Unit 6

MATRICES AND DETERMINANTS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about matrices.*
- *know specific ideas, methods and principles concerning matrices.*
- *perform operation on matrices.*
- *apply principles of matrices to solve problems.*

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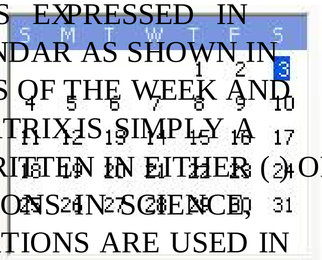
INTRODUCTION



MATRICES APPEAR WHEREVER INFORMATION IS EXPRESSED IN TABLES. ONE SUCH EXAMPLE IS A MONTHLY CALENDAR AS SHOWN IN THE FIGURE, WHERE THE COLUMNS GIVE THE DAYS OF THE WEEK AND THE ROWS GIVE THE DATES OF THE MONTH. A MATRIX IS SIMPLY A RECTANGULAR TABLE OR ARRAY OF NUMBERS WRITTEN IN EITHER ( ) OR [ ] BRACKETS. MATRICES HAVE MANY APPLICATIONS IN SCIENCE, ENGINEERING AND COMPUTING. MATRIX CALCULATIONS ARE USED IN CONNECTION WITH SOLVING LINEAR EQUATIONS.

IN THIS UNIT, YOU WILL STUDY MATRICES, OPERATIONS ON MATRICES, AND DETERMINANTS ALSO SEE HOW YOU CAN SOLVE SYSTEMS OF LINEAR EQUATIONS USING MATRICES.

HISTORICAL NOTE

Arthur Cayley (1821-95)

Many people have contributed to the development of the theory of matrices and determinants. Starting from the 2nd century BC, the Babylonians and the Chinese used the concepts in connection with solving simultaneous equations. The first abstract definition of a matrix was given by Cayley in 1858 in his book named Memoir on the theory of matrices.

He gave a matrix algebra defining addition, multiplication, scalar multiplication and inverses. He also gave an explicit construction of the inverse of a matrix in terms of the determinant of the matrix.

OPENING PROBLEM

CONSIDER A NUTRITIOUS DRINK WHICH CONSISTS OF WHOLE EGG, MILK AND ORANGE JUICE. FOOD ENERGY AND PROTEIN OF EACH OF THE INGREDIENTS ARE GIVEN BY THE FOLLOWING

	Food Energy (Calories)	Protein (Grams)
1 EGG	80	6
1 CUP OF MILK	160	9
1 CUP OF JUICE	110	2

HOW MUCH OF EACH DO YOU NEED TO PRODUCE A DRINK OF 540 CALORIES AND 25 GRAMS O PROTEIN?

6.1 MATRICES

Definition 6.1

LET ℝ BE THE SET OF REAL NUMBERS AND m AND n BE POSITIVE INTEGERS.

A RECTANGULAR ARRAY OF NUMBERS IN ℝ OF THE FORM,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

IS CALLED AN m BY n $(m \times n)$ **MATRIX** IN ℝ.

CONSIDER THE MATRIX A IN THE DEFINITION ABOVE:

- ✓ THE NUMBER m IS CALLED THE NUMBER OF ROWS OF A.
- ✓ THE NUMBER n IS CALLED THE NUMBER OF COLUMNS OF A.
- ✓ THE NUMBER a_{ij} IS CALLED THE ij^{th} ELEMENT OR ENTRY OF A WHICH IS AN ELEMENT IN THE i^{th} ROW AND j^{th} COLUMN OF A.
- ✓ A CAN BE ABBREVIATED BY: $A = (a_{ij})_{m \times n}$
- ✓ THE RECTANGULAR ARRAY OF ENTRIES IS ENCLOSED BY A ROUND BRACKET OR IN A SQUARE BRACKET.
- ✓ $m \times n$ (READ AS m BY n) IS CALLED THE **size or order** OF THE MATRIX.

Example 1 CONSIDER THE MATRIX.

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 0 & 3 \end{pmatrix}$$

THEN A IS A 2×3 MATRIX WITH $a_{11} = 1$, $a_{13} = 2$ AND $a_{23} = 3$.

Example 2 THE MATRIX $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 4 & 0 \end{pmatrix}$ IS A 3×2 MATRIX WITH:

$a_{11} = 3, a_{12} = -1, a_{21} = 1, a_{22} = 2,\ a_{31} = 4$ AND $a_{32} = 0$.

✍ Note:

- ✓ THE ENTRIES IN A GIVEN MATRIX NEED NOT BE DISTINCT.
- ✓ THE BEST WAY TO VIEW MATRICES IS AS THE CONTENTS OF A TABLE WHERE THE LABELS FOR ROWS AND COLUMNS HAVE BEEN REMOVED.

Example 3 THREE STUDENTS CHALTU, SOLOMON AND KALID HAVE 10, 50 AND 25 CENT COINS IN THEIR POCKETS. THE FOLLOWING TABLE SHOWS WHAT THEY HAVE.

		Student name		
No of coins		Chaltu	Kalid	Solomon
	10 CENT COIN	2	6	4
	50 CENT COIN	3	2	0
	25 CENT COIN	4	0	5

A REPRESENT THE TABLE IN MATRIX FORM.

B WHAT IS REPRESENTED BY THE COLUMNS?

C WHAT IS REPRESENTED BY EACH ROW?

D SUPPOSE a_{ij} DENOTES THE ENTRY IN THE i^{th} ROW AND j^{th} COLUMN. WHAT DOES a_{31} TELL YOU? WHAT ABOUT a_{23}?

Solution

A $A = \begin{pmatrix} 2 & 6 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}$

B THE COLUMNS REPRESENT THE NUMBER OF THE VARIOUS COINS EACH STUDENT HAS.

C THE ROWS REPRESENT THE NUMBER OF COINS OF A CERTAIN VALUE THAT THE STUDENTS HAVE.

D $a_{31} = 4$. IT MEANS CHALTU HAS FOUR 25-CENT COINS IN HER POCKET.

$a_{23} = 0$. THIS MEANS SOLOMON HAS NO 50-CENT COINS.

ACTIVITY 6.1

IN EACH OF THE FOLLOWING MATRICES, DETERMINE THE NUMBER OF ROWS AND THE NUMBER OF COLUMNS.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \\ 29 \end{pmatrix},\ C = \begin{pmatrix} 0 & -5 \\ 3 & 4 \\ 8 & 6 \end{pmatrix} \text{ AND } D = \begin{pmatrix} 0 & -6 & 7 \end{pmatrix}.$$

FROM ACTIVITY 6.1, YOU MAY HAVE OBSERVED THAT:

- ✓ THE NUMBER OF ROWS AND COLUMNS ARE EQUAL IN MATRIX A
- ✓ THE NUMBER OF COLUMNS IN MATRIX B IS ONE
- ✓ THE NUMBER OF ROWS IN MATRIX D IS ONE

Some important types of matrices

1. A MATRIX WITH ONLY ONE COLUMN IS CALLED A **column matrix**. IT IS ALSO CALLED A **column vector**.
2. A MATRIX WITH ONLY ONE ROW IS CALLED A **row matrix** (ALSO CALLED A ROW VECTOR).
3. A MATRIX WITH THE SAME NUMBER OF ROWS AND COLUMNS IS CALLED A **square matrix**.
4. A MATRIX WITH ALL ENTRIES 0 IS CALLED A **zero** MATRIX WHICH IS DENOTED BY 0.
5. A **diagonal matrix** IS A SQUARE MATRIX THAT HAS ZEROS EVERYWHERE EXCEPT POSSIBLY ALONG THE MAIN DIAGONAL (TOP LEFT TO BOTTOM RIGHT).
6. THE **identity** (**unit**) MATRIX IS A DIAGONAL MATRIX WHERE THE ELEMENTS OF THE PRINCIPAL DIAGONAL ARE ALL ONES.
7. A **scalar** MATRIX IS A DIAGONAL MATRIX WHERE ALL THE ELEMENTS OF THE PRINCIPAL DIAGONAL ARE EQUAL.
8. A **lower triangular matrix** IS A SQUARE MATRIX WHOSE ELEMENTS ABOVE THE MAIN DIAGONAL ARE ALL ZERO.
9. AN **upper triangular matrix** IS A SQUARE MATRIX WHOSE ELEMENTS BELOW THE MAIN DIAGONAL ARE ALL ZERO.

Example 4 GIVE THE TYPE(S) OF EACH MATRIX BELOW.

A $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **B** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **C** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

D $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ **E** $\begin{pmatrix} -35 & 0 & 4 \end{pmatrix}$ **F** $\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

G $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

A A ZERO MATRIX **B** IT IS A SQUARE, ZERO, DIAGONAL AND SCALAR MATRIX

C A DIAGONAL MATRIX **D** A COLUMN MATRIX **E** A ROW MATRIX

F A SCALAR MATRIX **G** AN IDENTITY MATRIX

Example 5 DECIDE WHETHER EACH MATRIX IS UPPER TRIANGULAR, LOWER TRIANGULAR OR NEITHER.

A $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 3 & 9 & 7 \end{pmatrix}$ **B** $\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$ **C** $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & 7 \end{pmatrix}$

D $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$ **E** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ **F** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 0 & 0 & 9 \end{pmatrix}$

Solution

A LOWER TRIANGULAR **B** LOWER TRIANGULAR **C** UPPER TRIANGULAR

D UPPER TRIANGULAR **E** BOTH (NOTICE THAT IT SATISFIES BOTH CONDITIONS)

F NEITHER

Equality of matrices

Definition 6.2

TWO MATRICES $A = (a_{ij})_{m\times n}$ AND $B = (b_{ij})_{m\times n}$ OF THE SAME ORDER ARE SAID TO BE **equal**, WRITTEN $A = B$, IF THEIR CORRESPONDING ELEMENTS ARE EQUAL, I.E. $a_{ij} = b_{ij}$ FOR ALL $1 \le i \le m$ AND $1 \le j \le n$.

Example 6 FIND x AND y IF THE MATRICES

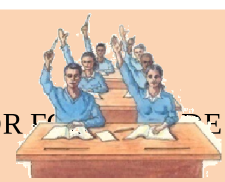
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & x+y & -1 \\ x & -7 & 2 \end{pmatrix} \text{ AND } B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 1 & -7 & 3+y \end{pmatrix} \text{ ARE EQUAL.}$$

Solution IF $A = B$, THEN $\begin{cases} x+y=0 \\ x=1 \\ 3+y=2 \end{cases}$

SOLVING THIS GIVES YOU $x = 1$ AND $y = -1$.

Addition and subtraction of matrices

ACTIVITY 6.2



A SCHOOL BOOK STORE HAS BOOKS IN FOUR SUBJECTS FOR F[illegible]E LEVELS. SOME NEWLY ORDERED BOOKS HAVE ARRIVED.

	Previous Books in Stock			
	Grade Level			
	7	8	9	10
Biology	101	89	72	75
Physics	62	58	70	43
Chemistry	57	65	71	94
Mathematics	81	87	91	93

	Newly arrived Books			
	Grade Level			
	7	8	9	10
Biology	60	65	54	45
Physics	27	35	50	27
Chemistry	55	66	65	44
Mathematics	75	68	70	51

HOW MANY OF EACH KIND DO THEY HAVE NOW?

Definition 6.3

LET $A = (a_{ij})_{m\times n}$ AND $B = (b_{ij})_{m\times n}$ BE TWO MATRICES. THEN THE SUM OF A AND B, DENOTED BY $A + B$, IS OBTAINED BY ADDING THE CORRESPONDING ELEMENTS, WHILE THE DIFFERENCE OF A AND B, DENOTED BY $A - B$, IS OBTAINED BY SUBTRACTING THE CORRESPONDING ELEMENTS I.E.

$A + B = (a_{ij} + b_{ij})_{m\times n}$ AND $A - B = (a_{ij} - b_{ij})_{m\times n}$.

Example 7 LET $A = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix}$ AND $B = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix}$.

FIND THE SUM AND DIFFERENCE OF A AND B, IF THEY EXIST.

