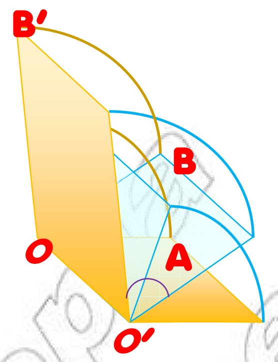
Unit 8



VECTORS AND TRANSFORMATION OF THE PLANE

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts and procedures about vectors and operation on vectors.*
- *know specific facts about vectors.*
- *apply principles and theorems about vectors in solving problems involving vectors.*
- *apply methods and procedures in transforming plane figures.*

Main Contents

INTRODUCTION

THE MEASUREMENT OF ANY PHYSICAL QUANTITY IS ALWAYS EXPRESSED IN TERMS OF A NU
A UNIT. IN PHYSICS, FOR EXAMPLE YOU COME ACROSS A NUMBER OF PHYSICAL QUANTITI
LENGTH, AREA, MASS, VOLUME, TIME, DENSITY, VELOCITY, FORCE, ACCELERATION, MOMEN
THUS, MOST OF THE PHYSICAL QUANTITIES CAN BE DIVIDED INTO TWO CATEGORIES AS GIVE

A PHYSICAL QUANTITIES HAVING MAGNITUDE ONLY

B QUANTITIES HAVING BOTH MAGNITUDE AND DIRECTION

Scalar quantities ARE COMPLETELY DETERMINED ONCE THE MAGNITUDE OF THE QUANTIT
GIVEN. HOWEVER, **vectors** ARE NOT COMPLETELY DETERMINED UNTIL *both a magnitude and a direction are specified*. FOR EXAMPLE, WIND MOVEMENT IS USUALLY DESCRIBED BY GIVING THE
SPEED AND THE DIRECTION, SAY 20 KM/HR NORTHEAST. THE WIND SPEED AND WIND DIRE
TOGETHER FORM A VECTOR QUANTITY - THE WIND VELOCITY.

IN THIS UNIT, YOU FOCUS ON VARIOUS GEOMETRIC AND ALGEBRAIC ASPECTS OF VECTOR REP
AND VECTOR ALGEBRA.

8.1 REVISION ON VECTORS AND SCALARS

ACTIVITY 8.1

1 BASED ON YOUR KNOWLEDGE, CLASSIFY THE MEASURES OF THE FOLLOWING SITUATIONS AS SCALAR OR VECTOR.

A THE WIDTH OF YOUR CLASSROOM.

B THE FLOW OF A RIVER.

C THE NUMBER OF STUDENTS IN YOUR CLASS ROOM.

D THE DIRECTION OF YOUR HOME FROM YOUR SCHOOL.

E WHEN AN OPEN DOOR IS CLOSED.

F WHEN YOU MOVE NOWHERE IN ANY DIRECTION.

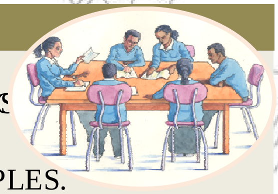
2 CLASSIFY EACH OF THE FOLLOWING QUANTITIES AS EITHER VECTOR OR SCALAR:

DISPLACEMENT, DISTANCE, SPEED, VELOCITY, WORK, ACCELERATION, AREA, TIME, WEIG
VOLUME, DENSITY, FORCE, MOMENTUM, TEMPERATURE, MASS.

8.1.1 Vectors and Scalars

IN GRADE 9, YOU DISCUSSED VECTORS AND THEIR REPRESENTATIONS. YOU ALSO DISCUSSED V
AND SCALARS. THE FOLLOWING GROUP WORK AND SUBSEQUENT ACTIVITIES WILL HELP YOU
THE CONCEPTS YOU LEARNT.

Group work 8.1



1. DISCUSS THE REPRESENTATION OF VECTORS AS POINTS, AS ARROWS AND AS COLUMN VECTORS.
2. DISCUSS EQUALITY OF VECTORS AND GIVE EXAMPLES.
3. WHEN IS A VECTOR SAID TO BE REPRESENTED IN STANDARD FORM?
4. IF $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ IS A VECTOR WHOSE INITIAL POINT IS THE ORIGIN, THEN FIND

   A. THE COMPONENTS OF $\mathbf{v}$. B. THE MAGNITUDE OF $\mathbf{v}$.

   C. THE DIRECTION OF $\mathbf{v}$.
5. DESCRIBE SCALAR AND VECTOR QUANTITIES FROM YOUR SURROUNDINGS.

Definition 8.1

A QUANTITY WHICH CAN BE COMPLETELY DESCRIBED BY ITS MAGNITUDE EXPRESSED IN
PARTICULAR UNIT IS CALLED A **scalar quantity.**

EXAMPLES OF SCALAR QUANTITIES ARE MASS, TIME, TEMPERATURE, ETC.

Definition 8.2

A QUANTITY WHICH CAN BE COMPLETELY DESCRIBED BY STATING BOTH ITS MAGN
EXPRESSED IN SOME PARTICULAR UNIT AND ITS DIRECTION IS CALLED A **vector quantity**.

EXAMPLES OF VECTOR QUANTITIES ARE VELOCITY, ACCELERATION, ETC.

8.1.2 Representation of a Vector

Definition 8.3 **Coordinate form of a vector in a plane**

IF $\mathbf{v}$ IS A VECTOR IN THE PLANE WHOSE INITIAL POINT IS THE ORIGIN AND WHOSE TERMINAL
IS (x, y), THEN THE COORDINATE FORM OF $\mathbf{v}$ IS GIVEN BY $\mathbf{v} = (x, y)$ OR $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

THE NUMBERS x AND y ARE CALLED **components** (OR **coordinates**) OF $\mathbf{v}$.

✍Note:

1 IF BOTH THE INITIAL AND TERMINAL POINTS LIE AT THE ORIGIN, THEN **v** IS THE ZERO VECTOR AND IS GIVEN BY **v** = (0, 0) OR $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2 THE ABOVE DEFINITION IMPLIES THAT TWO VECTORS ARE EQUAL IF THEIR CORRESPONDING COMPONENTS ARE EQUAL.

The following procedure can be used to convert directed line segments to coordinate form and vice versa.

1 IF P = (x_1, y_1) AND Q = (x_2, y_2), ARE TWO POINTS ON THE PLANE, THEN THE COORDINATE FORM OF THE VECTOR **v** REPRESENTED BY $\overrightarrow{PQ}$ IS **v** = $(x_2 - x_1, y_2 - y_1)$. MOREOVER, THE LENGTH OF **v** IS:

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2 IF **v** = (x, y), THEN **v** CAN BE REPRESENTED BY THE DIRECTED LINE SEGMENT IN STANDARD POSITION, FROM O = (0, 0) TO Q = (x, y).

Example 1 FIND THE COORDINATE FORM AND THE LENGTH OF THE VECTOR **v** THAT HAS INITIAL POINT (3, –7) AND TERMINAL POINT (–2, 5).

Solution LET P = (3, –7) AND Q = (–2, 5). THEN, THE COORDINATE FORM OF **v** IS:

$$\mathbf{v} = (-2 - 3, 5 - (-7)) = (-5, 12)$$

THE LENGTH OF **v** IS:

$$|\mathbf{v}| = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Exercise 8.1

Fill in the blank spaces with the appropriate answer.

1 A DIRECTED LINE SEGMENT HAS A _________ AND A _____. THE MAGNITUDE OF THE DIRECTED LINE SEGMENT $\overrightarrow{PQ}$, DENOTED BY __________, IS ITS __________.

2 A VECTOR WHOSE INITIAL POINT IS AT THE ORIGIN O(0, 0) CAN BE UNIQUELY REPRESENTED BY THE COORDINATES OF ITS TERMINAL POINT P(x, y). THIS IS THE ___________, WRITTEN **v** = (x, y), WHERE x AND y ARE THE _________ OF **v**.

3 THE COORDINATE FORM OF THE VECTOR WITH INITIAL POINT P = (p_1, p_2) AND TERMINAL POINT Q = (q_1, q_2) IS $\overrightarrow{PQ}$ = _____________ = **v**.

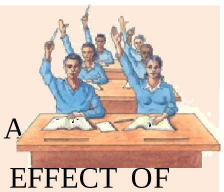
THE MAGNITUDE (OR LENGTH) OF **v** IS:

$$|\mathbf{v}| = \sqrt{\_\_\_\_\_\_}.$$

4 THE COORDINATE FORM AND MAGNITUDE OF THE VECTOR **v** THAT HAS A(1, 7) AS ITS INITIAL POINT AND B(4, 3) AS ITS TERMINAL POINT ARE __________ AND __________.

8.1.3 Addition of Vectors

ACTIVITY 8.2



1 CONSIDER A DISPLACEMENT $\overrightarrow{AB}$ OF 3M DUE N FOLLOWED BY A DISPLACEMENT $\overrightarrow{BC}$ OF 4M DUE E. FIND THE COMBINED EFFECT OF THESE TWO DISPLACEMENTS AS A SINGLE DISPLACEMENT.

2 CONSIDER THE FOLLOWING DISPLACEMENT VECTORS.

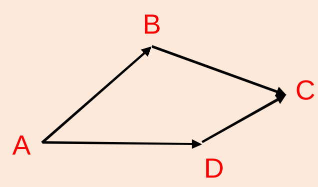


Figure 8.1

DISCUSS HOW TO DETERMINE THE COMBINED EFFECT OF THE VECTORS AS A SINGLE VEC

FROM ACTIVITY 8.2 YOU SEE THAT IT IS POSSIBLE TO ADD TWO VECTORS GEOMETRICALLY USIN TO TIP RULE.

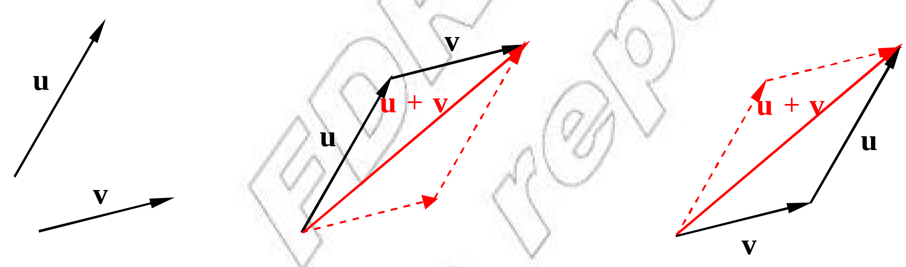


Figure 8.2

TO FIND **u** + **v** *Move the initial point of* **v** *to the terminal point of* **u**. **or** *Move the initial point of* **u** *to the terminal point of* **v**.

Definition 8.4 **Addition of vectors (tail-to-tip rule)**

IF **u** AND **v** ARE ANY TWO VECTORS, THE SUM **u** + **v** IS THE VECTOR DETERMINED AS FOLLOWS: TRANSLATE THE VECTOR **v** SO THAT ITS INITIAL POINT COINCIDES WITH THE TERMINAL POINT OF **u**. THE VECTOR **u** + **v** IS REPRESENTED BY THE **arrow from the initial point of u to the terminal point of v**.

Note:

1 ONE CAN EASILY SEE THAT **u**, **v** AND **u** + **v** ARE REPRESENTED BY THE SIDES OF A TRIANGLE, WHICH IS CALLED THE TRIANGLE LAW OF VECTOR ADDITION.

2 THE ADDITION OF VECTORS HAS PROPERTIES LIKE THE NUMBERS; THE TWO USEFUL PROPERTIES OF VECTOR ADDITION ARE GIVEN BELOW.

Theorem 8.1 Commutative property of vector addition

IF **u** AND **v** ARE ANY TWO VECTORS, THEN

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Proof: TAKE ANY POINT O AND DRAW THE VECTORS $\overrightarrow{OA} = \mathbf{u}$ AND $\overrightarrow{AB} = \mathbf{v}$ SUCH THAT THE TERMINAL POINT OF THE VECTOR **u** IS THE INITIAL POINT OF THE VECTOR **v** AS SHOWN IN FIGURE 8.3.

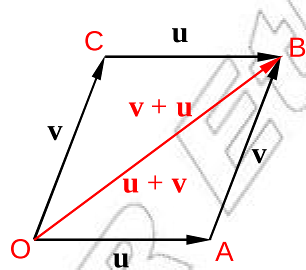


Figure 8.3

THEN, BY DEFINITION OF VECTOR ADDITION YOU HAVE:

$$\mathbf{u} + \mathbf{v} = \overrightarrow{OB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

NOW, COMPLETING THE PARALLELOGRAM *OABC* WHOSE ADJACENT SIDES ARE *OA* AND *AB*, YOU INFER THAT $\overrightarrow{OC} = \overrightarrow{AB} = \mathbf{v}$, AND $\overrightarrow{CB} = \overrightarrow{OA} = \mathbf{u}$

USING THE TRIANGLE LAW OF VECTOR ADDITION, YOU OBTAIN

$$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\mathbf{v} + \mathbf{u} = \overrightarrow{OB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

FROM 1 AND 2, WE HAVE:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

HENCE, VECTOR ADDITION IS *commutative*. THIS IS ALSO CALLED THE **parallelogram law of vectors**.

Theorem 8.2 Associative Property of Vector Addition

IF **u**, **v**, **w** ARE ANY THREE VECTORS, THEN

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

Proof: LET **u**, **v**, **w** BE THREE VECTORS REPRESENTED BY THE LINE SEGMENTS AS SHOWN IN FIGURE 8.4. I.E., **u** = $\overrightarrow{OA}$, **v** = $\overrightarrow{AB}$, **w** = $\overrightarrow{BC}$

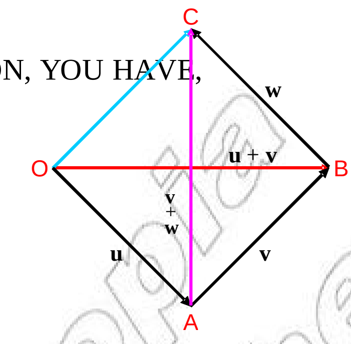


Figure 8.4

USING THE DEFINITION OF VECTOR ADDITION, YOU HAVE,

I.E., $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{OC}$ = (**u** + **v**) + **w**. **1**

AGAIN, YOU HAVE,

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$

I.E. $\overrightarrow{OC}$ = **u** + (**v** + **w**) **2**

COMPARING **1** AND **2**, YOU HAVE,

(**u** + **v**) + **w** = **u** + (**v** + **w**)

HENCE, VECTOR ADDITION HAS ASSOCIATIVE PROPERTY.

8.1.4 Multiplication of Vectors by Scalars

Group work 8.2

CONSIDER THE VECTOR $\overrightarrow{PQ}$

1 WHAT WILL BE $k\overrightarrow{PQ}$, WHEN $k > 0$ AND $k < 0$?

2 DISCUSS THE LENGTH AND DIRECTION OF $-\overrightarrow{PQ}$. WHAT WILL BE $\overrightarrow{PQ} + (-\overrightarrow{PQ})$?

3 DISCUSS $\overrightarrow{PQ} + (-\overrightarrow{PQ})$ AND $\overrightarrow{PQ} - \overrightarrow{PQ}$

4 IF **u** AND **v** ARE TWO VECTORS, THEN REPRESENT **u** − **v** GEOMETRICALLY.

GEOMETRICALLY, THE PRODUCT OF A VECTOR **v** AND A SCALAR k IS THE VECTOR THAT IS $|k|$ TIMES AS LONG AS **v**, AS SHOWN IN FIGURE 8.5.

