p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Unit 4

MATHEMATICAL REASONING

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about mathematical logic.*
- *know methods and procedures in combining and determining the validity of statements.*
- *know basic facts about argument and validity.*

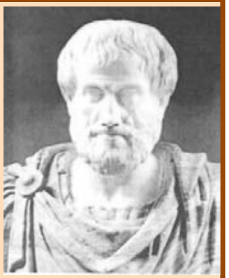
Main Contents

INTRODUCTION

MATHEMATICAL REASONING IS A TOOL FOR ORGANIZING EVIDENCE IN A SYSTEMATIC WAY MATHEMATICAL LOGIC. IN THIS UNIT, YOU WILL STUDY MATHEMATICAL LOGIC, THE SYSTEM OF THE ART OF REASONING. IN SOME WAYS, MATHEMATICS CAN BE THOUGHT OF AS A T LOGIC. LOGIC HAS A WIDE RANGE OF APPLICATIONS, PARTICULARLY IN JUDGING THE CORRE CHAIN OF REASONING, AS IN MATHEMATICAL PROOFS.

IN THE FIRST SUB-UNIT, LOGIC, YOU WILL STUDY THE FOLLOWING: STATEMENTS AN STATEMENTS, FUNDAMENTAL LOGICAL CONNECTIVES (OR LOGICAL OPERATORS), C PROPOSITIONS, PROPERTIES AND LAWS OF LOGICAL CONNECTIVES, CONTRADICTION AND CONVERSE, CONTRAPOSITIVE AND QUANTIFIERS. IN THE SECOND SUB-UNIT, YOU WIL ARGUMENTS, VALIDITY, AND RULES OF INFERENCES.

HISTORICAL NOTE



Aristotle (384 – 322 B.C.)

Aristotle was one of the greatest philosophers of ancient Greece. After studying for twenty years in Plato's Academy, he became tutor to Alexander the Great. Later, he founded his own school, the Lyceum, where he contributed to nearly every field of human knowledge. After Aristotle's death, his treatises on reasoning were grouped together and came to be known as the Organon.

The word "logic" did not acquire its modern meaning until the second century AD, but the subject matter of logic was determined by the content of the Organon.

OPENING PROBLEM

DO YOU THINK THAT THE FOLLOWING ARGUMENTS ARE ACCEPTABLE?

WAGES WILL INCREASE ONLY IF THERE IS INFLATION. IF THERE IS INFLATION, THEN THE COS WILL INCREASE. WAGES WILL INCREASE. THEREFORE, THE COST OF LIVING WILL INCREASE.

CONFUSED! DON'T WORRY! YOUR STUDY OF LOGIC WILL HELP YOU TO DECIDE WHETHER OR GIVEN ARGUMENT IS ACCEPTABLE.

4.1 LOGIC

IN THIS SUB-UNIT, YOU WILL LEARN MATHEMATICAL LOGIC AT ITS ELEMENTARY LEVEL, PROPOSITIONAL LOGIC. PROPOSITIONAL LOGIC IS THE STUDY OF ASSERTIVE OR DECLARATI WHICH CAN BE SAID TO BE EITHER TRUE DENOTED BY **T** OR FALSE DENOTED BY **F**, BUT NOT BOTH. THE VALUE T OR F THAT IS ASSIGNED TO A SENTENCE IS CALLED THE **truth value** OF THE SENTENCE.

4.1.1 "Statement" and "Open Statement"

WE BEGIN THIS SUBTOPIC BY IDENTIFYING WHETHER A GIVEN SENTENCE CAN BE SAID TO B FALSE OR NEITHER. WE DEFINE THOSE SENTENCES WHICH CAN BE SAID TO BE TRUE OR FA NOT BOTH, AS **statements** OR **propositions**. THE FOLLOWING GROUP WORK SHOULD LEAD TO THE DEFINITION.

Group Work 4.1

DISCUSS THE FOLLOWING ISSUES IN GROUPS AND JUSTI … S.

1 WHAT IS A SENTENCE?

2 IDENTIFY WHETHER THE FOLLOWING SENTENCES CAN BE TRUE, FALSE OR NEITHER AND GIVE YOUR REASONS.

A MAN IS MORTAL.

B WELCOME.

C $2 + 5 = 9$

D $4 + 5 = 9$

E GOD BLESS YOU.

F IT IS IMPOSSIBLE TO GET MEDICINE FOR HIV/AIDS.

G YOU CAN GET A GOOD GRADE IN MATHEMATICS.

H $x + 6 = 8$

I KING ABBA JIFAR WEIGHED 60 KG WHEN HE WAS 30 YEARS OLD.

J $x + 3 < 10$

K ____ IS A TOWN IN ETHIOPIA.

L x IS LESS THAN y

FROM THE ABOVE GROUP WORK, YOU MAY HAVE IDENTIFIED THE FOLLOWING:

- ✓ SENTENCES WHICH CAN BE SAID TO BE TRUE OR FALSE (BUT NOT BOTH).
- ✓ SENTENCES WITH ONE OR MORE VARIABLES OR BLANK SPACES.
- ✓ SENTENCES WHICH EXPRESS HOPES OR OPINIONS (AS OPPOSED TO FACTS).

Definition 4.1

I A SENTENCE WHICH CAN BE SAID TO BE TRUE OR FALSE (BUT NOT BOTH) IS SAID TO BE A **proposition** (OR **statement**).

II A SENTENCE WITH ONE OR MORE VARIABLES WHICH BECOMES A STATEMENT UPON REPLACING THE VARIABLE OR VARIABLES BY AN INDIVIDUAL OR INDIVIDUALS IS CALLED AN **open proposition** (**or open statement**).

III THE WORDS TRUE AND FALSE, DENOTED BY T AND F, RESPECTIVELY, ARE CALLED **truth values**.

Example 1 FROM GROUP WORK 4.1 ABOVE, YOU SEE THAT

I **A** MAN IS MORTAL. **C** $2 + 5 = 9$ **D** $4 + 5 = 9$

I KING ABBA JIFAR WEIGHED 60 KG WHEN HE WAS 30 YEARS OLD,

ARE ALL PROPOSITIONS.

II **H** $x + 6 = 8$ **J** $x + 3 < 10$ **L** x IS LESS THAN y

K ____ IS A TOWN IN ETHIOPIA, ARE ALL OPEN PROPOSITIONS.

III **B** WELCOME. **F** IT IS IMPOSSIBLE TO GET MEDICINE FOR HIV/AIDS.

G YOU CAN GET A GOOD GRADE IN MATHEMATICS. **E** GOD BLESS YOU,

ARE ALL NEITHER PROPOSITIONS NOR OPEN PROPOSITIONS.

Exercise 4.1

IDENTIFY EACH OF THE FOLLOWING AS A PROPOSITION, AN OPEN PROPOSITION OR NEITHER.

A ON HIS 35TH BIRTHDAY, EMPEROR TEWODROS INVITED 1000 PEOPLE FOR DINNER.

B SUDAN IS A COUNTRY IN AFRICA.

C IF x IS ANY REAL NUMBER, THEN $x^2 - 1 = (x - 1)(x + 1)$.

D YOU ARE A GOOD STUDENT.

E A SQUARE OF AN EVEN NUMBER IS EVEN.

F AMBO IS A TOWN IN OROMIYA.

G $8^{90} > 9^{80}$

H GOD HAVE MERCY ON MY SOUL!

I x IS LESS THAN 9.

J _______ IS THE STUDY OF PLANTS.

K FOR A REAL NUMBER x, $x^2 + 1 < 0$.

L NO WOMAN SHOULD DIE WHILE GIVING BIRTH.

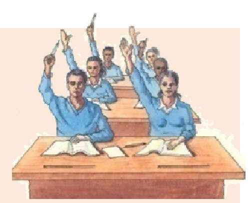
M LAWS AND ORDERS ARE DYNAMIC.

N EVERY CHILD HAS THE RIGHT TO BE FREE OF CORPORAL PUNISHMENT.

4.1.2 Fundamental Logical Connectives (Operators)

GIVEN TWO OR MORE PROPOSITIONS, YOU CAN USE CONNECTIVES TO JOIN THE SENTENC
FUNDAMENTAL CONNECTIVES IN LOGIC ARE: **or, and, if ... then, if and only if** AND **not**.
UNDER THIS SUBTOPIC, YOU LEARN HOW TO FORM A STATEMENT WHICH CONSISTS OF TWO
COMPONENT PROPOSITIONS CONNECTED BY LOGICAL CONNECTIVES OR LOGICAL OPERATO
THIS, YOU ALSO LEARN THE RULES THAT GOVERN US WHEN COMMUNICATING THROUGH
WILL BEGIN WITH THE FOLLOWING ACTIVITY.

ACTIVITY 4.1



CONSIDER THE FOLLOWING PROPOSITIONS.

WATER IS A NATURAL RESOURCE. (TRUE)

PLANTS DO NOT NEED WATER TO GROW. (FALSE)

WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

EVERYONE DOES NOT HAVE THE RIGHT TO HOLD OPINIONS WITHOUT INTERFERENCE. (F

DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING:

A WATER IS **not** A NATURAL RESOURCE.

B PLANTS NEED WATER TO GROW.

C WATER IS A NATURAL RESOURCE **and** PLANTS NEED WATER TO GROW.