Solution

$$A + B = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 5+3 & 2+1 & 2+4 \\ 4+5 & 4+0 & 1+3 \\ 6+6 & 0+0 & 3+2 \\ 3+4 & 6+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 8 & 3 & 6 \\ 9 & 4 & 4 \\ 12 & 0 & 5 \\ 7 & 6 & 4 \end{pmatrix}$$

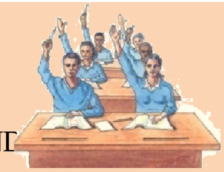
$$A - B = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 4 & -2 \\ 0 & 0 & 1 \\ -1 & 6 & -4 \end{pmatrix}$$

Example 8 LET $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 7 & 9 \end{pmatrix}$ AND $C = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$.

FIND $A - B$ AND $B + C$, IF THEY EXIST.

Solution $A - B = \begin{pmatrix} -1 & 1 & 0 \\ 6 & -2 & -5 \end{pmatrix}$, BUT SINCE B AND C HAVE DIFFERENT ORDERS, THEY CANNOT BE ADDED TOGETHER.

ACTIVITY 6.3



LET $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}$ AND $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. FIND

A $(A + B) + C$, **B** $A + (B + C)$ **C** $A - A$

D $A + 0$ **E** $A + B$ **F** $B + A$

FROM ACTIVITY 6.3, YOU CAN OBSERVE THE FOLLOWING PROPERTIES OF MATRIX ADDITION.

1. $A + B = B + A$ (COMMUTATIVE PROPERTY)
2. $(A + B) + C = A + (B + C)$ (ASSOCIATIVE PROPERTY)
3. $A + 0 = A = 0 + A$ (EXISTENCE OF ADDITIVE IDENTITY)
4. $A + (-A) = 0$ (EXISTENCE OF ADDITIVE INVERSE)

Multiplication of a matrix by a scalar

ACTIVITY 6.4

THE MARKS OBTAINED BY NIGIST AND HAGOS (OUT OF 50) EXAMINATIONS ARE GIVEN BELOW.

	Nigist	Hagos
ENGLISH	37	31
MATHEMAT	46	44
BIOLOGY	28	25

IF THE MARKS ARE TO BE CONVERTED OUT OF 100, THEN FIND THE MARKS OF NIGIST AND HA EACH SUBJECT OUT OF 100.

FROM ACTIVITY 6.4, YOU MAY HAVE OBSERVED THAT GIVEN A MATRIX, YOU CAN GET ANOTHER MATRIX BY MULTIPLYING EACH OF ITS ELEMENTS BY A CONSTANT.

Definition 6.4

IF r IS A SCALAR (I.E. A REAL NUMBER) AND $A = (a_{ij})_{m\times n}$ IS A GIVEN MATRIX, THEN THE MATRIX rA OBTAINED FROM A BY MULTIPLYING EACH ELEMENT OF A BY r. i.e. $rA = (ra_{ij})_{m\times n}$

Example 9 IF $A = \begin{pmatrix} 5 & -2 & -2 \\ 4 & 4 & -6.5 \end{pmatrix}$, THEN FIND $5A$, $\frac{1}{2}A$ AND $-3A$.

Solution $5A=\begin{pmatrix}5\times5 & 5\times(-2) & 5\times(-2)\\ 5\times4 & 5\times4 & 5\times(-6.5)\end{pmatrix}=\begin{pmatrix}25 & -10 & -10\\ 20 & 20 & -32.5\end{pmatrix}$

$$\frac{1}{2}A=\begin{pmatrix}\frac{1}{2}\times5 & \frac{1}{2}\times(-2) & \frac{1}{2}\times(-2)\\ \frac{1}{2}\times4 & \frac{1}{2}\times4 & \frac{1}{2}\times(-6.5)\end{pmatrix}=\begin{pmatrix}\frac{5}{2} & -1 & -1\\ 2 & 2 & -3.25\end{pmatrix} \text{ AND}$$

$$-3A=\begin{pmatrix}(-3)\times5 & (-3)\times(-2) & (-3)\times(-2)\\ (-3)\times4 & (-3)\times4 & (-3)\times(-6.5)\end{pmatrix}=\begin{pmatrix}-15 & 6 & 6\\ -12 & -12 & 19.5\end{pmatrix}$$

Example 10 ALEMITU PURCHASED COFFEE, SUGAR, WHEAT FLOUR, AND TEFF FLOUR FROM A AS SHOWN BY THE FOLLOWING MATRIX. ASSUME THE QUANTITIES ARE IN KG.

$A=\begin{pmatrix}6\\11\\60\\90\end{pmatrix}$. FIND THE NEW MATRIX, IF

A SHE DOUBLES HER ORDER **B** SHE HALVES HER ORDER

C SHE ORDERS 75% OF HER PREVIOUS ORDER

Solution

A $2A=\begin{pmatrix}12\\22\\120\\180\end{pmatrix}$ **B** $\frac{1}{2}A=\begin{pmatrix}3\\5.5\\30\\45\end{pmatrix}$ **C** $0.75A=\begin{pmatrix}4.5\\8.25\\45\\67.5\end{pmatrix}$

ACTIVITY 6.5

LET $A=\begin{pmatrix}-1 & 1 & -1\\ 6 & -2 & -1\end{pmatrix}$ AND $B=\begin{pmatrix}1 & 0 & 1\\ 2 & -1 & 3\end{pmatrix}$

IF $r=-7$ AND $s=4$, THEN FIND EACH OF THE FOLLOWING:

A $r(A+B)$ **B** $rA+rB$ **C** $(rs)A$ **D** $r(sA)$

E $(r+s)A$ **F** $rA+sA$ **G** $1A$ **H** $0A$

Properties of scalar multiplication

IF A AND B ARE MATRICES OF THE SAME ORDER AND r AND s ARE ANY SCALARS (I.E., REAL NUMBERS), THEN:

A $r(A+B)=rA+rB$ **B** $(r+s)A=rA+sA$

C $(rs)A=r(sA)$ **D** $1A=A$ *and* $0A=0$

Exercise 6.1

1 IF $A = \begin{pmatrix} 8 & 2 & 4.23 & -4 \\ 9 & 2 & 1 & 3 \\ 7.5 & 51 & 2 & 4 \\ 0 & 9 & 3 & 6 \end{pmatrix}$, THEN DETERMINE THE VALUES OF THE FOLLOWING:

A a_{21} **B** a_{33} **C** a_{42} **D** a_{32}

2 WHAT IS THE ORDER OF EACH OF THE FOLLOWING MATRICES?

A $\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}$ **B** $\begin{pmatrix} 1 & 4 & 7 \\ 5 & -6 & 3 \end{pmatrix}$ **C** $\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$

D $(1\ \ 2\ \ 3)$ **E** (7)

3 WHAT ARE THE DIAGONAL ELEMENTS OF EACH OF THE FOLLOWING SQUARE MATRICES?

A $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -4 & 7 \\ 0 & 7 & 1 \end{pmatrix}$ **B** $\begin{pmatrix} 0 & 1 & 3 & 1 \\ -4.5 & 1 & 8 & 2 \\ 54 & 1 & 71 & 3 \\ 2 & 1 & 5 & 4 \end{pmatrix}$

4 CONSTRUCT A 4×3 MATRIX $A = (a_{ij})$, WHERE $a_{ij} = 3i - 2j$.

5 GIVEN $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix}$ AND $B = \begin{pmatrix} -4 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$, FIND EACH OF THE FOLLOWING.

A $A + B$ **B** $A - B$ **C** $3B + 2A$

D $B + A$ **E** $2A + 3B$

6 GIVEN $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$ AND $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$, FIND MATRICES C THAT SATISFY THE FOLLOWING CONDITION:

A $A + C = B$ **B** $A + 2C = 3B$

7 GRADUATING STUDENTS FROM A CERTAIN HIGH SCHOOL SOLD TICKETS ON TWO DIFFERENT OCCASIONS, IN TWO KEBELES, IN ORDER TO RAISE MONEY THAT THEY W DONATE TO THEIR SCHOOL. THE FOLLOWING MATRICES SHOW THE NUMBER OF STUD ATTENDED THE OCCASIONS.

$$\begin{array}{lcc} & \multicolumn{2}{c}{1^{ST}\ occasion} \\ & kebele\ 1 & kebele\ 2 \\ Boys & 175 & 221 \\ Girls & 199 & 150 \end{array} \qquad \begin{array}{lcc} & \multicolumn{2}{c}{2^{ND}\ occasion} \\ & kebele\ 1 & kebele\ 2 \\ Boys & 120 & 150 \\ Girls & 199 & 181 \end{array}$$

A GIVE THE SUM OF THE MATRICES.

B IF THE TICKETS WERE SOLD FOR BIRR 2.50 A PIECE ON THE 1ST OCCASION AND BIRR 3.00 A PIECE ON THE SECOND OCCASION, HOW MUCH MONEY WAS RAISED FROM THE BO FROM THE GIRLS? IN KEBELE 1. WHAT IS THE TOTAL AMOUNT RAISED FOR THE SCH

Multiplication of matrices

TO STUDY THE RULE FOR MULTIPLICATION OF MATRICES, LET US DEFINE THE RULE FOR MATRICES OF ORDER $1 \times p$ AND $p \times 1$.

LET $A = (a_{11}\ \ a_{12}\ \dots\ a_{1p})$ AND $B = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix}$.

THEN THE PRODUCT AB IN THE GIVEN ORDER IS THE 1×1 MATRIX GIVEN BY

$$AB = (a_{11}\ \ a_{12}\ \dots\ a_{1p})\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix} = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \dots + a_{1P}b_{P1})$$

Example 11 IF $A = (1\ \ 2\ \ 3)$ AND $B = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, FIND AB.

Solution $AB = (1\ \ 2\ \ 3)\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = (1 \times 2) + (2 \times (-3)) + (3 \times 1) = -1.$

Note:

✓ THE NUMBER OF COLUMNS OF A = THE NUMBER OF ROWS OF B = p.

✓ THE OPERATION IS DONE ROW BY COLUMN IN SUCH A WAY THAT EACH ELEMENT OF T IS MULTIPLIED BY THE CORRESPONDING ELEMENT OF THE COLUMN AND THEN THE P ARE ADDED.

Notation:

LET $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m_2} & \dots & a_{mn} \end{pmatrix}$.

THEN YOU DENOTE THE i^{th} ROW AND j^{TH} COLUMN OF A BY A_i AND A^j, RESPECTIVELY.

Example 12 LET $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ -3 & 5 & 6 \end{pmatrix}$. THEN $A_1 = (1\ 2\ 3)$, $A_2 = (0\ \ 4\ \ 1)$,

$$A_3 = (-3\ \ 5\ \ 6),\quad A^1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, A^2 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \text{ AND } A^3 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}.$$

ACTIVITY 6.6

GIVEN $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ AND $B = \begin{pmatrix} 5 & 3 & 3 \\ 2 & 4 & 2 \\ 2 & 1 & 2 \end{pmatrix}$, FIND:

A A_1B^1 **B** A_1B^2 **C** A_1B^3

D A_2B^1 **E** A_2B^2 **F** A_2B^3

THE MATRIX $\begin{pmatrix} A_1B^1 & A_1B^2 & A_1B^3 \\ A_2B^1 & A_2B^2 & A_2B^3 \end{pmatrix}$ IN ACTIVITY 6.6 IS THE PRODUCT OF A AND B, DENOTED BY AB.

IN GENERAL, YOU HAVE THE FOLLOWING DEFINITION OF MULTIPLICATION OF MATRICES.