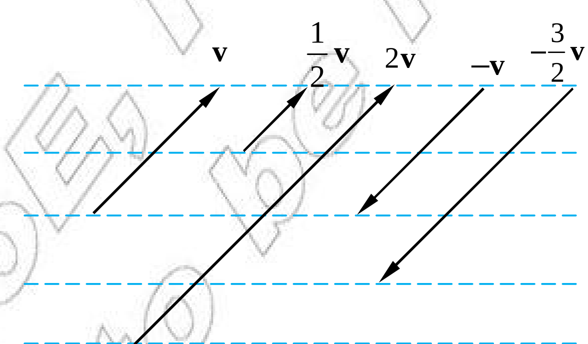


Figure 8.5

IF k IS POSITIVE, THEN k**v** HAS THE SAME DIRECTION AS **v**. IF k IS NEGATIVE, THEN k**v** HAS THE OPPOSITE DIRECTION.

Example 2 LET **v** BE ANY VECTOR. THEN 3**v** IS A VECTOR IN THE SAME DIRECTION AS **v** AND WITH LENGTH 3 TIMES THE LENGTH OF **v**.

Definition 8.5

IF **v** IS A NON-ZERO VECTOR AND k IS A NON-ZERO NUMBER (SCALAR), THEN THE PRODUCT k**v** IS DEFINED TO BE THE VECTOR WHOSE LENGTH IS $|k|$ TIMES THE LENGTH OF **v** AND WHOSE DIRECTION IS THE SAME AS THAT OF **v** IF $k > 0$ AND OPPOSITE TO THAT OF **v** IF $k < 0$.

$k\mathbf{v} = \mathbf{0}$ IF $k = 0$ OR $\mathbf{v} = \mathbf{0}$.

A VECTOR OF THE FORM k**v** IS CALLED A **scalar multiple** OF **v**.

Theorem 8.3

SCALAR MULTIPLICATION SATISFIES THE DISTRIBUTIVE LAWS, I.E., IF k_1 AND k_2 ARE ANY TWO SCALARS AND **u** AND **v** ARE TWO VECTORS, THEN YOU HAVE:

I $(k_1 + k_2)\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$ **II** $k_1(\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$

Note:

1. TO OBTAIN THE DIFFERENCE **u** – **v** WITHOUT CONSTRUCTING –**v**, POSITION **u** AND **v** SO THAT THEIR INITIAL POINTS COINCIDE; THE VECTOR FROM THE TERMINAL POINT OF **v** TO THE TERMINAL POINT OF **u** IS THEN THE VECTOR **u** – **v**.
2. IF **v** IS ANY NON-ZERO VECTOR AND –**v** IS THE NEGATIVE OF **v**, THEN **v** + (–**v**) = **0**
3. FOR ANY THREE VECTORS **u**, **v** AND **w**, IF **u** = **v** AND **v** = **w**, THEN **u** = **w**.
4. THE ZERO VECTOR **0** HAS THE FOLLOWING PROPERTY: FOR ANY VECTOR **u**, **u** + **0** = **0** + **u** = **u**.
5. FOR ANY VECTOR **u**, 1**u** = **u**
6. IF c AND d ARE SCALARS AND **u** A VECTOR, THEN $c(d\mathbf{u}) = (cd)\mathbf{u}$.

THE OPERATIONS OF VECTOR ADDITION AND MULTIPLICATION BY A SCALAR ARE EASY TO V TERMS OF COORDINATE FORMS OF VECTORS. FOR THE MOMENT, WE SHALL RESTRICT THE D VECTORS IN THE PLANE.

RECALL FROM GRADE 9 THAT IF $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ AND k IS A SCALAR, THEN

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2);\ k\mathbf{u} = (kx_1, ky_1)$$

Example 3 IF $\mathbf{u} = (1, -2)$, $\mathbf{v} = (7, 6)$ AND $k = 2$, FIND $\mathbf{u} + \mathbf{v}$ AND $2\mathbf{u}$.

Solution $\mathbf{u} + \mathbf{v} = (1 + 7, -2 + 6) = (8, 4)$, $2\mathbf{u} = (2(1), 2(-2)) = (2, -4)$

Definition 8.6

IF $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$, k IS A SCALAR, THEN

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2) \qquad k\mathbf{u} = (kx_1, ky_1)$$

Example 4 IF $\mathbf{u} = (1, -3)$ AND $\mathbf{w} = (4, 2)$, THEN $\mathbf{u} + \mathbf{w} = (5, -1)$

$2\mathbf{u} = (2, -6)$, $-\mathbf{w} = (-4, -2)$ AND $\mathbf{u} - \mathbf{w} = (-3, -5)$

Exercise 8.2

1. A STUDENT WALKS A DISTANCE OF 3KM DUE EAST, THEN ANOTHER 4KM DUE SOUTH. FIND THE DISPLACEMENT RELATIVE TO HIS STARTING POINT.
2. A CAR TRAVELS DUE EAST AT 60KM/HR FOR 15 MINUTES, THEN TURNS AND TRAVELS AT 100KM/HR ALONG A FREEWAY HEADING DUE NORTH FOR 15 MINUTES. FIND THE DISPLACEMENT FROM ITS STARTING POINT.
3. SHOW THAT IF $\mathbf{v}$ IS A NON-ZERO VECTOR AND m AND n ARE SCALARS SUCH THAT $m\mathbf{v} = n\mathbf{v}$, THEN $m = n$.
4. LET $\mathbf{u} = (1, 6)$ AND $\mathbf{v} = (-4, 2)$. FIND

   A $3\mathbf{u}$ **B** $3\mathbf{u} + 4\mathbf{v}$ **C** $\mathbf{u} - \frac{1}{2}\mathbf{v}$
5. WHAT IS THE RESULTANT OF THE DISPLACEMENTS 6M NORTH, 8M EAST AND 10M NORTH WEST?
6. DRAW DIAGRAMS TO ILLUSTRATE THE FOLLOWING VECTOR EQUATIONS.

   A $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC}$ **B** $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AD}$
7. IF $ABCDEF$ IS A REGULAR POLYGON IN WHICH $\overrightarrow{AB}$ REPRESENTS A VECTOR $\mathbf{v}$ AND $\overrightarrow{BC}$ REPRESENTS A VECTOR $\mathbf{w}$, EXPRESS EACH OF THE FOLLOWING VECTORS IN TERMS OF $\mathbf{v}$ AND $\mathbf{w}$: $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}$ AND $\overrightarrow{AF}$.
8. USING VECTORS PROVE THAT THE LINE SEGMENT JOINING THE MID POINTS OF THE SIDES OF A TRIANGLE IS HALF AS LONG AS AND PARALLEL TO THE THIRD SIDE.

8.2 REPRESENTATION OF VECTORS

ACTIVITY 8.3

1. IF $\mathbf{w}$ IS A VECTOR, THEN DISCUSS HOW YOU CAN EXPRESS THE VECTOR $\mathbf{w}$ AS SUM OF TWO OTHER VECTORS.
2. USING THE VECTORS $\mathbf{i} = (1, 0)$ AND $\mathbf{j} = (0, 1)$ DISCUSS THE FOLLOWING RULES OF VECTORS.

   I THE ADDITION RULE: $(a\mathbf{i} + b\mathbf{j}) + (c\mathbf{i} + d\mathbf{j}) = (a + c)\mathbf{i} + (b + d)\mathbf{j}$

   II THE SUBTRACTION RULE: $(a\mathbf{i} + b\mathbf{j}) - (c\mathbf{i} + d\mathbf{j}) = (a - c)\mathbf{i} + (b - d)\mathbf{j}$

   III MULTIPLICATION OF VECTORS BY SCALARS: $t(a\mathbf{i} + b\mathbf{j}) = (ta)\mathbf{i} + (tb)\mathbf{j}$

GIVEN A VECTOR **w**, YOU MAY WANT TO FIND TWO VECTORS **u** AND **v** WHOSE SUM IS **w**. THE VECTORS **u** AND **v** ARE CALLED **components** OF **w** AND THE PROCESS OF FINDING THEM IS CALLED **resolving**, OR REPRESENTING THE VECTOR INTO ITS VECTOR COMPONENTS.

WHEN YOU RESOLVE A VECTOR, YOU GENERALLY LOOK FOR PERPENDICULAR COMPONENTS. OFTEN (IN THE PLANE), ONE COMPONENT WILL BE PARALLEL TO THE x-AXIS AND THE OTHER WILL BE PARALLEL TO THE y-AXIS. FOR THIS REASON, THEY ARE OFTEN CALLED THE **horizontal** AND **vertical** COMPONENTS OF A VECTOR.

IN THE FIGURE 8.6 BELOW, THE VECTOR $\mathbf{w} = \overrightarrow{AC}$ IS RESOLVED AS THE SUM OF $\mathbf{u} = \overrightarrow{AB}$ AND $\mathbf{v} = \overrightarrow{BC}$.

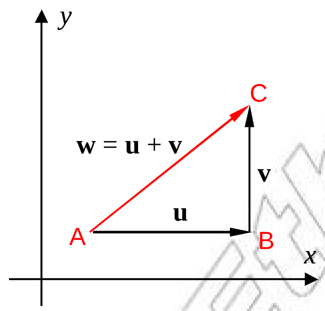


Figure 8.6

THE HORIZONTAL COMPONENT OF **w** IS **u** AND THE VERTICAL COMPONENT IS **v**.

Example 1 A CAR WEIGHTING 8000N IS ON A STRAIGHT ROAD THAT HAS A SLOPE OF 10° AS SHOWN IN FIGURE 8.7. FIND THE FORCE THAT KEEPS THE CAR FROM ROLLING DOWN.

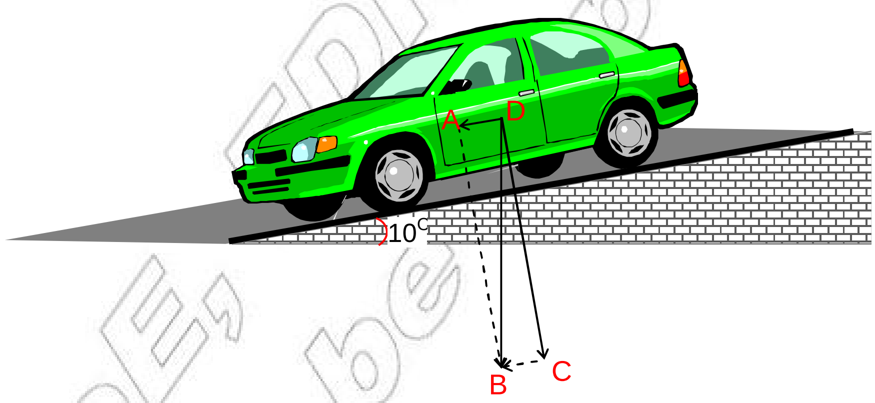


Figure 8.7

Solution THE FORCE VECTOR $\overrightarrow{DB}$ ACTS IN THE DOWNWARD DIRECTION.

$$\Rightarrow \left|\overrightarrow{DB}\right| = 8000\,\text{N}.$$

OBSERVE THAT $\overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{DB}$ AND $m(\measuredangle ABD) = 10^{\circ}$

$\Rightarrow$ THE FORCE THAT KEEPS THE CAR AT D FROM ROLLING DOWN IS IN THE OPPOSITE DIRECTION OF $\overrightarrow{DA}$.

$$\Rightarrow \quad \sin(\angle ABD) = \frac{|\overrightarrow{CB}|}{|\overrightarrow{DB}|} = \frac{|\overrightarrow{DA}|}{|\overrightarrow{DB}|} \Rightarrow \sin 10^\circ = \frac{\overrightarrow{DA}}{8000\text{N}}$$

$$\Rightarrow \quad |\overrightarrow{DA}| = 8000 \text{ N} \times \sin 10^\circ = 1389.185 \text{ N}$$

Note:

1. EVIDENTLY, A GIVEN VECTOR HAS AN INFINITE NUMBER OF PAIRS OF POSSIBLE COMPONENT VECTORS. HOWEVER, IF DIRECTIONS OF THE COMPONENT VECTORS ARE SPECIFIE PROBLEM OF RESOLVING THE VECTOR INTO COMPONENT VECTORS HAS A UNIQUE SOLU

2. LET **u** AND **v** BE TWO NON-ZERO VECTORS. IN THE EXPRESSION $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$
   a. THE VECTORS $k_1\mathbf{u}$ AND $k_2\mathbf{v}$ ARE SAID TO BE THE **components** OF **w** RELATIVE TO **u** AND **v**.
   b. THE SCALARS k_1 AND k_2 ARE CALLED THE **coordinates** OF THE VECTOR **w** RELATIVE TO **u** AND **v**.

Definition 8.7

TWO VECTORS **u** AND **v** ARE SAID TO BE **parallel (or collinear)**, IF **u** AND **v** LIE EITHER ON PARALLEL LINES OR ON THE SAME LINE.

Definition 8.8

ANY VECTOR WHOSE MAGNITUDE IS ONE IS CALLED A **unit vector.**

IF **v** IS ANY NON-ZERO VECTOR, THE UNIT VECTOR IN THE DIRECTION OF **v** IS OBTAINED BY MULTIPLYING **v** BY $\frac{1}{|\mathbf{v}|}$. THAT IS, THE UNIT VECTOR IN THE DIRECTION OF **v** IS $\frac{1}{|\mathbf{v}|}\mathbf{v}$.

THE UNIT VECTORS (1, 0) AND (0, 1) ARE CALLED THE **standard unit vectors in the plane.**

EVERY PAIR OF NON-COLLINEAR VECTORS CAN BE THOUGHT OF AS A BASIS. OF COURSE, THE COMPONENTS AND THE COORDINATES OF A GIVEN VECTOR IN THE PLANE WILL BE DIFF DIFFERENT BASES. FOR EXAMPLE, THE VECTOR **w** = (5, 8) CAN BE WRITTEN AS

(5, 8) = (3, 2) + (2, 6) = (1, 6) + (4, 2) = (5, 0) + (0, 8), ETC.

THEREFORE, (3, 2) AND (2, 6), (1, 6) AND (4, 2), AND (5, 0) AND (0, 8), ETC ARE COMPONENTS OF **w**.

YOUR MAIN INTEREST IN THIS SECTION IS TO FIND THE HORIZONTAL AND VERTICAL COMPO
VECTOR **w**, DENOTED BY w_x AND w_y.

The unit vectors i and j

VECTORS IN THE xy-PLANE ARE REPRESENTED BASED ON THE TWO SPECIAL VECTORS **i** = (1, 0) AND **j** = (0, 1). NOTICE THAT $|\mathbf{i}| = |\mathbf{j}| = 1$. **i** AND **j** POINT IN THE POSITIVE DIRECTIONS OF THE x AND y AXES, RESPECTIVELY, AS SHOWN IN FIGURE 8.8. THESE VECTORS ARE CALLED STANDARD UNIT BASE VECTORS.

ANY VECTOR **v** IN THE PLANE CAN BE EXPRESSED UNIQUELY IN THE FORM

$$\mathbf{v} = s\mathbf{i} + t\mathbf{j}$$

WHERE s AND t ARE SCALARS. IN THIS CASE, YOU SAY THAT **v** IS EXPRESSED AS A LINEAR COMBINATION OF **i** AND **j**.

CONSIDER A VECTOR **v** WHOSE INITIAL POINT IS THE ORIGIN AND WHOSE TERMINAL POINT IS THE POINT A = (x, y).

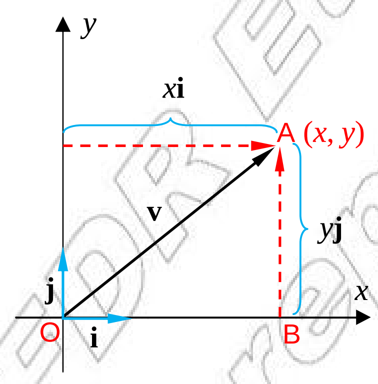


Figure 8.8

IF $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, THEN $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathrm{I} + y\mathrm{J}$

Note:

THE NORM OF **v** = $|\mathbf{v}| = \sqrt{x^2 + y^2}$

IF $\overrightarrow{PQ}$ IS A VECTOR WITH INITIAL POINT (x_1, y_1) AND TERMINAL POINT (x_2, y_2) AS SHOWN IN FIGURE 8.9, THEN ITS POSITION VECTOR **v** IS DETERMINED AS

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$

THUS, ($x_2 - x_1$) AND ($y_2 - y_1$) ARE THE COORDINATES OF **v** WITH RESPECT TO THE BASE {**i**, **j**}.