D WATER IS A NATURAL RESOURCE **or** PLANTS NEED WATER TO GROW.

- **E** If WATER IS A NATURAL RESOURCE, then PLANTS NEED WATER TO GROW.
- **F** WATER IS A NATURAL RESOURCE, if and only if PLANTS NEED WATER TO GROW.
- **G** WATER IS not A NATURAL RESOURCE or WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT.
- **H** WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT, if and only if EVERYONE DOES NOT HAVE THE RIGHT TO HOLD OPINIONS WITHOUT INTERFERENCE.
- **I** If WATER IS not A NATURAL RESOURCE, then PLANTS NEED WATER TO GROW.
- **J** If EVERYONE HAS NO RIGHT TO HOLD OPINIONS WITHOUT INTERFERENCE, then WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT.

TO FIND THE TRUTH-VALUE OF A STATEMENT WHICH IS COMBINED BY USING CONNECTIV NEED RULES WHICH GIVE THE TRUTH VALUE OF THE COMPOUND STATEMENT. YOU AL SYMBOLS FOR CONNECTIVES AND NOTATIONS FOR PROPOSITIONS. YOU USUALLY R PROPOSITIONS BY SMALL LETTERS, SUCH AS p, q, r, s, t, AND SO ON. NOW LET p REPRESENT ONE PROPOSITION AND q REPRESENT ANOTHER PROPOSITION.

Connective	Name of the connective	Symbol	How to write	How to read
not	NEGATION	$\neg$	$\neg p$	THE NEGATION OF p
and	CONJUNCTIO	$\wedge$	$p \wedge q$	p AND q
or	DISJUNCTION	$\vee$	$p \vee q$	p OR q
If…, then…	IMPLICATION	$\Rightarrow$	$p \Rightarrow q$	p IMPLIES q
If and only if	BI-IMPLICATIO	$\Leftrightarrow$	$p \Leftrightarrow q$	p IF AND ONLY IF q

Example 2 LET p REPRESENT THE PROPOSITION: WATER IS A NATURAL RESOURCE.

LET q REPRESENT THE PROPOSITION: PLANTS NEED WATER TO GROW. THEN,

- **A** $\neg p$ REPRESENTS: WATER IS NOT A NATURAL RESOURCE.
- **B** $p \wedge q$ REPRESENTS: WATER IS A NATURAL RESOURCE AND PLANTS NEED WATER TO
- **C** $p \vee q$ REPRESENTS: WATER IS A NATURAL RESOURCE OR PLANTS NEED WATER TO G
- **D** $p \Rightarrow q$ REPRESENTS: IF WATER IS A NATURAL RESOURCE, THEN PLANTS NEED WA GROW.
- **E** $p \Leftrightarrow q$ REPRESENTS: WATER IS A NATURAL RESOURCE, IF AND ONLY IF PLANTS WATER TO GROW.

NOW WE WILL SEE TO THE RULES THAT GOVERN US IN COMMUNICATING THROUGH LOGIC **truth tables** FOR EACH OF THE LOGICAL OPERATORS.

RULE 1 Rule for Negation (“¬”)

LET p BE A PROPOSITION.

THEN AS SHOWN FROM THE TABLE BELOW, ITS NEGATION IS REPRESENTED BY $\neg p$

> **Note:**
>
> $\neg p$ IS TRUE, IF AND ONLY IF p IS FALSE.

THIS IS BEST EXPLAINED BY THE FOLLOWING TABLE CALLED THE TRUTH TABLE FOR NEGATI

p	$\neg p$
T	F
F	T

Example 3 p: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

$\neg p$: WORK IS NOT AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (FALSE)

q: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

$\neg q$: NAIROBI IS NOT THE CAPITAL CITY OF ETHIOPIA. (TRUE)

> **Note:**
>
> THE WORD “NOT” DENOTED BY “¬” IS APPLIED TO A SINGLE STATEMENT AND DOES NOT CONNECT TWO STATEMENTS, AS A RESULT OF THIS, THE NAME LOGICAL OPERATOR IS APPROPRIATE FO

RULE 2 Rule for Conjunction (“∧”)

WHEN TWO PROPOSITIONS p AND q ARE JOINED WITH THE CONNECTIVE "and" (DENOTED BY $p \wedge q$), THE PROPOSITION FORMED IS A LOGICAL CONJUNCTION. p AND q ARE CALLED **the components of the conjunction**.

$p \wedge q$ IS TRUE, IF AND ONLY IF p AND q BOTH ARE TRUE.

TO DETERMINE THE TRUTH VALUE OF $p \wedge q$, WE HAVE TO KNOW THE TRUTH VALUE OF THE COMPONENTS p AND q.

The possibilities are as follows:

- p TRUE AND q TRUE
- p FALSE AND q TRUE
- p TRUE AND q FALSE
- p FALSE AND q FALSE.

THIS IS ILLUSTRATED BY THE FOLLOWING TRUTH TABLE.

THE TRUTH TABLE FOR CONJUNCTION IS GIVEN AS:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 4 CONSIDER THE FOLLOWING PROPOSITIONS:

p: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

q: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

r: $2 < 3$ (TRUE)

A $p \wedge q$: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT AND NAIROBI IS CAPITAL CITY OF ETHIOPIA. (FALSE)

B $p \wedge \neg q$: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT AND NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (TRUE)

C $p \wedge r$: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT AND $2 < 3$. (TRUE)

RULE 3 Rule for Disjunction ("∨")

WHEN TWO PROPOSITIONS p AND q ARE JOINED WITH THE CONNECTIVE **"or"** (DENOTED BY $p \vee q$), THE PROPOSITION FORMED IS A LOGICAL DISJUNCTION.

P ∨ Q IS FALSE, IF AND ONLY IF BOTH P AND Q ARE FALSE.

TO DETERMINE THE TRUTH VALUE OF $p \vee q$, WE HAVE TO KNOW THE TRUTH VALUE OF THE COMPONENTS p AND q. AS MENTIONED EARLIER, IF WE HAVE TWO PROPOSITIONS TO BE COMBINED THERE ARE FOUR POSSIBILITIES OF COMBINATIONS OF THE TRUTH VALUES OF COM PROPOSITIONS.

THE TRUTH TABLE FOR DISJUNCTION IS GIVEN AS:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5 CONSIDER THE FOLLOWING PROPOSITIONS

p: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

q: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

r: $2 < 3$ (TRUE)

A $p \vee q$: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT OR NAIROBI IS T
CAPITAL CITY OF ETHIOPIA. (TRUE)

B $q \vee r$: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA OR $2 < 3$. (TRUE)

C $q \vee \neg r$: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA OR $2 \geq 3$. (FALSE)

RULE 4 Rule for Implication ("⇒")

WHEN TWO PROPOSITIONS p AND q ARE JOINED WITH THE CONNECTIVE "implies" (DENOTED BY $p \Rightarrow q$) THE PROPOSITION FORMED IS A LOGICAL IMPLICATION.

$p \Rightarrow q$ IS FALSE, IF AND ONLY IF p IS TRUE AND q IS FALSE.

THIS IS ILLUSTRATED BY THE TRUTH TABLE FOR IMPLICATION WHICH IS GIVEN AS FOLLOWS:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 6 CONSIDER THE FOLLOWING PROPOSITIONS:

p: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

q: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

r: $2 < 3$ (TRUE)

A $p \Rightarrow q$: IF WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT, THEN NAIR
IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

B $q \Rightarrow r$: IF NAIROBI IS THE CAPITAL CITY OF ETHIOPIA, THEN $2 < 3$. (TRUE)

C $q \Rightarrow \neg r$: IF NAIROBI IS THE CAPITAL CITY OF ETHIOPIA, THEN $2 \geq 3$. (TRUE)

D $\neg q \Rightarrow r$: IF NAIROBI IS NOT THE CAPITAL CITY OF ETHIOPIA, THEN $2 < 3$. (TRUE)

RULE 5 Rule for Bi-implication ("if and only if")

WHEN TWO PROPOSITIONS p AND q ARE JOINED WITH THE CONNECTIVE **"bi-implication"** (DENOTED BY $p \Leftrightarrow q$) THE PROPOSITION FORMED IS A LOGICAL BI-IMPLICATION.

$p \Leftrightarrow q$ IS FALSE, IF AND ONLY IF p AND q HAVE DIFFERENT TRUTH VALUES.

THIS IS ILLUSTRATED BY THE TRUTH TABLE FOR BI-IMPLICATION, WHICH IS GIVEN AS FOLLOWS:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 7 CONSIDER THE FOLLOWING PROPOSITIONS

p: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT. (TRUE)

q: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

r: $2 < 3$ (TRUE)

A $p \Leftrightarrow q$: WORK IS AN INSTRUMENT FOR NATIONAL DEVELOPMENT, IF AND ONLY NAIROBI IS THE CAPITAL CITY OF ETHIOPIA. (FALSE)

B $q \Leftrightarrow r$: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA, IF AND ONLY IF $2 < 3$. (FALSE)

C $q \Leftrightarrow \neg r$: NAIROBI IS THE CAPITAL CITY OF ETHIOPIA, IF AND ONLY IF $2 \geq 3$. (TRUE)

D $\neg q \Leftrightarrow r$: NAIROBI IS NOT THE CAPITAL CITY OF ETHIOPIA, IF AND ONLY IF $2 < 3$. (TRUE)

Exercise 4.2

GIVEN THAT p: MAN IS MORTAL.

q: BOTANY IS THE STUDY OF PLANTS.

r: 6 IS A PRIME NUMBER.