Definition 6.5

LET $A = (a_{IJ})$ BE AN $m \times p$ MATRIX AND $B = (b_{JK})$ BE A $p \times n$ MATRIX SUCH THAT THE NUMBER OF COLUMNS OF A IS EQUAL TO THE NUMBER OF ROWS OF B. THEN THE PRODUCT AB IS A MATRIX $C = (c_{IK})$ OF ORDER $m \times n$, WHERE $c_{IK} = A_I B^k$, I.E. $c_{IK} = a_{I1} b_{1K} + a_{I2} b_{2K} + a_{I3} b_{3K} + \dots + a_{IP} b_{PK}$

Example 13 LET $A = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}$ AND $B = \begin{pmatrix} 2 & 5 & -4 \\ 3 & 2 & 6 \end{pmatrix}$. THEN FIND AB

Solution $AB = \begin{pmatrix} A_1B^1 & A_1B^2 & A_1B^3 \\ A_2B^1 & A_2B^2 & A_2B^3 \end{pmatrix}$

$$AB = \begin{pmatrix} (2 \quad 3)\begin{pmatrix}2\\3\end{pmatrix} & (2 \quad 3)\begin{pmatrix}5\\2\end{pmatrix} & (2 \quad 3)\begin{pmatrix}-4\\6\end{pmatrix} \\ (2 \quad -1)\begin{pmatrix}2\\3\end{pmatrix} & (2 \quad -1)\begin{pmatrix}5\\2\end{pmatrix} & (2 \quad -1)\begin{pmatrix}-4\\6\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 13 & 16 & 10 \\ 1 & 8 & -14 \end{pmatrix}$$

ACTIVITY 6.7

LET $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$ AND $C = \begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}$. FIND:

A $A(BC)$ **B** $(AB)C$ **C** $A(B + C)$

D $AB + AC$ **E** $(B + C)A$ **F** $BA + CA$

Properties of Multiplication of Matrices

IF A, B AND C HAVE THE RIGHT ORDER FOR MULTIPLICATION AND ADDITION I.E., THE OPERATI
DEFINED FOR THE GIVEN MATRICES, THE FOLLOWING PROPERTIES HOLD:

1 $A\,(BC) = (AB)\,C$ (ASSOCIATIVE PROPERTY)

2 $A\,(B + C) = AB + AC$ (DISTRIBUTIVE PROPERTY)

3 $(B + C)\,A = BA + CA$ (DISTRIBUTIVE PROPERTY)

Example 14 LET $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ AND $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. FIND AB AND BA.

Solution: $AB = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$ AND $BA = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$.

FROM EXAMPLE 14, YOU CAN CONCLUDE THAT MULTIPLICATION OF MATRICES IS NOT COMMUTA

Transpose of a matrix

Definition 6.6

The **Transpose** of a matrix $A = (a_{IJ})_{m\times n}$, denoted by A^T, is the $n \times m$ matrix found by interchanging the rows and columns of A. i.e., $A^T = B = (b_{ji})$ OF ORDER $n \times m$ SUCH THAT $b_{ji} = a_{ij}$.

Example 15 GIVE THE TRANSPOSE OF THE MATRIX $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

Solution $\quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

ACTIVITY 6.8

GIVEN $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 0 \end{pmatrix}$, FIND:

A	A^T	**B**	$(A^T)^T$	**C**	$3A^T$
D	$(3A)^T$	**E**	$(AB)^T$	**F**	B^TA^T

Properties of transposes of matrices

THE FOLLOWING ARE PROPERTIES OF TRANSPOSES OF MATRICES:

A $\quad (A^T)^T = A$

B $\quad (A + B)^T = A^T + B^T$, A AND B BEING OF THE SAME ORDER.

C $\quad (rA)^T = rA^T$, r ANY SCALAR

D $\quad (AB)^T = B^T A^T$; PROVIDED AB IS DEFINED

Definition 6.7

A SQUARE MATRIX A IS CALLED A **symmetric** MATRIX IF $A^T = A$.

Example 16 SHOW THAT $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$ IS SYMMETRIC.

Solution $\quad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix} = A$. SO, A IS SYMMETRIC.

Example 17 WHICH OF THE FOLLOWING ARE SYMMETRIC MATRICES?

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & 3 \end{pmatrix},\quad B = \begin{pmatrix} a & d & c & d \\ d & k & l & m \\ c & l & w & a \\ d & m & a & x \end{pmatrix} \text{ AND } C = \begin{pmatrix} 1 & 7 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

Solution $\quad A$ AND B ARE SYMMETRIC WHILE C IS NOT.

Exercise 6.2

1 FIND THE PRODUCTS AB AND BA, WHENEVER THEY EXIST.

A $A=\begin{pmatrix}3 & 1\\ 3 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 1 & -3\\ 3 & 1 & 6\end{pmatrix}$ **B** $A=\begin{pmatrix}2 & 2\end{pmatrix}$, $B=\begin{pmatrix}1 & 5\\ -2 & 3\\ 0 & 4\end{pmatrix}$

C $A=\begin{pmatrix}-1 & 2\\ 1 & 4\\ -3 & 0\end{pmatrix}$, $B=\begin{pmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{pmatrix}$ **D** $A=\begin{pmatrix}10 & 3 & 2\\ -8 & -5 & 9\\ -5 & 7 & 7\end{pmatrix}$, $B=\begin{pmatrix}3\\ -1\\ 1\end{pmatrix}$

2 LET $A=\begin{pmatrix}2 & -1 & 3\\ 1 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}$ AND $B=\begin{pmatrix}1 & -4\\ 2 & 3\\ 4 & 0\end{pmatrix}$

A WHAT IS THE ORDER OF AB? **B** IF $C = AB$, THEN FIND C_{32}, C_{11} AND C_{21}.

3 FOR THE MATRICES IN QUESTION 2 ABOVE, FIND AB, BA, AND $A(AB)$.

4 THE FIRST OF THE FOLLOWING TABLES GIVES THE POINT SYSTEM USED IN SOCCER (FOOTBALL) IN THE OLD DAYS AND THE POINT SYSTEM THAT IS IN USE NOW. THE SECOND TABLE GIV OVERALL RESULTS OF 4 TEAMS IN A GAME SEASON.

	Points	
	Old system	New system
Win	2	3
Draw	1	1
Loss	0	0

		Win	Draw	Loss
Teams	A	5	2	2
	B	3	6	0
	C	4	4	1
	D	6	0	3

LET $T=\begin{pmatrix}5 & 2 & 2\\ 3 & 6 & 0\\ 4 & 4 & 1\\ 6 & 0 & 3\end{pmatrix}$ AND $P=\begin{pmatrix}2 & 3\\ 1 & 1\\ 0 & 0\end{pmatrix}$. ANSWER THE FOLLOWING QUESTIONS:

A FIND THE PRODUCT TP. WHICH SYSTEM IS BETTER TO RANK THE TEAMS-THE OLD O NEW?

B WHICH TEAM STANDS FIRST? WHICH STANDS LAST?

5 IF $A = \begin{pmatrix} 3 & -1 \\ 0 & \frac{4}{3} \end{pmatrix}$ AND $B = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & -2 \\ -1 & 0 & 1 \end{pmatrix}$, THEN FIND $A + A^T$ AND $B + B^T$. CHECK WHETHER OR NOT THE RESULTING MATRICES ARE SYMMETRIC.

6 IF $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, THEN SHOW THAT $AA^T = A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

7 SHOW THAT, IF A IS A SQUARE MATRIX OF ORDER n, THEN $A + A^T$ IS SYMMETRIC MATRIX (HINT: SHOW THAT $(A + A^T)^T = A^T + A$)

8 A SQUARE MATRIX A IS CALLED SKEW-SYMMETRIC, IF AND ONLY IF $A^T = -A$. VERIFY THAT THE FOLLOWING MATRICES ARE SKEW-SYMMETRIC:

A $A = \begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 7 \\ -4 & -7 & 0 \end{pmatrix}$ **B** $B = \begin{pmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{pmatrix}$

9 IF A IS A SQUARE MATRIX, SHOW THAT $A - A^T$ IS A SKEW-SYMMETRIC MATRIX.

10 IF A IS A SKEW-SYMMETRIC MATRIX, SHOW THAT THE ELEMENTS IN THE MAIN DIAGONAL ARE ALL ZERO.

6.2 DETERMINANTS AND THEIR PROPERTIES

THE DETERMINANT OF A SQUARE MATRIX IS A REAL NUMBER ASSOCIATED WITH THE SQUARE MATRIX. IT IS HELPFUL IN SOLVING SIMULTANEOUS EQUATIONS. THE DETERMINANT OF A MATRIX A IS ASSOCIATED WITH A ACCORDING TO THE FOLLOWING DEFINITION.

Determinants of 2 × 2 matrices

Definition 6.8

1 THE DETERMINANT OF A 1 × 1 MATRIX $A = (a)$ IS THE REAL NUMBER a.

2 THE DETERMINANT OF A 2 × 2 MATRIX $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ IS DEFINED TO BE THE NUMBER $ad - bc$.

THE DETERMINANT OF A IS DENOTED BY $\det(A)$ OR $|A|$.

THUS, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example 1 FIND $|A|$ FOR $A = \begin{pmatrix} 1 & 2 \\ 6 & 4 \end{pmatrix}$.

Solution $|A| = \begin{vmatrix} 1 & 2 \\ 6 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 6 = 4 - 12 = -8$

Note:

✓ $|A|$ DENOTES DETERMINANT OF A WHILE A IS A MATRIX; THE SAME SYMBOL IS USED FOR ABSOLUTE VALUE OF A REAL NUMBER. IT IS THE CONTEXT THAT DECIDES THE MEANING.

✓ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ DENOTES A MATRIX, WHILE $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ DENOTES ITS DETERMINANT. THE DETERMINANT IS A REAL NUMBER.

ACTIVITY 6.9

LET $A = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ AND $B = \begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix}$.

1 CALCULATE

A $|A|$ **B** $|B|$ **C** $|A^T|$

2 CALCULATE AND COMPARE $|AB|$ AND $|A|\,|B|$.

3 CALCULATE AND COMPARE $|A + B|$ AND $|A| + |B|$.

Determinants of 3 × 3 matrices

TO DEFINE THE DETERMINANT OF A 3 × 3 MATRIX, IT IS USEFUL TO FIRST DEFINE THE CONCEPTS OF MINOR AND COFACTOR.

LET $A = (a_{ij})_{3\times3}$. THEN THE MATRIX A_{ij} IS A 2×2 MATRIX WHICH IS FOUND BY CROSSING OUT THE i^{th} ROW AND j^{th} COLUMN OF A.

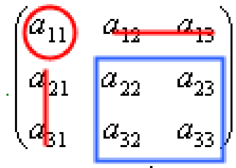
Example 2 IF $A = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 3 & 5 \\ 4 & 7 & 18 \end{pmatrix}$, THEN $A_{11} = \begin{pmatrix} 3 & 5 \\ 7 & 18 \end{pmatrix}$ AND $A_{23} = \begin{pmatrix} 0 & 1 \\ 4 & 7 \end{pmatrix}$.

Definition 6.9

LET $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. THEN $M_{ij} = |A_{ij}|$ IS CALLED THE **minor** OF THE ELEMENT a_{ij} AND $C_{ij} = (-1)^{i+j}|A_{ij}|$ IS CALLED THE **cofactor** OF THE ELEMENT a_{ij}.

Example 3 LET $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. GIVE THE MINORS AND COFACTORS OF a_{11}, a_{23} AND a_{32}.

Solution THE MINOR OF a_{11} = $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. IT IS FOUND BY CROSSING OUT THE FIRST ROW AND THE FIRST COLUMN AS IN THE FIGURE.



THUS, THE MINOR OF a_{11} = $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

THE COFACTOR OF a_{11} = $c_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

THE MINOR OF a_{23} = $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$, WHILE $c_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$.

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ AND $c_{32} = -M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$.

Example 4 FIND THE MINORS AND COFACTORS OF a_{22}, a_{33} AND a_{12} OF THE MATRIX $\begin{pmatrix} -3 & 4 & -7 \\ 1 & 2 & 0 \\ -4 & 8 & 11 \end{pmatrix}$.

Solution

$M_{22} = \begin{vmatrix} -3 & -7 \\ -4 & 11 \end{vmatrix} = -61$ AND $c_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} -3 & -7 \\ -4 & 11 \end{vmatrix} = (-3)(11) - (-4)(-7) = -61$

$M_{33} = \begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix} = -10$ AND $c_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix} = (-3)(2) - (1)(4) = -10$

$M_{12} = \begin{vmatrix} 1 & 0 \\ -4 & 11 \end{vmatrix} = 11$ AND $c_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 1 & 0 \\ -4 & 11 \end{vmatrix} = -11$

Note:

NOTE THAT THE SIGN $(-1)^{i+j}$ ACCOMPANYING THE MINORS FORM A CHESS BOARD PATTERN WITH '+' S ON THE MAIN DIAGONAL AS SHOWN: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

YOU CAN NOW DEFINE THE 3 × 3 DETERMINANT (DETERMINANT OF ORDER 3) AS FOLLOWS:

Definition 6.10

LET $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. THEN THE DETERMINANT OF A ALONG ANY ROW I OR ANY COLUMN J IS GIVEN BY ONE OF THE FORMULAS:

i^{th} ROW EXPANSION: $|A| = a_{i1}c_{i1}+a_{i2}c_{i2}+a_{i3}c_{i3}$, FOR ANY ROW i (i = 1,2 OR 3), OR

j^{th} COLUMN EXPANSION: $|A| = a_{1j}c_{1j}+a_{2j}c_{2j}+a_{3j}c_{3j}$, FOR ANY COLUMN j (j = 1,2 OR 3).