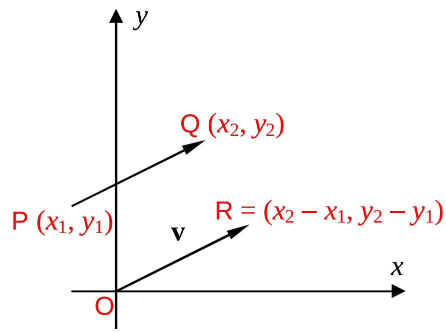


Figure 8.9

Example 2 EXPRESS THE FOLLOWING VECTORS IN TERMS OF THE **i** AND **j** VECTORS AND FIND THEIR NORM.

A (7, –8) **B** (–1, 5) **C** (–2, 3)

Solution

A (7, –8) = 7**i** – 8**j** AND ITS NORM (OR MAGNITUDE) IS

$$\sqrt{7^2 + (-8)^2} = \sqrt{49+64} = \sqrt{113}$$

B (–1, 5) = –1**i** + 5**j** AND ITS NORM (OR MAGNITUDE) IS

$$\sqrt{(-1)^2 + 5^2} = \sqrt{1+25} = \sqrt{26}$$

C (–2, 3) = –2**i** + 3**j** WITH NORM $\sqrt{13}$

Example 3 EXPRESS EACH OF THE FOLLOWING AS A VECTOR IN THE COORDINATE FORM.

A 3**i**+**j** **B** 2**i**– 2**j** **C** –**i** + 6**j**

Solution

A 3**i** +**j**= 3 (1, 0) + (0, 1) = (3, 0) + (0, 1) = (3, 1)

B 2**i** – 2**j**= 2 (1, 0) – 2 (0, 1) = (2, 0) + (0, –2) = (2, –2)

C –**i** + 6**j**= –(1, 0) + 6 (0, 1) = (–1, 0) + (0, 6) = (–1, 6)

Exercise 8.3

1 FIND **u** +**v** FOR EACH OF THE FOLLOWING PAIRS OF VECTORS

A **u** = (1, 4), **v** = (6, 2) **C** **u** = (2, –2), **v** = (–2, 3)

B **u** = (7, –8), **v** = (–1, 6) **D** **u** = (1 + $\sqrt{2}$, 0), **v** = (–$\sqrt{2}$, 2)

2 FIND THE NORM (OR MAGNITUDE) OF EACH OF THE FOLLOWING VECTORS.

A **u**= (1, 1) **B** **u** = ($\frac{3}{2}$, 0)

C **v** = (–2, 1) **D** **v** = x**i** + y**j** , $x, y \in \mathbb{R}$

3 IF $\mathbf{u} = 3\mathbf{i} + \frac{5}{2}\mathbf{j}$ AND $\mathbf{v} = \frac{7}{2}\mathbf{i} - \frac{1}{4}\mathbf{j}$, FIND

A $\mathbf{u} + \mathbf{v}$ **B** $\mathbf{u} - \mathbf{v}$ **C** $t\mathbf{u}, t \in \mathbb{R}$ **D** $2\mathbf{u} - \mathbf{v}$

4 **A** FIND A UNIT VECTOR IN THE DIRECTION OF THE VECTOR (2, 4).

B FIND A UNIT VECTOR IN THE DIRECTION OPPOSITE TO THE VECTOR (1, 2).

C FIND TWO UNIT VECTORS, ONE IN THE SAME DIRECTION AS, AND THE OTHER OPPOSITE TO, THE VECTOR $\mathbf{u} = (x, y) \neq 0$.

5 WHAT ARE THE COORDINATES OF THE ZERO VECTOR? USE COORDINATES TO SHOW THAT $\mathbf{u} + \mathbf{0} = \mathbf{u}$ FOR ANY VECTOR $\mathbf{u}$.

8.3 SCALAR (INNER OR DOT) PRODUCT OF VECTORS

SO FAR YOU HAVE STUDIED TWO VECTOR OPERATIONS, VECTOR ADDITION AND MULTIPLICATION BY A SCALAR, EACH OF WHICH YIELDS ANOTHER VECTOR. IN THIS SECTION, YOU WILL STUDY A THIRD OPERATION, **the dot product**. THIS PRODUCT YIELDS A SCALAR, RATHER THAN A VECTOR.

Group work 8.3

1 SUPPOSE A BODY IS MOVED FROM A TO B UNDER A CONSTANT FORCE **F** AS SHOWN IN FIGURE 8.10. DISCUSS THE USES OF **F**, AB AND .

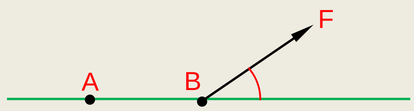


Figure 8.10

2 LET **u** AND **v** BE TWO VECTORS WITH THE SAME INITIAL POINT. THEN THE ANGLE BETWEEN **u** AND **v** IS FORMED AS SHOWN IN FIGURE 8.11.

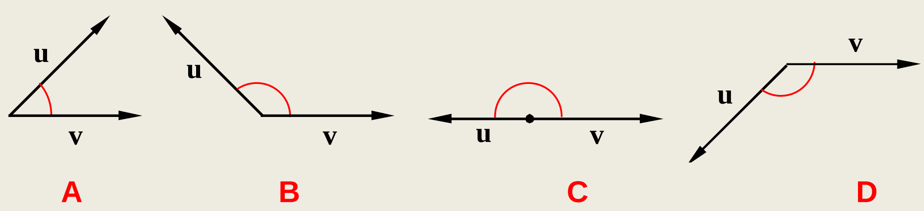


Figure 8.11

DISCUSS HOW TO EXPRESS IN TERMS OF |**u**| AND |**v**|.

8.3.1 Scalar (Dot or Inner) Product of Vectors

Definition 8.9

IF **u** AND **v** ARE VECTORS AND θ THE ANGLE BETWEEN **u** AND **v**, THEN THE DOT PRODUCT OF **u** AND **v**, DENOTED BY **u**·**v**, IS DEFINED BY:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

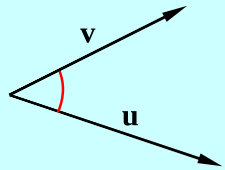


Figure 8.12

Example 1 FIND THE DOT PRODUCT OF THE VECTORS **u** AND **v** WHEN

A $\mathbf{u} = (0, 1)$ AND $\mathbf{v} = (0, 2)$

B $\mathbf{u} = (-2, 0)$ AND $\mathbf{v} = (\sqrt{3}, 3)$

Solution USING THE DEFINITION OF DOT PRODUCT, YOU HAVE

A $|\mathbf{u}| = 1, |\mathbf{v}| = 2$ AND $\theta = 0 \Rightarrow \mathbf{u}\cdot\mathbf{v} = 1 \times 2\cos 0° = 2$

B $|\mathbf{u}| = 2, |\mathbf{v}| = \sqrt{(\sqrt{3})^2 + 3^2} = 2\sqrt{3}$ AND $\theta = 120°$

$$\Rightarrow \mathbf{u}\cdot\mathbf{v} = 2\times 2\sqrt{3}\cos 120° = -2\sqrt{3}$$

Note:

- $\mathbf{i}\cdot\mathbf{j} = 0$, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$
- ✓ IF EITHER **u** OR **v** IS **0**, THEN **u**·**v** = 0.
- **u**·**v** = **v**·**u** (DOT PRODUCT OF VECTORS IS COMMUTATIVE)
- ✓ IF THE VECTORS **u** AND **v** ARE PARALLEL, THEN $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}|\,|\mathbf{v}|$. IN PARTICULAR, FOR ANY VECTOR **u**, WE HAVE $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$. HERE, WE WRITE $\mathbf{u}^2$ TO MEAN $|\mathbf{u}|^2$.
- ✓ IF THE VECTORS **u** AND **v** ARE PERPENDICULAR, THEN $\mathbf{u}\cdot\mathbf{v} = 0$ BECAUSE $\cos\left(\frac{\pi}{2}\right) = 0$.

FOR PURPOSES OF COMPUTATION, IT IS DESIRABLE TO HAVE A FORMULA THAT EXPRESSES THE DOT PRODUCT OF TWO VECTORS IN TERMS OF THE COMPONENTS OF THE VECTORS.

IN GENERAL, USING THE FORMULA IN THE DEFINITION OF THE DOT PRODUCT, YOU CAN FIND THE ANGLE BETWEEN TWO VECTORS. IF **u** AND **v** ARE NONZERO VECTORS, THEN THE COSINE OF THE ANGLE θ BETWEEN **u** AND **v** IS GIVEN BY:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

THE FOLLOWING THEOREM LISTS THE MOST IMPORTANT PROPERTIES OF THE DOT PRODUCT USEFUL IN CALCULATIONS INVOLVING VECTORS.

Theorem 8.4

LET **u**, **v** AND **w** BE VECTORS AND k BE A SCALAR. THEN,

I $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$. . . . . *associative property*

II $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ . . . .*distributive property*

$\mathbf{u}\cdot\mathbf{u} > 0$ IF $\mathbf{u} \neq \mathbf{0}$, AND $\mathbf{u}\cdot\mathbf{u} = 0$ IF $\mathbf{u} = \mathbf{0}$

Corollary 8.1

IF $\mathbf{u} = (u_1, u_2)$ AND $\mathbf{v} = (v_1, v_2)$ ARE VECTORS, THEN $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.

Proof:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j})\\ &= u_1\mathbf{i}\cdot(v_1\mathbf{i} + v_2\mathbf{j}) + u_2\,\mathbf{j}\cdot(v_1\mathbf{i} + v_2\mathbf{j})\\ &= u_1\mathbf{i}\cdot v_1\mathbf{i} + u_1\mathbf{i}\cdot v_2\mathbf{j} + u_2\mathbf{j}\cdot v_1\mathbf{i} + u_2\,\mathbf{j}\cdot v_2\,\mathbf{j}\\ &= u_1v_1\,\mathbf{i}\cdot\mathbf{i} + u_1v_2\,\mathbf{i}\cdot\mathbf{j} + u_2v_1\,\mathbf{j}\cdot\mathbf{i} + u_2v_2\,\mathbf{j}\cdot\mathbf{j}\\ &= u_1v_1 + u_2v_2. \text{ (SINCE } \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1 \text{ AND } \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0)\end{aligned}$$

Example 2 FIND THE DOT PRODUCT OF THE VECTORS $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ AND $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$

Solution $\mathbf{u}\cdot\mathbf{v} = (3\mathbf{i} + 2\mathbf{j})\cdot(5\mathbf{i} - 3\mathbf{j}) = 3\times 5 + 2\times(-3) = 9$

8.3.2 Application of the Dot Product of Vectors

THE DOT PRODUCT HAS MANY APPLICATIONS. THE FOLLOWING ARE EXAMPLES OF SOME OF THEM.

Example 3 FIND THE ANGLE BETWEEN $3\mathbf{i} + 5\mathbf{j}$ AND $-7\mathbf{i} + \mathbf{j}$.

Solution USING VECTOR METHOD,

$(3\mathbf{i} + 5\mathbf{j})\cdot(-7\mathbf{i} + \mathbf{j}) = 3(-7) + 5(1) = 16$

BUT BY DEFINITION,

$$(3\mathbf{i} + 5\mathbf{j})\cdot(7\mathbf{i} + \mathbf{j}) = |3\mathbf{i} + 5\mathbf{j}|\,|-7\mathbf{i} + \mathbf{j}| \cos\theta = \sqrt{9+25}\sqrt{49+1}\cos\theta$$

$$= \sqrt{34}\sqrt{50}\cos\theta = 16$$

$$\Rightarrow \cos\theta = \frac{16}{\sqrt{34}\sqrt{50}}$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{34}\sqrt{50}}\right)$$

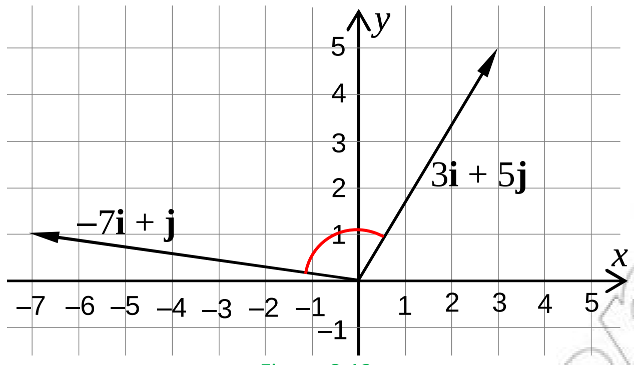


Figure 8.13

THE FOLLOWING ARE SOME OTHER IMPORTANT PROPERTIES OF THE DOT PRODUCT OF VECTO

Corollary 8.2

I $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}^2 - \mathbf{v}^2$

II $(\mathbf{u}\pm\mathbf{v})^2 = \mathbf{u}^2 \pm 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}^2$, WHERE $\mathbf{u}^2 = \mathbf{u}\cdot\mathbf{u}$

Example 4 SUPPOSE **a** AND **b** ARE VECTORS WITH $|\mathbf{a}| = 4$, $|\mathbf{b}| = 7$ AND THE ANGLE BETWEEN **a** AND **b** IS $\frac{}{3}$.

A EVALUATE $|3\mathbf{a}-2\mathbf{b}|$

B FIND THE COSINE OF THE ANGLE BETWEEN $3\mathbf{a}-2\mathbf{b}$ AND **a**

Solution USING THE PROPERTIES OF DOT PRODUCT WE HAVE,

A $|3\mathbf{a}-2\mathbf{b}|^2 = (3\mathbf{a}-2\mathbf{b})\cdot(3\mathbf{a}-2\mathbf{b}) = 9\mathbf{a}^2 - 12\mathbf{a}\cdot\mathbf{b} + 4\mathbf{b}^2$

$$= 9\times16 - 12|\mathbf{a}||\mathbf{b}|\cos\frac{}{3} + 4\times49 = 144 - 12\times4\times7\times\frac{1}{2} + 196$$

$$= 172$$

$$\Rightarrow |3\mathbf{a}-2\mathbf{b}| = \sqrt{172} = 2\sqrt{43}$$

B LET BE THE ANGLE BETWEEN $3\mathbf{a}-2\mathbf{b}$ AND **a**. THEN

$$(3\mathbf{a}-2\mathbf{b})\cdot\mathbf{a} = |3\mathbf{a}-2\mathbf{b}||\mathbf{a}|\cos \Rightarrow 3\mathbf{a}^2 - 2\mathbf{b}\cdot\mathbf{a} = 2\sqrt{43}\times4\cos$$

$$\Rightarrow 3\times16 - 2|\mathbf{b}||\mathbf{a}|\cos\frac{}{3} = 8\sqrt{43}\cos$$

$$\Rightarrow 48 - 2\times7\times4\times\frac{1}{2} = 8\sqrt{43}\cos$$

$$\Rightarrow \cos = \frac{5\sqrt{43}}{86}$$

THE FOLLOWING STATEMENT SHOWS HOW THE DOT PRODUCT CAN BE USED TO OBTAIN INF ABOUT THE ANGLE BETWEEN TWO VECTORS.