DETERMINE THE TRUTH VALUES OF EACH OF THE FOLLOWING.

A $p \wedge q$ **D** $\neg p \vee q$ **G** $\neg p \wedge \neg q$

B $(p \wedge q) \Rightarrow r$ **E** $\neg (p \vee q)$ **H** $\neg p \vee \neg q$

C $(p \wedge q) \Leftrightarrow \neg r$ **F** $\neg (p \wedge q)$ **I** $p \Leftrightarrow q$

4.1.3 Compound Statements

SO FAR, YOU HAVE DEFINED STATEMENTS AND LOGICAL CONNECTIVES (OR LOGICAL OPERA YOU HAVE SEEN THE RULES THAT GO WITH THE LOGICAL CONNECTIVES. NOW YOU ARE GOI A NAME FOR STATEMENTS FORMED FROM TWO OR MORE COMPONENT PROPOSITIONS BY USI LOGICAL OPERATORS. EACH 'SENTENCE' A – I IN EXERCISE 4.2 IS A STATEMENT FORMED BY USING ONE OR MORE CONNECTIVES.

Definition 4.2

A STATEMENT FORMED BY JOINING TWO OR MORE STATEMENTS BY A CONNECTIVE CONNECTIVES) IS CALLED A **compound statement**.

Example 8 CONSIDER THE FOLLOWING STATEMENTS:

p: 3 DIVIDES 81. (TRUE)

q: KHARTOUM IS THE CAPITAL CITY OF KENYA. (FALSE)

r: A SQUARE OF AN EVEN NUMBER IS EVEN. (TRUE)

s: $\frac{22}{7}$ IS AN IRRATIONAL NUMBER. (FALSE)

DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING:

A $(p \wedge q) \Rightarrow (r \vee s)$ **B** $(\neg p \vee q) \wedge (r \wedge s)$

C $(p \wedge r) \Leftrightarrow (q \wedge s)$ **D** $(r \vee S) \wedge (p \wedge \neg q)$

Solution:

A $p \wedge q$ HAS TRUTH VALUE F, $r \vee s$ HAS TRUTH VALUE T, THUS $(p \wedge q) \Rightarrow (r \vee s)$ HAS TRUTH VALUE T.

B $(\neg p \vee q)$ HAS TRUTH VALUE F, AND HENCE $(\neg p \vee q) \wedge (r \vee s)$ HAS TRUTH VALUE F.

C $(p \wedge r)$ HAS TRUTH VALUE T, $(q \wedge s)$ HAS TRUTH VALUE F AND HENCE $(p \wedge r) \Leftrightarrow (q \wedge s)$ HAS TRUTH VALUE F.

D $(r \vee S)$ HAS TRUTH VALUE T AND $(p \wedge \neg q)$ HAS TRUTH VALUE T, HENCE $(r \vee s) \wedge (p \wedge \neg q)$ HAS TRUTH VALUE T.

Example 9 LET p, q, r HAVE TRUTH VALUES T, F, T, RESPECTIVELY. DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING.

A $\neg p \vee q$ **B** $\neg p \wedge \neg q$ **C** $(p \vee q) \Rightarrow r$

Solution:

a SINCE p HAS TRUTH VALUE T, $\neg p$ HAS TRUTH VALUE F.

$\neg p$ HAS TRUTH VALUE F AND q ALSO HAS TRUTH VALUE F.

THUS $\neg p \vee q$ HAS TRUTH VALUE F BY THE RULE OF LOGICAL DISJUNCTION.

b FROM (A) $\neg p$ HAS TRUTH VALUE F.

q HAS TRUTH VALUE F, AND HENCE $\neg q$ HAS TRUTH VALUE T.

THUS $\neg p \wedge \neg q$ HAS TRUTH VALUE F BY THE RULE OF CONJUNCTION.

c SINCE p HAS TRUTH VALUE T AND q HAS TRUTH VALUE F.

$p \vee q$ HAS TRUTH VALUE T BY THE RULE OF DISJUNCTION.

SINCE r HAS TRUTH VALUE T, $(p \vee q) \Rightarrow r$ HAS TRUTH VALUE T BY THE RULE OF IMPLICATION.

Example 10 LET p AND q BE ANY TWO PROPOSITIONS. CONSTRUCT ONE TRUTH TABLE FOR EACH OF THE FOLLOWING PAIRS OF COMPOUND PROPOSITION AND COMPARE THEIR TRUTH VALUES.

a $p \Rightarrow q$, $\neg p \vee q$

b $\neg (p \vee q)$, $\neg p \wedge \neg q$

c $p \Rightarrow q$, $\neg q \Rightarrow \neg p$

d $p \Rightarrow q$, $q \Rightarrow p$

Solution WE CONSTRUCT THE TRUTH TABLE AS FOLLOWS:

a

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

OBSERVE THAT BOTH $p \Rightarrow q$ AND $\neg p \vee q$ HAVE THE SAME TRUTH VALUES.

b

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

OBSERVE THAT BOTH $\neg(p \vee q)$ AND $\neg p \wedge \neg q$ HAVE THE SAME TRUTH VALUES.

C

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

OBSERVE THAT BOTH $p \Rightarrow q$ AND $\neg q \Rightarrow \neg p$ HAVE THE SAME TRUTH VALUES.

D

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

OBSERVE THAT $p \Rightarrow q$ AND $q \Rightarrow p$ DO NOT HAVE THE SAME TRUTH TABLE. AS YOU HAVE SEEN FROM EXAMPLE 10, SOME COMPOUND PROPOSITIONS HAVE THE SAME TRUTH VALUES FOR EACH ASSIGNMENT OF THE TRUTH VALUES OF COMPONENT PROPOSITIONS. SUCH PAIRS OF COMPOUND PROPOSITIONS ARE CALLED **equivalent propositions**. WE USE THE SYMBOL "≡" IN-BETWEEN THE TWO PROPOSITIONS TO MEAN THEY ARE EQUIVALENT.

THUS, FROM OBSERVATION OF THE TABLES FOR EXAMPLE 10, WE HAVE:

A $p \Rightarrow q \equiv \neg p \vee q$

B $\neg (p \vee q) \equiv \neg p \wedge \neg q$

C $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

D $p \Rightarrow q$ AND $q \Rightarrow p$ ARE NOT EQUIVALENT.

Exercise 4.3

1 LET p, q, r HAVE TRUTH VALUES T, F, T RESPECTIVELY, THEN DETERMINE THE TRUTH VAL EACH OF THE FOLLOWING:

A $\neg (p \vee q)$ **B** $(\neg p \vee q) \Rightarrow r$ **C** $(p \wedge q) \Rightarrow r$

D $(p \vee q) \Rightarrow \neg r$ **E** $(p \wedge q) \Leftrightarrow r$

2 GIVEN p: THE SUN RISES DUE EAST.

q: 5 IS LESS THAN 2.

r: PIGEONS ARE BIRDS.

s: LAWS AND ORDERS ARE DYNAMIC.

t: LAKE TANA IS FOUND IN ETHIOPIA.

EXPRESS EACH OF THE FOLLOWING COMPOUND PROPOSITIONS IN GOOD ENGLISH
DETERMINE THE TRUTH VALUE OF EACH.

A $p \wedge r$ B $p \vee r$ C $(p \wedge r) \Rightarrow q$

D $(p \wedge \neg r) \Leftrightarrow \neg q$ E $p \Rightarrow (q \vee r)$ F $p \Leftrightarrow (q \wedge r)$

G $s \Rightarrow t$ H $s \Leftrightarrow t$ I $s \wedge t$

3 CONSTRUCT THE TRUTH TABLE FOR EACH OF THE FOLLOWING COMPOUND STATEMENTS.

A $p \Rightarrow (p \Rightarrow q)$ B $p \Rightarrow \neg (p \wedge r)$

C $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ D $(p \wedge q) \Leftrightarrow (p \vee q)$

4 SUPPOSE THE TRUTH VALUE OF $p \Leftrightarrow q$ IS T.

WHAT CAN BE SAID ABOUT THE TRUTH VALUE OF $(p \wedge q) \Leftrightarrow (p \vee q)$?

5 SUPPOSE THE TRUTH VALUE OF $p \Leftrightarrow q$ IS F.

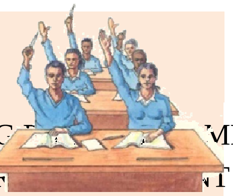
WHAT CAN BE SAID ABOUT THE TRUTH VALUES OF

A $p \Leftrightarrow \neg q$? B $\neg p \Leftrightarrow q$? C $\neg p \Leftrightarrow \neg q$?

4.1.4 Properties and Laws of Logical Connectives

UNDER THIS SUBTOPIC, YOU ARE GOING TO SEE SOME OF THE PROPERTIES OF LOGICAL CON AND DISCUSS COMMUTATIVE, ASSOCIATIVE AND DISTRIBUTIVE PROPERTIES IN THE SE EQUIVALENCE AND ALSO SEE OTHER PROPERTIES KNOWN AS DE MORGAN'S LAWS. THE FOL ACTIVITY WILL HELP YOU TO HAVE MORE UNDERSTANDING OF THESE PROPERTIES.

