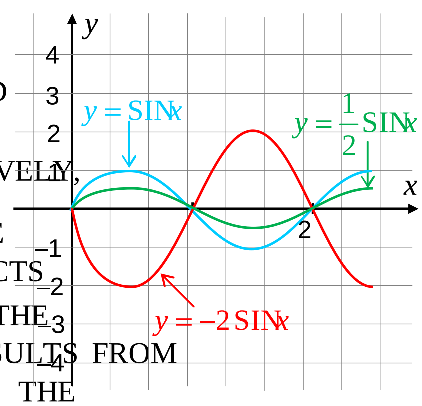


Figure 9.19 a

IN GENERAL, FOR ANY FUNCTION f, THE GRAPH OF af IS DRAWN BY EXPANDING OR COMPRESSING THE GRAPH OF f IN THE VERTICAL DIRECTION AND BY REFLECTING IN THE x-AXIS WHEN $a < 0$. FOR $a \neq \pm 1$, THE AMPLITUDE OF af IS DIFFERENT FROM THAT OF f, WHEREAS THE PERIOD DOESN'T CHANGE.

SIMILARLY, THE GRAPHS OF $y = \frac{1}{3}$ COS x, $y = -3$ COS x, $y =$ COS x, $0 \leq x \leq 2$ ARE AS SHOWN IN FIGURE 9.19B.

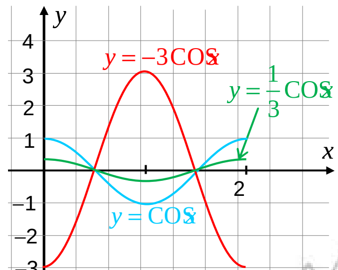


Figure 9.19b

9.3.1 The Graph of $f(x) = \sin kx$, $k > 0$

Group Work 9.3

1 FILL IN THE VALUES OF THE FOLLOWING FUNCTIONS FOR THE VALUES OF x GIVEN BELOW.

x	$2x$	$\frac{1}{2}x$	SIN (x)	SIN (2x)	SIN$\left(\frac{1}{2}x\right)$
0					
$\frac{}{4}$	$\frac{}{2}$	$\frac{}{8}$	$\frac{\sqrt{2}}{2}$	1	
$\frac{}{2}$					
⋮					
2					

COPY AND COMPLETE THE TABLE FOR

$$x = 0, \frac{}{4}, \frac{}{2}, \frac{3}{4}, \ , \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \ .$$

2 FIND THE MAXIMUM AND MINIMUM VALUES OF

A $f(x) =$ SIN (2x) **B** $g(x) =$ SIN$\left(\frac{1}{2}x\right)$

3 USING THE VALUES IN THE TABLE ABOVE, DRAW THE GRAPH OF

A $f(x) =$ SIN (2x) **B** $g(x) =$ SIN$\left(\frac{1}{2}x\right)$

FROM GROUP WORK 9.3, IT CAN BE OBSERVED THAT

✓ THE FUNCTION $f(x) = \text{SIN}(2x)$ COVERS ONE COMPLETE CYCLE ON THE INTERVAL [0,], BUT $g(x) = \text{SIN}\left(\frac{1}{2}x\right)$ COVERS EXACTLY HALF OF ONE CYCLE ON THE INTERVAL [0, 2].

✓ BOTH FUNCTIONS ARE PERIODIC AND THE SHAPE OF THEIR GRAPHS IS A SINE WAVE.

YOU CAN SKETCH THE GRAPHS OF $f(x) = \text{SIN}(2x)$ AND $g(x) = \text{SIN}\left(\frac{1}{2}x\right)$ BASED ON THESE PROPERTIES AND SOME OTHER STRATEGIC POINTS SUCH AS THE x- AND y-INTERCEPTS AND THE VALUES OF x WHICH GIVE MINIMUM VALUE OR MAXIMUM VALUE. THUS, FOR $0 \le x \le 2$,

- SIN ($2x$) = 0, $\Rightarrow$ $2x$ = 0, , 2 $\Rightarrow$ x = 0, $\frac{}{2}$,

$\Rightarrow$ THE GRAPH CROSSES THE x-AXIS AT (0, 0), $\left(\frac{}{2},\ 0\right)$ AND (, 0).

- SIN ($2x$) = 1 $\Rightarrow$ $2x = \frac{}{2}$ $\Rightarrow$ $x = \frac{}{4}$

$\Rightarrow$ THE FUNCTION ATTAINS ITS MAXIMUM VALUE AT $x = \frac{}{4}$

- SIN $2x$ = − 1 $\Rightarrow$ $2x = \frac{3}{2}$ $\Rightarrow$ $x = \frac{3}{4}$

$\Rightarrow$ THE FUNCTION ATTAINS ITS MINIMUM VALUE AT $x = \frac{3}{4}$

FROM ALL THESE, YOU HAVE THE FOLLOWING SKETCH OF THE CURVE OF $f(x) = \text{SIN}(2x)$ DRAWN TOGETHER WITH $y = \text{SIN}\,x$.

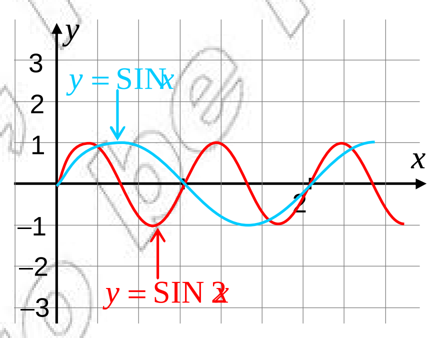


Figure 9.20

Note:

THE PERIOD OF $y = \text{SIN}(2x)$ IS . IT HAS TWO COMPLETE CYCLES ON [0, 2].

SIMILARLY, FOR $0 \le x \le 4$,

- $\sin\left(\frac{1}{2}x\right) = 0 \Rightarrow \frac{1}{2}x = 0, \ , 2$

$\Rightarrow x = 0, 2 \ , 4$

$\Rightarrow$ THE GRAPH OF $f(x) = \sin\left(\frac{1}{2}x\right)$ CROSSES THE x-AXIS AT (0, 0), (2 , 0) AND (4 , 0).

- $\sin\left(\frac{1}{2}x\right) = 1 \Rightarrow \frac{1}{2}x = \frac{}{2} \Rightarrow x = \ .$

$\Rightarrow$ THE GRAPH HAS A PEAK AT (, 1).

- $\sin\left(\frac{1}{2}x\right) = -1 \Rightarrow \frac{1}{2}x = \frac{3}{2} \Rightarrow x = 3$

$\Rightarrow$ THE GRAPH HAS A VALLEY AT (3 , –1).

BASED ON THE ABOVE FACTS, DRAW THE GRAPHS OF $f(x) = \sin\left(\frac{1}{2}x\right)$ AND $y = \sin x$ AS FOLLOWS

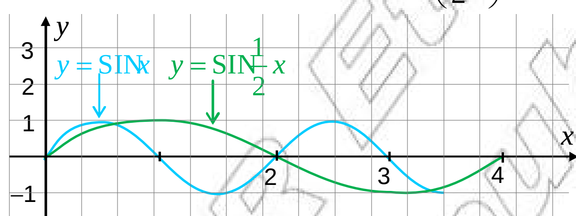


Figure 9.21

NOW INVESTIGATE THE EFFECT OF k BY COMPARING

$y = \sin x$ AND $y = \sin(kx)$, $k > 0$

WHERE BOTH HAVE THE SAME AMPLITUDE 1. SINCE $y = \sin x$ HAS PERIOD 2 , IT FOLLOWS THAT $y = \sin(kx)$ COMPLETES ONE CYCLE AS kx VARIES FROM $kx = 0$ TO $kx = 2$ OR AS x VARIES FROM $x = 0$ TO $x = \frac{2}{k}$.

THUS, THE PERIOD OF $y = \sin(kx)$ IS $\frac{2}{k}$.

A SIMILAR INVESTIGATION SHOWS THAT THE PERIOD OF $y = \cos(kx)$ IS $\frac{2}{k}$.

IF $k < 0$, REMOVE THE NEGATIVE SIGN FROM INSIDE THE FUNCTION BY USING THE IDENTITIES:

$\sin(-x) = -\sin x$ AND $\cos(-x) = \cos x$.

IN THE CASE $k \ne 0$, THE PERIOD OF $y = \sin(kx)$ AND $y = \cos(kx)$ IS $\frac{2}{|k|}$.

Graphs of $y = a \sin (kx)$ and $y = a \cos (kx)$

ALL THE ABOVE DISCUSSIONS MAY LEAD YOU TO THE FOLLOWING PROCEDURES OF DRAWING GRAPHS.

Procedures for drawing graphs

Step 1: DETERMINE THE PERIOD $P = \frac{2}{|k|}$ AND THE AMPLITUDE $|a|$.

Step 2: DIVIDE THE INTERVAL $[0, P]$ ALONG THE x-AXIS INTO FOUR EQUAL PARTS:

$$x = 0,\ \frac{P}{4},\ \frac{P}{2},\ \frac{3P}{4},\ P$$

Step 3: DRAW THE GRAPH OF THE POINTS CORRESPONDING TO $x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$.

x	0	$\frac{P}{4}$	$\frac{P}{2}$	$\frac{3P}{4}$	P
a SIN (kx)	0	a	0	$-a$	0
a COS (kx)	a	0	$-a$	0	a

Step 4: CONNECT THE POINTS FOUND IN STEP 3 BY A SINE WAVE.

Step 5: REPEAT THIS ONE CYCLE OF THE CURVE AS REQUIRED.

Example 2 DRAW THE GRAPH OF $y = 2$ SIN $(3x)$.

Solution

Step 1: THE PERIOD $P = \frac{2}{3}$ AND THE AMPLITUDE $|a| = 2$.

Step 2: THE CURVE COMPLETES ONE CYCLE ON THE INTERVAL $\left[0, \frac{2}{3}\right]$.

DIVIDE $\left[0, \frac{2}{3}\right]$ INTO FOUR EQUAL PARTS BY

$$x = 0,\ \frac{P}{4} = \frac{}{6},\ \frac{P}{2} = \frac{}{3},\ \frac{3P}{4} = \frac{}{2},\ P = \frac{2}{3}$$

Step 3:

x	0	$\frac{}{6}$	$\frac{}{3}$	$\frac{}{2}$	$\frac{2}{3}$
2 SIN $(3x)$	0	2	0	–2	0

Step 4: CONNECT THE POINTS FOUND IN STEP 3 BY A SINE WAVE.