Corollary 8.3

LET **u** AND **v** BE NONZERO VECTORS AND THE ANGLE BETWEEN THEM, THEN

IS **acute**, IF AND ONLY IF $\mathbf{u}\cdot\mathbf{v} > 0$

IS **obtuse**, IF AND ONLY IF $\mathbf{u}\cdot\mathbf{v} < 0$

$= \frac{\;}{2}$ IF AND ONLY IF $\mathbf{u}\cdot\mathbf{v} = 0$

Example 5 DETERMINE THE VALUE OF k SO THAT THE ANGLE BETWEEN THE VECTORS $\mathbf{u} = (k, 1)$ AND $\mathbf{v} = (-2, 3)$ IS

A ACUTE **B** OBTUSE

Solution USING A DIRECT APPLICATION OF COROLLARY 8.3, WE HAVE,

A $\mathbf{u}\cdot\mathbf{v} > 0 \Rightarrow (k, 1)\cdot(-2, 3) > 0 \Rightarrow -2k + 3 > 0 \Rightarrow k < \frac{3}{2}$

B $\mathbf{u}\cdot\mathbf{v} < 0 \Rightarrow k > \frac{3}{2}$.

OBSERVE THAT THE ABOVE VECTORS ARE PERPENDICULAR (ORTHOGONAL) IF $k = \frac{3}{2}$

Exercise 8.4

1. FIND THE VECTORS $\mathbf{z} = \mathbf{u} - 2(\mathbf{v} + \mathbf{w})$ AND $\mathbf{z}' = (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$, WHERE,
   - **A** $\mathbf{u} = (8, 3)$, $\mathbf{v} = (-1, 2)$, $\mathbf{w} = (1, -4)$
   - **B** $\mathbf{u} = \left(\frac{2}{3}, -\frac{1}{2}\right)$, $\mathbf{v} = \left(-3.5, -\frac{4}{5}\right)$, $\mathbf{w} = (-2, -1)$
2. VECTORS **u** AND **v** MAKE AN ANGLE $\frac{2}{3}$. IF $|\mathbf{u}| = 3$ AND $|\mathbf{v}| = 4$, CALCULATE
   - **A** $\mathbf{u}\cdot\mathbf{v}$ **B** $(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} + \mathbf{v})$ **C** $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} + \mathbf{v})$ **D** $|2\mathbf{u} + \mathbf{v}|$
3. USING PROPERTIES OF THE SCALAR PRODUCT, SHOW THAT FOR ANY VECTORS **u**, **v**, **w** AND **z**,
   - **A** $(\mathbf{u} + \mathbf{v})^2 = \mathbf{u}^2 + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}^2$ **B** $(\mathbf{u} - \mathbf{v})^2 = \mathbf{u}^2 - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}^2$
   - **C** $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = \mathbf{u}^2 - \mathbf{v}^2$ **D** $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{w} + \mathbf{z}) = \mathbf{u}\cdot\mathbf{w} + \mathbf{u}\cdot\mathbf{z} + \mathbf{v}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{z}$
4. LET $\mathbf{u} = (1, -1)$, $\mathbf{v} = (1, 1)$ AND $\mathbf{w} = (-2, 3)$. FIND THE COSINES OF THE ANGLES BETWEEN
   - **A** **u** AND **v** **B** **v** AND **w** **C** **u** AND **w**
5. PROVE THAT $\mathbf{u}\cdot\mathbf{v} = 0$ FOR ALL NON-ZERO VECTORS **v**, IF AND ONLY IF $\mathbf{u} = \mathbf{0}$.
6. SHOW THAT $\mathbf{u} + \mathbf{v}$ AND $\mathbf{u} - \mathbf{v}$ ARE PERPENDICULAR TO EACH OTHER, IF AND ONLY IF $|\mathbf{u}| = |\mathbf{v}|$

7 SHOW THAT $(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) \geq (\mathbf{u}\cdot\mathbf{v})^2$. WHEN IS $(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})^2$?

8 **A** SHOW THAT $\mathbf{u}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

B CONSIDER TRIANGLE ABC IN FIGURE 8.14. IF THE VECTORS $\mathbf{u} = \overrightarrow{BC}$ AND $\mathbf{v} = \overrightarrow{CA}$ ARE ORTHOGONAL, THEN WHAT IS THE GEOMETRIC MEANING OF THE RELATION IN A?

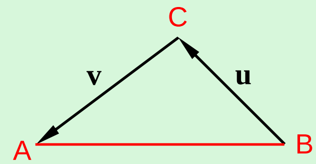


Figure 8.14

9 VECTORS **u** AND **v** MAKE AN ANGLE $\frac{\pi}{6}$. IF $|\mathbf{u}| = \sqrt{3}$ AND $|\mathbf{v}| = 1$, THEN FIND

A $|\mathbf{u} + \mathbf{v}|$ **B** $|\mathbf{u} - \mathbf{v}|$

10 LET $|\mathbf{u}| = 13$, $|\mathbf{v}| = 19$ AND $|\mathbf{u} + \mathbf{v}| = 24$. CALCULATE

A $\mathbf{u}\cdot\mathbf{v}$ **B** $|\mathbf{u} - \mathbf{v}|$ **C** $|3\mathbf{u} + 4\mathbf{v}|$

8.4 APPLICATION OF VECTOR

FROM PREVIOUS KNOWLEDGE, YOU NOTICE THAT VECTORS HAVE MANY APPLICAT GEOMETRICALLY, ANY TWO POINTS IN THE PLANE DETERMINE A STRAIGHT LINE. ALSO A ST IN THE PLANE IS COMPLETELY DETERMINED IF ITS SLOPE AND A POINT THROUGH WHICH IT P KNOWN. THESE LINES HAVE BEEN DETERMINED TO HAVE A CERTAIN DIRECTION. THUS, RE VECTORS, YOU WILL SEE HOW ONE CAN WRITE EQUATIONS OF LINES AND CIRCLES USING VE

Example 1 SHOW THAT, IN A RIGHT ANGLED TRIANGLE, THE SQUARE OF THE HYPOTENU EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

Solution LET ΔABC BE A GIVEN RIGHT-ANGLED TRIANGLE WITH $\angle ACB = 90^{\circ}$.

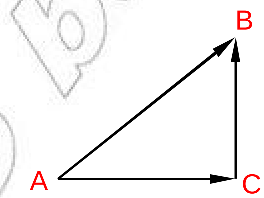


Figure 8.15

CONSIDER THE VECTORS $\overrightarrow{AC}$, $\overrightarrow{CB}$ AND $\overrightarrow{AB}$ AS SHOWN IN FIGURE 8.15

SINCE $\angle C = 90^{\circ}$, $\overrightarrow{AC}.\overrightarrow{CB} = 0$. BY VECTOR ADDITION YOU HAVE $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$. THUS

$$\overrightarrow{AB}^2 = \overrightarrow{AB}.\overrightarrow{AB} = (\overrightarrow{AC}+\overrightarrow{CB}).(\overrightarrow{AC}+\overrightarrow{CB}) = \overrightarrow{AC}^2 + 2\overrightarrow{CB}.\overrightarrow{AC} + \overrightarrow{CB}^2$$

$$= \overrightarrow{AC}^2 + \overrightarrow{CB}^2 \text{ . . . . SINCE } \overrightarrow{CB}.\overrightarrow{AC} = 0$$

HENCE, $\overrightarrow{AB}^2 = \overrightarrow{AC}^2 + \overrightarrow{CB}^2$.

Example 2 SHOW THAT THE PERPENDICULARS FROM THE VERTICES OF A TRIANGLE TO THE OPPOSITE SIDES ARE CONCURRENT (I.E. THEY INTERSECT AT A SINGLE POINT).

Solution LET ABC BE A GIVEN TRIANGLE AND AD AND BE BE PERPENDICULARS TO BC AND CA RESPECTIVELY. SUPPOSE AD AND BE MEET AT O AS SHOWN IN FIGURE 8.16.

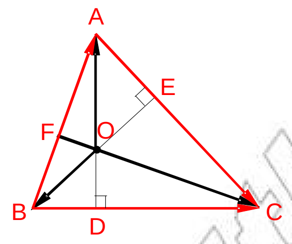


Figure 8.16

CONSIDER THE VECTORS $\overrightarrow{OA}$, $\overrightarrow{OB}$ AND $\overrightarrow{OC}$ AND $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$.

OBSERVE THAT $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$ AND $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

ACCORDING TO OUR HYPOTHESIS, $\overrightarrow{BC}$ AND $\overrightarrow{AD}$ ARE PERPENDICULAR. THUS

$$\overrightarrow{BC}.\overrightarrow{AD} = 0$$

$$\Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{AD} = 0 \quad \Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{OA} = 0$$

$$\Rightarrow \overrightarrow{OC}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OA} \text{ . . . . } \mathbf{1}$$

SIMILARLY, WE CAN WRITE FOR $\overrightarrow{BE}$ AND $\overrightarrow{CA}$, I.E., $\overrightarrow{BE}.\overrightarrow{CA} = 0$

$$\Rightarrow \overrightarrow{BE}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0 \Rightarrow \overrightarrow{OB}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0$$

$$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC} \text{ . . . . } \mathbf{2}$$

BY ADDING 1 AND 2, WE OBTAIN

$$\overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OB}.\overrightarrow{OC} \Rightarrow \overrightarrow{OC}.(\overrightarrow{OB} - \overrightarrow{OA}) = 0 \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$$

HENCE $\overrightarrow{BA}$ AND $\overrightarrow{OF}$ ARE PERPENDICULAR.

THUS, THE PERPENDICULARS FROM A, B AND C TO THE OPPOSITE SIDES ARE CONCURRENT.

Example 3 PROVE THAT THE PERPENDICULAR BISECTORS OF THE SIDES OF A TRIANGLE ARE CONCURRENT.

Solution LET ABC BE A TRIANGLE AND D, E, F THE MID-POINTS OF BC, CA, AND AB, RESPECTIVELY.

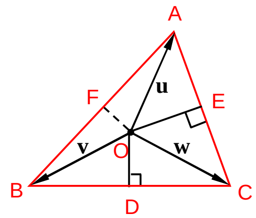


Figure 8.17

DO AND EO ARE PERPENDICULAR TO BC AND CA RESPECTIVELY. JOIN O TO THE MID-POINT F OF AB.

LET $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ BE THE VECTORS $\overrightarrow{OA}$, $\overrightarrow{OB}$ AND $\overrightarrow{OC}$ RESPECTIVELY.

THEN, $\overrightarrow{BC} = \mathbf{w} - \mathbf{v}$ AND $\overrightarrow{OD} = \dfrac{\mathbf{v} + \mathbf{w}}{2}$

SINCE $\overrightarrow{OD}$ AND $\overrightarrow{BC}$ ARE PERPENDICULAR, YOU HAVE

$$\overrightarrow{OD}.\overrightarrow{BC} = 0. \text{ I.E. } \left(\frac{\mathbf{v}+\mathbf{w}}{2}\right).(\mathbf{w}-\mathbf{v}) = 0 \text{ . . . . } 1$$

SIMILARLY, SINCE $\overrightarrow{OE}$ AND $\overrightarrow{CA}$ ARE PERPENDICULAR, YOU GET

$$\left(\frac{\mathbf{w}+\mathbf{u}}{2}\right).(\mathbf{u}-\mathbf{w}) = 0 \text{ . . . . } 2$$

FROM 1 AND 2, YOU OBTAIN $\mathbf{u}^2 - \mathbf{v}^2 = 0$ OR $\mathbf{v}^2 - \mathbf{u}^2 = 0$

$$\Rightarrow \frac{1}{2}(\mathbf{v}+\mathbf{u}).(\mathbf{v}-\mathbf{u}) = 0 \Rightarrow \overrightarrow{OF} \text{ AND } \overrightarrow{BA} \text{ ARE PERPENDICULAR.}$$

APART FROM THE APPLICATIONS DISCUSSED ABOVE, VECTORS HAVE MANY PRACTICAL APPLI
SOME ARE PRESENTED IN THE FOLLOWING SUBUNITS.

8.4.1 Vectors and Lines

LET $P_0(x_0, y_0)$ AND $P_1(x_1, y_1)$ BE TWO POINTS IN THE PLANE. THEN, THE VECTOR FROM P_0 TO P_1 IS
$P_1 - P_0 = (x_1 - x_0, y_1 - y_0)$ (*see* FIGURE 8.18)

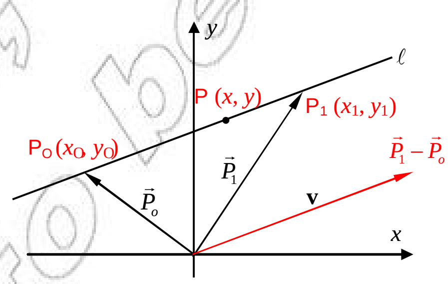


Figure 8.18

$$\vec{P}_1 - \vec{P}_o = (x_1 - x_0, y_1 - y_0)$$

WHERE $\vec{P}_o$ AND $\vec{P}_1$ ARE POSITION VECTORS CORRESPONDING TO P_0 AND P_1, RESPECTIVELY.

AS YOU CAN SEE FROM FIGURE 8.18 THE LINE ℓ THROUGH P_o AND P_1 IS PARALLEL TO THE VECTOR

$\vec{P}_1 - \vec{P}_o = (x_1 - x_O, y_1 - y_O)$.

LET P(x, y) BE ANY POINT ON ℓ. THEN THE POSITION VECTOR OF P IS OBTAINED FROM THE RELATION:

$$\overline{P} - \overline{P}_o = \overline{P_oP} = \quad (\overline{P}_1 - \overline{P}_o)$$

I.E., $\overline{P} - \overline{P}_o = \quad (\overline{P}_1 - \overline{P}_o)$, WHERE IS A SCALAR.

OBSERVE THAT YOU HAVE NOT USED THE POINT $P_1(x_1, y_1)$ IN THE ABOVE EQUATION EXCEPT FOR FINDING THE VECTOR $\mathbf{v} = \overline{P}_1 - \overline{P}_o$, WHICH IS OFTEN REFERRED TO AS A **direction** VECTOR OF THE LINE. THUS, IF A DIRECTION VECTOR **v** AND A POINT $P_O(x_O, y_O)$ ARE GIVEN, THEN THE VECTOR EQUATION OF THE LINE DETERMINED BY P_O AND **v** IS:

$$P = P_o + \quad \mathbf{v}; \quad \in \mathbb{R}, \mathbf{v} \neq 0.$$

IF **v** = (a, b), P (x, y) AND $P_O(x_O, y_O)$, THEN THE ABOVE EQUATION CAN BE WRITTEN AS:

$$(x, y) = (x_O, y_O) + \quad (a, b)$$

$$or \begin{cases} x = x_o + \quad a \\ y = y_o + \quad b \end{cases}; \quad \in \mathbb{R}, (a, b) \neq (0, 0)$$

THIS SYSTEM OF EQUATIONS IS CALLED THE **parametric equation of the line** ℓ, THROUGH $P_O(x_O, y_O)$, WHOSE DIRECTION IS THAT OF THE VECTOR **v** = (a, b), IS CALLED A PARAMETER.

NOW IF a AND b ARE BOTH DIFFERENT FROM 0, THEN

$$\frac{x - x_o}{a} = \quad \text{AND} \quad \frac{y - y_O}{b} = \quad \Rightarrow \frac{x - x_O}{a} = \frac{y - y_O}{b},$$

WHICH IS CALLED THE **standard equation of the line**.