Note:

NOTE THAT THE DEFINITION STATES THAT TO FIND THE DETERMINANT OF A SQUARE MATRIX

- ✓ CHOOSE A ROW OR COLUMN;
- ✓ MULTIPLY EACH ENTRY IN IT BY ITS COFACTOR;
- ✓ ADD UP THESE PRODUCTS.

Example 5 FIND THE DETERMINANT OF THE FOLLOWING MATRIX A BY EXPANDING ALONG THE 1ST ROW AND THEN EXPANDING ALONG THE 2ND COLUMN, WHERE

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{pmatrix}$$

Solution

Along row 1:

$$|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 2(-1)^2 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}$$

$$= 2(1\times5-2\times4)+(-1)(1\times5-4\times(-3))+0(1\times2-1\times(-3))$$

$$= 2(-3)-1(17)+0(5) = -6-17 = -23$$

$$\therefore \ |A| = -23$$

Along Column 2:

$$|A| = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 1(-1) \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} + 1(1) \begin{vmatrix} 2 & 0 \\ -3 & 5 \end{vmatrix} + 2(-1) \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= -1(1\times5-4\times(-3)) + 1(2\times5-0\times(-3)) - 2(2\times4-0\times1)$$

$$= -1(17) + 1(10) - 2(8) = -17 + 10 - 16 = -23$$

$$\therefore |A| = -23,$$

BOTH METHODS GIVE THE SAME RESULT.

Group Work 6.1

FOR THE MATRIX $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{pmatrix}$ DO EACH OF THE FOLLOWING IN

1 **A** CALCULATE $|A|$ AND $|A^T|$

B WHAT CAN YOU CONCLUDE FROM THESE RESULTS?

2 LET B BE THE MATRIX FOUND BY INTERCHANGING ROW 1 AND ROW 3 OF MATRIX A, I.E.,

$$B = \begin{pmatrix} 2 & 5 & 2 \\ 4 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

A FIND $|B|$

B COMPARE IT WITH $|A|$. WHAT RELATIONSHIP DO YOU SEE BETWEEN $|B|$ AND $|A|$?

3 LET C BE THE MATRIX FOUND BY MULTIPLYING ROW 2 BY 5. I.E.,

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 5\times 4 & 5\times 1 & 5\times 3 \\ 2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 20 & 5 & 15 \\ 2 & 5 & 2 \end{pmatrix}$$

A FIND $|C|$

B COMPARE IT WITH $|A|$. WHAT RELATIONSHIP DO YOU SEE BETWEEN $|C|$ AND $|A|$?

4 LET D BE THE MATRIX FOUND BY ADDING 10 TIMES COLUMN 1 ON COLUMN 3. I.E.,

$$D = \begin{pmatrix} 1 & 3 & 2+10\times 1 \\ 4 & 1 & 3+10\times 4 \\ 2 & 5 & 2+10\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 12 \\ 4 & 1 & 43 \\ 2 & 5 & 22 \end{pmatrix}$$

A FIND $|D|$

B COMPARE IT WITH $|A|$. WHAT RELATIONSHIP DO YOU SEE BETWEEN $|D|$ AND $|A|$?

Properties of determinants

THE FOLLOWING PROPERTIES HOLD. ALL THE MATRICES CONSIDERED ARE SQUARE MATRICES

1 $|A| = |A^T|$

VERIFY THIS PROPERTY BY CONSIDERING A 2 × 2 MATRIX

I.E., IF $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, THEN $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

HENCE, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. ALSO, $|A^T| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

THEREFORE, $|A| = |A^T|$.

2 IF B IS FOUND BY INTERCHANGING TWO ROWS (COLUMNS) OF A, THEN $|B| = -|A|$.

3 IF B IS FOUND BY MULTIPLYING ONE ROW (OR A COLUMN) OF A BY A SCALAR r, THEN $|B| = r|A|$.

4 IF B IS A MATRIX OBTAINED BY ADDING A MULTIPLE OF A ROW (COLUMN) OF A TO ANOTHER ROW (COLUMN) OF A, THEN $|B| = |A|$.

5 IF A HAS A ROW (OR A COLUMN) OF ZEROS, THEN THE DETERMINANT OF A IS ZERO.

6 IF A HAS TWO IDENTICAL ROWS (OR COLUMNS), THEN THE DETERMINANT OF A IS ZERO.

WE OMIT THE PROOFS OF THE ABOVE PROPERTIES; HOWEVER, WE SHALL ILLUSTRATE THESE PROPERTIES WITH EXAMPLES.

Example 6 COMPUTE THE DETERMINANT OF $\begin{pmatrix} 4 & 0 & -5 \\ 10 & 0 & 7 \\ -14 & 0 & 1 \end{pmatrix}$

Solution BY EXPANDING USING THE 2ND COLUMN, WE GET

$$\begin{vmatrix} 4 & 0 & -5 \\ 10 & 0 & 7 \\ -14 & 0 & 1 \end{vmatrix} = -0\begin{vmatrix} 10 & 7 \\ -14 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & -5 \\ -14 & 1 \end{vmatrix} - 0\begin{vmatrix} 4 & -5 \\ 10 & 7 \end{vmatrix} = 0$$

Example 7 IF $\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = 2$, GIVE THE VALUES OF EACH OF THE FOLLOWING.

A $\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix}$ **B** $\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix}$ **C** $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

D $\begin{vmatrix} p & x & 0 \\ q & y & 0 \\ r & z & 0 \end{vmatrix}$ **E** $\begin{vmatrix} 4a & 12x & 4p \\ b & 3y & q \\ c & 3z & r \end{vmatrix}$ **F** $\begin{vmatrix} a & x & p \\ b & y & q \\ 3b+c & 3y+z & 3q+r \end{vmatrix}$

Solution:

A 0 (1ST COLUMN AND 3RD COLUMN ARE THE SAME.)

B –2 (COLUMN INTERCHANGE RESULTS IN CHANGE OF SIGN.)

C 2 (A MATRIX AND ITS TRANSPOSE HAVE THE SAME DETERMINANT.)

D 0 (0 COLUMN.)

E 24 (FACTOR 4 OUT AND THEN 3; 12 × ORIGINAL DETERMINANT.)

F 2 (ADDING A CONSTANT MULTIPLE OF A ROW TO ANOTHER ROW GIVES THE SAME RESULT.)

Exercise 6.3

1 COMPUTE EACH OF THE FOLLOWING DETERMINANTS:

A $\begin{vmatrix} 1 & 5 \\ 7 & 3 \end{vmatrix}$ **B** $\begin{vmatrix} 1 & 3 & 3 \\ 0 & 2 & -1 \\ 2 & 1 & 2 \end{vmatrix}$ **C** $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

2 SOLVE EACH OF THE FOLLOWING EQUATIONS:

A $\begin{vmatrix} 2x & x \\ 4 & x \end{vmatrix} = 0$ **B** $\begin{vmatrix} 2 & -2 & 1 \\ x & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 1$ **C** $\begin{vmatrix} x+1 & 2 & 1 \\ 1 & 1 & 2 \\ x-1 & 1 & x \end{vmatrix} = 0$

3 FOR THE GIVEN MATRIX A, CALCULATE THE COFACTOR OF THE GIVEN ENTRY:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 9 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

A a_{32} **B** a_{22} **C** a_{23}

4 **A** COMPUTE THE DETERMINANT $\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix}$

B VERIFY THAT THE EQUATION OF A STRAIGHT LINE THROUGH THE DISTINCT POINTS (a, b) AND (c, d) IS GIVEN BY $\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = 0$

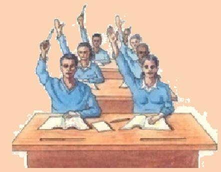
5 VERIFY THAT EACH OF THE FOLLOWING STATEMENTS IS TRUE (*Assume that all letters represent non-zero real number*).

A $\begin{vmatrix} x & t+w \\ y & s+u \end{vmatrix} = \begin{vmatrix} x & t \\ y & s \end{vmatrix} + \begin{vmatrix} x & w \\ y & u \end{vmatrix}$ **B** $\begin{vmatrix} a+rb & b \\ c+rd & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

C $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

6.3 INVERSE OF A SQUARE MATRIX

ACTIVITY 6.10



LET $A = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$ AND $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. FIND:

A AI_2 **B** I_2A

C FIND A MATRIX C (IF IT EXISTS) SUCH THAT $AC = I_2$

D IS THERE A MATRIX D SUCH THAT $BD = I_2$?

FROM ACTIVITY 6.10, THE MATRIX C OBTAINED IN (C) IS CALLED THE INVERSE OF MATRIX A.

Definition 6.11

A SQUARE MATRIX A IS SAID TO BE **invertible** OR **non-singular**, IF AND ONLY IF THERE IS A SQUARE MATRIX B SUCH THAT $AB = BA = I$, WHERE I IS THE IDENTITY MATRIX THAT HAS THE SAME ORDER AS A.

Remark

THE INVERSE OF A SQUARE MATRIX, IF IT EXISTS, IS UNIQUE.

Proof: LET A BE AN INVERTIBLE SQUARE MATRIX AND SUPPOSE B AND C ARE INVERSES OF A.

THEN $AB = BA = I$. AND $AC = CA = I$ (BY DEFINITION OF INVERSE)

NOW, $B = BI = B(AC) = (BA)C = IC = C$.

HENCE, THE INVERSE OF A IS UNIQUE.

Note:

- ✓ ONLY A SQUARE MATRIX CAN HAVE AN INVERSE.
- ✓ THE INVERSE OF MATRIX A, WHENEVER IT EXISTS, IS DENOTED BY A^{-1}.
- ✓ A AND A^{-1} HAVE THE SAME ORDER.
- ✓ A MATRIX THAT DOES NOT HAVE AN INVERSE IS CALLED **singular**.

Example 1

A SHOW THAT $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ AND $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ ARE INVERSES OF EACH OTHER.

B GIVEN $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, FIND A^{-1} (IF IT EXISTS.)

Solution

A $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

THUS, THEY ARE INVERSES OF EACH OTHER.

B SUPPOSE $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. THEN $AA^{-1} = I_2$.

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+c & b+d \\ 2a+3c & 2b+3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \begin{cases} a+c=1 \\ 2a+3c=0 \end{cases} \text{ AND } \begin{cases} b+d=0 \\ 2b+3d=1 \end{cases}$$

SOLVING THESE GIVES YOU $a = 3, b = -1, c = -2$ AND $d = 1$

HENCE $A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

IN THE ABOVE EXAMPLE, YOU HAVE SEEN HOW TO FIND THE INVERSES OF INVERTIBLE MA SOMETIMES, THIS METHOD IS TIRESOME AND TIME CONSUMING. THERE IS ANOTHER METH FINDING INVERSES OF INVERTIBLE MATRICES, USING THE ADJOINT.

Definition 6.12

THE **adjoint** OF A SQUARE MATRIX $A = (a_{ij})$ IS DEFINED AS THE TRANSPOSE OF THE MATRIX $C = (c_{ij})$ WHERE c_{ij} ARE THE COFACTORS OF THE ELEMENTS a_{ij}. ADJOINT OF A IS DENOTED BY **adj** A, I.E., ADJ $A = (c_{ij})^{\mathrm{T}}$.

Example 2 FIND ADJ A IF $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$.