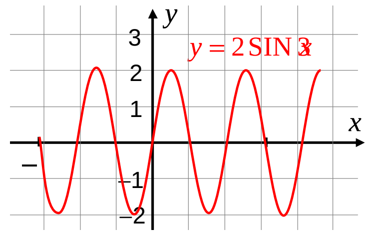


Figure 9.22

Example 3 DRAW THE GRAPH OF $f(x) = -3\cos\left(\frac{2}{3}x\right)$

SOLUTION

Step 1: PERIOD, $P = \frac{2}{\left(\frac{2}{3}\right)} = 3$ AND AMPLITUDE, $|a| = |-3| = 3$

Step 2: DIVIDE [0, 3] INTO FOUR EQUAL PARTS BY

$$x = 0,\ \frac{P}{4} = \frac{3}{4},\ \frac{P}{2} = \frac{3}{2},\ \frac{3P}{4} = \frac{9}{4},\ P = 3$$

Step 3:

x	0	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3
$-3\cos\left(\frac{2}{3}x\right)$	–3	0	3	0	–3

Step 4:

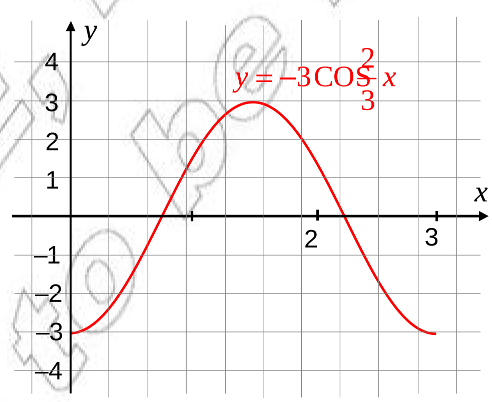


Figure 9.23

Exercise 9.3

1 DRAW THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = 4\,\text{SIN}\,x$ **B** $f(x) = -2\,\text{COS}\,x$

C $f(x) = \frac{2}{3}\,\text{SIN}\,x$ **D** $f(x) = \frac{1}{4}\,\text{COS}\,x$

2 DRAW THE GRAPH OF EACH OF THE FOLLOWING TRIGONOMETRIC FUNCTIONS FOR ONE CYCLE. INDICATE THE AMPLITUDE AND THE PERIOD.

A $f(x) = \text{SIN}\,(4x)$ **B** $f(x) = -2\,\text{SIN}\left(\frac{1}{3}x\right)$

C $f(x) = \frac{2}{3}\,\text{COS}\,(2x)$ **D** $f(x) = 5\,\text{SIN}\left(-\frac{2}{3}x\right)$

E $f(x) = 4\,\text{COS}\left(\frac{1}{4}x\right)$ **F** $f(x) = \frac{1}{2}\,\text{COS}\left(-\frac{3}{2}x\right)$

9.3.2 Graphs of $f(x) = a\sin(kx + b) + c$ and $f(x) = a\cos(kx + b) + c$

YOU HAVE ALREADY SKETCHED GRAPHS OF $f(x) = a\,\text{SIN}\,(kx)$ AND $f(x) = a\,\text{COS}\,(kx)$.

HERE YOU ARE INVESTIGATING THE GEOMETRIC EFFECT OF THE CONSTANTS b AND c IN DRAWING THE GRAPH OF THE FUNCTIONS.

CONSIDER THE FUNCTION $y = a\,\text{SIN}\,(kx + b) + c$

$$\Rightarrow\ y - c = a\,\text{SIN}\left(k\left(x + \frac{b}{k}\right)\right)$$

THIS IS SIMPLY THE FUNCTION $y = a\,\text{SIN}\,(kx)$ AFTER IT HAS BEEN SHIFTED $-\frac{b}{k}$ UNITS IN THE x-DIRECTION AND c UNITS IN THE y-DIRECTION.

IN PARTICULAR, IT IS SHIFTED TO THE POSITIVE x-DIRECTION IF $-\frac{b}{k} > 0$ AND TO THE NEGATIVE x-DIRECTION IF $-\frac{b}{k} < 0$. ALSO, IT IS SHIFTED TO THE POSITIVE y-DIRECTION IF $c > 0$ AND TO THE NEGATIVE y-DIRECTION IF $c < 0$. FOR EXAMPLE, IF YOU WANT TO DRAW THE GRAPH OF $y = 3\text{SIN}\left(2x - \frac{\ }{3}\right) - 2$, REWRITE THE EQUATION IN THE FORM

$$y + 2 = 3\,\text{SIN}\left(2\left(x - \frac{\ }{6}\right)\right)$$

THUS, THE GRAPH OF THIS FUNCTION IS OBTAINED BY SHIFTING THE GRAPH OF $y = 3\sin(2x)$ IN THE POSITIVE x-DIRECTION BY $\frac{}{6}$ UNITS AND 2 UNITS IN THE NEGATIVE y-DIRECTION AS SHOWN IN FIGURE 9.22

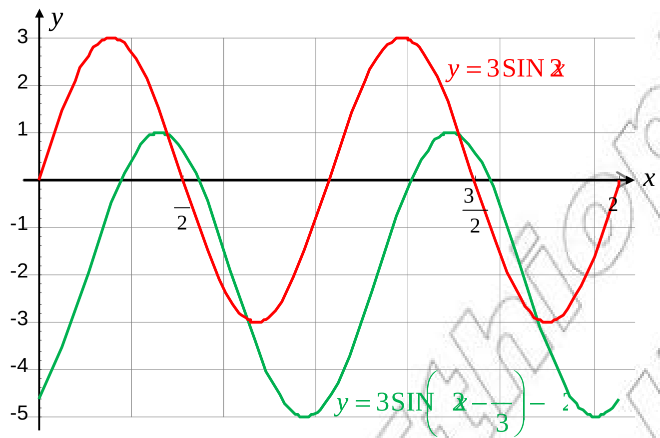
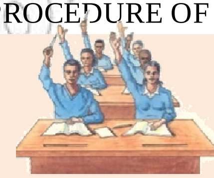


Figure 9.24

THE FOLLOWING ACTIVITY INTRODUCES A SIMPLIFIED PROCEDURE OF DRAWING GRAPHS.

ACTIVITY 9.3



1 IF $3x + 5$ VARIES FROM 0 TO 2, I.E., $0 \le 3x + 5 \le 2$, THEN,

$-5 \le 3x \le 2 \quad -5 \Rightarrow -\frac{5}{3} \le x \le \frac{2}{3} - \frac{5}{3}$; I.E. x VARIES FROM $-\frac{5}{3}$ TO $\frac{2}{3} - \frac{5}{3}$

BASED ON THIS EXAMPLE, FIND THE INTERVALS OVER WHICH x VARIES, IF EACH OF THE FOLLOWING EXPRESSIONS VARIES FROM 0 TO 2

A $2x + 1$ **B** $3x - 1$ **C** $2x - \frac{}{3}$ **D** $x + \frac{}{2}$

2 FIND THOSE VALUES OF x THAT DIVIDE THE GIVEN INTERVAL INTO FOUR EQUAL PARTS.

A $[0, 2\ ]$ **B** $\left[\frac{1}{4}, \frac{}{2} + \frac{1}{4}\right]$

3 FILL IN THE FOLLOWING TABLE

x	$\frac{1}{4}$	$\frac{1}{4}+\frac{}{8}$	$\frac{1}{4}+\frac{}{4}$	$\frac{1}{4}+\frac{3}{8}$	$\frac{1}{4}+\frac{}{2}$
3 sin (4x – 1)					
3 cos (4x – 1)					

FROM ACTIVITY 9.3, YOU HAVE THE FOLLOWING PROPERTIES.

1 IF $kx + b$ VARIES FROM 0 TO 2, I.E., $0 \le kx + b \le 2$, THEN,

$$-b \le kx \le -b + 2 \quad \Rightarrow \frac{-b}{k} \le x \le \frac{-b}{k} + \frac{2}{k} \quad (k > 0)$$

SO THAT x VARIES FROM $\frac{-b}{k}$ TO $\frac{-b}{k} + \frac{2}{k}$

THEREFORE, $f(x) = \text{SIN } (kx + b)$ GENERATES ONE CYCLE OF SINE WAVE AS $kx + b$ VARIES FROM 0 TO 2, OR AS x VARIES OVER THE INTERVAL $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2}{k}\right]$.

2 THE GRAPH "STARTS" AT $x = -\frac{b}{k}$ WHICH IS SAID TO BE THE PHASE SHIFT BECAUSE THE PHASE OF THE BASIC WAVE IS SHIFTED BY A FACTOR OF b.

Furthermore, you have the following procedures for drawing graphs:

ASSUME THAT $k > 0$. (*If $k < 0$, use the symmetric properties of sine and cosine*).

Step 1: DETERMINE THE PERIOD, $P = \frac{2}{k}$, THE AMPLITUDE $|a|$ AND PHASE SHIFT $= -\frac{b}{k}$

Step 2: DIVIDE THE INTERVAL $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2}{k}\right]$ ALONG THE x-AXIS INTO FOUR EQUAL PARTS.

THE LENGTH OF EACH INTERVAL IS $\frac{}{2k}$. WHY? EXPLAIN!

THE DIVIDING VALUES OF x ARE:

$$x = \frac{-b}{k},\ x = \frac{-b}{k} + \frac{}{2k},\ x = -\frac{b}{k} + \frac{}{k},\ x = \frac{-b}{k} + \frac{3}{2k} \text{ AND } x = \frac{-b}{k} + \frac{2}{k}$$

Step 3: DRAW THE GRAPH OF THE POINTS CORRESPONDING TO THE VALUES OF x IN THE TABLE

x	$-\frac{b}{k}$	$\frac{-b}{k} + \frac{}{2k}$	$\frac{-b}{k} + \frac{}{k}$	$\frac{-b}{k} + \frac{3}{2k}$	$\frac{-b}{k} + \frac{2}{k}$
$a \text{ SIN } (kx + b)$	0	a	0	$-a$	0
$a \text{ COS } (kx + b)$	a	0	$-a$	0	a

Step 4: CONNECT THE POINTS FOUND IN STEP 3 BY A SINE WAVE.

Step 5: REPEAT THIS PORTION OF THE GRAPH INDEFINITELY TO THE LEFT AND TO THE RIGHT EVERY $\frac{2}{k}$ UNITS ON THE x-AXIS.

Example 4 DRAW THE GRAPH OF $f(x) = 3\,\text{SIN}\left(\frac{1}{2}x - \frac{}{3}\right) + 1$.

Solution FIRST DRAW THE GRAPH OF $y = 3\,\text{SIN}\left(\frac{1}{2}x - \frac{}{3}\right)$ AND THEN SHIFT IT IN THE POSITIVE y-DIRECTION BY 1 UNIT.

Step 1: THE PERIOD, $P = \frac{2}{\left(\frac{1}{2}\right)} = 4$

AMPLITUDE, $|a| = 3$

PHASE SHIFT, $\frac{-b}{k} = \frac{\frac{}{3}}{\frac{1}{2}} = \frac{2}{3}$

Step 2: $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2}{k}\right] = \left[\frac{2}{3}, \frac{2}{3} + 4\right] = \left[\frac{2}{3}, \frac{14}{3}\right]$.

THE GRAPH COMPLETES FULL CYCLE ON $\left[\frac{2}{3}, \frac{14}{3}\right]$

DIVIDE $\left[\frac{2}{3}, \frac{14}{3}\right]$ INTO FOUR EQUAL PARTS BY $\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \frac{14}{3}$

Step 3:

x	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{8}{3}$	$\frac{11}{3}$	$\frac{14}{3}$
$3\,\text{SIN}\left(\frac{1}{2}x - \frac{}{3}\right)$	0	3	0	–3	0

Step 4, 5:

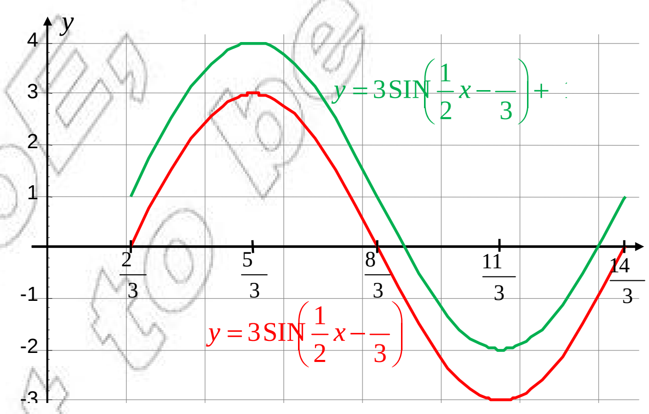


Figure 9.25

Example 5 DRAW THE GRAPH OF $f(x) = -5\cos(3x + 2) - 2$

Solution FIRST DRAW THE GRAPH OF $y = -5\cos(3x + 2)$ AND THEN SHIFT IT IN THE NEGATIVE y-DIRECTION BY 2 UNITS.