THE ABOVE EQUATION CAN ALSO BE WRITTEN AS:

$$\frac{1}{a}x - \frac{1}{a}x_o = \frac{1}{b}y - \frac{1}{b}y_o \qquad \Rightarrow \frac{1}{a}x - \frac{1}{b}y + \left(\frac{1}{b}y_o - \frac{1}{a}x_o\right) = 0$$

$$\Rightarrow Ax + By + C = 0 \qquad \text{WHERE } A = \frac{1}{a},\ B = -\frac{1}{b} \text{ AND } C = \frac{1}{b}y_o - \frac{1}{a}x_o$$

Example 4 FIND THE VECTOR EQUATION OF THE LINE THROUGH (–1, –1) AND (1, 3)

Solution HERE YOU MAY TAKE $P_O = (1, 3)$ AND $P_1 = (–1, –1)$. THUS, THE VECTOR EQUATION OF THE LINE IS:

$$(x, y) = (1, 3) + \quad ((–1, –1) – (1, 3)) = (1, 3) + \quad (–2, –4) = (1 – 2\ , 3 – 4\ )$$

THE PARAMETRIC VECTOR EQUATION IS $x = 1 – 2$, $y = 3 – 4$, $\in \mathbb{R}$, AND

THE STANDARD EQUATION IS: $\dfrac{x-1}{-2} = \dfrac{y-3}{-4}$

Example 5 FIND THE VECTOR EQUATIONS OF THE LINE THROUGH (1, –2) AND WITH DIRECTION V (3, 1)

Solution YOU HAVE P_0 = (1, –2) AND v = (3, 1). THUS, THE VECTOR EQUATION OF THE LINE IS:

$(x, y) = (1, -2) + t(3, 1) = (1 + 3t, -2 + t)$

THE PARAMETRIC VECTOR EQUATION IS: $x = 1 + 3t$, $y = -2 + t$, $t \in \mathbb{R}$,

THE STANDARD EQUATION IS GIVEN BY $\dfrac{x-1}{3} = \dfrac{y+2}{1}$

Example 6 FIND THE VECTOR EQUATION OF THE LINE PASSING THROUGH THE POINTS (2, 3) AND (–1,1).

Solution THE VECTOR EQUATION OF THE LINE PASSING THROUGH TWO POINTS A AND B WITH POSITION VECTORS **a** AND **b**, RESPECTIVELY, IS $P = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ OR $P = \mathbf{b} + t(\mathbf{b} - \mathbf{a})$.

USING THIS RESULT, $P = (2, 3) + t(3, 2)$ OR $P = (-1, 1) + t(3, 2)$

8.4.2 Vectors and Circles

A CIRCLE WITH CENTRE C(x_0, y_0) AND RADIUS $r > 0$ IS THE SET OF ALL POINTS P(x, y) IN THE PLANE SUCH THAT $|\overline{P} - \overline{C}| = r$

WHERE $\overline{P}$ AND $\overline{C}$ ARE POSITION VECTORS OF P(x, y) AND C(x_0, y_0) RESPECTIVELY. (*See* FIGURE 8.19)

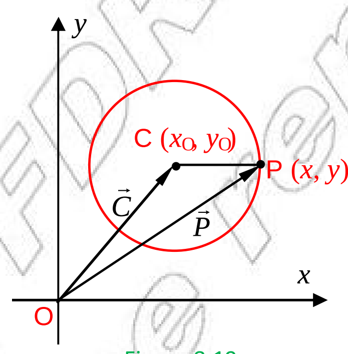


Figure 8.19

BY SQUARING BOTH SIDES OF THE EQUATION, WE OBTAIN,

$$|\overline{P} - \overline{C}|^2 = r^2 \quad ....\mathbf{1}$$

$$(\overline{P} - \overline{C}).(\overline{P} - \overline{C}) = r^2$$

$$\overline{P}.\overline{P} - 2\overline{P}.\overline{C} + \overline{C}.\overline{C} = r^2 \quad ....\mathbf{2}$$

THE ABOVE EQUATION IS SATISFIED BY A POSITION VECTOR OF ANY POINT ON THE CIRCLE. 2 REPRESENTS THE EQUATION OF THE CIRCLE CENTRED AT C(x_0, y_0) AND RADIUS r.

SUBSTITUTING THE CORRESPONDING COMPONENTS OF P AND C IN EQUATION 1, WE OBTAIN:

$$(x - x_o)^2 + (y - y_o)^2 = r^2$$

WHICH IS CALLED **the standard equation of a circle**.

BY EXPANDING AND REARRANGING THE TERMS, THIS EQUATION CAN BE EXPRESSED AS:

$x^2 + y^2 + Ax + By + C = 0$, WHERE $A = -2x_0$, $B = -2y_0$ AND $C = x_0^2 + y_0^2$.

Example 7 FIND AN EQUATION OF THE CIRCLE CENTRED AT C(–1, –2) AND OF RADIUS 2.

Solution LET $P(x, y)$ BE A POINT ON THE CIRCLE.

LET $\overline{P}$ AND $\overline{C}$ BE THE POSITION VECTORS OF P AND C, RESPECTIVELY.

THEN, FROM EQUATION (2), WE HAVE,

$$(x, y).(x, y) - 2(x, y).(-1,-2) + (-1,-2).(-1,-2) = 2^2$$

$$\Rightarrow x^2 + y^2 - 2(-x-2y) + (1+4) = 4 \Rightarrow x^2 + y^2 + 2x + 4y + 1 = 0$$

Example 8 FIND THE EQUATION OF THE CIRCLE WITH A DIAMETER THE SEGMENT FROM A (5, 3) TO B (3, –1).

Solution THE CENTRE OF THE CIRCLE IS $C(x_0, y_0) = C\left(\frac{5+3}{2}, \frac{3+(-1)}{2}\right) = C(4,1)$

THE RADIUS OF THE CIRCLE IS GIVEN BY $r = \frac{1}{2}\left(\sqrt{(5-3)^2 + (3+1)^2}\right) = \frac{1}{2}\sqrt{4+16}$

$$= \frac{1}{2}\sqrt{20} = \frac{2\sqrt{5}}{2} = \sqrt{5}$$

LET $P(x, y)$ BE A POINT ON THE CIRCLE AND $\overline{P}$ AND $\overline{C}$ BE POSITION VECTORS OF P AND C RESPECTIVELY. THEN, THE EQUATION OF THE CIRCLE IS:

$$(x, y).(x, y) - 2(x, y).(4, 1) + (4, 1).(4, 1) = \left(\sqrt{5}\right)^2,$$

$$\Rightarrow x^2 + y^2 - 2(4x + y) + 16 + 1 = 5$$

$$\Rightarrow x^2 + y^2 - 8x - 2y + 12 = 0$$

8.4.3 Tangent Line to a Circle

A LINE TANGENT TO A CIRCLE IS CHARACTERIZED BY THE FACT THAT THE RADIUS AT TH TANGENCY IS PERPENDICULAR (ORTHOGONAL) TO THE LINE.

LET THE CIRCLE BE GIVEN BY

$$(x - x_0)^2 + (y - y_0)^2 = r^2,\ r > 0$$

LET ℓ BE THE LINE TANGENT TO THE CIRCLE AT $P_1(x_1, y_1)$.

IF $P(x, y)$ IS AN ARBITRARY POINT ON ℓ, $\overrightarrow{P_1C}\cdot\overrightarrow{P_1P} = 0$

THEREFORE, THE EQUATION OF THE TANGENT LINE MUST BE:

$$(x - x_1, y - y_1).(x_1 - x_0, y_1 - y_0) = 0$$

$$\Rightarrow (x - x_1)(x_1 - x_0) + (y - y_1)(y_1 - y_0) = 0$$

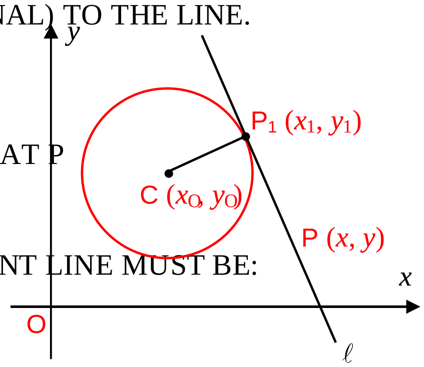


Figure 8.20

BY ADDING $(x_1 - x_O)^2 + (y_1 - y_O)^2 = r^2$ TO BOTH SIDES,
WE OBTAIN

$$(x - x_1)(x_1 - x_O) + (y - y_1)(y_1 - y_O) + (x_1 - x_O)^2 + (y_1 - y_O)^2 = r^2$$

$$\Rightarrow (x - x_1 + x_1 - x_O)(x_1 - x_O) + (y - y_1 + y_1 - y_O)(y_1 - y_O) = r^2$$

$$\Rightarrow (x - x_O)(x_1 - x_O) + (y - y_O)(y_1 - y_O) = r^2$$

Note:

IF THE CIRCLE IS CENTRED AT THE ORIGIN, THEN THE ABOVE EQUATION BECOMES:

$$x \cdot x_1 + y \cdot y_1 = r^2$$

Example 9 FIND THE EQUATION OF THE TANGENT LINE TO THE CIRCLE $x^2 + y^2 = 8$ AT THE POINT $P_1(2, -2)$.

Solution THE CIRCLE IS CENTRED AT THE ORIGIN WITH RADIUS $2\sqrt{2}$. HENCE THE EQUATION OF THE TANGENT LINE IS $2x - 2y = 8$.

Example 10 FIND THE EQUATION OF THE TANGENT LINE TO THE CIRCLE $x^2 + y^2 - 4x + 6y + 4 = 0$ AT (2,0).

Solution BY COMPLETING THE SQUARE, THE EQUATION OF THE CIRCLE CAN BE WRITTEN AS $(x - 2)^2 + (y + 3)^2 = 9$. THE CIRCLE HAS ITS CENTRE AT (2, –3) AND RADIUS $r = 3$.
THUS, THE EQUATION OF THE TANGENT LINE IS:

$$(x - 2)(2 - 2) + (y + 3)(0 + 3) = 9 \Rightarrow 0 + 3y + 9 = 9 \Rightarrow 3y = 0 \Rightarrow y = 0$$

THE TANGENT LINE TO THE GRAPH OF THE CIRCLE AT (2,0) IS THE HORIZONTAL LINE $y = 0$ (THE x-AXIS).

Practical application of vectors

PREVIOUSLY, YOU SAW HOW VECTORS ARE USEFUL IN DETERMINING THE EQUATIONS OF A LINE, AND THE EQUATIONS OF A TANGENT LINE TO A CIRCLE. NOW, YOU WILL CONSIDER PRACTICAL AND APPLICATIONS INVOLVING VECTORS.

Example 11 SHOW THAT THE VECTORS $\mathbf{u} = (1, 2)$ AND $\mathbf{v} = (0.5, 1)$ ARE TWO PARALLEL VECTORS WHICH ARE OF THE SAME DIRECTION WHEREAS THE VECTORS $\mathbf{u}_1 = (-1, 2)$ AND $\mathbf{v}_1 = (0.5, -1)$ ARE IN OPPOSITE DIRECTIONS.

Solution CONSIDER $\mathbf{u}.\mathbf{v}$ AND $\mathbf{u}_1.\mathbf{v}_1$.

$$\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \Rightarrow \frac{5}{2} = \sqrt{5} \times \frac{\sqrt{5}}{2}.\cos\theta \Rightarrow \cos\theta = 1 \text{ AND HENCE } \theta = 0.$$

THUS, $\mathbf{u}$ AND $\mathbf{v}$ ARE PARALLEL AND HAVE THE SAME DIRECTION.

SIMILARLY, $\mathbf{u}_1.\mathbf{v}_1 = |\mathbf{u}_1||\mathbf{v}_1|\cos\theta \Rightarrow -\frac{5}{2} = \sqrt{5} \times \frac{\sqrt{5}}{2}\cos\theta$

$$\Rightarrow \cos\theta = -1 \text{ AND HENCE } \theta = \pi.$$

THEREFORE, $\mathbf{u}_1$ AND $\mathbf{v}_1$ ARE PARALLEL AND HAVE OPPOSITE DIRECTIONS.

Example 12 IF **u**, **v**, **w** AND **z** ARE VECTORS FROM THE ORIGIN *O* TO THE POINTS *A*, *B*, *C* AND *D*, RESPECTIVELY, AND **v** – **u** = **w** – **z**, PROVE THAT *ABCD* IS A PARALLELOGRAM.

Solution LET *O* BE THE FIXED ORIGIN OF THESE VECTORS.

SINCE $\mathbf{v} - \mathbf{u} = \overrightarrow{AB}$ AND $\mathbf{w} - \mathbf{z} = \overrightarrow{DC}$, YOU HAVE $\overrightarrow{AB} = \overrightarrow{DC}$.

$\Rightarrow$ THE VECTORS $\overrightarrow{AB}$ AND $\overrightarrow{DC}$ ARE PARALLEL AND EQUAL.

ALSO, $\mathbf{v} - \mathbf{u} = \mathbf{w} - \mathbf{z} \Rightarrow \mathbf{w} - \mathbf{v} = \mathbf{z} - \mathbf{u} \Rightarrow \overrightarrow{BC} = \overrightarrow{AD}$

THUS, $\overrightarrow{BC}$ AND $\overrightarrow{AD}$ ARE PARALLEL AND EQUAL. HENCE, *ABCD* IS A PARALLELOGRAM.

Example13 PROVE THAT THE SUM OF THE THREE VECTORS DETERMINED BY THE MEDIANS OF A TRIANGLE DIRECTED FROM THE VERTICES IS ZERO.

Solution LET *ABC* BE A TRIANGLE AND *D*, *E*, *F* THE MID-POINTS OF THE SIDES *BC*, *CA* AND *AB*, RESPECTIVELY, AS SHOWN IN FIGURE 8.21.

FIRST, CONSIDER THE TRIANGLE *ABD*. YOU HAVE

$$\overrightarrow{AD} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \quad \ldots\ldots\ldots\ldots\ldots 1$$

IN THE SAME WAY, YOU SEE THAT

$$\overrightarrow{BE} = \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA} \quad \ldots\ldots\ldots\ldots\ldots 2$$

AND $$\overrightarrow{CF} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} \quad \ldots\ldots\ldots\ldots\ldots 3$$

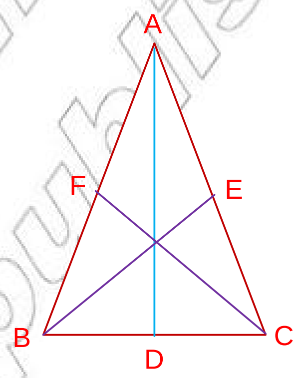


Figure 8.21

ADDING UP 1, 2 AND 3, YOU GET

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = \frac{3}{2}.0 = 0$$

Example14 A VIDEO CAMERA WEIGHING 15 POUNDS IS GOING TO BE SUSPENDED BY TWO WIRES FROM THE CEILING OF A ROOM AS SHOWN IN FIGURE 8.22. WHAT IS THE RESULTING TENSION IN EACH WIRE?

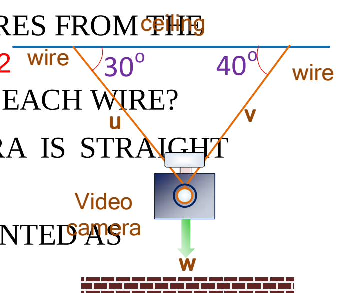


Figure 8.22

Solution THE FORCE VECTOR OF THE CAMERA IS STRAIGHT DOWN, SO **w** = (0, –15).

VECTOR **u** HAS MAGNITUDE |**u**| AND CAN BE REPRESENTED AS

$(-|\mathbf{u}|\cos 30^0,\ |\mathbf{u}|\sin (30^0))$.

SIMILARLY, $\mathbf{v} = (|\mathbf{v}|\cos 40^0,\ |\mathbf{v}|\sin 40^0)$.

SINCE THE SYSTEM IS IN EQUILIBRIUM, THE SUM OF THE FORCE VECTORS IS **0**.

$\Rightarrow \mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} = (-|\mathbf{u}|\cos 30^0 + |\mathbf{v}|\cos 40^0 + 0,\ |\mathbf{u}|\sin 30^0 + |\mathbf{v}|\sin 40^0 - 15)$

FROM THE COMPONENTS OF THE VECTOR EQUATION, YOU HAVE TWO EQUATIONS,

$$\begin{cases} 0 = -|\mathbf{u}|\cos 30^\circ + |\mathbf{v}|\cos 40^\circ \\ 0 = |\mathbf{u}|\sin 30^\circ + |\mathbf{v}|\sin 40^\circ - 15 \end{cases}$$

THAT YOU WANT TO SOLVE FOR THE TENSIONS $|\mathbf{u}|$ AND $|\mathbf{v}|$.