ACTIVITY 4.2



CONSTRUCT TRUTH TABLES FOR EACH OF THE FOLLOWING MPOUND PROPOSITIONS AND CHECK WHETHER THE GIVEN PAIRS A T OR NOT.

A $p \wedge q, q \wedge p$ B $p \vee q, q \vee p$

C $p \wedge (q \wedge r), (p \wedge q) \wedge r$ D $p \vee (q \vee r), (p \wedge q) \vee r$

E $p \wedge (q \vee r), (p \wedge q) \vee (p \vee r)$ F $p \vee (q \wedge r), (p \vee q) \wedge (p \vee r)$

G $\neg p \vee \neg q, \neg (p \wedge q)$ H $\neg p \wedge \neg q, \neg (p \vee q)$

FROM THE ABOVE ACTIVITY, YOU SHOULD HAVE OBSERVED THAT THE FOLLOWING PROPE TRUE.

1 CONJUNCTION IS **commutative**; THAT MEANS FOR ANY PROPOSITIONS p AND q, WE HAVE

$$p \wedge q \equiv q \wedge p$$

2 DISJUNCTION IS **commutative**; THAT MEANS FOR ANY PROPOSITIONS p AND q, WE HAVE

$$p \vee q \equiv q \vee p$$

3 CONJUNCTION IS **associative**; THAT MEANS FOR ANY PROPOSITIONS p, q AND r, WE HAVE

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

4 DISJUNCTION IS **associative**; THAT MEANS FOR ANY PROPOSITIONS p, q AND r, WE HAVE

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

5 CONJUNCTION IS **distributive over disjunction**; THAT MEANS FOR ANY PROPOSITIONS p, q AND r, WE HAVE

$$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

6 DISJUNCTION IS **distributive over conjunction**; THAT MEANS FOR ANY PROPOSITIONS p, q AND r, WE HAVE

$$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

7 YOU HAVE ALSO SEEN THAT

$$\neg p \vee \neg q \equiv \neg (p \wedge q)$$

$$\neg p \wedge \neg q \equiv \neg (p \vee q)$$

THESE TWO PROPERTIES ARE CALLED **De Morgan's Laws**.

4.1.5 Contradiction and Tautology

BEGIN THIS SUBSECTION BY DOING THE FOLLOWING GROUP WORK.

Group Work 4.2

COMPLETE THE TRUTH TABLE FOR EACH OF THE FOL ND PROPOSITIONS IN THE FOLLOWING TABLES AND DISCUS

A $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ **B** $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$

C $(p \vee q) \Leftrightarrow (p \vee \neg q)$

A

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$	$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$
T	T				
T	F				
F	T				
F	F				

I FROM THE ABOVE TRUTH TABLE, WHAT DID YOU OBSERVE ABOUT THE TRUTH VALUES OF $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$?

II IS THE LAST COLUMN ALWAYS TRUE?

III IS THE LAST COLUMN ALWAYS FALSE?

B

p	q	$\neg q$	$p \Rightarrow q$	$p \wedge \neg q$	$(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T				
T	F				
F	T				
F	F				

I FROM THE ABOVE TRUTH TABLE, WHAT DID YOU OBSERVE ABOUT THE TRUTH VALUES OF $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$?

II IS THE LAST COLUMN ALWAYS TRUE?

III IS THE LAST COLUMN ALWAYS FALSE?

C

p	q	$\neg q$	$p \vee q$	$p \vee \neg q$	$(p \vee q) \Leftrightarrow (p \vee \neg q)$
T	T				
T	F				
F	T				
F	F				

i FROM THE ABOVE TRUTH TABLE WHAT DID YOU OBSERVE ABOUT THE TRUTH VALUES OF $(p \vee q) \Leftrightarrow (p \vee \neg q)$?

ii IS THE LAST COLUMN ALWAYS TRUE?

iii IS THE LAST COLUMN ALWAYS FALSE?

THE FOLLOWING DEFINITION REFERS TO THE OBSERVATIONS MADE IN GROUP WORK 4.2 ABOVE:

Definition 4.3

A A COMPOUND PROPOSITION IS A **tautology**, IF AND ONLY IF FOR EVERY ASSIGNMENT OF TRUTH VALUES TO THE COMPONENT PROPOSITIONS OCCURRING IN IT, THE COMP PROPOSITION ALWAYS HAS TRUTH VALUE T.

B A COMPOUND PROPOSITION IS A **contradiction**, IF AND ONLY IF FOR EVERY ASSIGNMENT OF TRUTH VALUES TO THE COMPONENT PROPOSITIONS OCCURRING IN COMPOUND PROPOSITION ALWAYS HAS TRUTH VALUE F.

NOTE THAT IN THE ABOVE GROUP WORK (C) IS NEITHER A TAUTOLOGY NOR A CONTRADICTIO

Exercise 4.4

DETERMINE WHETHER EACH OF THE FOLLOWING COMPOUND PROPOSITIONS IS A TAUTO CONTRADICTION OR NEITHER.

A $(p \wedge q) \Leftrightarrow (q \wedge p)$

B $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

C $[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r]$

D $[p \vee (q \vee r)] \Leftrightarrow [\neg (p \wedge q) \wedge \neg r]$

E $[p \wedge (q \vee r)] \Leftrightarrow [\neg (p \wedge q) \vee \neg (p \vee r)]$

F $[\neg p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$

G $(\neg p \vee \neg q) \Leftrightarrow (p \wedge q)$

H $(\neg p \wedge \neg q) \Rightarrow \neg (p \vee q)$

4.1.6 Converse and Contrapositive

MATHEMATICAL STATEMENTS (OR ASSERTIONS) ARE USUALLY GIVEN IN THE FORM OF A C STATEMENT $p \Rightarrow q$. YOU WILL NOW EXAMINE SUCH CONDITIONAL STATEMENTS.

ACTIVITY 4.3

CONSIDER THE FOLLOWING STATEMENTS.

p: A CHILD HAS THE RIGHT TO BE FREE FROM CORPORAL PUNISHMENT.

q: THE SUN RISES DUE NORTH.

WRITE THE FOLLOWING IN GOOD ENGLISH.

A $p \Rightarrow q$ **B** $q \Rightarrow p$ **C** $\neg q \Rightarrow \neg p$

YOU MAY RECALL FROM EXAMPLE 10 THAT $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$ AND $p \Rightarrow q \not\equiv q \Rightarrow p$.

NOW YOU WILL LEARN THE NAME OF THESE RELATIONS IN THE FOLLOWING DEFINITION.

Definition 4.4

GIVEN A CONDITIONAL STATEMENT $p \Rightarrow q$

A $q \Rightarrow p$ IS CALLED THE CONVERSE OF $p \Rightarrow q$

B $\neg q \Rightarrow \neg p$ IS CALLED THE CONTRAPOSITIVE OF $p \Rightarrow q$.

C IN $p \Rightarrow q$, p IS SAID TO BE A HYPOTHESIS OR SUFFICIENT CONDITION FOR q; q IS SAID TO BE THE CONCLUSION OR NECESSARY CONDITION FOR p.

Example 11 CONSIDER THE FOLLOWING:

p: A QUADRILATERAL IS A SQUARE.

q: A QUADRILATERAL IS A RECTANGLE.

WRITE THE FOLLOWING CONDITIONAL STATEMENTS IN GOOD ENGLISH AND DETERM TRUTH VALUES OF EACH.

A $p \Rightarrow q$ **B** $q \Rightarrow p$ **C** $\neg q \Rightarrow \neg p$

Solution:

A IF A QUADRILATERAL IS A SQUARE, THEN IT IS A RECTANGLE. (TRUE)

B IF A QUADRILATERAL IS A RECTANGLE, THEN IT IS A SQUARE. (FALSE)

C IF A QUADRILATERAL IS NOT A RECTANGLE, THEN IT IS NOT A SQUARE. (TRUE)

OFTEN MATHEMATICAL STATEMENTS (OR THEOREMS) ARE GIVEN IN THE FORM OF CON STATEMENTS. TO PROVE SUCH STATEMENTS YOU CAN ASSUME THAT THE HYPOTHESIS IS SHOW THAT THE CONCLUSION IS ALSO TRUE. BUT IF THIS APPROACH BECOMES DIFFICULT, USE A KIND OF PROOF CALLED "**proof by contrapositive**". YOU CAN APPRECIATE THIS METHOD OF PROOF IF YOU COMPARE THE CONDITIONAL STATEMENT

$p \Rightarrow q$ WITH ITS CONTRAPOSITIVE $\neg q \Rightarrow \neg p$.

THE FOLLOWING EXAMPLE ILLUSTRATES THIS.

Example 12 PROVE THE FOLLOWING ASSERTIONS.

A IF A NATURAL NUMBER k IS ODD, THEN ITS SQUARE IS ALSO ODD.

B IF A NATURAL NUMBER k IS EVEN, THEN ITS SQUARE IS ALSO EVEN.

C IF k IS A NATURAL NUMBER AND k^2 IS EVEN, THEN k IS EVEN.

Proof:

A FIRST YOU IDENTIFY THE HYPOTHESIS AND THE CONCLUSION

HYPOTHESIS p: k IS AN ODD NATURAL NUMBER.

CONCLUSION q: k^2 IS ODD.