Solution

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} = 0, \qquad c_{12} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = -4,$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12, \qquad c_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0,$$

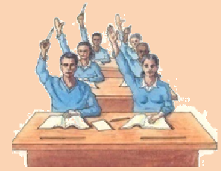
$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4, \qquad c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} = 0,$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3, \qquad c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3,$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3.$$

THEN MATRIX $C = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -4 & 0 \\ -3 & 3 & 3 \end{pmatrix}^{\mathrm{T}}$ AND, ADJ $A = C^T = \begin{pmatrix} 0 & 0 & -3 \\ -4 & -4 & 3 \\ -12 & 0 & 3 \end{pmatrix}$

ACTIVITY 6.11



1 SHOW THAT $\text{ADJ}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2 SHOW THAT $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \text{ADJ}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3 IF $A = \begin{pmatrix} 5 & -3 \\ 4 & 2 \end{pmatrix}$, THEN

- **A** FIND A^{-1}.
- **B** FIND ADJ A
- **C** FIND $|A|$.
- **D** COMPARE A^{-1} AND $\frac{1}{|A|}$ ADJ A

FROM ACTIVITY 6.11, YOU MAY HAVE OBSERVED THAT FOR A 2×2 MATRIX A,

$$A(\text{ADJ}A) = |A|\, I_2 = (\text{ADJ}A)A.$$

IF $|A| \neq 0$, THEN $A\frac{1}{|A|}\text{ADJ}A = I_2$

THEREFORE, $A^{-1} = \frac{1}{|A|}\text{ADJ}A$

> **Theorem 6.1**
>
> A SQUARE MATRIX A IS INVERTIBLE, IF AND ONLY IF $|A| \neq 0$. IF A IS INVERTIBLE, THEN
>
> $A^{-1} = \frac{1}{|A|}\text{ADJ}A$.

Example 3 FIND THE INVERSE OF $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$

Solution FIRST FIND ADJA.

$c_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1;$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} = -4;$ $\quad C_{13} = +\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8$

$c_{21} = -\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} = 19;$ $\quad C_{22} = +\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14;$ $\quad C_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3$

$c_{31} = +\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = -8;$ $\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1;$ $\quad C_{33} = +\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$

THUS, ADJ $A = \begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix}$

NEXT, FIND $|A|$.

$|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = (-1)(-1) + (-2)(-4) + (3)(8) = 31$. SINCE $|A| \neq 0$,, THEN A IS INVERTIBLE AND

$$A^{-1} = \frac{1}{|A|}\text{ADJ}(A) = \frac{1}{31}\begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{31} & \frac{19}{31} & \frac{-8}{31} \\ \frac{-4}{31} & \frac{14}{31} & \frac{-1}{31} \\ \frac{8}{31} & \frac{3}{31} & \frac{2}{31} \end{pmatrix}$$

Example 4 SHOW THAT $\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$ IS NOT INVERTIBLE

Solution $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = (1)(-6) - (3)(-2) = 0$. THUS, THE INVERSE DOES NOT EXIST.

Theorem 6.2

IF A AND B ARE TWO INVERTIBLE MATRICES OF THE SAME ORDER, THEN $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

IF A AND B ARE INVERTIBLE MATRICES OF THE SAME ORDER, THEN $|A| \neq 0$ and $|B| \neq 0$.

$\Rightarrow |AB| = |A||B| \neq 0$

HENCE, AB IS INVERTIBLE WITH INVERSE $(AB)^{-1}$. ON THE OTHER HAND,

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$ AND SIMILARLY

$(B^{-1}A^{-1})(AB) = I$.

THEREFORE $B^{-1}A^{-1}$ IS AN INVERSE OF AB AND INVERSE OF A MATRIX IS UNIQUE.

HENCE $B^{-1}A^{-1} = (AB)^{-1}$.

Example 5 VERIFY THAT $(AB)^{-1} = B^{-1}A^{-1}$, FOR THE FOLLOWING MATRICES:

$A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ AND $B = \begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix}$

Solution $|A| = 2$ AND $|B| = -9$. TO FIND ADJ(A), INTERCHANGE THE DIAGONAL ELEMENTS AND TAKE THE NEGATIVES OF THE NON-DIAGONAL ELEMENTS. THUS,

$$\text{ADJ}(A) = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \text{ AND ADJ}(B) = \begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix}$$

IT FOLLOWS THAT, $A^{-1} = \frac{1}{|A|}\text{ADJ}A = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}$, WHILE

$$B^{-1} = \frac{1}{|B|}\text{ADJ}B = -\frac{1}{9}\begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

THIS GIVES US $B^{-1}A^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 \\ \frac{-5}{2} & 2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

ON THE OTHER HAND, $AB = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 10 \\ -6 & 13 \end{pmatrix}$, SO THAT

$$|AB| = -18 \text{ AND ADJ}(AB) = \begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix}.$$

$$(AB)^{-1} = -\frac{1}{18}\begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

THEREFORE, $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 6.4

1 SHOW THAT $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$ AND $\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$ ARE INVERSES OF EACH OTHER.

2 FIND THE INVERSE, IF IT EXISTS, FOR EACH OF THE FOLLOWING MATRICES:

A $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ **B** $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

3 SHOW THAT THE MATRIX $A = \begin{pmatrix} 3-k & 6 \\ 2 & 4-k \end{pmatrix}$ IS SINGULAR WHEN $k = 0$ OR $k = 7$. WHAT IS THE INVERSE WHEN $k = 1$?

4 GIVEN $A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, SHOW THAT $A^{-1} = A^T$.

5 USING $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ AND $B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, VERIFY THAT $(AB)^{-1} = B^{-1}A^{-1}$.

6 PROVE THAT IF A IS NON-SINGULAR, THEN $AB = AC$ IMPLIES $B = C$. DOES THIS NECESSARILY HOLD IF A IS SINGULAR? IF NOT, TRY TO PRODUCE AN EXAMPLE TO THE CONTRARY.

6.4 SYSTEMS OF EQUATIONS WITH TWO OR THREE VARIABLES

MATRICES ARE MOST USEFUL IN SOLVING SYSTEMS OF LINEAR EQUATIONS. SYSTEMS OF EQUATIONS ARE USED TO GIVE MATHEMATICAL MODELS OF ELECTRICAL NETWORKS, TRAFFIC, AND MANY OTHER REAL LIFE SITUATIONS.

Definition 6.13

AN EQUATION $a_1x_1 + a_2x_2 + ... + a_nx_n = b$, WHERE $a_1, a_2, ..., a_n, b$ ARE CONSTANTS AND $x_1, x_2, ..., x_n$ ARE VARIABLES IS CALLED A LINEAR EQUATION. IF $b = 0$, THEN THE LINEAR EQUATION IS SAID TO BE **homogeneous**.

A LINEAR SYSTEM OF m EQUATIONS IN n UNKNOWNS (VARIABLES), $x_1, x_2, ..., x_n$ IS A SET OF EQUATIONS OF THE FORM

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \qquad (*)$$

THE SYSTEM OF EQUATIONS (*) IS EQUIVALENT TO $AX = B$, WHERE

$$A = (a_{ij})_{m\times n},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \text{ AND } B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}.$$

MATRIX A IS CALLED **the coefficient matrix** OF THE SYSTEM AND THE MATRIX

$$(A/B)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$ IS CALLED **the augmented matrix** OF THE SYSTEM.

Example 1 WHICH OF THE FOLLOWING ARE SYSTEMS OF LINEAR EQUATIONS?

A $\begin{cases} 5x-23y=6 \\ x+14y=12 \end{cases}$ **B** $\begin{cases} 5x^2-23y=6 \\ x+14y=12 \end{cases}$ **C** $\begin{cases} 5x-23y+z=6 \\ x+14y-4z=18 \end{cases}$

Solution A AND C ARE SYSTEMS OF LINEAR EQUATIONS. B IS NOT A LINEAR EQUATION BECAUSE THE FIRST EQUATION IN THE SYSTEM IS NOT LINEAR IN x.

Example 2 GIVE THE AUGMENTED MATRIX OF THE FOLLOWING SYSTEMS OF EQUATIONS.

A $\begin{cases} 2x+5y=1 \\ 3x-8y=4 \end{cases}$ **B** $\begin{cases} 2x-y+z=3 \\ 3x-2y+8z=-24 \\ x+3y+4y=-2 \end{cases}$ **C** $\begin{cases} x+y=0 \\ 2x-y+3z=3 \\ x-2y-z=3 \end{cases}$

Solution

A $\begin{pmatrix} 2 & 5 & 1 \\ 3 & -8 & 4 \end{pmatrix}$ **B** $\begin{pmatrix} 2 & -1 & 1 & 3 \\ 3 & -2 & 8 & -24 \\ 1 & 3 & 4 & -2 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & -2 & -1 & 3 \end{pmatrix}$

Elementary operations on matrices

ACTIVITY 6.12

SOLVE EACH OF THE FOLLOWING SYSTEMS OF LINEAR EQUA...

A $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$ **B** $\begin{cases} 2x-y=4 \\ -x+y=-1 \end{cases}$ **C** $\begin{cases} 3x-5y=-5 \\ x+2y=2 \end{cases}$

FROM ACTIVITY 6.12, EQUATIONS A AND B HAVE THE SAME SOLUTION SET. YOU HAVE THE FOLLOWING DEFINITION FOR EQUATIONS HAVING THE SAME SOLUTION SET.

Definition 6.14

TWO SYSTEMS OF LINEAR EQUATIONS ARE **equivalent**, IF AND ONLY IF THEY HAVE EXACTLY THE SAME SOLUTION.

TO SOLVE SYSTEMS OF LINEAR EQUATIONS, YOU MAY RECALL, WE USE EITHER THE SUBS METHOD OR THE ELIMINATION METHOD. THE METHOD OF ELIMINATION IS MORE SYSTEMAT THE METHOD OF SUBSTITUTION. IT CAN BE EXPRESSED IN MATRIX FORM AND MATRIX OPE CAN BE DONE BY COMPUTERS. THE METHOD OF ELIMINATION IS BASED ON EQUIVALENT SYST EQUATIONS.

TO CHANGE A SYSTEM OF EQUATIONS INTO AN EQUIVALENT SYSTEM, WE USE ANY OF THE FO
THREE **elementary** (ALSO CALLED **Gaussian**) **operations.**

Swapping	INTERCHANGE TWO EQUATIONS OF THE SYSTEM.
Rescaling	MULTIPLY AN EQUATION OF THE SYSTEM BY A NON-ZERO CONSTANT.
Pivoting	ADD A CONSTANT MULTIPLE OF ONE EQUATION TO ANOTHER EQUATION OF THE SY

Note:

✓ IN THE ELIMINATION METHOD, THE ARITHMETIC INVOLVES ONLY THE NUMERICAL COEFFICIENTS. THUS IT IS BETTER TO WORK WITH THE NUMERICAL COEFFICIENTS ONLY.

✓ THE NUMERICAL COEFFICIENTS AND THE CONSTANT TERMS OF A SYSTEM OF EQUATIONS CAN BE EXPRESSED IN MATRIX FORM, CALLED THE **augmented matrix,** AS SHOWN BELOW IN EXAMPLE 3.

Elementary row operations

Swapping	INTERCHANGING TWO ROWS OF A MATRIX
Rescaling	MULTIPLYING A ROW OF A MATRIX BY A NON-ZERO CONSTANT
Pivoting	ADDING A CONSTANT MULTIPLE OF ONE ROW OF THE MATRIX ONTO ANOTHER RO

Elementary column operations

Swapping	INTERCHANGING TWO COLUMNS OF A MATRIX
Rescaling	MULTIPLYING A COLUMN OF A MATRIX BY A NON-ZERO CONSTANT
Pivoting	ADDING A CONSTANT MULTIPLE OF ONE COLUMN OF THE MATRIX ONTO ANOTHER C

Definition 6.15

TWO MATRICES ARE SAID TO BE ROW (OR COLUMN) **equivalent**, IF AND ONLY IF ONE IS OBTAINED FROM THE OTHER BY PERFORMING ANY OF THE ELEMENTARY OPERATIONS.