FROM THE FIRST, YOU GET $|\mathbf{u}|\cos 30^\circ = |\mathbf{v}|\cos 40^\circ \Rightarrow |\mathbf{v}| = |\mathbf{u}|\dfrac{\cos 30^\circ}{\cos 40^\circ}$

SUBSTITUTING THIS VALUE FOR $|\mathbf{v}|$ INTO THE SECOND EQUATION YOU HAVE

$$0 = |\mathbf{u}|\sin 30^\circ + |\mathbf{u}|\frac{\cos 30^\circ}{\cos 40^\circ}.\sin 40^\circ - 15$$

$$\Rightarrow |\mathbf{u}| = \frac{15}{\sin 30^\circ + (\cos 30^\circ)(\tan 40^\circ)} \cong 12.2 \text{ POUNDS}$$

PUTTING THIS VALUE BACK INTO

$|\mathbf{v}| = |\mathbf{u}|\dfrac{\cos 30^\circ}{\cos 40^\circ}$, YOU GET $|\mathbf{v}| = (12.2)\dfrac{\cos(30^\circ)}{\cos(40^\circ)} \cong 13.9$ POUNDS.

Exercise 8.5

1 FIND THE VECTOR EQUATION OF THE LINE THAT PASSES THROUGH THE POINT P_0 AND IS PARALLEL TO THE VECTOR $\mathbf{v}$ WHERE

A $P_O = (-2, 1)$; $\mathbf{v} = (-1,1)$ B $P_O = (1,1)$; $\mathbf{v} = (2,2)$

2 FIND AN EQUATION OF THE CIRCLE CENTRED AT C(1, –2) WITH RADIUS $r = \dfrac{3}{2}$

3 GIVEN AN EQUATION OF A LINE ℓ BY $P = (1, 0) + t(2, 2)$, $t \in \mathbb{R}$, FIND OUT WHETHER THE POINTS A (1, 0), B (2, 2), C(–5, –6) AND D (3, 0) LIE ON ℓ. FOR THOSE OF THEM LYING ON ℓ FIND THE RESPECTIVE VALUES OF THE PARAMETER t.

4 ARE THE POINTS A, B AND C COLLINEAR?

A A (1, –4), B (–2, –3), C (11, –11) B A(–2, –3), B(4, 9), C (–11, –21)

5 FIND THE EQUATION (BOTH IN PARAMETRIC FORM AND STANDARD FORM) OF THE LINE THROUGH THE POINTS (3, 5) AND (–2, 3).

6 SHOW THAT THE GIVEN POINT LIES ON THE CIRCLE AND FIND THE EQUATION OF THE TANGENT LINE AT THE POINT.

A $x^2 + y^2 - 2x - 4y - 9 = 0$ AT $P_1(1,4)$ B $(x+2)^2 + y^2 = 3$ AT $P_1(-1, \sqrt{2})$

7 IF **u**, **v**, **w**, **z** ARE VECTORS FROM THE ORIGIN TO THE POINTS A, B, C, D, RESPECTIVELY, AN
v – **u** = **w** – **z**, THEN SHOW THAT ABCD IS A PARALLELOGRAM.

8 FIGURE 8.23 SHOWS THE MAGNITUDES AND DIRECTIONS OF SIX COPLANAR FORCES (FORCE
THE SAME PLANE).

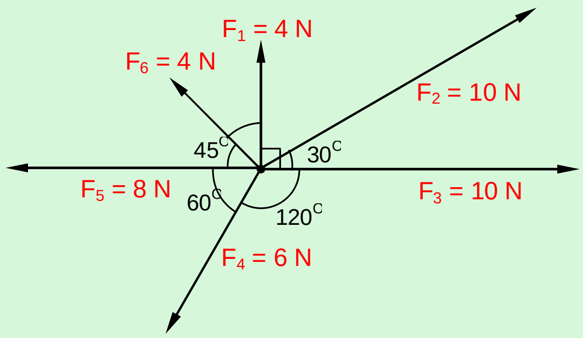


Figure 8.23

FIND EACH OF THE FOLLOWING DOT PRODUCTS.

A $F_1 . F_2$ **B** $F_5 . F_6$ **C** $(F_1 + F_2 - F_3) . (F_4 + F_5 - F_6)$

9 LET **a** = 3**i** + **j**, **b** = 2**i** – 2**j** AND **c** = **i** + 3**j** BE VECTORS. FIND THE UNIT VECTORS IN THE DIRECTION OF EACH OF THE FOLLOWING VECTORS.

A **a** + **b** **B** $2\mathbf{a} + \mathbf{b} - \frac{3}{2}\mathbf{c}$.

10 THREE FORCES $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + 2\mathbf{j}$ AND $\mathbf{F}_3 = 3\mathbf{i} - \mathbf{j}$ MEASURED IN NEWTON ACT ON A PARTICLE CAUSING IT TO MOVE FROM A = **i** + 2**j** TO B = 3**i** + 4**j** WHERE AB IS MEASURED IN METERS. FIND THE TOTAL WORKDONE BY THE COMBINED FORCES.

8.5 TRANSFORMATION OF THE PLANE

TRANSFORMATIONS ARE OF PRACTICAL IMPORTANCE, ESPECIALLY IN **solving problems** AND **describing difficulties** IN SIMPLER FORMS. TRANSFORMATIONS CAN BE MANAGED IN DIFFEREN FORMS, THOSE THAT **maintain direction** AND THOSE THAT **change direction**. THERE ARE MANY VERSIONS OF TRANSFORMATIONS, BUT, IN THIS SECTION, YOU ARE GOING TO CONSIDER THR TRANSFORMATIONS NAMELY **translations**, **reflections** AND **rotations**.

Group work 8.4

1 WHEN YOU BLOW UP A BALLOON, ITS SHAPE AND S

Figure 8.24

IN WHICH OF THE FOLLOWING CONDITIONS DOES THE SHAPE OR SIZE OR BOTH OF THE CHANGE.

A WHEN A RUBBER IS STRETCHED.

B WHEN A COMMERCIAL JET FLIES FROM PLACE TO PLACE AT A SPECIFIC TIME.

C WHEN THE EARTH ROTATES ABOUT ITS AXIS.

D WHEN YOU SEE YOUR IMAGE IN A PLANE MIRROR.

E WHEN YOU DRAW THE MAP OF YOUR SCHOOL COMPOUND.

2 LET T BE A MAPPING OF THE PLANE ONTO ITSELF GIVEN BY T((x, y)) = (x + 1, –y).

FOR EXAMPLE, T((4, 3)) = (4 + 1, – 3) = (5, –3).

IF A = (0, 1), B = (–3, 2) AND C = (2, 0), FIND THE COORDINATES OF THE IMAGE OF A, B AND C.

FIND THE IMAGE OF ΔABC UNDER T. IS ΔABC CONGRUENT TO ITS IMAGE?

3 SUPPOSE T IS A MAPPING OF THE PLANE ONTO ITSELF WHICH SENDS POINT P TO POINT P'

LET A = (2, –3) AND B = (5, 4). COMPARE THE LENGTHS OF AB AND A'B' WHEN

A T ((x, y)) = (x, 0) **B** T ((x, y)) = (x, –y)

C T ((x, y)) = (x + 1, y – 3) **D** T ((x, y)) = $\left(\frac{1}{2}x, 2y\right)$

4 CAN YOU LIST SOME OTHER TRANSFORMATIONS?

IN THIS GROUP WORK, YOU SAW THAT SOME MAPPINGS CALLED **Transformations** OF THE PLANE ONTO ITSELF PRESERVE SHAPE, SIZE OR DISTANCE BETWEEN ANY TWO POINTS. BASED ON TH TRANSFORMATIONS ARE CLASSIFIED AS EITHER RIGID MOTION OR NON RIGID MOTION.

Definition 8.10 Rigid motion

A MOTION IS SAID TO BE **rigid motion**, IF IT PRESERVES DISTANCE. THAT IS, FOR P ≠ Q, PQ = P'Q' WHERE P' AND Q' ARE THE IMAGES OF P AND Q, RESPECTIVELY. OTHERWISE IT IS SAID TO BE NON-RIGID MOTION.

A TRANSFORMATION IS SAID TO BE AN **identity transformation**, IF THE IMAGE OF EVERY POINT IS ITSELF. FOR EXAMPLE, IF AN OBJECT IS ROTATED 360^{o} IT IS AN IDENTITY TRANSFORMATION.

Note:

✓ RIGID MOTION CARRIES ANY PLANE FIGURE TO A CONGRUENT PLANE FIGURE, I.E., IT CARRIES TRIANGLES TO CONGRUENT TRIANGLES, RECTANGLES TO CONGRUENT RECTANGLES, ET

AN **identity transformation** IS A RIGID MOTION.

IN THIS TOPIC THREE DIFFERENT TYPES OF RIGID MOTIONS ARE PRESENTED.

Translations **Reflections** **Rotations**

Figure 8.25

8.5.1 Translation

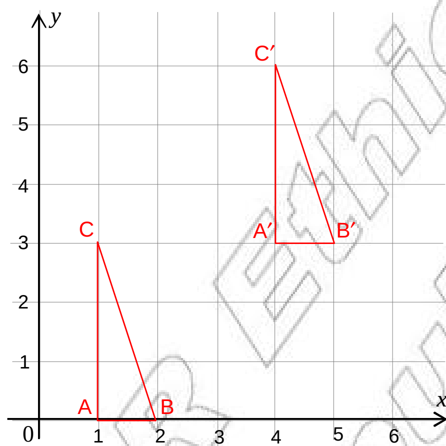


Figure 8.24

WHEN ΔABC IS TRANSFORMED TO $\Delta A'B'C'$, AB AND $A'B'$ ARE PARALLEL TO THE x-AXIS, AND AC AND $A'C'$ ARE PARALLEL TO THE y-AXIS. MOREOVER, ΔABC AND $\Delta A'B'C'$ HAVE THE SAME ORIENTATION. I.E., THE WAY THEY FACE IS THE SAME. THIS TYPE OF TRANSFORMATION IS SAID TO BE A **translation**.

Definition 8.11

IF EVERY POINT OF A FIGURE IS MOVED ALONG THE SAME DIRECTION THROUGH THE SAME DISTANCE, THEN THE TRANSFORMATION IS CALLED A **translation** OR **parallel movement**.

IF POINT P IS TRANSLATED TO THE POINT P', THEN THE VECTOR $\overrightarrow{PP'}$ IS SAID TO BE THE **translation vector**.

IF $\mathbf{u} = (h, k)$ IS A TRANSLATION VECTOR, THEN THE IMAGE OF THE POINT $P(x, y)$ UNDER THIS TRANSLATION WILL BE THE POINT $P'(x + h, y + k)$.

Example 1 LET T BE A TRANSLATION THAT TAKES THE ORIGIN TO (1, 2). DETERMINE THE TRANSLATION VECTOR AND FIND THE IMAGES OF THE FOLLOWING POINTS.

A (2,–1) **B** (–3, 5) **C** (1, 2)

Solution T ((0, 0)) = (1, 2) $\Rightarrow$ **u** = (1, 2) IS THE TRANSLATION VECTOR.

$\Rightarrow x \mapsto x + 1$ AND $y \mapsto y + 2$

THUS,

A T ((2, –1)) = (2 + 1, –1 + 2) = (3, 1)

B T ((–3, 5)) = (–3 + 1, 5 + 2) = (–2, 7)

C T ((1, 2)) = (1 + 1, 2 + 2) = (2, 4).

Example 2 LET THE POINTS P(x_1, y_1) AND Q(x_2, y_2) BE TRANSLATED BY THE VECTOR **u** = (h, k). SHOW THAT $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{P'Q'}\right|$.

Solution CLEARLY P' = ($x_1 + h$, $y_1 + k$) AND Q' = ($x_2 + h$, $y_2 + k$).

THEN, $\left|\overrightarrow{P'Q'}\right| = \sqrt{(x_2 + h - x_1 - h)^2 + (y_2 + k - y_1 - k)^2}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \overrightarrow{PQ}$.

THE ABOVE EXAMPLE SHOWS THAT A TRANSLATION IS A RIGID MOTION. YOU CAN STATE A T FORMULA IN TERMS OF COORDINATES AS FOLLOWS:

1 IF (h, k) IS A THE TRANSLATION VECTOR, THEN

A THE ORIGIN IS TRANSLATED TO (h, k), I.E., (0, 0) → (h, k)

B THE POINT P(x, y) IS TRANSLATED TO P'(x + h, y + k).

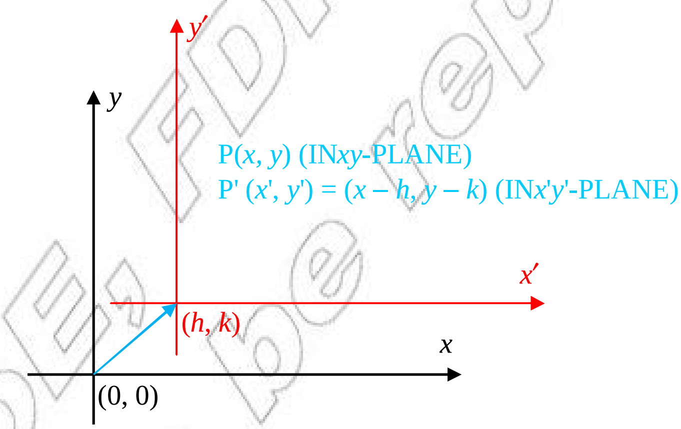


Figure 8.25

2 IF THE TRANSLATION VECTOR IS $\overrightarrow{AB}$ WHERE A = (a, b) AND B = (c, d), THEN

A THE ORIGIN IS TRANSLATED TO (c – a, d – b), AND

B THE POINT P(x, y) IS TRANSLATED TO P'(x + c – a, y + d – b)

Example 3 IF A TRANSLATION T TAKES THE ORIGIN TO P(1, 2), THEN

T (x, y) = (x + 1, y + 2) AND T (–2, 3) = (–2 + 1, 3 + 2) = (–1, 5).

Example 4 IF A TRANSLATION T TAKES THE ORIGIN TO (–1, 1), THEN FIND

A THE IMAGES OF THE POINTS P (1, 3) AND Q (–3, 6)

B THE IMAGE OF THE TRIANGLE WITH VERTICES A(2, –2), B(–3, 2) AND C(4, 1)

C THE EQUATION OF THE IMAGE FOR THE CIRCLE WHOSE EQUATION IS $x^2 + y^2 = 4$

Solution

A THE IMAGE OF THE POINT P (1, 3) IS T (1, 3) = (1 – 1, 3 + 1) = (0, 4).

THE IMAGE OF THE POINT Q (–3, 6) IS T (–3, 6) = (–3 – 1, 6 +1) = (–4, 7)

B T (2, –2) = (2 + (–1), –2 + 1) = (1,–1)

T (–3, 2) = (–3 + (–1), 2 + 1) = (–4, 3)

T (4, 1) = (4 + (–1), 1 + 1) = (3, 2)

THUS, A' = (1, –1), B' = (–4, 3) AND C' = (3, 2)

THE IMAGE OF ΔABC IS ΔA'B'C'.

C THE IMAGE OF (x, y) UNDER T IS $T(x, y) = (x – 1, y + 1)$.

THE CENTRE OF THE CIRCLE (0, 0) IS TRANSLATED TO (–1, 1)

THUS, THE IMAGE OF $x^2 + y^2 = 4$ IS $(x + 1)^2 + (y – 1)^2 = 4$

Example 5 IF A TRANSLATION T TAKES THE POINT (–1, 3) INTO (4, 2), THEN FIND THE IMAGES OF THE FOLLOWING LINES UNDER THE TRANSLATION T.