THE STATEMENT IS IN THE FORM OF $p \Rightarrow q$

NOW k IS ODD IMPLIES THAT $k = 2n - 1$, FOR SOME NATURAL NUMBER n

$\Rightarrow k^2 = (2n - 1)^2 = 4n^2 - 4n + 1 = 2(2n^2 - 2n + 1) - 1.$

$\Rightarrow k^2 = 2m - 1$, WHERE $m = 2n^2 - 2n + 1$ IS A NATURAL NUMBER.

$\Rightarrow k^2$ IS ODD.

THEREFORE, THE ASSERTION IS PROVED.

B HYPOTHESIS p: k IS AN EVEN NATURAL NUMBER.

CONCLUSION q: k^2 IS EVEN.

THE STATEMENT IS IN THE FORM OF $p \Rightarrow q$

NOW k IS EVEN IMPLIES THAT $k = 2n$ FOR SAME NATURAL NUMBER n.

$\Rightarrow k^2 = (2n)^2 = 4n^2 = 2(2n^2)$

$\Rightarrow k^2 = 2m$, WHERE $m = 2n^2$ IS ALSO A NATURAL NUMBER.

$\Rightarrow k^2$ IS EVEN.

THEREFORE, THE ASSERTION IS PROVED.

C HYPOTHESIS p: k IS NATURAL NUMBER AND k^2 IS EVEN.

CONCLUSION q: k IS EVEN.

THE STATEMENT IS IN THE FORM OF $p \Rightarrow q$

YOU MAY USE PROOF BY CONTRAPOSITIVE.

ASSUME THAT k IS NOT EVEN; THAT IS $\neg q$ IS TRUE.

k IS NOT EVEN IMPLIES THAT k IS ODD.

$\Rightarrow k^2$ IS ODD, BY (A)

$\Rightarrow \neg p$ IS TRUE

$\Rightarrow p$ IS FALSE

THIS CONTRADICTS THE GIVEN HYPOTHESIS AND HENCE THE ASSUMPTION THAT k IS NOT EVEN IS FALSE.

THEREFORE, k MUST BE EVEN.

Exercise 4.5

1 CONSTRUCT THE TRUTH TABLE OF THE FOLLOWING AND COMPARE THE TRUTH VALUES OF EACH:

A $\neg (p \Rightarrow q)$ **B** $\neg p \Rightarrow \neg q$ **C** $p \wedge \neg q$

WHICH ONE IS EQUIVALENT TO $\neg (p \Rightarrow q)$?

2 FOR EACH OF THE FOLLOWING CONDITIONAL STATEMENTS, GIVE THE CONVERSE, INVERSE AND CONTRAPOSITIVE.

A IF 2 > 3, THEN 6 IS PRIME.

B IF ETHIOPIA IS IN ASIA, THEN SUDAN IS IN AFRICA.

C IF ETHIOPIA WERE IN EUROPE, THEN LIFE WOULD BE SIMPLER.

3 PROVE THAT, IF k IS A NATURAL NUMBER SUCH THAT k^2 IS ODD, THEN k IS ODD.

4.1.7 Quantifiers

OPEN STATEMENTS CAN BE CONVERTED INTO STATEMENTS BY REPLACING THE VARIABLE BY AN INDIVIDUAL ENTITY. IN THIS SECTION, YOU ARE GOING TO SEE HOW OPEN STATEMENTS CAN BE CONVERTED INTO STATEMENTS BY USING QUANTIFIERS.

ACTIVITY 4.4

CONSIDER THE FOLLOWING OPEN STATEMENTS.

P(x): $x + 5 = 7$; WHERE x IS A NATURAL NUMBER.

Q(x): $x^2 \geq 0$; WHERE x IS A REAL NUMBER.

CAN YOU DETERMINE THE TRUTH VALUE OF THE FOLLOWING?

A THERE IS A NATURAL NUMBER x SUCH THAT $x + 5 = 7$.

B FOR ALL NATURAL NUMBERS x, $x + 5 = 7$.

C THERE IS A REAL NUMBER x FOR WHICH $x^2 \geq 0$.

D FOR EVERY REAL NUMBER x, $x^2 \geq 0$.

YOU USE THE SYMBOL $\exists$ FOR THE PHRASE **"there is"** OR **"there exists"** AND CALL IT AN **existential** QUANTIFIER; YOU USE THE SYMBOL $\forall$ FOR THE PHRASE **"for all"** OR **"for every"** OR **"for each"** AND CALL IT A **universal quantifier.**

THUS, YOU CAN REWRITE THE ABOVE STATEMENTS A TO D AS FOLLOWS USING THE SYMBOLS AND READ THEM AS FOLLOWS:

A $(\exists x)\ P(x)$ ≡ THERE IS **some natural number** WHICH SATISFIES PROPERTY *P*.

OR THERE IS **at least one natural number** WHICH SATISFIES PROPERTY *P*.

B $(\forall x)\ P(x)$ ≡ **all natural numbers** SATISFY PROPERTY *P*

OR **every natural number** SATISFIES PROPERTY *P*

OR **each natural number** SATISFIES PROPERTY *P*

C $(\exists x)\ Q(x)$ ≡ THERE IS **some real number** WHICH SATISFIES PROPERTY *Q*.

D $(\forall x)\ Q(x)$ ≡ **every real** NUMBER SATISFIES PROPERTY *Q*

ACTUALLY, WHEN WE ATTACH QUANTIFIERS TO OPEN PROPOSITIONS, THEY ARE NO LON PROPOSITIONS. FOR EXAMPLE, $(\exists x)\ P(x)$ IS **true,** IF THERE IS SOME INDIVIDUAL IN THE GIVEN UNIVERSE WHICH SATISFIES PROPERTY *P*, IT BECOMES **false** IF THERE IS NO SUCH INDIVIDUAL IN THE UNIVERSE WHICH SATISFIES PROPERTY *P*. LIKEWISE, $(\forall x)\ P(x)$ IS **true,** IF ALL INDIVIDUALS IN THE UNIVERSE SATISFY PROPERTY *P*; IT BECOMES **false** IF THERE IS AT LEAST ONE INDIVIDUAL IN THE UNIVERSE WHICH DOES NOT SATISFY PROPERTY *P*. THAT MEANS, $(\exists x)\ P(x)$ AND $(\forall x)\ P(x)$ HAVE GOT TRUTH VALUES AND THEY BECOME PROPOSITIONS.

Example 13 LET S = {2, 4, 5, 6, 8, 10} AND $P(x)$: x IS A MULTIPLE OF 2 WHERE $x \in S$.

DETERMINE THE TRUTH VALUES OF THE FOLLOWING.

A $(\exists x)\ P(x)$ **B** $(\forall x)\ P(x)$

Solution:

A $(\exists x)\ P(x)$ IS TRUE, SINCE 8 SATISFIES PROPERTY *P*. [ARE THERE OTHER ELEMENTS OF S WHICH SATISFY PROPERTY *P*?]

B $(\forall x)\ P(x)$ IS FALSE, SINCE 5 DOES NOT SATISFY PROPERTY *P*.

Exercise 4.6

DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING ASSUMING THAT THE UNIVER OF REAL NUMBERS.

A $(\exists x)\ (4x - 3 = -2x + 1)$ **B** $(\exists x)\ (x^2 + x + 1 = 0)$

C $(\exists x)\ (x^2 + x + 1 > 0)$ **D** $(\exists x)\ (x^2 + x + 1 < 0)$

E $(\forall x)\ (x^2 > 0)$ **F** $(\forall x)\ (x^2 + x + 1 \neq 0)$

G $(\forall x)\ (4x - 3 = -2x + 1)$

Relations between quantifiers

GIVEN A PROPOSITION, IT IS OBVIOUS THAT ITS NEGATION IS ALSO A PROPOSITION. THIS LEA QUESTION:

What is the form of the negation of $(\exists x)P(x)$ and the form of the negation of $(\forall x)P(x)$?

Group Work 4.3

LET $P(x)$ BE AN OPEN STATEMENT.

DISCUSS THE FOLLOWING: WHEN DO YOU SAY THAT

1. $(\exists x)P(x)$ IS TRUE?
2. $(\forall x)P(x)$ IS TRUE?
3. $(\exists x)P(x)$ IS FALSE?
4. $(\forall x)P(x)$ IS FALSE?

FROM THE ABOVE GROUP WORK YOU SHOULD BE ABLE TO SUMMARIZE THE FOLLOWING:

THE PROPOSITION $(\forall x)P(x)$ WILL BE FALSE ONLY IF WE CAN FIND AN INDIVIDUAL "a" SUCH THAT $P(a)$ IS FALSE, WHICH MEANS $(\exists x)\neg P(x)$ IS TRUE. IF WE SUCCEED IN GETTING SUCH AN INDIVIDUAL "a", THEN $(\forall x)P(x)$ IS FALSE. THEREFORE, THE NEGATION OF $(\forall x)P(x)$ BECOMES $(\exists x)\neg P(x)$. IN SYMBOLS, THIS IS

$$\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$$

TO FIND THE SYMBOLIC FORM OF THE NEGATION OF $(\exists x)P(x)$, PROCEED AS FOLLOWS: $(\exists x)P(x)$ IS FALSE IF THERE IS NO INDIVIDUAL "a" FOR WHICH $P(a)$ IS TRUE.