Note:

✓ SINCE EACH ROW OF AN AUGMENTED MATRIX CORRESPONDS TO AN EQUATION OF A SYSTEM OF EQUATIONS, WE WILL USE ELEMENTARY ROW OPERATIONS ONLY

✓ WE SHALL USE THE FOLLOWING NOTATIONS:

- SWAPPING OF i^{th} AND j^{th} ROWS WILL BE DENOTED BY: $R_i \leftrightarrow R_j$
- RESCALING OF i^{th} ROW BY NON-ZERO NUMBER r WILL BE DENOTED BY: $R_i \to rR_i$
- PIVOTING OF i^{th} ROW BY r TIMES THE j^{th} ROW WILL BE DENOTED BY: $R_i \to R_i + rR_j$

Example 3 SOLVE THE SYSTEM OF EQUATIONS GIVEN BELOW BY USING THE AUGMENTED MATRIX

$$\begin{cases} x-2y+z=7 \\ 3x+y-z=2 \\ 2x+3y+2z=7 \end{cases}$$

Solution

Write the augmented matrix	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 3 & 1 & -1 & 2 \\ 2 & 3 & 2 & 7 \end{pmatrix}$	THE OBJECTIVE IS TO GET AS MANY ZEROS AS POSSIBLE IN THE COEFFICIENTS.
$R_2 \longrightarrow R_2 + -3R_1$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 2 & 3 & 2 & 7 \end{pmatrix}$	A ZERO IS OBTAINED IN THE a_{21} POSITION. NOTE THAT THE OTHER ELEMENTS OF ROW 2 ARE ALSO CHANGED.
$R_3 \longrightarrow R_3 + -2R_1$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 0 & 7 & 0 & -7 \end{pmatrix}$	A ZERO IS OBTAINED IN THE a_{31} POSITION. NOTE THAT THE OTHER ELEMENTS OF ROW 3 ARE ALSO CHANGED.
$R_3 \longrightarrow R_3 + -1.R_2$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 0 & 0 & 4 & 12 \end{pmatrix}$	A ZERO IS OBTAINED IN THE a_{32} POSITION. NOTE THAT THE OTHER ELEMENTS OF ROW 3 ARE ALSO CHANGED.

THE LAST MATRIX CORRESPONDS TO THE SYSTEM OF EQUATION:

$$\begin{cases} x-2y+z=7 \\ 7y-4z=-19 \\ 4z=12 \end{cases}$$

SINCE THIS EQUATION AND THE GIVEN EQUATION ARE EQUIVALENT, THEY HAVE THE SAME SOLUTIONS. THUS BY BACK-SUBSTITUTING $z = 3$ FROM THE 3RD EQUATION INTO THE 2ND, WE GET $y = -1$ AND BACK-SUBSTITUTING $z = 3$ AND $y = -1$ IN THE 1ST EQUATION, WE GET $x = 2$.

THE SOLUTION SET IS $\{(2, -1, 3)\}$.

Definition 6.16

A MATRIX IS SAID TO BE IN **Row Echelon Form** IF,

1. A ZERO ROW (IF THERE IS) COMES AT THE BOTTOM.
2. THE FIRST NONZERO ELEMENT IN EACH NON-ZERO ROW IS 1.
3. THE NUMBER OF ZEROS PRECEDING THE FIRST NON-ZERO ELEMENT IN EACH NON-ZERO ROW EXCEPT THE FIRST ROW IS GREATER THAN THE NUMBER OF SUCH ZEROS IN THE PRECEDI

Example 4 WHICH OF THE FOLLOWING MATRICES ARE IN ECHELON FORM?

$$A=\begin{pmatrix}1&-2&-4\\0&1&-2\\0&0&0\end{pmatrix},\ B=\begin{pmatrix}0&0&1&2\\2&3&0&-2\\3&3&6&-9\end{pmatrix},\ C=\begin{pmatrix}1&-2&1&7\\0&7&-4&-19\\2&3&2&7\end{pmatrix},\ D=\begin{pmatrix}2&3&1&2\\0&0&0&0\\3&3&-6&-9\end{pmatrix}$$

Solution

A IS IN ECHELON FORM.

B IS NOT IN ECHELON FORM BECAUSE THE NUMBER OF ZEROS PRECEDING THE FIRST NO ELEMENT IN THE FIRST ROW IS GREATER THAN THE NUMBER OF SUCH ZEROS IN THE ROW. *C* IS NOT IN ECHELON FORM FOR THE SAME REASON. *D* IS NOT IN ECHELON FORM BECAUSE THE ZERO ROW IS NOT AT THE BOTTOM.

Example 5 SOLVE THE SYSTEM OF EQUATIONS

$$\begin{cases} z = 2\\ 2x+3y = -2\\ 3x+3y+6z = -9\end{cases}$$

Solution

Write the augmented matrix	$\begin{pmatrix}0&0&1&2\\2&3&0&-2\\3&3&6&-9\end{pmatrix}$	THE OBJECTIVE IS TO GET AS MANY ZEROS AS POSSIBLE IN THE COEFFICIENTS.
$R_1 \leftrightarrow R_3$	$\begin{pmatrix}3&3&6&-9\\2&3&0&-2\\0&0&1&2\end{pmatrix}$	MORE ZEROS MOVED TO LAST ROW.
$R_1 \longrightarrow \frac{1}{3}R_1$	$\begin{pmatrix}1&1&2&-3\\2&3&0&-2\\0&0&1&2\end{pmatrix}$	A LEADING ENTRY 1 IS OBTAINED IN ROW 1. NOTE THAT THE OTHER ELEMENTS OF ROW 1 ARE ALSO CHANGED.
$R_2 \longrightarrow R_2 + -2R_1$	$\begin{pmatrix}1&1&2&-3\\0&1&-4&4\\0&0&1&2\end{pmatrix}$	A ZERO IS OBTAINED AT POSITION a_{21}. NOTE THAT THE OTHER ELEMENTS OF ROW 2 ARE ALSO CHANGED.
$R_1 \longrightarrow R_1 + -1R_2$	$\begin{pmatrix}1&0&6&-7\\0&1&-4&4\\0&0&1&2\end{pmatrix}$	A ZERO IS OBTAINED AT POSITION a_{12}. NOTE THAT THE OTHER ELEMENTS OF ROW 1 ARE ALSO CHANGED.

THE LAST MATRIX CORRESPONDS TO THE SYSTEM OF EQUATION:

$$\begin{cases} x+6z=-7\\ y-4z=4\\ z=2\end{cases}$$

SINCE THIS LAST EQUATION AND THE GIVEN EQUATION ARE EQUIVALENT, WE GET THE S

$x = -19$, $y = 12$ AND $z = 2$.

THE SOLUTION SET IS $\{(-19, 12, 2)\}$. THE SYSTEM HAS EXACTLY ONE SOLUTION.

THE LAST MATRIX WE OBTAINED IS SAID TO BE IN **reduced-echelon form**, AS GIVEN IN THE FOLLOWING DEFINITION:

Definition 6.17

A MATRIX IS IN **Row Reduced Echelon** FORM, IF AND ONLY IF,

1. IT IS IN ECHELON FORM
2. THE FIRST NON-ZERO ELEMENT IN EACH NON-ZERO ROW IS THE ONLY NON-ZERO ELEMENT IN ITS COLUMN.

Example 6 SOLVE THE SYSTEM OF EQUATIONS $\begin{cases} x + 2y = 0 \\ 2x + y = 1 \\ x - y = 2 \end{cases}$

Solution

Augmented matrix	$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$	
$R_2 \longrightarrow R_2 + -2R_1$ $R_3 \longrightarrow R_3 + -1R_1$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & -3 & 2 \end{pmatrix}$	
$R_3 \longrightarrow R_3 + -1R_2$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
$R_2 \longrightarrow -\frac{1}{3}R_2$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 1 \end{pmatrix}$	NOTICE THAT THIS MATRIX IS IN ROW ECHELON FORM.

IN THE LAST ROW, THE COEFFICIENT ENTRIES ARE 0, WHILE THE CONSTANT IS 1. THIS THAT $0x + 0y = 1$. BUT, THIS HAS NO SOLUTION.

THUS, $\begin{cases} x+2y=0 \\ 2x+y=1 \\ x-y=2 \end{cases}$ HAS NO SOLUTION.

I.E., THE SOLUTION SET IS EMPTY SET.

✍Note:

WHEN THE AUGMENTED MATRIX IS CHANGED INTO EITHER ECHELON FORM OR REDUCED-ECI
FORM AND IF THE LAST NON-ZERO ROW HAS NUMERICAL COEFFICIENTS WHICH ARE ALL ZER
HAVING NON-ZERO CONSTANT PART, THEN THE SYSTEM HAS NO SOLUTION.

Example 7 SOLVE THE FOLLOWING SYSTEM OF EQUATIONS

$$\begin{cases} x-2y-4z=0 \\ -x+y+2z=0 \\ 3x-3y-6z=0 \end{cases}$$

Solution

Augmented matrix	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & -3 & -6 & 0 \end{pmatrix}$	
$R_2 \longrightarrow R_2 + R_1$ $R_3 \longrightarrow R_3 + -3R_1$	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$	
$R_2 \longrightarrow -1R_2$	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$	
$R_3 \longrightarrow R_3 + -3R_2$ $R_1 \longrightarrow R_1 + 2R_2$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	THE MATRIX IS NOW IN REDUCED-ECHELON FORM.

THE LAST MATRIX GIVES THE SYSTEM $\begin{cases} x=0 \\ y+2z=0 \end{cases}$

THIS HAS SOLUTION $x = 0$, $y = -2z$.

THE SOLUTION SET IS $\{(0, -2z, z) \mid z$ A REAL NUMBER$\}$.

NOTICE THAT THE SOLUTION SET IS INFINITE.

Note:

WHEN THE AUGMENTED MATRIX IS CHANGED INTO EITHER ECHELON FORM OR REDUCED-ECH FORM AND IF THE NUMBER OF NON-ZERO ROWS IS LESS THAN THE NUMBER OF VARIABLES, T SYSTEM HAS AN INFINITE SOLUTIONS.

THE METHOD OF SOLVING A SYSTEM OF LINEAR EQUATIONS BY REDUCING THE AUGMENTED OF THE SYSTEM INTO REDUCED-ECHELON FORM IS CALLED **Gaussian Elimination Method**.

NOTE THAT EXAMPLES 3 - 7 ABOVE GIVE ALL THE POSSIBILITIES FOR SOLUTION SETS OF SYSTEM OF LINEAR EQUATIONS.

Case 1: THERE IS **exactly one solution**—SUCH A SYSTEM OF LINEAR EQUATIONS IS CALLED **consistent.**

Case 2: THERE IS **no solution**—SUCH A SYSTEM OF LINEAR EQUATIONS IS CALLED **inconsistent.**

Case 3: THERE IS AN **infinite number of solutions**—SUCH A SYSTEM OF LINEAR EQUATIONS IS CALLED **dependent**.

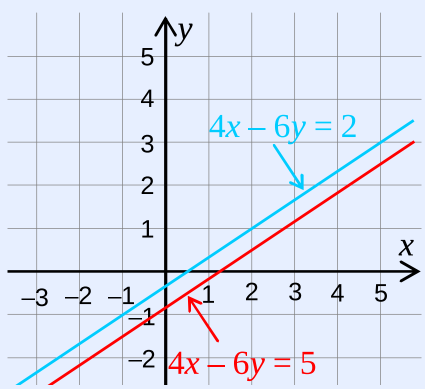
Example 8 GIVE THE SOLUTION SETS OF EACH OF THE FOLLOWING SYSTEM OF LINE EQUATIONS. SKETCH THEIR GRAPHS AND INTERPRET THEM.

A $\begin{cases} 4x - 6y = 2 \\ 4x - 6y = 5 \end{cases}$ **B** $\begin{cases} 5x - 4y = 6 \\ x + 2y = -3 \end{cases}$ **C** $\begin{cases} 3x - y = 2 \\ 6x - 2y = 4 \end{cases}$

Solution

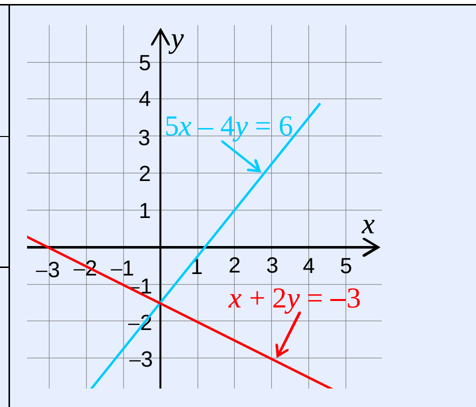
A

Augmented matrix	$\begin{pmatrix} 4 & -6 & 2 \\ 4 & -6 & 5 \end{pmatrix}$	
$R_2 \longrightarrow R_2 + -1.R_1$	$\begin{pmatrix} 4 & -6 & 2 \\ 0 & 0 & 3 \end{pmatrix}$	



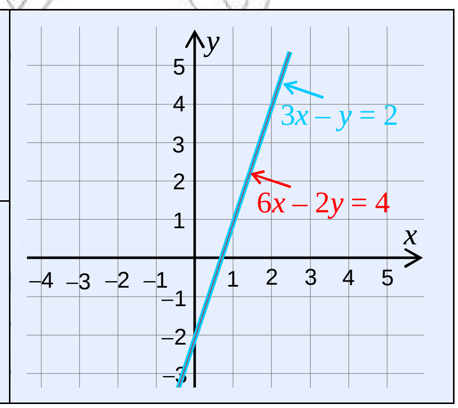
THE SYSTEM HAS NO SOLUTION. AS YOU CAN SEE FROM THE FIGURE, THE TWO LINES AF PARALLEL I.E., THE TWO LINES DO NOT INTERSECT.