Step 1: PERIOD, $P = \frac{2}{3}$, AMPLITUDE, $|a| = |-5| = 5$.

PHASE SHIFT $= \frac{-2}{3}$, PHASE ANGLE = –2

Step 2: DIVIDE THE INTERVAL $\left[\frac{-2}{3}, \frac{2}{3} - \frac{2}{3}\right]$ INTO FOUR EQUAL INTERVALS OF LENGTH $\frac{}{6}$

Step 3:

x	$-\frac{2}{3}$	$\frac{-2}{3}+\frac{}{6}$	$\frac{-2}{3}+\frac{}{3}$	$\frac{-2}{3}+\frac{}{2}$	$-\frac{2}{3}+\frac{2}{3}$
–5 COS (3x + 2)	–5	0	5	0	–5
–5 COS (3x + 2) –2	–7	–2	3	–2	–7

Step 4, 5

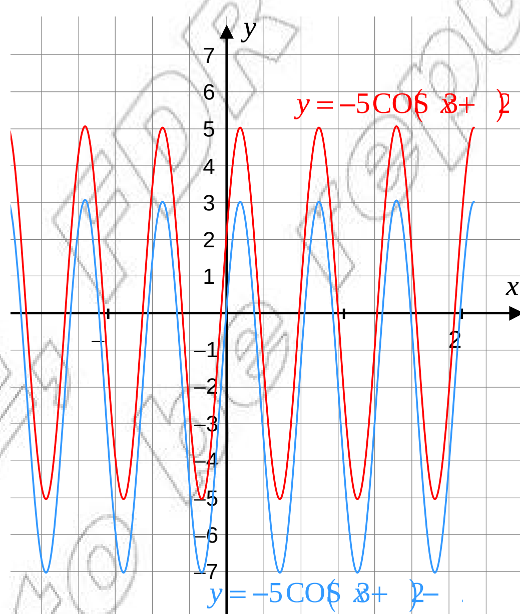


Figure 9.26

Example 6 GRAPH $f(x) = \frac{1}{2}\cos\left(\frac{}{2}x + \frac{}{2}\right)$ FOR ONE CYCLE.

Solution AS $\frac{}{2}x+\frac{}{2}$ VARIES FROM 0 TO 2, x VARIES FROM –1 TO 3.

THE GRAPH COMPLETES ONE FULL CYCLE ON THE INTERVAL [–1, 3].

x = –1, 0, 1, 2, 3 DIVIDES [–1, 3] INTO FOUR EQUAL PARTS.

USING THE FOLLOWING TABLE, SKETCH THE GRAPH FOR ONE CYCLE.

x	–1	0	1	2	3
$\frac{1}{2}\cos\left(\frac{}{2}x+\frac{}{2}\right)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

Step 4, 5

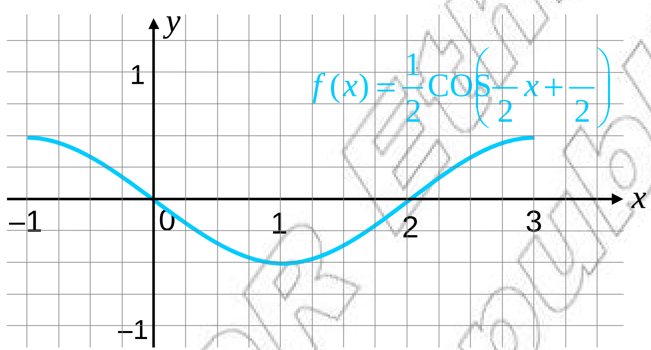


Figure 9.27

Exercise 9.4

DRAW THE GRAPHS OF EACH OF THE FOLLOWING TRIGONOMETRIC FUNCTIONS FOR ON INDICATE THE AMPLITUDE, PERIOD, AND PHASE SHIFT.

1 $f(x)=-\frac{1}{2}\sin(2x-1)$

2 $f(x)=\frac{1}{2}\cos(3x+2)$

3 $f(x)=3\sin\left(\frac{1}{2}x+3\right)-2$

4 $f(x)=\sin(x)+3$

5 $f(x)=2\cos(2x-)$

6 $f(x)=3-2\cos\left(\frac{x}{2}\right)$

7 $f(x)=-\frac{3}{2}\sin\left(3x+\frac{3}{4}\right)$

8 $f(x)=2-\frac{1}{2}\cos\left(\frac{3}{2}x+\frac{}{4}\right)$

9.3.3 Applications of Graphs in Solving Trigonometric Equations

General solutions of trigonometric equations

IF YOU DRAW THE GRAPH OF $y = \text{SIN } x$ AND THE LINE $y = \frac{1}{2}$ IN THE SAME COORDINATE SYSTEM AND FOR $0 \le x \le 2$, THEY MEET AT TWO PARTICULAR POINTS, $x = \frac{\ }{6}$ AND $x = \frac{5}{6}$.

BUT YOU KNOW THAT THE LINE $y = \frac{1}{2}$ CROSSES THE GRAPH OF $f(x) = \text{SIN } x$ INFINITELY MANY TIMES AS SHOWN IN THE FIGURE BELOW.

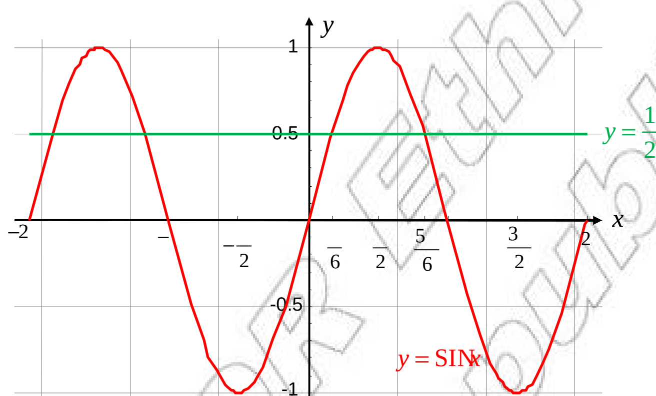
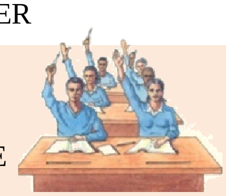


Figure 9.28

IN THIS SECTION, YOU WILL DETERMINE ALL THOSE INFINITE POINTS IN TERMS OF THE PA
POINTS, THE PERIOD OF THE SINE FUNCTION AND AN INTEGER n.

ACTIVITY 9.4



1 DRAW THE GRAPHS OF $f(x) = \text{TAN } x$ AND THE LINE $y = 1$ USING THE SAME COORDINATE SYSTEM. USING THE GRAPHS

A DETERMINE THE PARTICULAR SOLUTION IN THE RANGE $-\frac{\ }{2} < x < \frac{\ }{2}$ THAT SATISFIES THE EQUATION TAN $x = 1$.

B FIND THE GENERAL SOLUTION OF THE EQUATION TAN $x = 1$.

C IF x_1 IS A PARTICULAR SOLUTION OF THE EQUATION TAN $x = 1$ IN THE RANGE $-\frac{\ }{2} < x < \frac{\ }{2}$, DETERMINE THE GENERAL SOLUTION IN TERMS OF x_1 AND n.

2 DRAW THE GRAPHS OF $y = \frac{1}{2}$ AND y = COS x USING THE SAME COORDINATE SYSTEM.

DETERMINE A PARTICULAR SOLUTION OF THE EQUATION COS $x = \frac{1}{2}$ IN THE RANGE 0 ≤ x ≤ .

3 DETERMINE THE GENERAL SOLUTION OF COS $x = \frac{1}{2}$ USING THE PARTICULAR SOLUTIONS, n AND 2

FROM ACTIVITY 9.4 IT IS CLEAR THAT THE GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS IS EXPRESSED IN TERMS OF THE PARTICULAR SOLUTIONS, THE PERIOD AND n. THE FOLLOWING ARE THE TECHNIQUES OF FINDING THE GENERAL SOLUTION OF SOME TRIGONOMETRIC EQUATIONS.

I TAN $x = t$; $t \in \mathbb{R}$.

THE PERIOD OF TANGENT FUNCTION IS

IF x_1 IS THE PARTICULAR SOLUTION IN THE RANGE $-\frac{}{2} < x < \frac{}{2}$, THEN THE GENERAL SOLUTION SET IS $\{x_1 + n\}$.

Example 7 SOLVE TAN $x = -\frac{1}{\sqrt{3}}$.

Solution: $x_1 = -\frac{}{6} \Rightarrow S.S. = \left\{-\frac{}{6} + n\right\}$

II COS $x = b$; $|b| \le 1$. IF x_1 IS A PARTICULAR SOLUTION IN THE RANGE $- \le x \le$, THEN $-x_1$ IS A PARTICULAR SOLUTION IN THE SAME RANGE.

$\Rightarrow$ S.S = $\{2n \pm x_1\}$.

Example 8 SOLVE COS $x = -\frac{\sqrt{3}}{2}$.

Solution: $x_1 = \frac{5}{6} \Rightarrow S.S. = \left\{2n \pm \frac{5}{6}\right\}$

III SIN $x = b$, $|b| \le 1$

IF $b = 0$, THEN SIN $x = 0 \Rightarrow S.S. = \{n\}$,

SIN x = 1 $\Rightarrow$ S.S = $\left\{\frac{}{2} + 2n\right\}$

SIN x = –1 $\Rightarrow$ S.S = $\left\{-\frac{}{2} + 2n\right\}$

SUPPOSE $0 < |b| < 1$. AS IT IS DONE IN THE ACTIVITY, THE LINE $y = b$ CROSSES THE GRAPH OF $y = \sin x$ AT EXACTLY TWO POINTS IN THE INTERVAL $[0, 2]$.

IF x_1 AND x_2 ARE THE PARTICULAR SOLUTIONS, THEN THE GENERAL SOLUTION SET IS

$$\{x_1 + 2n , x_2 + 2n \}.$$

Example 9 SOLVE $\sin x = \frac{\sqrt{2}}{2}$.

Solution: YOU KNOW THAT $\sin \frac{}{4} = \frac{\sqrt{2}}{2}$ AND $\sin \frac{3}{4} = \frac{\sqrt{2}}{2}$.

$$\Rightarrow S.S. = \left\{\frac{}{4} + 2n , \frac{3}{4} + 2n \right\}.$$

✍Note:

$\sin x_1 = \sin(- x_1) \Rightarrow x_2 = - \frac{}{4} = \frac{3}{4}$. ALSO, IF x_1 IS A PARTICULAR SOLUTION IN THE INTERVAL $[0, 2]$, THEN THE GENERAL SOLUTION SET OF THE EQUATION $\sin x = b$, $|b| \le 1$ IS $\left\{(-1)^n x_1 + n \right\}$.

Example 10 SOLVE $\sin(4x) = -\frac{1}{2}$.