A $\ell : y = 2x – 3$ **B** $\ell : 5y + x = 1$

Solution THE TRANSLATION VECTOR IS (h, k) = (4 – (–1), 2 – 3) = (5, –1). THUS, THE POINT P(x, y) IS TRANSLATED TO THE POINT P'$(x + 5, y – 1)$. A TRANSLATION MAPS LINES ONTO PARALLEL LINES. LET ℓ' BE THE IMAGE OF ℓ UNDER T. THEN,

A $\ell' : y – (–1) = 2(x – 5) – 3$

$\Rightarrow \ell' : y = 2x – 14$

B $\ell' : 5(y + 1) + (x – 5) = 1$

$\Rightarrow \ell' : 5y + x = 1 \Rightarrow \ell' = \ell$. *Explain!*

Example 6 DETERMINE THE EQUATION OF THE CURVE $2x^2 + 3y^2 - 8x + 6y = 7$ WHEN THE ORIGIN IS TRANSLATED TO THE POINT A(2, –1).

Solution THE TRANSLATION VECTOR IS $(h, k) = (2, –1)$. THUS, THE POINT P(x, y) IS TRANSLATED TO THE POINT P'$(x + 2, y – 1)$. SUBSTITUTING $x – 2$ AND $y + 1$ IN THE EQUATION, YOU OBTAIN $2(x-2)^2 + 3(y+1)^2 - 8(x-2) + 6(y+1) = 7$.

EXPANDING AND SIMPLIFYING, THE EQUATION OF THE CURVE BECOMES

$$2x^2 + 3y^2 - 16x + 12y + 26 = 0$$

Exercise 8.6

1 IF A TRANSLATION T TAKES THE ORIGIN TO THE POINT A(–3, 2), FIND THE IMAGE OF RECTANGLE ABCD WITH VERTICES A(3, 1), B(5, 1), C(5, 4) AND D(3, 4).

2 TRIANGLE ABC IS TRANSFORMED INTO TRIANGLE A'B'C' BY THE TRANSLATION VECTOR IF A = (2, 1), B = (3, 5) AND C = (–1, –2), FIND THE COORDINATES OF A', B' AND C'.

3 QUADRILATERAL ABCD IS TRANSFORMED INTO A'B'C'D' BY A TRANSLATION VECTOR (3, –

IF A = (1, 2), B = (3, 4), C = (7, 4) AND D = (2, 5), THEN FIND A', B', C' AND D' AND DRAW THE QUADRILATERALS ABCD AND A'B'C'D' ON GRAPH PAPER.

4 WHAT IS THE IMAGE OF A CIRCLE UNDER A TRANSLATION?

5 FIND THE EQUATION OF THE IMAGE OF THE CIRCLE $(x + 1)^2 + (y - 3)^2 = 5$ WHEN TRANSLATED BY THE VECTOR $\overrightarrow{PQ}$, WHERE P = (1, –1) AND Q = (–4, 3).

6 A TRANSLATION T TAKES THE ORIGIN TO A(3, –2). A SECOND TRANSLATION S TAKES ORIGIN TO B(–2, –1). FIND WHERE T FOLLOWED BY S TAKES THE ORIGIN, AND WHERE FOLLOWED BY T TAKES THE ORIGIN.

7 IF A TRANSLATION T TAKES (2, –5) TO (–2, 1), FIND THE IMAGE OF THE LINE $\ell: 2x - 3y = 7$.

8 IF A TRANSLATION T TAKES THE ORIGIN TO (4, –5), FIND THE IMAGE OF EACH OF THE FOLLOWING LINES.

A $y = 3x + 7$ B $4y + 5x = 10$

9 IF THE POINT A(3, –2) IS TRANSLATED TO THE POINT A'(7, 10), THEN FIND THE EQUATION OF THE IMAGE OF

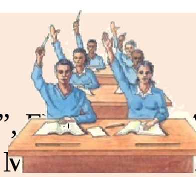
A THE ELLIPSE $x^2 + 4\,3y^2 - 2x + 6y = 0$ B THE PARABOLA $y^2 = 4x$

C THE HYPERBOLA $xy = 1$ D THE FUNCTION $f(x) = x^3 - 3x^2 + 4$

8.5.2 Reflections

AS THE NAME INDICATES, REFLECTION TRANSFORMS AN OBJECT USING A REFLECTING MATE

ACTIVITY 8.4



1 USING THE CONCEPT “REFLECTION BY A PLANE MIRROR”, F… AGES OF THE FOLLOWING FIGURES BY CONSIDERING LINE L AS A M… .

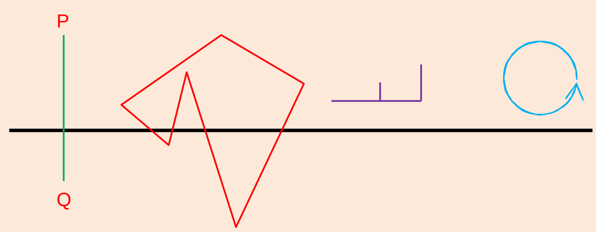


Figure 8.26

2 IN FIGURE 8.27A BELOW A' IS THE MIRROR IMAGE OF A. COPY THE FIGURE AND DRAW THE REFLECTING LINE.

3 IN FIGURE 8.27B BELOW A' AND B' ARE THE IMAGES OF A AND B, RESPECTIVELY. COPY THE FIGURE AND DETERMINE THE REFLECTION LINE.

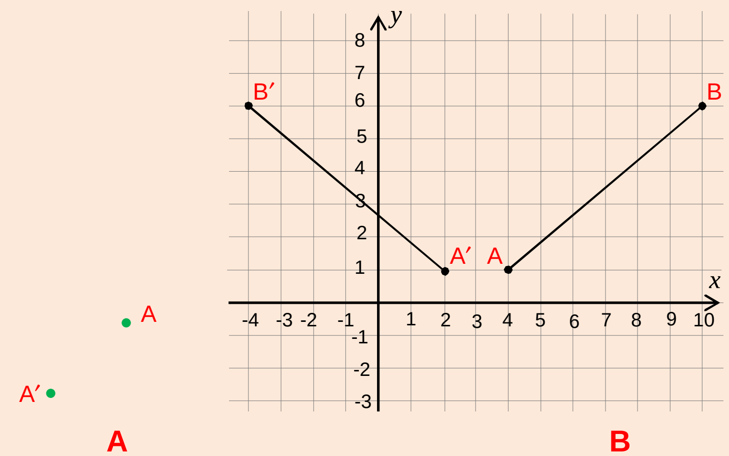


Figure 8.27

4 DISCUSS THE CONDITIONS THAT ARE NECESSARY TO DEFINE REFLECTION.

> **Definition 8.12**
>
> LET L BE A FIXED LINE IN THE PLANE. A REFLECTION M ABOUT A LINE L IS A TRANSFORMAT[ION OF] THE PLANE ONTO ITSELF WHICH CARRIES EACH POINT P OF THE PLANE INTO THE POINT P' [OF THE] PLANE SUCH THAT L IS THE **perpendicular bisector** OF PP'.

THE LINE L IS SAID TO BE THE LINE OF REFLECTION OR THE AXIS OF REFLECTION.

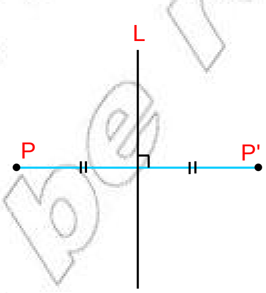


Figure 8.28

> **Note:**
>
> EVERY POINT ON THE AXIS OF REFLECTION IS ITS OWN IMAGE.

> **NOTATION:**
>
> THE REFLECTION OF POINT P ABOUT THE LINE L IS DENOTED BY $M_L(P)$.

REFLECTION HAS THE FOLLOWING PROPERTIES:

1. A REFLECTION ABOUT A LINE ℓ HAS THE PROPERTY THAT, IF FOR TWO POINTS P AND Q IN THE PLANE, P = Q, THEN M(P) = M(Q). HENCE, REFLECTION IS A FUNCTION FROM THE SET OF POINTS IN THE PLANE INTO THE SET OF POINTS IN THE PLANE.
2. A REFLECTION ABOUT A LINE ℓ MAPS DISTINCT POINTS TO DISTINCT POINTS, I.E., IF P ≠ Q, THEN M(P) ≠ M(Q). EQUIVALENTLY, IT HAS THE PROPERTY THAT IF, FOR TWO POINTS P, Q IN THE PLANE, M(P) = M(Q), THEN P = Q. THUS, REFLECTION IS A ONE-TO-ONE MAPPING.
3. FOR EVERY POINT P' IN THE PLANE, THERE EXISTS A POINT P SUCH THAT M(P) = P'. IF THE POINT P' IS ON L, THEN THERE EXISTS P = P' SUCH THAT M(P) = P'. THUS, REFLECTION IS AN ONTO MAPPING.

Theorem 8.5

A REFLECTION M IS A RIGID MOTION. THAT IS, IF P' = M(P) AND Q' = M(Q), THEN PQ = P'Q'.

WE NOW CONSIDER REFLECTIONS WITH RESPECT TO THE AXES AND THE LINES $y = mx + b$.

A Reflection in the x and y-axes

ACTIVITY 8.5

1. FIND THE IMAGE OF $f(x) = e^x$, WHEN IT IS REFLECTED
   - **A** IN THE y-AXIS
   - **B** IN THE x-AXIS
   - **C** IN THE LINE $y = x$
2. DISCUSS HOW TO DETERMINE THE IMAGES OF POINTS P(x, y), LINES $y = mx + b$ AND CIRCLES $(x-h)^2 + (y-k)^2 = r^2$, WHEN THEY ARE REFLECTED IN EACH OF THE FOLLOWING LINES
   - **A** $y = 0$ (x-AXIS)
   - **B** $x = 0$ (y-AXIS)
   - **C** $y = x$
   - **D** $y = -x$

B Reflection in the line $y = mx$, where $m = \tan$

LET ℓ BE A LINE PASSING THROUGH THE ORIGIN AND MAKING AN ANGLE WITH THE POSITIVE x-AXIS. THEN THE SLOPE OF ℓ IS GIVEN BY m = TAN AND ITS EQUATION IS $y = mx$. *See* FIGURE 8.29.

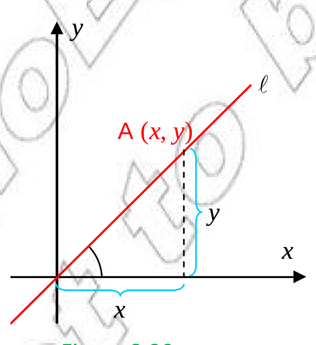


Figure 8.29

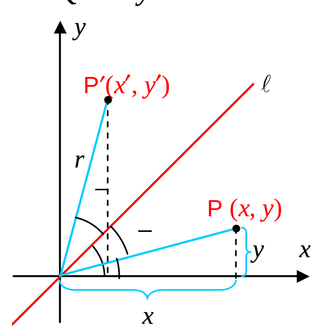


Figure 8.30

YOU WILL NOW FIND THE IMAGE OF A POINT P(x, y) WHEN IT IS REFLECTED ABOUT THIS LINE. *See* FIGURE 8.30

LET P'(x', y') BE THE IMAGE OF P(x, y).

THE COORDINATES OF P ARE:

$x = r$ COS AND $y = r$ SIN

THE COORDINATES OF P' ARE:

$x' = r$ COS (2 –) AND $y' = r$ SIN (2 –)

EXPANDING COS (2 –) AND SIN (2 –),

NOW, USE THE FOLLOWING TRIGONOMETRIC IDENTITIES THAT YOU WILL LEARN IN SECTION 9.4.2

1 Sine of the sum and the difference

- ✓ SIN ($x + y$) = SIN x COS y + COS x SIN y
- ✓ SIN ($x – y$) = SIN x COS y – COS x SIN y

2 Cosine of the sum and difference

- ✓ COS ($x + y$) = COS x COS y – SIN x SIN y
- ✓ COS ($x – y$) = COS x COS y + SIN x SIN y

USING THESE TRIGONOMETRIC IDENTITIES, YOU OBTAIN:

$x' = r$[COS 2 COS + SIN 2 SI] = (r COS) COS 2 + (r SIN) SI

= x COS 2 + y SIN 2 AND

$y' = r$[SIN 2 COS – SIN CO] = (r COS) SIN 2 – r (SIN) C(

= x SIN 2 – y COS

THUS, THE COORDINATES OF P'(x', y'), THE IMAGE OF THE POINT P(x, y) WHEN REFLECTED ABOUT THE LINE $y = mx$ IS:

> x' = x COS 2 + y SIN 2
>
> y' = x SIN 2 – y COS 2
>
> WHERE IS THE ANGLE OF INCLINATION OF THE LINE $y = mx$.

BASED ON THE VALUE OF , YOU WILL HAVE THE FOLLOWING FOUR SPECIAL CASES:

1 WHEN = 0, YOU WILL HAVE REFLECTION IN THE x-AXIS. HENCE (x, y) IS MAPPED TO (x, –y)

2 WHEN = $\frac{}{4}$, YOU WILL HAVE REFLECTION ABOUT THE LINE $y = x$ AND HENCE (x, y) IS MAPPED TO (y, x).

3 WHEN $\theta = \frac{\pi}{2}$, YOU WILL HAVE REFLECTION IN THE y-AXIS AND (x, y) IS MAPPED TO $(-x, y)$.

4 WHEN $\theta = \frac{3\pi}{4}$, YOU WILL HAVE REFLECTION ABOUT THE LINE $y = -x$ AND (x, y) IS MAPPED TO $(-y, -x)$.

Example 7 FIND THE IMAGES OF THE POINTS (3, 2), (0, 1) AND (–5, 7) WHEN REFLECTED ABOUT THE LINE $y = mx$, WHERE $m = \tan\theta$ AND $\theta = \frac{\pi}{4}$

Solution: THIS IS ACTUALLY A REFLECTION ABOUT THE LINE $y = x$. THUS, THE IMAGES OF (3, 2), (0, 1) AND (–5, 7) ARE (2, 3), (1, 0) AND (7, –5), RESPECTIVELY.

Example 8 FIND THE IMAGES OF THE POINTS P (3, 2), Q (0, 1) AND R (–5, 7) WHEN REFLECTED ABOUT THE LINE $y = \frac{1}{\sqrt{3}}x$

Solution SINCE $\tan\theta = \frac{1}{\sqrt{3}}$, YOU HAVE $\theta = \frac{\pi}{6}$. THUS, IF $P'(x', y')$ IS THE IMAGE OF P, THEN

$$x' = x\cos 2\theta + y\sin 2\theta = 3\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) = 3\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{2}$$

$$y' = x\sin 2\theta - y\cos 2\theta = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) - 2\times\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - 1$$

HENCE, THE IMAGE OF P (3, 2) IS $P'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} - 1\right)$

SIMILARLY, YOU CAN SHOW THAT THE IMAGES OF Q (0, 1) AND R(–5, 7) ARE $Q'\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ AND $R'\left(\frac{-5+7\sqrt{3}}{2}, \frac{-5\sqrt{3}-7}{2}\right)$, RESPECTIVELY.

Example 9 FIND THE IMAGE OF A = (1,–2) AFTER IT HAS BEEN REFLECTED IN THE LINE $y = 2x$.

Solution $y = 2x \Rightarrow y = (\tan\theta)x \Rightarrow \theta = \tan^{-1}(2)$.

BUT, FROM TRIGONOMETRY, YOU HAVE

$$\sin\theta = \frac{2}{\sqrt{5}} \text{ AND } \cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5},$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{4}{5} \Rightarrow x' = -\frac{3}{5}x + \frac{4}{5}y \text{ AND } y' = \frac{4}{5}x + \frac{3}{5}y$$

$$\Rightarrow M((1, -2)) = \left(-\frac{11}{5}, -\frac{2}{5}\right)$$

Note:

1 IF A LINE ℓ' IS PERPENDICULAR TO THE AXIS OF REFLECTION ℓ, THEN ℓ' IS ITS OWN IMAGE.

2 IF THE CENTRE OF A CIRCLE C IS ON THE LINE OF REFLECTION L, THEN THE IMAGE OF C IS ITSELF.

3 IF THE CENTRE O OF A CIRCLE C HAS IMAGE O' WHEN REFLECTED ABOUT A LINE L, THEN THE IMAGE CIRCLE HAS CENTRE O' AND RADIUS THE SAME AS C.