B

Augmented matrix	$\begin{pmatrix} 5 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix}$	
$R_1 \leftrightarrow R_2$	$\begin{pmatrix} 1 & 2 & -3 \\ 5 & -4 & 6 \end{pmatrix}$	
$R_2 \rightarrow R_2 + -5R_1$	$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -14 & 21 \end{pmatrix}$	

HERE BY BACK-SUBSTITUTION, $y = -\frac{3}{2}$ AND $x = 0$. YOU CAN SEE THAT THE LINES INTERSECT AT EXACTLY ONE POINT, $\left(0, -\frac{3}{2}\right)$, WHICH IS THE SOLUTION.

C

Augmented matrix	$\begin{pmatrix} 3 & -1 & 2 \\ 6 & -2 & 4 \end{pmatrix}$	
$R_2 \rightarrow R_2 + -2.R_1$	$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	

THE SYSTEM HAS INFINITE SOLUTION. IN ECHELON FORM, THERE IS ONLY ONE EQUATION HAVING TWO VARIABLES. IN THE GRAPH, THERE IS ONLY ONE LINE, I.E., BOTH EQUATION REPRESENT THIS SAME LINE.

Exercise 6.5

1 STATE THE ROW OPERATIONS YOU WOULD USE TO LOCATE A ZERO IN THE SECOND COLU ROW ONE.

A $\begin{pmatrix} 5 & 3 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ **B** $\begin{pmatrix} 1 & -1 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{pmatrix}$

2 REDUCE EACH OF THE FOLLOWING MATRICES INTO ECHELON FORM.

A $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ **B** $\begin{pmatrix} 1 & -1 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & -1 & 3 & -6 \\ 5 & 3 & -2 & 4 \\ 1 & 3 & 4 & 11 \end{pmatrix}$

3 REDUCE EACH OF THE FOLLOWING MATRICES INTO REDUCED - ECHELON FORM.

A $\begin{pmatrix} 3 & 5 & -1 & -4 \\ 2 & 5 & 4 & -9 \\ -1 & 1 & -2 & 11 \end{pmatrix}$ **B** $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{pmatrix}$

4 **A** WRITE $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ IN THE FORM $AX = B$, WHERE

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ AND } B = \begin{pmatrix} e \\ f \end{pmatrix}.$$

B IF A IS NON-SINGULAR, SHOW THAT $X = A^{-1}B$ IS THE SOLUTION.

C USING **A** AND **B** ABOVE, SOLVE $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$

5 SOLVE EACH SYSTEM OF EQUATIONS USING AUGMENTED MATRICES.

A $\begin{cases} 2x-2y=12 \\ -2x+3y=10 \end{cases}$ **B** $\begin{cases} 2x-5y=8 \\ 6x+15y=18 \end{cases}$ **C** $\begin{cases} \frac{x}{3}+\frac{3y}{5}=4 \\ \frac{x}{6}-\frac{y}{2}=-3 \end{cases}$

D $\begin{cases} x-3y+z=-1 \\ 2x+y-4z=-1 \\ 6x-7y+8z=7 \end{cases}$ **E** $\begin{cases} 4x+2y+3z=6 \\ 2x+7y=3z \\ -3x-9y+13=-2z \end{cases}$

6 FIND THE VALUE OF c FOR WHICH THIS SYSTEM HAS AN INFINITE NUMBER OF SOLUTIONS.

$$\begin{cases} 2x-4y=6 \\ -3x+6y=c \end{cases}$$

7 FOR WHAT VALUES OF k DOES

$\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases}$ HAVE A UNIQUE SOLUTION?

8 FIND THE VALUES OF c AND d FOR WHICH BOTH THE GIVEN POINTS LIE ON THE GIVEN STRAIGHT LINE.

$cx + dy = 2$; (0, 4) AND (2, 16)

9 FIND A QUADRATIC FUNCTION $y = ax^2 + bx + c$, THAT CONTAINS THE POINTS (1, 9), (4, 6) AND (6, 14).

6.5 CRAMER'S RULE

DETERMINANTS CAN BE USED TO SOLVE SYSTEMS OF LINEAR EQUATIONS WITH EQUAL NUM EQUATIONS AND UNKNOWNS.

THE METHOD IS PRACTICABLE, WHEN THE NUMBER OF VARIABLES IS EITHER 2 OR 3.

CONSIDER THE SYSTEM $\begin{cases} a_1x + b_1y = c \\ a_2x + b_2y = d \end{cases}$.

$\begin{cases} a_1b_2x + b_1b_2y = b_2c \\ b_1a_2x + b_1b_2y = b_1d \end{cases}$	MULTIPLYING THE 1ST EQUATION BY b_2 AND THE 2ND EQUATION BY b_1.
$(a_1b_2 - b_1a_2)x = b_2c - b_1d$	SUBTRACTING THE FIRST EQUATION FROM THE SECOND.
$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}$	EXPRESSING THE ABOVE EQUATION IN DETERMINANT NOTATION.

LET $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ AND $D_x = \begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}$. THEN, IF $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$,

$$x = \frac{\begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}.$$

A SIMILAR CALCULATION GIVES:

$$y = \frac{\begin{vmatrix} a_1 & c \\ a_2 & d \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$$

THE METHOD IS CALLED **Cramer's rule** FOR A SYSTEM WITH TWO EQUATIONS AND TWO UNKNOWNS.

Note:

- ✓ D_x AND D_y ARE OBTAINED BY REPLACING THE FIRST AND SECOND COLUMNS BY THE CONSTANT COLUMN VECTOR, RESPECTIVELY.
- ✓ UNDER SIMILAR CONDITIONS, THE RULE HOLDS FOR THREE UNKNOWNS.

THE SYSTEM OF EQUATIONS $\begin{cases} a_1x + b_1y + c_1z = d \\ a_2x + b_2y + c_2z = e \\ a_3x + b_3y + c_3z = f \end{cases}$ HAS EXACTLY ONE SOLUTION, PROVIDED THAT

THE DETERMINANT OF THE COEFFICIENT MATRIX IS NON-ZERO. IN THIS CASE THE SOLUTION I

$$x = \frac{\begin{vmatrix} d & b_1 & c_1 \\ e & b_2 & c_2 \\ f & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_x}{D}, \quad y = \frac{\begin{vmatrix} a_1 & d & c_1 \\ a_2 & e & c_2 \\ a_3 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_y}{D} \text{ AND } z = \frac{\begin{vmatrix} a_1 & b_1 & d \\ a_2 & b_2 & e \\ a_3 & b_3 & f \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_z}{D}$$

Example 1 USE CRAMER'S RULE TO FIND THE SOLUTION SET OF $\begin{cases} 3x - 4y = 2 \\ 7x + 7y = 3 \end{cases}$

Solution $D = \begin{vmatrix} 3 & -4 \\ 7 & 7 \end{vmatrix} = 49 \neq 0.$

THUS, BY CRAMER'S RULE, $x = \frac{D_x}{D} = \frac{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}}{49} = \frac{26}{49}$ AND $y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix}}{49} = -\frac{5}{49}$

THE SOLUTION OF THE SYSTEM IS $x = \frac{26}{49}$, $y = -\frac{5}{49}$

Example 2 USING CRAMER'S RULE SOLVE THE FOLLOWING SYSTEM: $\begin{cases} 2x - 2y + 3z = 0 \\ 7y - 9z = 1 \\ 5x - 2y + 6z = -2 \end{cases}$

Solution $D = \begin{vmatrix} 2 & -2 & 3 \\ 0 & 7 & -9 \\ 5 & -2 & 6 \end{vmatrix} = 33 \neq 0.$

USING CRAMER'S RULE:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 0 & -2 & 3 \\ 1 & 7 & -9 \\ -2 & -2 & 6 \end{vmatrix}}{33} = \frac{4}{11}, \qquad y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 0 & 3 \\ 0 & 1 & -9 \\ 5 & -2 & 6 \end{vmatrix}}{33} = -\frac{13}{11}$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 2 & -2 & 0 \\ 0 & 7 & 1 \\ 5 & -2 & -2 \end{vmatrix}}{33} = -\frac{34}{33}$$

THEREFORE, THE SOLUTION OF THE SYSTEM IS $x = \frac{4}{11}$, $y = -\frac{13}{11}$, $z = -\frac{34}{33}$

Example 3 ONE SOLUTION OF THE FOLLOWING SYSTEM IS $x = y = z = 0$ (WHICH IS KNOWN AS THE TRIVIAL SOLUTION). IS THERE ANY OTHER SOLUTION?

$$\begin{cases} 2x - 2y + 3z = 0 \\ 7y - 9z = 0 \\ 5x - 2y + 6z = 0 \end{cases}$$

Solution AS SHOWN IN THE PREVIOUS EXAMPLE, $D = \begin{vmatrix} 2 & -2 & 3 \\ 0 & 7 & -9 \\ 5 & -2 & 6 \end{vmatrix} = 33 \neq 0.$

THUS, THE SYSTEM HAS A UNIQUE SOLUTION. BUT WE ALREADY HAVE ONE SOLUTION, NAMELY, $x = 0$, $y = 0$, $z = 0$. SO, IT IS THE ONLY SOLUTION.

Remark

IN THE PREVIOUS SECTIONS, YOU HAVE SEEN THAT THE DETERMINANT OF A MATRIX CAN BE USED TO FIND THE INVERSE OF A NON-SINGULAR MATRIX NOW YOU WILL USE IT IN FINDING THE SOLUTION OF A SYSTEM OF LINEAR EQUATIONS WHEN THE NUMBER OF EQUATIONS AND THE NUMBER OF VARIABLES ARE EQUAL.

CONSIDER THE LINEAR SYSTEM (IN MATRIX FORM), $AX = B$

IF $|A| \neq 0$, THEN A IS INVERTIBLE AND $A^{-1}(AX) = A^{-1}B$

$$\Rightarrow \left(A^{-1}A\right) X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

THEREFORE, THE SYSTEM HAS A UNIQUE SOLUTION.