Solution: NOTICE THAT THE LINE $y = -\frac{1}{2}$ CROSSES THE GRAPH OF $y = \sin(4x)$ TWICE IN THE INTERVAL $\left[0, \frac{}{2}\right]$.

$$\sin(4x) = -\frac{1}{2} \Rightarrow \sin(-4x) = \frac{1}{2} \Rightarrow -4x_1 = \frac{}{6}, -4x_2 = \frac{5}{6}$$

$$\Rightarrow x_1 = -\frac{}{24}, x_2 = -\frac{5}{24}$$

THUS, THE PARTICULAR SOLUTIONS IN THE INTERVAL $\left[0, \frac{}{2}\right]$ ARE

$$-\frac{}{24} + \frac{}{2} = \frac{11}{24}, -\frac{5}{24} + \frac{}{2} = \frac{7}{24}$$

$$\Rightarrow S.S = \left\{\frac{11}{24} + \frac{n}{2}, \frac{7}{24} + \frac{n}{2}\right\}$$

Exercise 9.5

1 FIND THE GENERAL SOLUTION SET FOR EACH OF THE FOLLOWING TRIGONOMETRIC EQUATIONS.

A $\sin x = -\frac{1}{2}$ **B** $\cos x = \frac{\sqrt{3}}{2}$ **C** $\tan x = \sqrt{3}$

D $2\cos^2 x + 3\sin x = 0$ **E** $\cos 2x + \sin^2 x = 0$ **F** $\sin(6x) = \frac{\sqrt{3}}{2}$

2 SOLVE $\sin^2 x - \sin x \cos x = 0$ OVER [0, 2]

3 FIND THE GENERAL SOLUTION SETS FOR EACH OF THE FOLLOWING TRIGONOMETRIC EQUATIONS ON THE GIVEN INTERVALS.

A $\cos x = \frac{\sqrt{3}}{2}$ AND $\tan x = -\frac{\sqrt{3}}{3}$ ON [0, 2].

B $\cos\left(\frac{}{3}x - 2\right) = \frac{1}{2}$ ON [–6, 6].

C $\sec\left(\frac{3}{2}x - \frac{}{3}\right) = 2$ AND $\cot x < 0$ ON [0, 2].

D $2\sin^2 x + \cos^2 x - 1 = 0$ ON [0, 2].

9.4 APPLICATION OF TRIGONOMETRIC FUNCTIONS

IN THIS TOPIC, YOU STUDY SOME OF THE APPLICATIONS OF TRIGONOMETRIC FUNCTIONS TO SCIENCE, NAVIGATION, WAVE MOTIONS AND OPTICS. THE LAWS OF SINES, AND COSINES, THE ANGLE AND HALF ANGLE FORMULAS ARE INCLUDED IN THIS TOPIC.

MANY APPLIED PROBLEMS CAN BE SOLVED BY USING RIGHT-ANGLE TRIANGLE TRIGONOM WILL SEE A NUMBER OF ILLUSTRATIONS OF THIS FACT IN THIS SECTION.

9.4.1 Solving Triangles

IN THE APPLICATIONS OF TRIGONOMETRY THAT YOU CONSIDER IN THIS SECTION, IT IS NECES ALL SIDES AND ANGLES OF A RIGHT-ANGLED TRIANGLE. TO SOLVE A TRIANGLE MEANS LENGTHS OF ALL ITS SIDES AND THE MEASURES OF ALL ITS ANGLES. FIRST SOLVE A RIGHT-AN

Example 1 SOLVE THE RIGHT-ANGLED TRIANGLE SHOWN BELOW FOR UNKNOWN SIDES AND ANGLES.

Solution BECAUSE C = 90°, IT FOLLOWS THAT A + B = 90°, AND B = 55.8°

TO SOLVE FOR a, USE THE FACT THAT

$$\text{TAN } A = \frac{opp}{adj} = \frac{a}{b} \text{ WHICH IMPLIES}$$

$$a = b \text{ TAN } A$$

SO, $a = 19.4 \times \text{TAN } 34.2^\circ \approx 13.18$.

SIMILARLY, TO SOLVE FOR c, USE THE FACT THAT

$$\text{COS } A = \frac{adj}{hyp} = \frac{b}{c} \text{ WHICH IMPLIES}$$

$$c = \frac{b}{\text{COS } A} = \frac{19.4}{\text{COS } 34.2^\circ} \approx 23.46$$

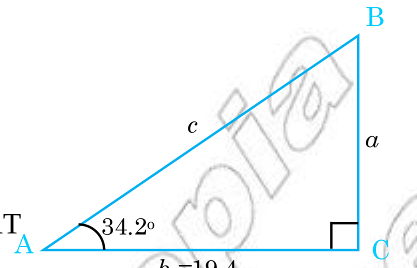


Figure 9.29

IN MANY SITUATIONS, TRIGONOMETRIC FUNCTIONS CAN BE USED TO DETERMINE A DISTAN DIFFICULT TO MEASURE DIRECTLY. TWO SUCH CASES ARE ILLUSTRATED BELOW.

A

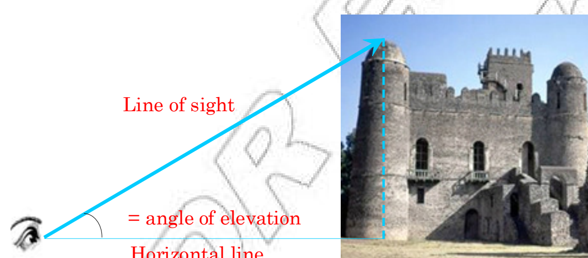


Figure 9.30

B

Figure 9.31

EACH ANGLE IS FORMED BY TWO LINES: A HORIZONTAL LINE AND A LINE OF SIGHT. IF THE MEASURED UPWARD FROM THE HORIZONTAL, AS IN A, THEN THE ANGLE IS CALLED AN **angle of elevation**. IF IT IS MEASURED DOWNWARD, AS IN B, IT IS CALLED AN **angle of depression**.

Example 2 A SURVEYOR IS STANDING 50 M FROM THE BASE OF A LARGE TREE, AS SHOWN BELOW. THE SURVEYOR MEASURES THE ANGLE OF ELEVATION TO THE TOP OF TH AS 15°. HOW TALL IS THE TREE IF THE SURVEYOR IS 1.72 M TALL?

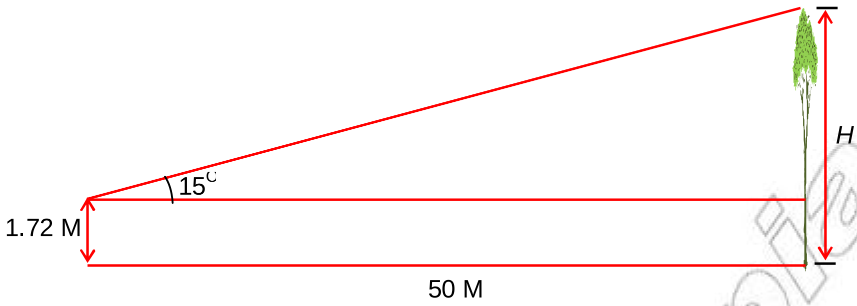


Figure 9.32

Solution THE INFORMATION GIVEN SUGGESTS THE USE OF THE TANGENT FUNCTION.

LET THE HEIGHT OF THE TREE BE h METRES. THEN,

$$\tan 15° = \frac{(h-1.72)}{50}$$

$$0.268 \approx \frac{(h-1.72)}{50}$$

$\Rightarrow h = (50\,(0.2679) + 1.72)$ M

$\Rightarrow h = 15.115$ M

THUS, THE TREE IS ABOUT 15 M TALL.

Example 3 A WOMAN STANDING ON TOP OF A CLIFF SPOTS A BOAT IN THE SEA, AS GIVEN FIGURE 9.33. IF THE TOP OF THE CLIFF IS 70 M ABOVE THE WATER LEVEL, HER EYE LEVEL IS 1.6 M ABOVE THE TOP OF THE CLIFF AND IF THE ANGLE OF DEPRESSIO 30^{O}, HOW FAR IS THE BOAT FROM A POINT AT SEA LEVEL THAT IS DIRECTLY BELO OBSERVER?

Figure 9.33

Solution IN THE FIGURE, THE OBSERVER'S EYES ARE 71.6 M ABOVE THE WATER LEVEL. USING TRIANGLE BCD, COMPUTE

$$\text{TAN } 30^{\circ} = \frac{\text{BD}}{\text{DC}} \Rightarrow \frac{1}{\sqrt{3}} = \frac{71.6\text{M}}{\text{DC}}$$

$$\Rightarrow \text{DC} = 71.6\sqrt{3}\text{ M}$$

$\therefore$ THE BOAT IS $71.6\sqrt{3}$ M FAR AWAY FROM THE BOTTOM OF THE CLIFF.

Example 4 IN ORDER TO MEASURE THE HEIGHT OF A HILL, A SURVEYOR TAKES TWO SIGHTINGS FROM A TRANSIT 1.5 M HIGH. THE SIGHTINGS ARE TAKEN 1000 M APART FROM THE SAME GROUND ELEVATION. THE FIRST MEASURED ANGLE OF ELEVATION IS 51°, AND THE SECOND IS 29°. TO THE NEAREST METRE, WHAT IS THE HEIGHT OF THE HILL (ABOVE GROUND LEVEL)?

Solution FIRST DRAW THE FIGURE AND LABEL THE KNOWN PARTS. (*See* FIGURE 9.34)

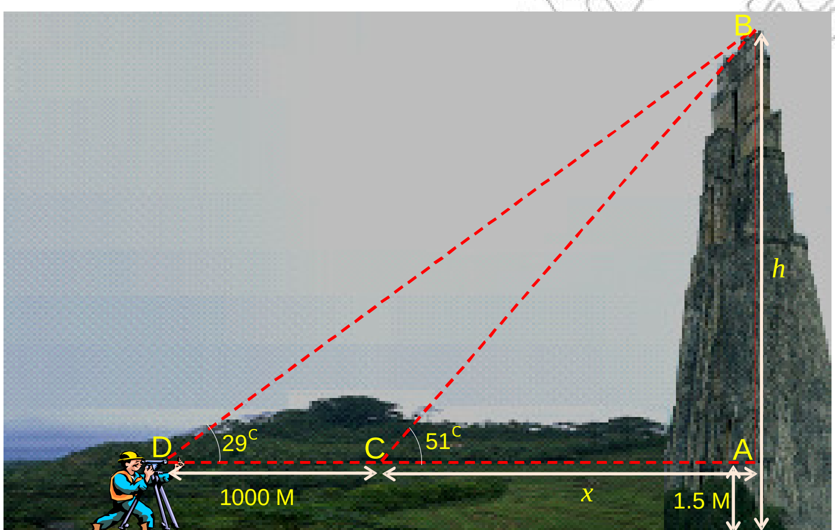


Figure 9.34

THE HEIGHT OF THE HILL = $AB + 1.5\text{ M} = h$

BUT, $\text{TAN } 51^{\circ} = \frac{AB}{x}$ AND $\text{TAN } 29^{\circ} = \frac{AB}{x+1000}$

$$AB = x\,\text{TAN } 51^{\circ} \text{ AND } AB = (x + 1000)\,\text{TAN } 29^{\circ}$$

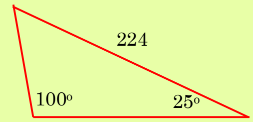
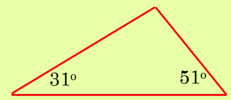
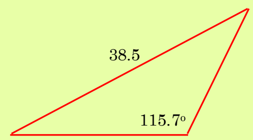
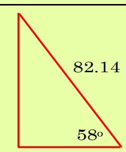
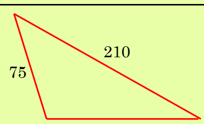
$$AB = 1.235x \text{ and } AB = (x+1000)(0.5543) = (0.5543x + 554.3)$$

EQUATING THE TWO EXPRESSIONS FOR $\overline{AB}$, YOU HAVE

$$1.235x = 0.5543x + 554.3 \Rightarrow x \approx 814.31$$

THUS, $AB = 1.235 \times 814.31 \approx 1005.67$ AND HENCE $h = AB + 1.5\text{ M} \approx 1007\text{ M}$.