THUS FOR EVERY x, $P(x)$ IS FALSE, WHICH MEANS FOR EVERY x, THE NEGATION OF $P(x)$ IS TRUE. THEREFORE, THE NEGATION OF $(\exists x)P(x)$ BECOMES $(\forall x)\neg P(x)$. IN SYMBOLS, THIS IS

$$\neg(\exists x)P(x) \equiv (\forall x)\neg P(x)$$

Example 14 GIVE THE NEGATION OF EACH OF THE FOLLOWING STATEMENTS AND DETERMINE THE TRUTH VALUES OF EACH ASSUMING THAT THE UNIVERSE IS THE SET OF ALL REA NUMBERS.

A $(\exists x)(x^2 < 0)$ **B** $(\forall x)(2x - 1 = 0)$

Solution

A $\neg(\exists x)(x^2 < 0) \equiv (\forall x)\neg(x^2 < 0) \equiv (\forall x)(x^2 \geq 0)$

$(\exists x)(x^2 < 0)$ IS FALSE; AND $(\forall x)(x^2 \geq 0)$ IS TRUE.

B $\neg(\forall x)(2x - 1 = 0) \equiv (\exists x)\neg(2x - 1 = 0) \equiv (\exists x)(2x - 1 \neq 0)$

$(\forall x)(2x - 1 = 0)$ IS FALSE; AND $(\exists x)(2x - 1 \neq 0)$ IS TRUE.

Exercise 4.7

1 GIVE THE NEGATION OF EACH OF THE FOLLOWING STATEMENTS AND DETERMINE THE TRUTH VALUES FOR EACH, ASSUMING THAT THE UNIVERSE IS THE SET OF REAL NUMBERS.

A $(\exists x)\ (4x - 3 = -2x + 1)$ **B** $(\exists x)\ (x^2 + 1 = 0)$

C $(\forall x)\ (x^2 + 1 > 0)$ **D** $(\forall x)\ (x^2 < 0)$

E $(\exists x)\ (x^2 + x + 1 = 0)$

2 LET U = {1, 2, 3, 4, 5} BE A GIVEN UNIVERSE.

$P(x)$: x IS AN EVEN NUMBER

$H(x)$: x IS A MULTIPLE OF 2

$R(x)$: x IS AN ODD PRIME NUMBER

$Q(x)$: $x \leq 5$.

DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING

A $(\exists x)\, P(x)$ **B** $(\exists x)\, (P(x) \wedge \text{H}(x))$

C $(\exists x)\ (P(x) \Rightarrow \text{H}(x))$ **D** $(\forall x)\ (R(x) \Rightarrow \text{P}(x))$

E $\neg\left[(\forall x)\,(P(x) \Rightarrow \text{H}(x))\right]$ **F** $(\forall x)\ Q(x)$ **G** $(\exists x)\ R(x)$

Quantifiers occurring in combinations

UNDER THIS SUBTOPIC, YOU ARE GOING TO SEE HOW TO CONVERT AN OPEN STATEMENT INV TWO VARIABLES INTO A STATEMENT. IT INVOLVES THE USE OF TWO QUANTIFIERS TOGETHE THE QUANTIFIERS TWICE. TO BEGIN WITH, THE FOLLOWING ACTIVITY MAY HELP YOU.

ACTIVITY 4.5

ANSWER THE FOLLOWING QUESTIONS:

1 FOR EACH NATURAL NUMBER, CAN YOU FIND A NATURAL NUMBER THAT IS GREATER THAN IT?

2 FOR EACH NATURAL NUMBER, CAN YOU FIND A NATURAL NUMBER THAT IS LESS THAN IT?

3 FOR EACH INTEGER, CAN YOU FIND AN INTEGER THAT IS LESS THAN IT?

4 GIVEN AN INTEGER x, CAN YOU FIND AN INTEGER y SUCH THAT $x + y = 0$?

5 IS THERE AN INTEGER x FOR EVERY INTEGER Y SUCH THAT:

A $x + y = y$? **B** $x + y = x$?

OBSERVE THAT EACH QUESTION IN THE ACTIVITY ABOVE INVOLVES TWO VARIABLES AND HENCE YOU NEED EITHER ONE QUANTIFIER TWICE OR THE TWO QUANTIFIERS TOGETHER TO COVER STATEMENTS INTO STATEMENTS.

SUPPOSE YOU HAVE AN OPEN PROPOSITION INVOLVING TWO VARIABLES, SAY

$P(x, y):\ x + y = 5$, WHERE x AND y ARE NATURAL NUMBERS.

THIS OPEN PROPOSITION CAN BE CHANGED TO A PROPOSITION EITHER BY REPLACING BOTH V BY CERTAIN NUMBERS EXPLICITLY OR BY USING QUANTIFIERS. TO USE QUANTIFIERS, EITHER TO USE ONE OF THE QUANTIFIER TWICE OR BOTH QUANTIFIERS IN COMBINATION. SO IT IS I TO KNOW HOW TO READ AND WRITE SUCH QUANTIFIERS. THE FOLLOWING WILL GIVE YOU PRACTICE!

$(\exists x)(\exists y)P(x, y)$ ≡ THERE IS SOME x AND SOME y SO THAT PROPERTY P IS SATISFIED.

THIS STATEMENT IS TRUE IF ONE CAN SUCCEED IN FINDING ONE INDIVIDUAL x AND ONE INDIVIDUAL y WHICH SATISFY PROPERTY P.

$(\exists x)(\forall y)P(x, y)$ ≡ THERE IS SOME x SO THAT PROPERTY P IS SATISFIED FOR EVERY y.

≡ THERE IS SOME x WHICH STANDS FOR ALL y SO THAT PROPERTY P IS SATISFIED.

THIS STATEMENT IS TRUE, IF ONE CAN SUCCEED IN FINDING AN INDIVIDUAL x FOR WHICH PROPERTY P IS SATISFIED BY EVERY VALUE OF y.

$(\forall x)(\exists y)P(x, y)$ ≡ FOR EVERY x THERE IS SOME y SO THAT PROPERTY P IS SATISFIED.

≡ GIVEN x WE CAN FIND y SO THAT PROPERTY P IS SATISFIED.

THIS STATEMENT IS TRUE IF ONE CAN SUCCEED IN FINDING ONE y CORRESPONDING TO A GIVEN x SO THAT PROPERTY P IS SATISFIED.

$(\forall x)(\forall y)P(x, y)$ ≡ FOR EVERY x AND EVERY y PROPERTY P IS SATISFIED.

THIS STATEMENT IS FALSE IF ONE CAN SUCCEED IN FINDING AN INDIVIDUAL x OR AN INDIVIDUAL y WHICH DOES NOT SATISFY PROPERTY P.

THUS, IF WE APPLY THIS FOR THE OPEN STATEMENT:

$P(x, y):\ x + y = 5$, WHERE x AND y ARE NATURAL NUMBERS, WE HAVE.

$(\exists x)(\exists y)P(x, y)$, HAS TRUTH VALUE T. (YOU CAN TAKE $x = 1$ AND $y = 4$)

$(\exists x)(\forall y)P(x, y)$, HAS TRUTH VALUE F.

$(\forall x)(\exists y)P(x, y)$, HAS TRUTH VALUE F, SINCE IF x IS GIVEN TO BE 6, FOR EXAMPLE, WE CANNOT FIND A NATURAL NUMBER y SO THAT $6 + y = 5$.

$(\forall x)(\forall y)P(x, y)$, HAS TRUTH VALUE F.

BUT IF WE CHANGE THE UNIVERSE FROM NATURAL NUMBERS TO INTEGERS AS:

$P(x, y):\ x + y = 5$, WHERE x AND y ARE INTEGERS, THEN

$(\exists x)(\exists y)P(x, y)$, HAS TRUTH VALUE T.

$(\exists x)(\forall y)P(x, y)$, HAS TRUTH VALUE F.

$(\forall x)(\exists y)P(x, y)$, HAS TRUTH VALUE T, SINCE FOR A GIVEN x WE CAN TAKE $y = 5 - x$ WHICH IS ALSO AN INTEGER, AND P SATISFIES

$(\forall x)(\forall y)P(x, y)$, HAS TRUTH VALUE F.

Exercise 4.8

1 GIVEN $Q(x, y)$: $x = y$ AND $H(x, y)$: $x > y$, DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING ASSUMING THE UNIVERSE TO BE THE SET OF NATURAL NUMBERS.

A $(\exists x)(\forall y)Q(x, y)$ B $(\forall x)(\forall y)H(x, y)$ C $(\forall x)(\forall y)Q(x, y)$

D $(\forall y)(\forall x)Q(x, y)$ E $(\exists x)(\forall y)H(x, y)$ F $(\exists x)(\exists y)H(x, y)$

G $(\forall x)(\exists y)H(x, y)$

2 GIVEN $P(x, y)$: $y = x + 5$; $Q(x, y)$: $x = y$ AND $H(x, y)$: $x > y$; DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING, IF THE UNIVERSE IS THE SET OF REAL NUMBERS.