Example 4 SOLVE THE SYSTEM $\begin{cases} x + y = 7 \\ 2x + 3y = -3 \end{cases}$

Solution THE SYSTEM IS EQUIVALENT TO $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

THE COEFFICIENT MATRIX IS $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ WITH $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ IS INVERTIBLE WITH INVERSE $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

HENCE THE SOLUTION IS: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 24 \\ -17 \end{pmatrix}$, I.E. $x = 24$ AND $y = -17$

Exercise 6.6

1 USE CRAMER'S RULE TO SOLVE EACH OF THE FOLLOWING SYSTEMS.

A $\begin{cases} -3x+5y=4 \\ 7x+2y=6 \end{cases}$ **B** $\begin{cases} 4x+y=0 \\ x-6y=7 \end{cases}$

C $\begin{cases} 3x+2y-z=5 \\ x-y+3z=-15 \\ 2x+y+7z=-28 \end{cases}$ **D** $\begin{cases} 2x+3y=5 \\ x+3z=6 \\ 5y-z=11 \end{cases}$

2 USE CRAMER'S RULE TO DETERMINE WHETHER EACH OF THE FOLLOWING HOMOGENEOUS SYSTEMS HAS EXACTLY ONE SOLUTION (NAMELY, THE TRIVIAL ONE):

A $\begin{cases} -3x+5y=0 \\ 7x+2y=0 \end{cases}$ **B** $\begin{cases} 3x+2y-z=0 \\ 2x+y+z=0 \\ 5x-2y-z=0 \end{cases}$

Key Terms

adjoint	elementary row operations	scalar matrix
augmented matrix	inconsistent	singular and non-singular matrix
cofactor	inverse	skew-symmetric matrix
column	matrix order	square matrix
consistent	minor	symmetric matrix
dependent	reduced-echelon form	transpose
determinant	row	triangular matrix
diagonal matrix	scalar	zero matrix
echelon form		

Summary

1 A **matrix** IS A RECTANGULAR ARRAY OF ENTRIES ARRANGED IN ROWS AND COLUMNS.

2 THE size OR order OF A MATRIX IS WRITTEN AS **rows × columns**.

3 A MATRIX WITH ONLY ONE COLUMN IS CALLED A **column matrix** (**column vector**).

4 A MATRIX WITH ONLY ONE ROW IS CALLED A **row matrix** (**row vector**).

5 A MATRIX WITH THE SAME NUMBER OF ROWS AND COLUMNS IS CALLED A **square matrix**.

6 A MATRIX WITH ALL ENTRIES 0 IS CALLED A **zero matrix**.

7 A **diagonal matrix** IS A SQUARE MATRIX THAT HAS ZEROS EVERYWHERE EXCEPT POSSIBLY ALONG THE MAIN DIAGONAL.

8 THE **identity** (**unity**) MATRIX IS A DIAGONAL MATRIX WHERE ALL THE ELEMENTS ON THE MAIN DIAGONAL ARE ONES.

9 A **scalar matrix** IS A DIAGONAL MATRIX WHERE ALL ELEMENTS ON THE MAIN DIAGONAL ARE EQUAL.

10 A **lower triangular matrix** IS A SQUARE MATRIX WHOSE ELEMENTS ABOVE THE MAIN DIAGONAL ARE ALL ZERO.

11 AN **upper triangular matrix** IS A SQUARE MATRIX WHOSE ELEMENTS BELOW THE MAIN DIAGONAL ARE ALL ZERO.

12 LET $A = (a_{ij})_{m\times n}$ AND $B = (b_{ij})_{m\times n}$ BE TWO MATRICES. THEN,

$A + B = (a_{ij} + b_{ij})_{m\times n}$ AND $A - B = (a_{ij} - b_{ij})_{m\times n}$.

13 IF r IS A SCALAR AND A IS A GIVEN MATRIX, THEN rA IS THE MATRIX OBTAINED FROM A BY MULTIPLYING EACH ELEMENT OF A BY r.

14 IF $A=(a_{ij})$ IS AN $m \times p$ MATRIX AND $B = (b_{jk})$ IS A $p \times n$ MATRIX, THEN THE PRODUCT AB IS A MATRIX $C=(C_{ik})$ OF ORDER $m \times n$, WHERE

$C_{ik} = a_{ij}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3} b_{3j} + \ldots + a_{ip} b_{pj}$.

15 THE **transpose of a matrix** A IS THE MATRIX FOUND BY INTERCHANGING THE ROWS AND COLUMNS OF A. IT IS DENOTED BY A^T.

16 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

17 A **minor of** a_{ij}, DENOTED BY M_{ij}, IS THE DETERMINANT THAT RESULTS FROM THE MATRIX WHEN THE ROW AND COLUMN THAT CONTAIN a_{ij} ARE DELETED.

18 THE **cofactor of** a_{ij} IS $(-1)^{i+j} M_{ij}$. DENOTE THE COFACTOR OF a_{ij} BY C_{ij}.

19 LET $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. THEN WE CAN EXPAND THE DETERMINANT ALONG ANY ROW i OR ANY COLUMN j. THUS WE HAVE THE FORMULAE:

i^{th} ROW EXPANSION: $|A| = a_{i1}C_{i1}+a_{i2}C_{i2}+a_{i3}C_{i3}$, FOR ANY ROW i (i = 1,2 OR 3) OR

j^{th} column expansion: $|A| = a_{1j}C_{1j}+a_{2j}C_{2j}+a_{3j}C_{3j}$, FOR ANY COLUMN j (j = 1,2 OR 3)

20 THE **adjoint of a square matrix** $A = (a_{ij})$ IS DEFINED AS THE TRANSPOSE OF THE MATRIX $C = (C_{ij})$ WHERE C_{ij} ARE THE COFACTORS OF THE ELEMENTS a_{ij}. ADJOINT OF A IS DENOTED BY $adj\,A$.

21 WHEN A IS INVERTIBLE OR NON-SINGULAR, THEN $A^{-1} = \frac{1}{|A|} adj(A)$.

22 ***Elementary Row operations:***

Swapping: INTERCHANGING TWO ROWS OF A MATRIX.

Rescaling: MULTIPLYING A ROW OF A MATRIX BY A NON-ZERO CONSTANT.

Pivoting: ADDING A CONSTANT MULTIPLE OF ONE ROW OF A MATRIX ON ANOTHER ROW

23 A MATRIX IS IN **echelon form**, IF AND ONLY IF

A THE LEADING ENTRY (THE FIRST NON-ZERO ENTRY) IN EACH ROW AFTER THE FIRST IS TO THE RIGHT OF THE LEADING ENTRY IN THE PREVIOUS ROW.

B IF THERE ARE ANY ROWS WITH NO LEADING ENTRIES (ROWS HAVING ZEROS ENTIRELY) THEY ARE AT THE BOTTOM.

24 A MATRIX IS IN **reduced-echelon form**, IF AND ONLY IF

A IT IS IN ECHELON FORM

B THE LEADING ENTRY IS 1.

C EVERY ENTRY OF A COLUMN THAT HAS A LEADING ENTRY IS ZERO (EXCEPT THE LEADING ENTRY).

25 IF $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, THE SOLUTIONS OF $\begin{cases} a_1x + b_1y = c \\ a_2x + b_2y = d \end{cases}$ ARE GIVEN BY

$$x = \frac{\begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}, \qquad y = \frac{\begin{vmatrix} a_1 & c \\ a_2 & d \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}.$$

26 IF $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, THEN THE SOLUTIONS OF $\begin{cases} a_1x + b_1y + c_1z = d \\ a_2x + b_2y + c_2z = e \\ a_3x + b_3y + c_3z = f \end{cases}$ ARE

$$x = \frac{\begin{vmatrix} d & b_1 & c_1 \\ e & b_2 & c_2 \\ f & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_x}{D}, \quad y = \frac{\begin{vmatrix} a_1 & d & c_1 \\ a_2 & e & c_2 \\ a_3 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_y}{D} \text{ AND } z = \frac{\begin{vmatrix} a_1 & b_1 & d \\ a_2 & b_2 & e \\ a_3 & b_3 & f \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_z}{D}.$$

Review Exercises on Unit 6

1 IF $\begin{pmatrix} a & 6 \\ 10 & d \\ e & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 10 & -1 \\ 3 & 0 \end{pmatrix}$, FIND a, d, AND e.

2 IF $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 6 \\ 5 & 8 & 9 \end{pmatrix}$ AND $B = \begin{pmatrix} 3 & 0 & 5 \\ 5 & 3 & 2 \\ 0 & 4 & 7 \end{pmatrix}$, FIND $3A - 2B$.

3 GIVEN $A = \begin{pmatrix} 3 & 3 & 5 \\ 0 & -1 & 2 \\ 4 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ 2 & -3 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, FIND WHERE POSSIBLE:

A AB **B** BA **C** BC **D** CB **E** CX

F X^TCC^T **G** $B^TA - 2B$ **H** X^TX **I** $B^TB + 4C$

4 SOFIA SELLS CANNED FOOD PRODUCED BY FOUR DIFFERENT PRODUCERS A, B, C AND D. HER MONTHLY ORDER IS:

	A	B	C	D
Beef Meat	300	400	500	600
Tomato	500	400	700	750
Soya Beans	400	400	600	500

FIND HER ORDER, TO THE NEAREST WHOLE NUMBER, IF

A SHE INCREASES HER TOTAL ORDER BY 25%.

B SHE DECREASES HER ORDER BY 15%.

5 KELECHA WANTS TO BUY 1 HAMMER, 1 SAW AND 2 KG OF NAILS, WHILE ALEMU WANTS TO BUY 1 HAMMER, 2 SAWS AND 3 KG OF NAILS. THEY WENT TO TWO HARDWARE SHOPS AND LEARNED THE PRICES IN BIRR TO BE:

	Hammer	Saw	Nails
SHOP 1	30	35	7
SHOP 2	28	37	6

a WRITE THE ITEMS MATRIX I AS A 3×2 MATRIX

b WRITE THE PRICES MATRIX P AS A 2×3 MATRIX

c FIND PI.

d WHAT ARE KELECHA'S COST AT SHOP 1 & ALEMU'S COST AT SHOP 2?

e SHOULD THEY BUY FROM SHOP 1 OR SHOP 2?

6 IF $\begin{pmatrix} 0 & -3 & -4 \\ m & 0 & 8 \\ 4 & -8 & 0 \end{pmatrix}$ IS A SKEW-SYMMETRIC MATRIX, WHAT IS THE VALUE OF m?

7 **a** FOR ANY SQUARE MATRIX A, CHECK THAT $\frac{A + A^T}{2}$ IS SYMMETRIC, WHILE $\frac{A - A^T}{2}$ IS SKEW-SYMMETRIC.

b USING **a** ABOVE, SHOW THAT ANY SQUARE MATRIX IS EXPRESSIBLE AS THE SUM OF A SYMMETRIC MATRIX AND A SKEW-SYMMETRIC MATRIX

8 COMPUTE THE DETERMINANTS OF EACH OF THE FOLLOWING MATRICES

a $\begin{pmatrix} 4 & 3.5 \\ -7 & -20 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 & 4 \\ -7 & 0 & 5 \\ -2 & 5 & 8 \end{pmatrix}$

9 IF $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, SHOW THAT DET$(rA) = r^2$.DET$(A)$.

10 PROVE THAT $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$

11 IN EACH OF THE FOLLOWING, FIND x

a $\begin{vmatrix} 3x & -1 \\ x & -3 \end{vmatrix} = \frac{3}{2}$ **b** $\begin{vmatrix} -3 & -x \\ 3x & 4 \end{vmatrix} = 15$

12 FIND THE INVERSE OF THE FOLLOWING MATRIX: $\begin{pmatrix} 2 & 4 & 2 \\ 3 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

13 REDUCE THE MATRIX $A=\begin{pmatrix}0 & -1 & 5\\ 1 & 3 & -2\\ 2 & 1 & 4\end{pmatrix}$ TO REDUCED-ECHELON FORM.

14 DETERMINE THE VALUES OF a AND b FOR WHICH THE SYSTEM

$$\begin{cases}3x - ay = 1\\ bx + 4y = 6\end{cases}$$

A HAS ONLY ONE SOLUTION;

B HAS NO SOLUTION;

C HAS INFINITELY MANY SOLUTIONS.

15 DETERMINE THE VALUES OF a AND b FOR WHICH THE SYSTEM

$$\begin{cases}3x - 2y + z = b\\ 5x - 8y + 9z = 3\\ 2x + y + az = -1\end{cases}$$

A HAS ONLY ONE SOLUTION;

B HAS INFINITELY MANY SOLUTIONS;

C HAS NO SOLUTION.

16 FOR WHAT VALUES OF k DOES THE FOLLOWING SYSTEM OF EQUATIONS HAVE NO SOLUTION?

$$\begin{cases}x + 2y - z = 12\\ 2x - y - 2z = 2\\ x - 3y + kz = 11\end{cases}.$$

17 SOLVE EACH OF THE FOLLOWING.

A $\begin{pmatrix}5 & 2 & 1\\ 3 & 2 & 0\\ 1 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}8\\ 5\\ 3\end{pmatrix}$ **B** $\begin{pmatrix}2+ & -\\ - & 1+\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}5\\ 0\end{pmatrix}$

18 USE CRAMER'S RULE TO SOLVE EACH OF THE FOLLOWING.

A $\begin{cases}2x + y = 7\\ 3x - 2y = 0\end{cases}$ **B** $\begin{cases}-x + 4y - z = 1\\ 2x - y + z = 0\\ x + y + z = 1\end{cases}$

19 SOLVE THE ABOVE BY FIRST FINDING A^{-1} AND THEN USING $X = A^{-1}B$.