THE TRIGONOMETRIC FUNCTIONS CAN ALSO BE USED TO SOLVE TRIANGLES THAT ARE NOT RIGHT TRIANGLES. SUCH TRIANGLES ARE CALLED OBLIQUE TRIANGLES. ANY TRIANGLE, RIGHT OR OBLIQUE, CAN BE SOLVED IF AT LEAST ONE SIDE AND ANY OTHER TWO MEASURES ARE KNOWN. THE FOLLOWING ARE THE DIFFERENT POSSIBLE CONDITIONS.

1	**AAS**: TWO ANGLES OF A TRIANGLE AND A SIDE OPPOSITE TO ONE OF THEM ARE KNOWN.	**A**	
2	**ASA**: TWO ANGLES OF A TRIANGLE AND THE INCLUDED SIDE ARE KNOWN.	**B**	
3	**SSA**: TWO SIDES OF A TRIANGLE AND AN ANGLE OPPOSITE TO ONE OF THEM ARE KNOWN (IN THIS CASE THERE MAY BE NO SOLUTION, ONE SOLUTION OR TWO SOLUTIONS. THE LATTER IS KNOWN AS THE AMBIGUOUS CASE)	**C**	
4	**SAS:** TWO SIDES OF A TRIANGLE AND THE INCLUDED ANGLE ARE KNOWN.	**D**	
5	**SSS:** ALL THREE SIDES OF THE TRIANGLE ARE KNOWN.	**E**	 Figure 9.35

IN ORDER TO SOLVE OBLIQUE TRIANGLES, YOU NEED THE **law of sines** AND THE **law of cosines**.

THE LAW OF SINES APPLIES TO THE FIRST THREE SITUATIONS LISTED ABOVE. THE LAW OF COSINES APPLIES TO THE LAST TWO SITUATIONS.

The law of sines

IN ANY TRIANGLE ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

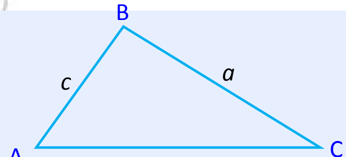


Figure 9.36

Note:

IN ANY TRIANGLE, THE SIDES ARE PROPORTIONAL TO THE SINE OF THE OPPOSITE ANGLES.

Example 5 IN ΔEFG, $FG = 4.56$, $m(\angle E) = 43^{O}$, AND $m(\angle G) = 57^{O}$. SOLVE THE TRIANGLE.

Solution FIRST DRAW THE TRIANGLE AND LABEL THE KNOWN PARTS. YOU KNOW THREE OF SIX MEASURES.

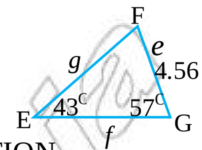


Figure 9.37

$\angle E = 43^{O}$ $e = 4.56$ $\angle G = 57^{O}$

$\angle F = ?$ $f = ?$ $g = ?$

FROM THE FIGURE, YOU HAVE THE AAS SITUATION.

YOU BEGIN BY FINDING $m(\angle F)$.

$$m(\angle F) = 180^{O} - (43^{O} + 57^{O}) = 80^{O}$$

YOU CAN NOW FIND THE OTHER TWO SIDES, USING THE LAW OF SINES:

$$\frac{f}{\sin F} = \frac{e}{\sin E} \Rightarrow \frac{f}{\sin 80^{O}} = \frac{4.56}{\sin 43^{O}}$$

$$\Rightarrow f \cong 6.58$$

ALSO, $\dfrac{g}{\sin G} = \dfrac{e}{\sin E} \Rightarrow \dfrac{g}{\sin 57^{O}} = \dfrac{4.56}{\sin 43^{O}}$

$$\Rightarrow g \cong 5.61$$

THUS, YOU HAVE SOLVED THE TRIANGLE:

$\angle E = 43^{O}$ $e = 4.56$,

$\angle F = 80^{O}$ $f \cong 6.58$

$\angle G = 57^{O}$ $g = 5.61$

Example 6 IN ΔQRS, $q = 15$, $r = 28$ AND $\angle Q = 43.6^{O}$. SOLVE THE TRIANGLE.

Solution DRAW THE TRIANGLE AND LIST THE KNOWN MEASURES:

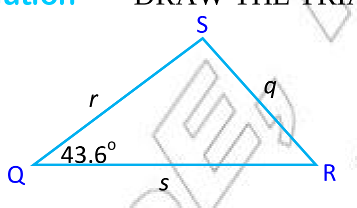


Figure 9.38

$\angle Q = 43.6^{O}$ $q = 15$

$\angle R = ?$ $r = 28$

$\angle S = ?$ $s = ?$

YOU HAVE THE SSA SITUATION AND USE THE LAW OF SINES TO FIND R.

$$\frac{q}{\sin Q} = \frac{r}{\sin R} \Rightarrow \frac{15}{\sin 43.6^{O}} = \frac{28}{\sin R}$$

$$\Rightarrow \sin R \cong 1.2873.$$

SINCE THERE IS NO ANGLE WITH A SINE GREATER THAN 1, THERE IS NO SOLUTION.

The law of cosines

IN ANY TRIANGLE ABC,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

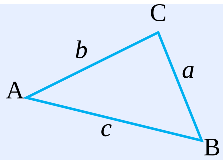


Figure 9.39

☑ Remark:

WHEN THE INCLUDED ANGLE IS 90°, THE LAW OF COSINES IS REDUCED TO THE PYTHAGOREAN THEOREM.

Example 7 SOLVE $\triangle ABC$, IF $a = 32$, $c = 48$ AND $\angle B = 125.2^{\circ}$

Solution YOU FIRST LABEL A TRIANGLE WITH THE KNOWN AND UNKNOWN MEASURES.

$\angle A = ?$	$a = 32$
$\angle B = 125.2^{\circ}$	$b = ?$
$\angle C = ?$	$c = 48$

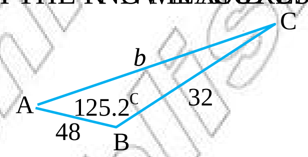


Figure 9.40

YOU CAN FIND THE THIRD SIDE USING THE LAW OF COSINES, AS FOLLOWS:

$b^2 = a^2 + c^2 - 2ac \cos B$

$\Rightarrow b^2 = 32^2 + 48^2 - 2(32)(48) \cos 125.2^{\circ}$

$\Rightarrow b^2 \cong 5089.8$

$\Rightarrow b \cong 71.34$

YOU NOW HAVE $a = 32$, $b \cong 71.34$ AND $c = 48$, AND YOU NEED TO FIND THE MEASURES OF THE OTHER TWO ANGLES. AT THIS POINT, YOU CAN FIND THEM IN TWO WAYS, EITHE THE LAW OF SINES OR THE LAW OF COSINES. THE ADVANTAGE OF USING THE LAW OF C THAT IF YOU SOLVE FOR THE COSINE AND FIND THAT ITS VALUE IS NEGATIVE, THEN Y THAT THE ANGLE IS OBTUSE. IF THE VALUE OF THE COSINE IS POSITIVE, THEN THE A ACUTE. THUS YOU USE THE LAW OF COSINES:

TO FIND ANGLE A, YOU USE

$a^2 = b^2 + c^2 - 2bc \cos A$

$32^2 = (71.34)^2 + 48^2 - 2(71.34)(48) \cos A$

$\angle A \approx 21.55^{\circ}$

THE THIRD IS NOW EASY TO FIND:

$\angle C \approx 180^{\circ} - (125.2^{\circ} + 21.55^{\circ}) \approx 33.25^{\circ}$

9.4.2 Trigonometric Formulae for the Sum and Differences

IN GRADE 10, YOU HAVE SEEN THE FUNDAMENTAL TRIGONOMETRIC IDENTITIES FOR A SINGLE VARIABLE. IN THIS TOPIC, YOU HAVE TRIGONOMETRIC IDENTITIES INVOLVING THE SUM OR DIFFERENC VARIABLES.

FOR EXAMPLE, USING YOUR KNOWLEDGE OF THE TRIGONOMETRIC VALUES OF 30^{o} AND 45^{o}, YOU WILL THEN BE ABLE TO DETERMINE THE TRIGONOMETRIC VALUES OF $45^{o} + 30^{o} = 75^{o}$ AND $45^{o} - 30^{o} = 15^{o}$.

Theorem 9.1 Sum and Difference Formulae

1 Sine of the Sum and the Difference

- ✓ $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- ✓ $\sin(x - y) = \sin x \cos y - \cos x \sin y$

2 Cosine of the Sum and Difference

- ✓ $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- ✓ $\cos(x - y) = \cos x \cos y + \sin x \sin y$

3 Tangent of the Sum and Difference

- ✓ $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- ✓ $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Example 8 FIND THE EXACT VALUES OF $\sin 75^{o}$ AND $\sin 15^{o}$

Solution $\sin 75^{o} = \sin(30^{o} + 45^{o}) = \sin 30^{o} \cos 45^{o} + \cos 30^{o} \sin 45^{o}$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\sin 15^{o} = \sin\left(45^{o} - 30^{o}\right) = \sin 45^{o} \cos 30^{o} - \cos 45^{o} \sin 30^{o}$$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

Example 9 FIND THE EXACT VALUE OF $\cos 105^{o}$

Solution $\cos 105^{o} = \cos(60^{o} + 45^{o}) = \cos 60^{o} \cos 45^{o} - \sin 60^{o} \sin 45^{o}$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

Example 10 Find the exact values of

a $\tan 150^\circ$ **b** $\tan 195^\circ$

Solution

a $\tan 150^\circ = \tan (180^\circ - 30^\circ)$

$$= \frac{\tan 180^\circ - \tan 30^\circ}{1 + \tan 180^\circ \times \tan 30^\circ} = \frac{0 - \frac{1}{\sqrt{3}}}{1 + 0 \times \frac{1}{\sqrt{3}}} = -\frac{1}{\sqrt{3}}$$

b $\tan 195^\circ = \tan (150^\circ + 45^\circ) = \dfrac{\tan 150^\circ + \tan 45^\circ}{1 - \tan 150^\circ \times \tan 45^\circ}$

$$= \frac{-\frac{1}{\sqrt{3}} + 1}{1 - \left(-\frac{1}{\sqrt{3}}\right) \times 1} = 2 - \sqrt{3}$$

Theorem 9.2 Double Angle and Half Angle Formulas

1 Double Angle Formula.

✓ $\sin (2x) = 2 \sin x \cos x$

✓ $\cos (2x) = \cos^2 x - \sin^2 x$

✓ $\tan (2x) = \dfrac{2 \tan x}{1 - \tan^2 x}$

2 Half Angle Formula

✓ $\cos^2 \left(\dfrac{x}{2}\right) = \dfrac{1 + \cos x}{2}$; $\cos \left(\dfrac{x}{2}\right) = \pm \sqrt{\dfrac{1 + \cos x}{2}}$

✓ $\sin^2 \left(\dfrac{x}{2}\right) = \dfrac{1 - \cos x}{2}$; $\sin \left(\dfrac{x}{2}\right) = \pm \sqrt{\dfrac{1 - \cos x}{2}}$

✓ $\tan^2 \left(\dfrac{x}{2}\right) = \dfrac{1 - \cos x}{1 + \cos x}$ for $\cos x \neq -1$;

$$\tan \left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

The sign is determined by the quadrant that contains $\dfrac{x}{2}$.