A $(\exists x)(\exists y)P(x, y)$ B $(\exists x)(\forall y)P(x, y)$ C $(\forall x)(\forall y)P(x, y)$

D $(\forall x)(\exists y)P(x, y)$ E $(\exists x)(\forall y)Q(x, y)$ F $(\forall x)(\forall y)H(x, y)$

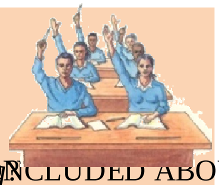
G $(\forall x)(\forall y)Q(x, y)$ H $(\forall y)(\forall x)Q(x, y)$ I $(\exists x)(\forall y)H(x, y)$

J $(\exists x)(\exists y)H(x, y)$

4.2 ARGUMENTS AND VALIDITY

THE MOST IMPORTANT PART OF MATHEMATICAL LOGIC AS A SYSTEM OF LOGIC IS TO PRO RULES OF INFERENCES WHICH PLAY A CENTRAL ROLE IN THE GENERAL THEORY OF THE REASONING. WE ARE CONCERNED HERE WITH A PROBLEM OF DECISION, WHETHER A CERTAI REASONING WILL BE ACCEPTED AS CORRECT OR INCORRECT ON THE BASIS OF ITS FORM. BY REASONING WE MEAN A FINITE SEQUENCE OF STATEMENTS OF WHICH THE LAST STATEM SEQUENCE, CALLED THE **conclusion** MAY BE INFERRED FROM THE INITIAL SET OF STATEMENTS CALLE **premises.** THE THEORY OF INFERENCE MAY BE APPLIED TO TEST THE VALIDITY OF AN ARGU EVERYDAY LIFE.

ACTIVITY 4.6



1 WHAT CAN BE CONCLUDED ABOUT q, IF p IS TRUE AND $p \Rightarrow q$ IS TRUE?

2 IF p AND $p \wedge q$ HAVE TRUTH VALUES T, WHAT CAN BE CONCLUDED ABOUT q?

3 IF p AND $p \vee q$ HAVE TRUTH VALUES T, WHAT CAN BE SAID ABOUT q?

AS YOU HAVE SEEN FROM THE ABOVE ACTIVITY, IN ORDER TO COME TO THE CONCLUSION OF THE TRUTH VALUES OF q, YOU EVALUATE THE TRUTH VALUES OF CERTAIN CONDITIONS CALLED **hypotheses or premises**. THEN YOU CAN FIND THE TRUTH VALUE OF ANOTHER STATEMENT CALLED THE **conclusion.**

FOR EXAMPLE, IN ACTIVITY 4.6 QUESTION 2 GIVEN THAT p HAS TRUTH VALUE T AND $p \wedge q$ HAS TRUTH VALUE T, YOU ARE ASKED TO FIND THE TRUTH VALUE OF q. ACTUALLY, ONE CAN SEE FROM THE RULE FOR CONJUNCTION q MUST HAS TRUTH VALUE T; THIS IS KNOWN AS LOGICAL DEDUCTION, OR ARGUMENT FORM.

Definition 4.5

A **logical deduction (argument form)** IS AN ASSERTION THAT A GIVEN SET OF STATEMENTS $P_1, P_2, \ldots, P_n$, CALLED HYPOTHESES OR PREMISES YIELD ANOTHER STATEMENT Q, CALLED THE **conclusion**. SUCH A LOGICAL DEDUCTION IS DENOTED BY:

$P_1, P_2, \ldots, P_n \vdash Q$ **Or**

$$\begin{array}{l} P_1 \\ P_2 \\ . \\ . \\ . \\ \underline{P_n} \\ Q \end{array}$$

Example 1 WE CAN WRITE THE LOGICAL DEDUCTION IN ACTIVITY 4.6 QUESTION 2 AS:

$p,\ p \wedge q \vdash q$ OR $\begin{array}{l} p \\ \underline{p \wedge q} \\ q \end{array}$

AN ARGUMENT FORM IS ACCEPTED TO BE EITHER CORRECT OR INCORRECT (ACCEPTED OR RE VALID OR INVALID (FALLACY).

When do we say that an argument is valid or invalid?

Definition 4.6

AN ARGUMENT FORM $P_1, P_2, \ldots, P_n \vdash Q$ IS SAID TO BE **valid** IF Q IS TRUE, WHENEVER ALL THE PREMISES $P_1, P_2, \ldots, P_n$, ARE TRUE; OTHERWISE IT IS **invalid.**

Example 2 INVESTIGATE THE VALIDITY OF THE FOLLOWING ARGUMENT FORMS:

A $p,\ p \Rightarrow q \vdash q$

Solution NOW FOR THE ARGUMENT TO BE VALID, WE ASSUME ALL THE PREMISES TO BE TRUE AND SHOW THAT THE CONCLUSION IS ALSO TRUE; OTHERWISE IT IS INVALID.

1 p IS TRUE ----------- PREMISE

2 $p \Rightarrow q$ IS TRUE ------ PREMISE

THEREFORE, q MUST BE TRUE FROM 1, 2 AND RULE FOR "$\Rightarrow$"

THEREFORE, THE ARGUMENT FORM $p, p \Rightarrow q \vdash q$ IS VALID.

YOU CAN USE TRUTH TABLE TO TEST VALIDITY AS FOLLOWS:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

THE PREMISES p AND $p \Rightarrow q$ ARE TRUE SIMULTANEOUSLY IN ROW 1 ONLY. SINCE IN THIS CASE q IS ALSO TRUE, THE ARGUMENT IS VALID.

B IF YOU STUDY HARD, THEN YOU WILL PASS THE EXAM. YOU DID NOT PASS THE EXAM.

THEREFORE, YOU DID NOT STUDY HARD.

Solution:

LET p: YOU STUDY HARD.

q: YOU WILL PASS THE EXAM.

$\neg p$: YOU DID NOT STUDY HARD.

$\neg q$: YOU DID NOT PASS THE EXAM.

THE ARGUMENT FORM FOR B IS, THEREFORE WRITTEN AS,

$p \Rightarrow q$

$\underline{\neg q}$

$\neg p$

THUS TO CHECK THE VALIDITY, YOU HAVE THE FOLLOWING REASONING:

1 $\neg q$ IS TRUE --------------PREMISE

2 q IS FALSE -------------- USING (1)

3 $p \Rightarrow q$ IS TRUE ---------- PREMISE

4 p IS FALSE FROM (2) AND (3), AND RULE OF "$\Rightarrow$"

5 $\neg p$ IS TRUE FROM (4)

THEREFORE, THE ARGUMENT FORM IS VALID.

ALTERNATIVELY, YOU CAN USE THE FOLLOWING TRUTH TABLE, TO DECIDE WHE
ARGUMENT IS VALID OR NOT.

p	q	$\neg q$	$\neg p$	$p \Rightarrow q$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

THE PREMISES $p \Rightarrow q$ AND $\neg q$ ARE TRUE SIMULTANEOUSLY IN ROW 4 ONLY. SINCE IN THIS CASE $\neg p$ IS ALSO TRUE, THE ARGUMENT IS VALID.

C $p \Rightarrow q, \neg q \Rightarrow r \vdash p$

Solution USE THE FOLLOWING TRUTH TABLE:

p	q	r	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow r$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	F

THE PREMISES $p \Rightarrow q, \neg q \Rightarrow r$ ARE TRUE IN THE 1ST, 2ND, 5TH, 6TH AND 7TH ROWS, BUT THE CONCLUSION p IS FALSE IN THE 5TH, 6TH AND 7TH ROWS.

THEREFORE, THE ARGUMENT FORM IS INVALID.

NOTE THAT WE CAN SHOW WHETHER AN ARGUMENT FORM IS VALID OR INVALID BY TWO METHODS ILLUSTRATED BY EXAMPLE 2 ABOVE. ONE IS BY USING A TRUTH TABLE AND THE OTHER IS WITHOUT USING A TRUTH TABLE. THE PROOF PROVIDED WITHOUT USING A TRUTH TABLE, JUST BY A S… REASONING, IS CALLED A **formal proof.**

Exercise 4.9

1 DECIDE WHETHER EACH OF THE FOLLOWING ARGUMENT FORMS IS VALID OR INVALID.

A $\neg p \Rightarrow q, q \vdash p$ **B** $p \Rightarrow \neg q,\ p,\ r \Rightarrow q \vdash \neg r$

C $p \Rightarrow q,\ \neg r \Rightarrow \neg q\ \vdash \neg r \Rightarrow \neg p$ **D** $p \Rightarrow q,\ q \vdash p$

E $p \vee q,\ p \vdash q$

2 FOR THE FOLLOWING ARGUMENT FORMS GIVEN BELOW (I) AND (II):

A IDENTIFY THE PREMISES AND THE CONCLUSION.

B USE APPROPRIATE SYMBOLS TO REPRESENT THE STATEMENTS IN THE ARGUMENTS.

C WRITE THE ARGUMENT FORMS USING SYMBOLS.

D CHECK THE VALIDITY.

I IF THE RAIN DOES NOT COME, THEN THE CROPS ARE RUINED AND THE PEOPLE WILL STARVE. THE CROPS ARE NOT RUINED OR THE PEOPLE WILL NOT STARVE.

THEREFORE, THE RAIN COMES.

II IF THE TEAM IS LATE, THEN IT CANNOT PLAY THE GAME. IF THE REFEREE IS HERE, THEN THE TEAM CAN PLAY THE GAME. THE TEAM IS LATE.

THEREFORE, THE REFEREE IS NOT HERE.