✍Note:

I COS (2x) = COS²x – SIN²x
= COS²x – (1 – COS²x)
GIVING COS(2x) = 2 COS²x – 1

II COS (2x) = COS²x – SIN²x
= (1 – SIN²x) – SIN²x
GIVING COS(2x) = 1 – 2 SIN²x

Example 11 FIND THE EXACT VALUES OF

A $\text{SIN}\frac{\ }{8}$ **B** COS 15° **C** $\text{TAN}\frac{\ }{8}$

Solution

A $\text{SIN}^2\frac{\ }{8} = \frac{1-\text{COS}\frac{\ }{4}}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$

$\Rightarrow \text{SIN}\frac{\ }{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$ SINCE $\text{SIN}\frac{\ }{8} > ($

B $\text{COS}^2 15^{\circ} = \frac{1+\text{COS } 30^{\circ}}{2} = \frac{2+\sqrt{3}}{4} \Rightarrow \text{COS } 15^{\circ} = \frac{\sqrt{2+\sqrt{3}}}{2}$

C $\frac{\ }{4} = \frac{\ }{8} + \frac{\ }{8} \Rightarrow \text{TAN}\frac{\ }{4} = \frac{2\,\text{TAN}\frac{\ }{8}}{1-\text{TAN}^2\frac{\ }{8}}$

$\Rightarrow 1 = \frac{2\,\text{TAN}\frac{\ }{8}}{1-\text{TAN}^2\frac{\ }{8}} \Rightarrow \text{TAN}^2\frac{\ }{8} + 2\,\text{TAN}\frac{\ }{8} = 1$

SOLVING THE QUADRATIC EQUATION GIVES $\text{TAN}\frac{\ }{8} = -1 \pm \sqrt{2}$

$\Rightarrow \text{TAN}\frac{\ }{8} = \sqrt{2} - 1$, BECAUSE $\text{TAN}\frac{\ }{8}$

9.4.3 Navigation

IN NAVIGATION, DIRECTIONS TO AND FROM A REFERENCE POINT ARE OFTEN GIVEN IN T BEARINGS. A BEARING IS AN ACUTE ANGLE BETWEEN A LINE OF TRAVEL OR LINE OF SIGH NORTH-SOUTH LINE. BEARINGS ARE USUALLY GIVEN ANGLES IN DEGREES SUCH AS EAST (NORTH, SO THAT IT IS READ AS EAST OF NORTH, AND SO ON.

Example 12 THE TWO BEARINGS IN FIGURE 9.41 BELOW ARE RESPECTIVELY,

A N30°E **B** S10°E.

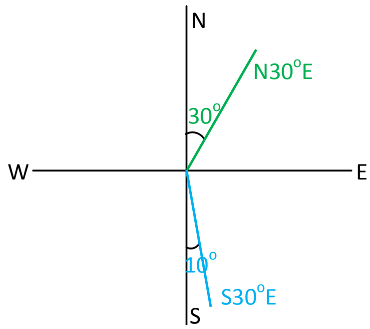


Figure 9.41

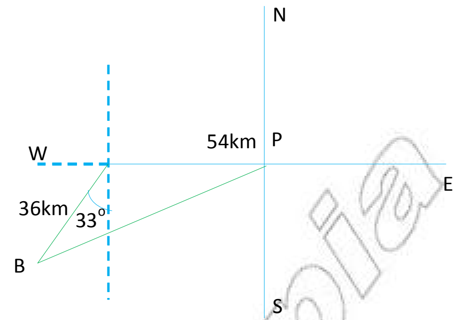


Figure 9.42

Example 13 A SHIP LEAVES A PORT AND TRAVELS 54 KM DUE WEST. IT THEN CHANGES COURSE AND SAILS 36 KM ON A BEARING OF S33°W. HOW FAR IS IT FROM THE PORT AT THIS POINT? (*See* FIGURE 9.42)

Solution THE SHIP IS AT POINT *B*. YOU MUST CALCULATE THE DISTANCE $\overline{PB}$. USING THE LAW OF COSINES,

$$(\overline{PB})^2 = 54^2 + 36^2 - 2\times54\times36\times\cos 123^\circ = 2916 + 1296 - 3888 \times (-0.5446)$$

$$= 6329.4048$$

$\Rightarrow$ $PB = 79.5576$

$\Rightarrow$ THE SHIP IS ABOUT 80 KM FROM THE PORT.

9.4.4 Optics Problem

Snell's law of refraction, WHICH WAS DISCOVERED BY DUTCH PHYSICIST WILLEBRORD SNELL (1591 – 1626), STATES THAT A LIGHT RAY IS REFRACTED (BENT) AS IT PASSES FROM A FIRST MEDIUM INTO A SECOND MEDIUM ACCORDING TO THE EQUATION:

$$\frac{\sin\alpha}{\sin\beta} = \mu$$

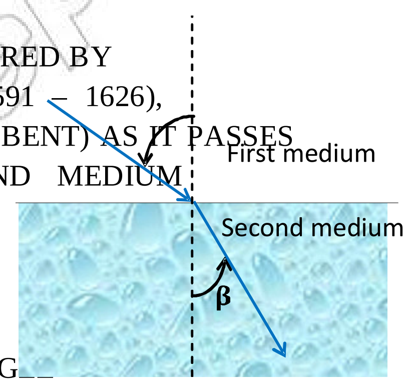


Figure 9.43

WHERE α IS THE ANGLE OF INCIDENCE AND β IS THE ANGLE OF REFRACTION.

THE GREEK LETTER μ (mu), IS CALLED THE **index of refraction** OF THE SECOND MEDIUM WITH RESPECT TO THE FIRST.

Example 14 THE INDEX OF REFRACTION OF WATER WITH RESPECT TO AIR IS $\mu = 1.33$. DETERMINE THE ANGLE OF **refraction**, IF A RAY OF LIGHT PASSES THROUGH WATER WITH AN ANGLE OF INCIDENCE $\alpha = 30^\circ$.

Solution $= \frac{\text{SIN}}{\text{SIN}} \Rightarrow 1.33 = \frac{\text{SIN } 30^0}{\text{SIN}}$

$$\Rightarrow \text{SIN} = \frac{0.5}{1.33} \approx 0.3759 \Rightarrow \quad = \text{SIN}^{-1}(0.3759)$$

$$\Rightarrow \quad = 22.1^{O}$$

9.4.5 Simple Harmonic Motion

THE PERIODIC NATURE OF THE TRIGONOMETRIC FUNCTIONS IS USEFUL FOR DESCRIBING THE A POINT ON AN OBJECT THAT VIBRATES, OSCILLATES, ROTATES, OR IS MOVED BY WAVE MOT

IN PHYSICS, BIOLOGY, AND ECONOMICS, MANY QUANTITIES ARE PERIODIC. EXAMPLES INC THE VIBRATION OR OSCILLATION OF A PENDULUM OR A SPRING, PERIODIC FLUCTUATIC POPULATION OF A SPECIES, AND PERIODIC FLUCTUATIONS IN A BUSINESS CYCLE. MANY C QUANTITIES CAN BE DESCRIBED BY HARMONIC FUNCTIONS.

Definition 9.5

A **harmonic function** IS A FUNCTION THAT CAN BE WRITTEN IN THE FORM

$$g(t) = a\,\text{COS}\ t + b\,\text{SIN}\ t. \qquad 1$$

NOTE THAT 1 CAN BE WRITTEN IN THE FORMS

$$a\,\text{COS}\ t + b\,\text{SIN}\ t = A\,\text{COS}\,(\ t - \delta) \qquad 2$$

$$a\,\text{COS}\ t + b\,\text{SIN}\ t = A\,\text{SIN}\,(\ t + \ ) \qquad 3$$

WHERE $A = \sqrt{a^2 + b^2}$, $(\text{COS}\ , \text{SIN}\ ) = \left(\frac{a}{A}, \frac{b}{A}\right)$, AND $(\text{COS}\ , \text{SIN}\ ) = \left(\frac{b}{A}, \frac{a}{A}\right)$

IN 2 OR 3, THE PERIOD IS $\frac{2}{}$. THE FREQUENCY f OF THE FUNCTION IS THE NUMBER OF COMPLETE PERIODS PER UNIT TIME. SINCE $y = A\,\text{COS}\,(\ t - \Delta)$ OR $y = A\,\text{SIN}\,(\ t + \Delta)$ RETURNS TO THE SAME y VALUE IN ONE PERIOD EQUAL TO $\frac{2}{}$ TIME UNITS, YOU HAVE:

Natural frequency of a function

$$f = \frac{}{2}$$

UNITS OF FREQUENCY ARE CYCLES/SEC (ALSO CALLED HERTZ)

Example 15 *A simple electric circuit*

IN AN ELECTRIC CIRCUIT, SUCH AS THE ONE IN THE FIGURE ON THE RIGHT, AN ELECTROMOTIVE FORCE (EMF) E (VOLTS) A BATTERY OR GENERATOR, DRIVES AN ELECTR Q (COULOMBS) AND PRODUCES A CURRENT I (AMPER CIRCUIT SHOWN IN FIGURE 9.44, A RESISTOR OF RESISTANC (OHMS) IS A COMPONENT OF THE CIRCUIT THAT OPPOSES THE CURRENT, DISSIPATING THE ENERGY IN THE FORM OF HEAT. IT PRODUCES A DROP IN THE VOLTAGE GIVEN BY OHM'S LAW:

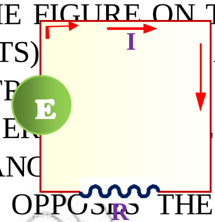


Figure 9.44

$$E = RI$$

THE ELECTROMOTIVE FORCE (EMF) MAY BE DIRECT OR ALTERNATING. A DIRECT EM GIVEN BY A CONSTANT VOLTAGE. AN ALTERNATING EMF IS USUALLY GIVEN AS A FUNCTION:

$$E = E_o \text{ SIN } _o t,\ E_o > 0$$

SINCE $-1 \le$ SIN $_o t \le 1$, YOU SEE THAT

$$-E_o \le E \le E_o$$

THUS, E_O IS THE MAXIMUM VOLTAGE AND $-E_o$ IS THE MINIMUM VOLTAGE.

Example 16 SUPPOSE THAT AN EMF OF $E = 10$ SIN $\frac{}{4}t$ VOLTS IS CONNECTED IN THE CIRCUIT OF FIGURE 9.45 ABOVE WITH A RESISTANCE OF 5 OHMS.

A WHAT IS THE PERIOD OF THE EMF?

B WHAT IS THE FREQUENCY?

C WHAT IS THE MAXIMUM CURRENT IN THE SYSTEM?

Solution

A PERIOD $= \frac{2}{} = \frac{2}{\frac{}{4}} = \frac{8}{} = 8$

B FREQUENCY $= \frac{}{2} = \frac{1}{8}$ CYCLES/

C FROM THE EQUATION E = RI, WE HAVE:

$$I = \frac{E}{R} = \frac{10 \text{ SIN} \frac{}{4}t}{5} = 2 \text{ SIN} \frac{}{4}t \text{ AMPERE.}$$

THE MAXIMUM CURRENT IS 2 AMPERES.

Example 17 GIVEN THE EQUATION FOR SIMPLE HARMONIC MOTION $d = 6\cos\frac{3\pi}{4}t$, FIND

A THE MAXIMUM DISPLACEMENT

B THE FREQUENCY

C THE VALUE OF d WHEN $t = 4$

D THE LEAST POSITIVE VALUE OF t FOR WHICH $d = 0$.

Solution

A THE MAXIMUM DISPLACEMENT IS 6, BECAUSE THE MAXIMUM DISPLACEMENT FROM THE POINT OF EQUILIBRIUM IS THE AMPLITUDE.