Rules of inferences

YOU HAVE SEEN HOW TO TEST THE VALIDITY OF ARGUMENTS BY USING TRUTH TABLES AND FORMAL PROOF. BUT IN PRACTICE, TESTING THE VALIDITY OF AN ARGUMENT USING A TRUTH TABI MORE DIFFICULT AS THE NUMBER OF COMPONENT STATEMENTS INCREASES. THEREFORE CASES, WE ARE FORCED TO USE THE FORMAL PROOF. THE FORMAL PROOF REGARDING THE AN ARGUMENT RELIES ON LOGICAL RULES CALLED **rules of inferences**. A FORMAL PROOF CONSISTS OF A SEQUENCE OF FINITE STATEMENTS COMPRISING THE PREMISES AND THE CONSEQUENC PREMISES CALLED **the conclusion.** THE PRESENCE OF EACH STATEMENT MUST BE JUSTIFIED BY A RULE OF INFERENCES. IT IS OBVIOUS THAT WE REPEATEDLY APPLY THESE RULES TO JUSTIF OF COMPLEX ARGUMENTS. BELOW ARE A FEW EXAMPLES OF SOME OF THESE RULES TOGETI THEIR CLASSICAL NAMES.

1 **Modes Ponens**

$$\begin{array}{c} P \\ P \Rightarrow Q \\ \hline Q \end{array}$$

2 **Modes Tollens**

$$\begin{array}{c} \neg Q \\ P \Rightarrow Q \\ \hline \neg P \end{array}$$

3 **Principle of Syllogism**

$$\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \\ \hline P \Rightarrow R \end{array}$$

4 **Principle of adjunction**

A
$$\begin{array}{c} P \\ Q \\ \hline P \wedge Q \end{array}$$

B
$$\begin{array}{c} P \\ \hline P \vee Q \end{array}$$

5 **Principle of detachment**

$$\begin{array}{c} P \wedge Q \\ \hline P,\ Q \end{array}$$

6	**Modes Tollendo ponens**	$\dfrac{\neg P \quad P \vee Q}{Q}$
7	**Principle of equivalence**	$\dfrac{P \Leftrightarrow Q \quad P}{Q}$
8	**Principle of conditioning**	$\dfrac{P}{Q \Rightarrow P}$

LET US SEE AN EXAMPLE TO ILLUSTRATE HOW TO USE THE RULES OF INFERENCES IN TESTING

Example 3 GIVE A FORMAL PROOF OF THE VALIDITY OF THE ARGUMENT GIVEN BELOW.

$P \wedge Q,\ (P \vee R) \Rightarrow S \vdash P \wedge S$

Proof:

1. $P \wedge Q$, HAS TRUTH VALUE T………….. PREMISE.
2. $(P \vee R) \Rightarrow S$, HAS TRUTH VALUE T …… PREMISE
3. P HAS TRUTH VALUE T ……. PRINCIPLE OF DETACHMENT FROM (1).
4. $P \vee R$, HAS TRUTH VALUE T….. PRINCIPLE OF ADJUNCTION (B) FROM (3)
5. S HAS TRUTH VALUE T………. MODES PONENS FROM (2) AND (4).
6. $P \wedge S$ HAS TRUTH VALUE T….PRINCIPLE OF ADJUNCTION (A) FROM (3) AND (5).

THEREFORE, THE ARGUMENT FORM $P \wedge Q,\ (P \vee R) \Rightarrow S \vdash P \wedge S$ IS VALID.

Exercise 4.10

1. USE THE RULES OF INFERENCES TO TEST THE VALIDITY OF EACH OF THE FOLLOWING ARGUMENT FORMS.

   A. $P \Rightarrow Q,\ R \Rightarrow P,\ R \vdash Q$ B. $\neg P \wedge \neg Q,\ (Q \vee R) \Rightarrow P \vdash R$

   C. $P \Rightarrow \neg Q,\ P,\ R \Rightarrow Q \vdash \neg R$ D. $\neg P \wedge \neg Q,\ (\neg Q \Rightarrow R) \Rightarrow P \vdash \neg R$

2. GIVEN AN ARGUMENT FORM:

   IF A PERSON STAYS UP LATE TONIGHT, THEN HE/SHE WILL BE DULL TOMORROW. IF HE/S NOT STAY UP LATE TONIGHT, THEN HE/SHE WILL FEEL THAT LIFE IS NOT WORTH LIVING THEREFORE, EITHER THE PERSON WILL BE DULL TOMORROW OR WILL FEEL THAT LIFE I WORTH LIVING.

   A. IDENTIFY THE PREMISES AND THE CONCLUSION.
   B. USE APPROPRIATE SYMBOLS TO REPRESENT THE STATEMENTS IN THE ARGUMENT.
   C. WRITE THE ARGUMENT FORM USING SYMBOLS.
   D. CHECK THE VALIDITY USING RULES OF INFERENCES.

Key Terms

arguments	**logical connectives (or logical operators)**
compound proposition	**open proposition (or open statement)**
contra positive of a conditional statement	**proposition (or statement)**
contradiction	**quantifiers; both existential and universal**
converse of a conditional statement	**rules of inferences**
equivalent compound propositions	**tautology**
invalid arguments	**valid arguments**

Summary

1 **Mathematical reasoning** IS A TOOL TO ORGANIZE EVIDENCE IN A SYSTEMATIC WAY THROUGH MATHEMATICAL LOGIC.

2 A SENTENCE WHICH HAS A TRUTH VALUE IS SAID TO BE A **proposition** (OR STATEMENT).

3 A SENTENCE WITH ONE OR MORE VARIABLES WHICH BECOMES A STATEMENT ON REPLACING THE VARIABLE(S) BY INDIVIDUAL (S) IS CALLED AN **open proposition** (OR **open statement**).

4 THE USUAL CONNECTIVES IN LOGIC ARE: **or**, **and**, **not**, **if.... then** AND **if and only if**.

5 A STATEMENT FORMED BY JOINING TWO OR MORE STATEMENTS BY A CONNECTIVE (OR CONNECTIVES) IS CALLED A **compound statement**.

6 A COMPOUND STATEMENT IS A **tautology**, IF AND ONLY IF FOR EVERY ASSIGNMENT OF TRUTH VALUES TO THE COMPONENT PROPOSITIONS OCCURRING IN IT, THE COMPOUND PROPOS ALWAYS HAS TRUTH VALUE T. IT IS A **contradiction**, IF THE COMPOUND PROPOSITION ALWAYS HAS TRUTH VALUE F.

7 WE USE THE SYMBOLS $\exists$ AND $\forall$ FOR THE PHRASE "**there is**", (**existential quantifier**) AND FOR THE PHRASE "**for all**", (**universal quantifier**) RESPECTIVELY.

8 A LOGICAL DEDUCTION (ARGUMENT FORM) IS AN ASSERTION THAT A GIVEN SET OF STATEMENTS $P_1, P_2,.., P_n$, CALLED HYPOTHESES OR PREMISES, YIELD ANOTHER STATEMENT Q, CALLED THE **conclusion**.

9 TO DECIDE WHETHER AN ARGUMENT IS VALID OR INVALID, WE USE A TRUTH TABLE OR FORMAL PROOF.

10 THE FORMAL PROOF REGARDING THE VALIDITY OF AN ARGUMENT RELIES ON LOGICAL RULES **called rules of inferences**.

Review Exercises on Unit 4

1 WHICH OF THE FOLLOWING COMPOUND PROPOSITIONS ARE TAUTOLOGIES, CONTRADICTIONS OR NEITHER.

A $(p \Rightarrow \neg q) \wedge (p \Rightarrow q)$ **B** $(\neg p \vee q) \Rightarrow (p \wedge \neg q)$

C $[(p \Rightarrow q) \vee (p \Rightarrow r)] \Leftrightarrow [p \Rightarrow (q \vee r)]$ **D** $(p \Rightarrow q) \Leftrightarrow \neg(\neg q \Rightarrow \neg p)$

2 GIVEN $P(x)$: $\sqrt{x^2} = |x|$;

$Q(x)$: $x - 1 = 3$;

$R(x, y)$: $x + y = 0$

$T(x, y)$: $x + y = y$

DETERMINE THE TRUTH VALUE OF EACH OF THE FOLLOWING, ASSUMING THAT THE UN
THE SET OF REAL NUMBERS.

A $(\exists x)\ P(x)$ **B** $(\forall x)\ P(x)$ **C** $(\exists x)\ Q(x)$

D $(\forall x)\ Q(x)$ **E** $(\exists x)(\forall y)\ R(x, y)$ **F** $(\forall x)\ (\exists y)\ R(x, y)$

G $(\forall x)\ (\forall y)\ R(x, y)$ **H** $(\exists x)\ (\forall y)\ T(x, y)$ **I** $(\forall x)(\exists y) T(x, y)$

3 CHECK THE VALIDITY OF EACH OF THE FOLLOWING ARGUMENTS.

A $\neg p \wedge q,\ (q \vee r) \Rightarrow p \vdash \neg r$ **B** $p \Rightarrow (q \vee r),\ \neg r,\ p \vdash q$

C IF MATHEMATICS IS A GOOD SUBJECT, THEN IT IS WORTH LEARNING. EITHER THE GRADING SYSTEM IS NOT FAIR OR MATHEMATICS IS NOT WORTH LEARNING. BU
GRADING SYSTEM IS FAIR. THEREFORE, MATHEMATICS IS NOT A GOOD SUBJECT.