B FREQUENCY $= \frac{\omega}{2\pi} = \frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8}$ CYCLE/UNIT

C $d = 6\cos\left(\frac{3\pi}{4}(4)\right) = 6\cos 3\pi = 6(-1) = -6$

D TO FIND THE LEAST POSITIVE VALUE OF t FOR WHICH $d = 0$, SOLVE THE EQUATION $d = 6\cos\frac{3\pi}{4}t = 0$ TO OBTAIN

$$\frac{3\pi}{4}t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \text{ WHICH IMPLIES } t = \frac{2}{3}, 2, \frac{10}{3}, \ldots$$

THUS, THE LEAST POSITIVE VALUE OF t IS $t = \frac{2}{3}$.

VIBRATIONS, SUCH AS THOSE CREATED BY PLUCKING A VIOLIN STRING OR STRIKING A TUBE, CAUSE SOUND WAVES, WHICH MAY OR MAY NOT BE AUDIBLE TO THE HUMAN OFTEN, SOUND WAVES ARE **sinusoidal** AND CAN THEREFORE BE WRITTEN IN THE FORM

$$y = a\sin\omega t$$

HERE YOU ASSUME THAT THERE IS NO PHASE SHIFT [= 0 IN EQUATION 3]. THE AMPLITUDE a IS RELATED TO THE LOUDNESS OF THE SOUND, WHICH IS MEASURED IN DECIB

Example 18 NIDDLE C IS STRUCK ON A PIANO WITH AMPLITUDE $a = 2$. THE FREQUENCY OF NIDDLE C IS 264 CYCLES/SEC. WRITE AN EQUATION FOR THE RESULTING SOUN WAVE.

Solution WITH $a = 2$, WE HAVE

$$y = 2\sin\omega t$$

BUT, FREQUENCY $= \frac{\omega}{2\pi} = 264$

SO $\omega = 264(2\pi) = 528\pi$. THUS $y = 2\sin 528\pi t$ IS THE EQUATION OF THE SOUND WAVE.

Exercise 9.6

1. A FLYING AIRPLANE IS SIGHTED IN A LINE FROM TWO STATIONS *A* AND *B*. THE ANGLE OF ELEVATION OF THE AIRPLANE FROM *A* IS 30° AND FROM *B* IS 60°. *A* AND *B* ARE ON THE SAME SIDE OF THE AIRPLANE. IF THE DISTANCE BETWEEN *A* AND *B* IS 200 M, FIND THE ALTITUDE OF THE AIRPLANE.

2. SOLVE EACH OF THE FOLLOWING TRIANGLES. APPROXIMATE THE ANSWERS TO TWO DECIMAL PLACES.

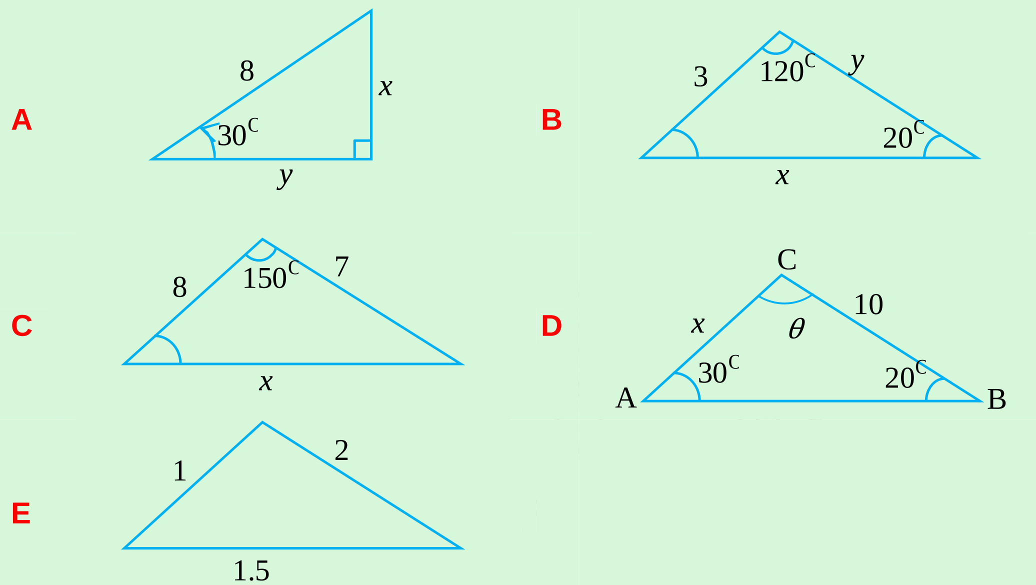


Figure 9.45

3. THE ANGLE OF ELEVATION OF THE TOP OF A BUILDING IS 45° AS MEASURED FROM A POINT ON A LEVEL GROUND. IF THE ANGLE OF ELEVATION OF A POINT ON THE BUILI IS 3 M BELOW THE TOP IS 60° AS MEASURED FROM THE SAME POINT ON THE GROUND, FIND THE HEIGHT OF THE BUILDING.

4. GIVEN BELOW IS AN ISOSCELES TRAPEZIUM WITH *b* UNITS AND THE CONGRUENT SIDES *a* UNITS LONG. IF THE BASE ANGLE MEASURES θ°, EXPRESS THE AREA OF THE TRAPEZIUM IN TERMS OF *a*, *b*, SIN θ AND COS θ.

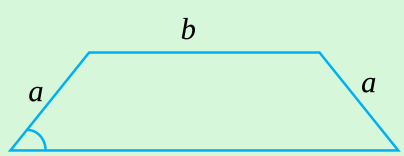


Figure 9.46

5 TWO BOATS A AND B LEAVE THE SAME PORT P AT THE SAME TIME. A TRAVELS 60 KM IN THE DIRECTION N 75°W TO PORT Q AND B TRAVELS 80 KM IN THE DIRECTION S 45°W TO PORT R. FIND THE DISTANCE BETWEEN PORT Q AND PORT R.

6 THE REFRACTION INDEX OF WATER WITH RESPECT TO AIR IS = 1.33. DETERMINE THE ANGLE OF REFRACTION OF A RAY OF LIGHT THAT STRIKES THE WATER BODY WITH AN ANGLE OF INCIDENCE = 45°.

7 FIND THE EXACT VALUES OF THE FOLLOWING TRIGONOMETRIC FUNCTIONS WITHOUT USING A CALCULATOR OR TABLES.

A $\sin 165^{\circ}$ **B** $\cos 105^{\circ}$ **C** $\tan \frac{17}{12}$

D $\sec \frac{11}{12}$ **E** $\cot \frac{19}{12}$ **F** $\csc \frac{13}{12}$

8 SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS.

A $\dfrac{\tan 175^{\circ} - \tan 130}{1 + \tan 175^{\circ} \times \tan 1^{\circ}}$ **B** $\dfrac{\sin x + \tan x}{\csc x + \cot x} \times \cot x$

C $\dfrac{\sin(2x) + \sin(4x)}{\cos(2x) + \cos(4x)}$ **D** $\dfrac{\cot x}{1 - \tan x} + \dfrac{\tan x}{1 - \cot x} - \dfrac{2}{\sin x}$

E $\sin\left(\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right)$

9 AN ALTERNATING CURRENT GENERATOR GENERATES A CURRENT GIVEN BY THE FORMULA I = 20 SIN 40 t, WHERE t IS TIME IN SECONDS.

A DETERMINE THE AMPLITUDE AND THE PERIOD.

B WHAT IS THE FREQUENCY OF THE CURRENT?

10 AN AEROPLANE IS FLYING IN A DIRECTION N 15°E AT AN AIR SPEED OF 1403 KM/HR. A STEADY WIND OF 56 KM/HR IS BLOWING IN THE DIRECTION S 30°W. FIND THE VELOCITY OF THE AEROPLANE RELATIVE TO THE GROUND.

11 A BOAT DIRECTED N 75°E IS CROSSING A RIVER AT A SPEED OF 20 KM/HR RELATIVE TO THE WATER. THE RIVER IS FLOWING IN THE DIRECTION S 30°E AT 6 KM/HR. FIND THE VELOCITY OF THE BOAT RELATIVE TO THE GROUND.

12 IN ΔXYZ, $x = 23.5$, $y = 9.8$, $\angle X = 39.7^{\circ}$. SOLVE THE TRIANGLE.

13 IN ΔABC, $b = 15$, $c = 20$, AND $\angle B = 29^{\circ}$. SOLVE THE TRIANGLE.

14 IF $x = a \cos - b \sin$ AND $y = a \sin + b \cos$, EXPRESS $x^2 + y^2$ IN TERMS OF a AND b

15 **Simple pendulum:** AN OBJECT CONSISTING OF A POINT MASS m SUSPENDED BY A WEIGHTLESS STRING OF LENGTH ℓ AS SHOWN IN FIGURE 9.47. IF IT IS PULLED TO ONE SIDE OF ITS VERTICAL POSITION AND RELEASED, IT MOVES PERIODICALLY TO THE RIGHT AND TO

LET y DENOTE THE DISPLACEMENT OF THE MASS FROM ITS VERTICAL POSITION, MEAS ALONG THE ARC OF THE SWING. SUPPOSE THAT $y = a$ WHEN $t = 0$, THE INSTANCE OF RELEASE. THEN, IF a IS NOT TOO LARGE, THE QUANTITY y WILL APPROXIMATELY OSCILLATE ACCORDING TO THE SIMPLE HARMONIC MODEL $y = a \cos \omega t$ WITH PERIOD $T = 2\sqrt{\frac{l}{g}}$,

WHERE g IS THE ACCELERATION OF GRAVITY.

$g \approx 32$ FEET/SEC^2 OR $g \approx 9.8$ M/SEC^2

IF $\ell = 1.2$ M AND $a = 0.06$ M, DETERMINE THE EQUATION FOR y AS A FUNCTION OF t AND FIND

A THE PERIOD.

B THE ANGULAR FREQUENCY.

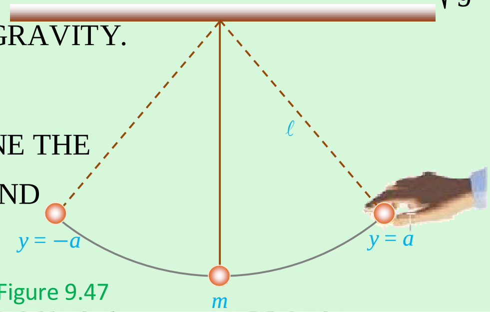


Figure 9.47

Key Terms

arccosecant	cosecant	secant
arccosine	cosine	sine
arccotangent	cotangent	sinusoidal
arcsecant	harmonic motion	tangent
arcsine	laws of cosines	trigonometric identities
arctangent	laws of sines	

Summary

1 ***The Reciprocal Trigonometric Functions:***

I ***The Cosecant Function:*** THE RECIPROCAL OF SINE FUNCTION, $y = \csc x$.

✓ $\csc x = \dfrac{1}{\sin x}$

✓ DOMAIN $= \mathbb{R} \setminus \{k\ : k \in \mathbb{Z}\}$

✓ RANGE $= (-\infty, -1] \cup [1, \infty)$

✓ PERIOD $= 2$

II ***The Secant Function:*** THE RECIPROCAL OF COSINE FUNCTION, $y = \text{SEC } x$.

- ✓ $\text{SEC } x = \dfrac{1}{\text{COS } x}$
- ✓ DOMAIN $= \mathbb{R} \setminus \left\{ (2k+1)\dfrac{\;}{2} \ : \ k \in \mathbb{Z} \right\}$
- ✓ RANGE $= (-\infty, \ -1] \cup [1, \infty)$
- ✓ PERIOD = 2

III ***The Cotangent Function:*** THE RECIPROCAL OF TANGENT FUNCTION, $y = \text{COT } x$.

- ✓ $\text{COT } x = \dfrac{1}{\text{TAN } x}$
- ✓ DOMAIN $= \mathbb{R} \setminus \{ k \ : k \in \mathbb{Z} \}$
- ✓ RANGE $= \mathbb{R}$
- ✓ PERIOD =

2 *Inverse Trigonometric Functions*

I ***The Inverse Sine or Arcsine***

$\text{SIN}^{-1} x = y$, IF AND ONLY IF $x = \text{SIN } y$ AND $-\dfrac{\;}{2} \le y \le \dfrac{\;}{2}$.

II ***The Inverse Cosine or Arccosine***

$\text{COS}^{-1} x = y$, IF AND ONLY IF $x = \text{COS } y$ AND $0 \le y \le$

III ***The Inverse Tangent or Arctangent***

$\text{TAN}^{-1} x = y$, IF AND ONLY IF $x = \text{TAN } y$ AND $-\dfrac{\;}{2} < y < \dfrac{\;}{2}$

IV ***The Inverse Cosecant or Arccosecant***

$\text{CSC}^{-1} x = y$, IF AND ONLY IF $x = \text{CSC } y$ AND $-\dfrac{\;}{2} \le y \le \dfrac{\;}{2}$ WITH $y \ne 0$.

$\text{CSC}^{-1} x = \text{SIN}^{-1}\left(\dfrac{1}{x}\right); |x| \ge 1$

V ***The Inverse Secant or Arcsecant***

$\text{SEC}^{-1} x = y$, IF AND ONLY IF $x = \text{SEC } y$ AND $0 \le y \le$ WITH $y \ne \dfrac{\;}{2}$.

$\text{SEC}^{-1} x = \text{COS}^{-1}\left(\dfrac{1}{x}\right); \ |x| \ge 1$

VI ***The Inverse Cotangent or Arccotangent***

COT^{-1} $x = y$ IF AND ONLY IF x = COT y AND $0 < y <$.

COT^{-1} $x = \frac{}{2}$ – TAN^{-1} x

3 Graphs of some trigonometric functions.

$y = a$ SIN $k(x + b) + c$ AND $y = a$ COS $k(x + b) + c$,

I **Amplitude** $= |a|$

II **Period**, $P = \frac{2}{k}$; $k > 0$

WHEN $k < 0$, USE THE SYMMETRIC PROPERTY

III **Range** $= \left[c - |a|,\ c + |a|\right]$

IV **Phase angle** $= -b$

V **Phase shift** $= \frac{-b}{k}$

4 Applications of Trigonometric Functions

Solving a triangle

I **The Law of Sines** $\frac{a}{\text{SIN } A} = \frac{b}{\text{SIN } B} = \frac{c}{\text{SIN } C}$

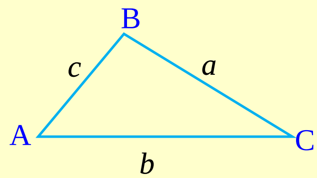


Figure 9.48

II **The Law of Cosines**

$c^2 = a^2 + b^2 - 2ab$ COS C, $b^2 = a^2 + c^2 - 2ac$ COS B,

$a^2 = b^2 + c^2 - 2bc$ COS A.

III TRIGONOMETRIC FORMULAE FOR THE SUM AND DIFFERENCE

The addition and difference identities

✓ SIN $(x \pm y)$ = SIN x COS $y \pm$ COS x SIN y

✓ COS $(x \pm y)$ = COS x COS $y \mp$ SIN x SIN y

✓ TAN $(x \pm y) = \frac{\text{TAN } x \pm \text{TAN } y}{1 \mp \text{TAN } x \text{ TAN } y}$

Double – Angle Formulas

✓ COS $(2x)$ = COS$^2 x$ – SIN$^2 x$

COS $(2x)$ = 2 COS$^2 x - 1$

✓ COS $(2x) = 1 - 2$ SIN$^2 x$

✓ SIN $(2x)$ = 2 SIN x COS x

- ✓ $\text{TAN}(2x) = \dfrac{2\,\text{TAN}\,x}{1-\text{TAN}^2 x}$

Half Angle Formulas

- ✓ $\text{COS}^2\left(\dfrac{x}{2}\right) = \dfrac{1+\text{COS}\,x}{2}$
- ✓ $\text{SIN}^2\left(\dfrac{x}{2}\right) = \dfrac{1-\text{COS}\,x}{2}$
- ✓ $\text{TAN}^2\left(\dfrac{x}{2}\right) = \dfrac{1-\text{COS}\,x}{1+\text{COS}\,x}$; $\text{COS}\,x \neq -1$

5 *Simple Harmonic Motion*

$g(t) = a\,\text{COS}(\ \ t) + b\,\text{SIN}(\ \ t)$

- ✓ **period**, $P = \dfrac{2}{\ }$
- ✓ **frequency**, $f = \dfrac{\ }{2}$

Review Exercises on Unit 9

1 PROVE THE FOLLOWING IDENTITIES.

A $\text{COT}(x + \ ) = \text{COT}\,x$ **B** $\text{COT}(\ - x) = -\text{COT}\,x$

C $\text{SEC}(-x) = \text{SEC}\,x$ **D** $\text{CSC}(-x) = -\text{CSC}\,x$

2 FIND EACH VALUE.

A $\text{SEC}\dfrac{\ }{4}$ **B** $\text{CSC}\dfrac{\ }{6}$ **C** $\text{COT}\dfrac{\ }{2}$

3 EXPLAIN HOW THE GRAPH OF $y = \text{CSC}\,x$ IS RELATED TO THE GRAPH OF $y = \text{SEC}\,x$.

4 FIND A FUNCTION f OF THE FORM $f(x) = a\,\text{SIN}(kx)$ SATISFYING THE GIVEN PROPERTIES

A AMPLITUDE 3 AND PERIOD $\dfrac{2}{5}$

B AMPLITUDE $\dfrac{2}{5}$ AND $f(3) = 0$

C PEAK AT $\left(\dfrac{\ }{3}, 5\right)$

D AMPLITUDE 2, THE GRAPH PASSES THROUGH $\left(\dfrac{\ }{3}, 0\right)$

5 REPEAT PROBLEM NUMBER 4, IF $f(x) = a\,\text{COS}\,k(x)$.

6 FIND EACH VALUE.

A $\text{SIN}^{-1}\left(\frac{-\sqrt{2}}{2}\right)$ **B** TAN^{-1} (**C** $\text{TAN}^{-1}\left(-\sqrt{\ }\right)$

7 USING A CALCULATOR OR TABLES, FIND EACH VALUE

A ARCSIN (0.0941) **B** ARCCOS (0.5525) **C** ARCTAN (−2.4147)

8 FIND THE EXACT VALUES OF EACH OF THE FOLLOWING WITHOUT USING A CALCULATOR OR TABLES.

A $\text{SIN}\left(\text{SIN}^{-1}\left(\frac{3}{5}\right)\right)$ **B** $\text{SIN}\left(\text{SIN}^{-1}(0.025)\right)$

C $\text{COS}\left(\text{COS}^{-1}\left(-\frac{3}{4}\right)\right)$ **D** $\text{SIN}\left(\text{COS}^{-1}\left(\frac{1}{8}\right)\right)$

E $\text{COS}\left(\text{SIN}^{-1}x\right)$ FOR $|x| \le$ **F** $\text{SIN}\left(\text{COS}^{-1}x\right)$ FOR $|x| \le$

G $\text{TAN}\left(\text{COS}^{-1}\frac{4}{9}\right)$ **H** $\text{SIN}\left(2\,\text{TAN}^{-1}\left(-\frac{4}{5}\right)\right)$

9 IF $\text{SIN}(\ +\ ) = \frac{55}{73}$ AND $\text{SIN}\ = \frac{3}{5}$ FIND SIN

10 IF $\text{SIN}\,x = -\frac{12}{37}$, $< x < \frac{3}{2}$, FIND $\text{COS}\left(\frac{x}{2}\right)$

11 DRAW THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS FOR ONE CYCLE.

A $f(x) = 2\,\text{SIN}\left(x - \frac{}{2}\right)$ **B** $f(x) = \text{COS}\left(-\frac{1}{2}x + \frac{}{4}\right)$

C $f(x) = 3 - \text{SIN}\left(\frac{1}{2}x + \frac{}{4}\right)$ **D** $f(x) = 2\,\text{COS}\left(\frac{}{4}x\right) +$

12 USE THE LAW OF SINES TO SOLVE ΔABC IF

A $a = 5$, $= 50^{O}$, $= 70^{O}$

B $a = 5$, $b = 3$, $= 45^{O}$

C $a = 11$, $b = 24$, $= 59.5^{O}$

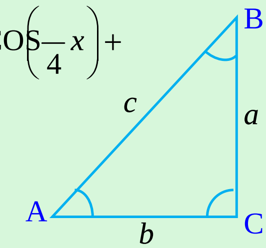


Figure 9.49

13 USE THE LAW OF COSINES TO SOLVE ΔABC IF

A $a = 5$, $b = 6$, $= 60^{O}$ **B** $b = 8$, $c = 7$, $= 30^{O}$

C $a = 20$, $c = 30$, $= 110^{O}$

14 SOLVE EACH OF THE FOLLOWING TRIGONOMETRIC EQUATIONS.

A $\sin(2x) = \sqrt{3}\sin x$ **B** $\sin(2x) = -\frac{1}{\sqrt{2}}$

C $\tan\left(x - \frac{\ }{4}\right) = \sqrt{3}$ **D** $2\sin x = \sin(2x)$

E $\tan\left(\frac{x}{2}\right) - 2\sin x =$

15 TWO DRIVERS A AND B LEAVE THE SAME PLACE AT THE SAME TIME. A DRIVES 80 KM/HR IN THE DIRECTION OF N 30° E AND B DRIVES 90 KM/HR IN THE DIRECTION OF S 60° W. HOW FAR APART ARE THEY AFTER $1\frac{1}{2}$ HOURS?

16 A TOWER 15 M HIGH IS ON THE BANK OF A RIVER. IT IS OBSERVED THAT THE ANGLE OF DEPRESSION FROM THE TOP OF THE TOWER TO A POINT ON THE OPPOSITE SHORE IS 30° AND THE ANGLE OF DEPRESSION FROM THE BASE OF THE TOWER TO THE SAME POINT ON THE OPPOSITE SHORE IS OBSERVED TO BE 15°. FIND THE WIDTH OF THE RIVER.

17 THE REFRACTION INDEX OF ICE WITH RESPECT TO AIR IS 1.309. DETERMINE THE ANGLE OF REFRACTION OF A RAY OF LIGHT THAT STRIKES A BLOCK OF ICE WITH AN ANGLE OF INCIDENCE $= 40^{\circ}$.

18 PROVE EACH OF THE FOLLOWING TRIGONOMETRIC IDENTITIES.

A $\cos^4 x - \sin^4 x = \cos(2x)$ **B** $\frac{\cos(2x)}{1+\sin(2x)} = \frac{\cot x - 1}{\cot x + 1}$

19 SIMPLIFY $y = \tan(2\sin^{-1} x)$ IN TERMS OF x.

20 THE POPULATION (IN HUNDREDS) OF A SPECIES OF A BIRD IN AN AREA IS MODELLED BY THE FUNCTION

$$P(t) = 5 + 3\sin\left(\frac{2\pi t}{5}\right);\ 0 \le t \le 12$$

WHERE t IS THE TIME IN MONTHS,

DETERMINE:

A THE INITIAL POPULATION.

B THE LARGEST AND SMALLEST POPULATIONS.

C THE FIRST TIME IN WHICH THE POPULATION REACHES 350 BIRDS.

D THE POPULATION AFTER ONE YEAR.