4 IF ℓ' IS A LINE PARALLEL TO THE LINE OF REFLECTION L, TO FIND THE IMAGE OF L' WHEN REFLECTED ABOUT L, WE FOLLOW THE FOLLOWING STEPS.

Step a: CHOOSE ANY POINT P ON ℓ'

Step b: FIND THE IMAGE OF P, M(P) = P'

Step c: FIND THE EQUATION OF ℓ'' WHICH IS THE LINE PASSING THROUGH P' WITH SLOPE EQUAL TO THE SLOPE OF ℓ'

C Reflection in the line $y = mx + b$

LET $\ell : y = mx + b$ BE THE LINE OF REFLECTION, WHERE $m \in \mathbb{R}\setminus\{0\}$.

LET P(x, y) BE A POINT IN THE PLANE, NOT ON ℓ.

LET P'(x', y') BE THE IMAGE OF P(x, y) WHEN REFLECTED ABOUT THE LINE ℓ.

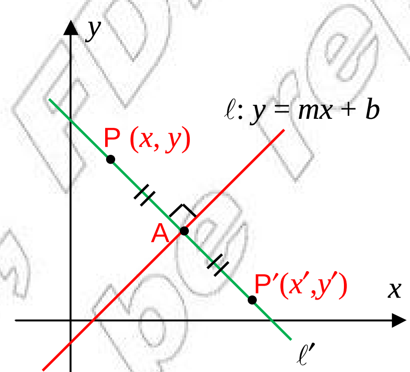


Figure 8.31

LET ℓ' BE THE LINE PASSING THROUGH THE POINTS P(x, y) AND P'(x', y'). THEN, ℓ' IS PERPENDICULAR TO ℓ, SINCE ℓ IS PERPENDICULAR TO $\overline{PP'}$. SINCE THE SLOPE OF ℓ IS m, THE SLOPE OF ℓ' IS $-\frac{1}{m}$.

THUS, ONE CAN DETERMINE THE EQUATION OF ℓ'. IF A IS THE POINT OF INTERSECTION OF ℓ AND ℓ', TAKING A AS THE MID-POINT OF $\overline{PP'}$, WE CAN FIND THE COORDINATES OF P'.

THUS, TO FIND THE IMAGE OF A POINT P(x, y) WHEN REFLECTED ABOUT A LINE ℓ, WE FOLLOW THE FOLLOWING FOUR STEPS.

Step 1: FIND THE SLOPE OF THE LINE ℓ, SAY m.

Step 2: FIND THE EQUATION OF THE LINE ℓ' WHICH PASSES THROUGH THE POINT P(x, y) AND HAS SLOPE $-\frac{1}{m}$.

Step 3: FIND THE POINT OF INTERSECTION OF ℓ AND ℓ' WHICH SERVES AS THE MIDPOINT OF $\overline{PP'}$.

Step 4: USING A AS THE MID-POINT OF $\overline{PP'}$, FIND THE COORDINATES OF P'.

Example 10 FIND IMAGES OF THE FOLLOWING LINES AND CIRCLES AFTER REFLECTION IN THE LINE $y = 2x - 3$.

A $2y + x = 1$ **B** $y = 2x + 1$ **C** $y = 3x + 4$

D $x^2 + y^2 - 4x - 2y + 4 = 0$ **E** $x^2 + y^2 - 2x + 3y = 8$

Solution

A THE IMAGE OF ℓ: $2y + x = 1$ IS ITSELF. EXPLAIN!

B ℓ: $y = 2x + 1$ IS PARALLEL TO THE REFLECTING AXIS.

HENCE ℓ' : $y = 2x + b$. WE NEED TO DETERMINE b.

LET (a, b) BE ANY POINT ON ℓ, SAY (0, 1), SO THAT ITS IMAGE LIES ON ℓ'.

BY THE ABOVE REFLECTING PROCEDURE,

$$M((0, 1)) = (a', b') \Rightarrow \frac{b'-1}{a'-0} = -\frac{1}{2} \Rightarrow a' = -2b' + 2$$

ALSO, THE MIDPOINT OF (0, 1) AND (a', b') WHICH IS $\left(\frac{a'}{2}, \frac{b'+1}{2}\right)$ LIES ON THE REFLECTING AXIS $\Rightarrow \frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}$,

BUT $a' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2}$

$$\Rightarrow b' = -\frac{3}{5} \Rightarrow a' = \frac{16}{5} \quad \Rightarrow \left(\frac{16}{5}, -\frac{3}{5}\right) \text{ LIES ON } \ell'$$

$$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b \quad b = -7 \qquad \Rightarrow \ell': y = 2x - 7$$

C $\ell: y = 3x + 4$ AND THE AXIS OF REFLECTION $y = 2x - 3$ MEET AT $(-7, -17)$

NEXT, TAKE A POINT ON ℓ SAY (0, 4) AND FIND ITS IMAGE M(0, 4), SO THAT ℓ' PASSES THROUGH (a', b'). PERFORM THE TECHNIQUE SIMILAR TO THE PROBLEM IN **B**.

THUS, $\frac{b'-4}{a'-0} = -\frac{1}{2}$ AND $\frac{4+b'}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{28}{5}$ AND $b' = \frac{6}{5}$

$$\Rightarrow \ell': y = \frac{91}{63}x - \frac{434}{63}$$

D $x^2 + y^2 - 4x - 2y + 4 = 0 \Rightarrow (x-2)^2 + (y-1)^2 = 1$

THIS IS A CIRCLE OF RADIUS 1 UNIT WITH CENTRE (2, 1) THAT IS ON $y = 2x - 3$.

$\Rightarrow$ THE CENTRE OF THE CIRCLE LIES ON THE AXIS OF REFLECTION. THEREFORE, THE CI ITS OWN IMAGE.

E $x^2 + y^2 - 2x + 3y = 8 \Rightarrow (x-1)^2 + (y+\frac{3}{2})^2 = \frac{45}{4}$

THE CENTRE $(1, -\frac{3}{2})$ HAS IMAGE $\left(\frac{3}{5}, -\frac{13}{10}\right)$

$\Rightarrow$ THE IMAGE CIRCLE IS $\left(x - \frac{3}{5}\right)^2 + \left(y + \frac{13}{10}\right)^2 = \frac{45}{4}$

Example 11 FIND THE IMAGE OF (–1, 5) WHEN REFLECTED ABOUT THE LINES

A $y = -1$ **B** $x = 1$ **C** $y = x + 2$ **D** $y = 2x + 5$

Solution

A THE IMAGE OF THE POINT (–1, 5) WHEN REFLECTED ABOUT THE LINE $y = -1$ IS (–1, –7)

B THE IMAGE OF THE POINT (–1, 5) WHEN REFLECTED ABOUT THE LINE $x = 1$ IS (3, 5)

C THE SLOPE OF $y = x + 2$ IS 1.

LET P'(x', y') BE THE IMAGE OF P (–1, 5). IF ℓ' IS THE LINE PASSING THROUGH P AND P', THEN ITS SLOPE IS $\frac{-1}{1} = -1$. THUS, THE EQUATION OF ℓ' IS

$$\frac{y-5}{x+1} = -1 \Rightarrow \ell': y = -x + 4$$

THE POINT OF INTERSECTION OF ℓ AND ℓ' IS (1, 3). TAKING (1, 3) AS A MIDPOINT OF $\overline{PP'}$, WE GET,

$$\frac{-1+x'}{2} = 1 \text{ AND } \frac{5+y'}{2} = 3 \Rightarrow -1 + x' = 2 \text{ AND } 5 + y' = 6$$

$$\Rightarrow x' = 3 \text{ AND } y' = 1$$

THEREFORE, THE IMAGE OF P (–1, 5) IS P' (3, 1).

D THE SLOPE OF $y = 2x + 5$ IS 2. IF P'(x', y') IS THE IMAGE OF P(–1, 5) AND ℓ' IS THE LINE THROUGH P AND P', THEN ITS SLOPE IS $\frac{-1}{2}$. THUS, THE EQUATION OF ℓ' IS:

$$\frac{y-5}{x+1} = \frac{-1}{2} \Rightarrow \ell' : y = \frac{-1}{2}x + \frac{9}{2}$$

THE POINT OF INTERSECTION OF ℓ AND ℓ' IS A$\left(\frac{-1}{5}, \frac{23}{5}\right)$. TAKING A AS THE MIDPOINT OF $\overline{PP'}$, FIND THE COORDINATES OF P' AS:

$$\frac{-1+x'}{2} = \frac{-1}{5} \text{ AND } \frac{5+y'}{2} = \frac{23}{5} \Rightarrow -5 + 5x' = -2 \text{ AND } 25 + 5y' = 46$$

$$\Rightarrow 5x' = 3 \text{ AND } 5y' = 46 - 25 = 21 \Rightarrow x' = \frac{3}{5} \text{ AND } y' = \frac{21}{5}$$

HENCE, THE IMAGE OF P (–1, 5) IS P'$\left(\frac{3}{5}, \frac{21}{5}\right)$.

Example 12 GIVEN THE EQUATION OF THE CIRCLE $x^2 + (y - 1)^2 = 1$, FIND THE EQUATION OF ITS GRAPH AFTER A REFLECTION ABOUT THE LINE $y = x$.

Solution THE CENTRE OF THE CIRCLE IS (0, 1). THE REFLECTION OF (0, 1) ABOUT THE LINE $y = x$ IS (1, 0), WHICH IS THE CENTRE OF THE IMAGE CIRCLE. THEREFORE, THE EQUATION OF THE IMAGE CIRCLE IS $(x - 1)^2 + y^2 = 1$

Example 13 FIND THE IMAGE OF THE LINE ℓ': $y = -3x - 7$ AFTER A REFLECTION ABOUT THE LINE ℓ: $y = -3x + 1$

Solution PICK A POINT P ON ℓ', SAY P (1, –10).

TO FIND THE IMAGE OF THE POINT P (1, –10) WHEN REFLECTED ABOUT THE LINE $y = -3x + 1$, PROCEED AS FOLLOWS:

SINCE SLOPE OF ℓ IS –3, THE SLOPE OF THE PERPENDICULAR LINE IS $\frac{1}{3}$. THUS, THE EQUATION OF THE LINE THROUGH (1, –10) WITH SLOPE $\frac{1}{3}$ IS: $\frac{y+10}{x-1} = \frac{1}{3}$

$$\Rightarrow y = \frac{1}{3}x - \frac{31}{3}$$

THE POINT OF INTERSECTION OF $y = -3x + 1$ AND $y = \frac{1}{3}x - \frac{31}{3}$ IS A$\left(\frac{34}{10}, \frac{-92}{10}\right)$.

TAKING A AS A MID-POINT OF $\overline{PP'}$, FIND THE COORDINATES OF THE IMAGE $P'(x', y')$ OF P, I.E.,

$$\frac{1+x'}{2} = \frac{34}{10} \quad \text{AND} \quad \frac{-10+y'}{2} = \frac{-92}{10}$$

$$\Rightarrow \; 10 + 10x' = 68 \quad \text{AND} \quad -100 + 10y' = -184$$

$$\Rightarrow \; x' = \frac{58}{10} \qquad \text{AND} \quad y' = \frac{-84}{10}$$

THEREFORE, THE IMAGE OF $P(1, -10)$ IS P' $\left(\frac{58}{10}, \frac{-84}{10}\right)$.

NOW, YOU NEED TO FIND THE EQUATION THE LINE PASSING THROUGH P' WITH SLOPE -3,

$$\frac{y + \frac{84}{10}}{x - \frac{58}{10}} = -3 \Rightarrow \frac{10y + 84}{10x - 58} = -3$$

$$\Rightarrow 10y + 84 = -30x + 174$$

$$\Rightarrow 10y = -30x + 174 - 84$$

$$\Rightarrow 10y = -30x + 90$$

$$\Rightarrow y = -3x + 9$$

HENCE, THE IMAGE OF THE LINE $y = -3x - 7$ WHEN REFLECTED ABOUT THE LINE $y = -3x + 1$ IS $y = -3x + 9$.

Example 14 FIND THE IMAGE OF THE CIRCLE $(x - 1)^2 + (y + 5)^2 = 1$, WHEN IT IS REFLECTED ABOUT THE LINE $y = 2x - 1$.

Solution THE CENTRE OF THE CIRCLE IS $(1, -5)$. THE IMAGE OF THE POINT $(1, -5)$ WHEN REFLECTED ABOUT THE LINE $\ell: y = 2x - 1$ IS $\left(\frac{-19}{5}, \frac{-13}{5}\right)$

THUS, THE EQUATION OF THE IMAGE CIRCLE IS $\left(x + \frac{19}{5}\right)^2 + \left(y + \frac{13}{5}\right)^2 = 1$

Exercise 8.7

1 THE VERTICES OF TRIANGLE ABC ARE A (2, 1), B (3, –2) AND C (5, –3). GIVE THE COORDINATES OF THE VERTICES AFTER:

A A REFLECTION IN THE x-AXIS **B** A REFLECTION IN THE y-AXIS

C A REFLECTION IN THE LINE $x + y = 0$ **D** A REFLECTION IN THE LINE $y = x$

2 FIND THE IMAGE OF THE POINT (–4, 3) AFTER A REFLECTION ABOUT THE LINE $\ell: y = -2$

3 IF THE IMAGE OF THE POINT (–1, 2) UNDER REFLECTION IS (1, 0), FIND THE LINE OF REFLECTION.

4 FIND OUT SOME OF THE FIGURES WHICH ARE THEIR OWN IMAGES IN REFLECTION ABOUT THE LINE $y = x$.

5 FIND THE IMAGE OF THE LINE $\ell: y = x + 4$ AFTER IT HAS BEEN REFLECTED ABOUT THE LINE $L: y = x - 3$

6 FIND THE IMAGE OF THE LINE $\ell: y = 2x + 1$ AFTER IT HAS BEEN REFLECTED ABOUT THE LINE $L: y = 3x + 2$

7 GIVEN AN EQUATION OF A CIRCLE $(x - 2)^2 + (y - 3)^2 = 25$, FIND THE EQUATION OF THE IMAGE CIRCLE AFTER A REFLECTION ABOUT THE LINE $y = x + 3$.

8 THE IMAGE OF THE CIRCLE $x^2 + y^2 - x + 2y = 0$ WHEN IT IS REFLECTED ABOUT THE LINE L IS $x^2 + y^2 - 2x + y = 0$. FIND THE EQUATION OF L.

9 IF T IS A TRANSLATION THAT SENDS (0, 0) TO (3, –2) AND M IS A REFLECTION THAT MAPS (0, 0) TO (2, 4), FIND

A T(M(1, 3)) **B** M(T(1, 3))

10 IN A REFLECTION, THE IMAGE OF THE LINE $y - 2x = 3$ IS THE LINE $y - 2x = 9$. FIND THE AXIS OF REFLECTION.

8.5.3 Rotations

ROTATION IS A TYPE OF TRANSFORMATION IN WHICH FIGURES TURN AROUND A POINT CALLED THE CENTRE OF ROTATION. THE GROUP WORK WILL INTRODUCE YOU THE IDEA OF ROTATION.

Group work 8.5

1 IN THE FOLLOWING FIGURE, A, B, C AND D ARE POINTS ON A CIRCLE WITH CENTRE AT THE ORIGIN. THE CHORDS $\overline{AC}$ AND $\overline{BD}$ ARE PERPENDICULAR.

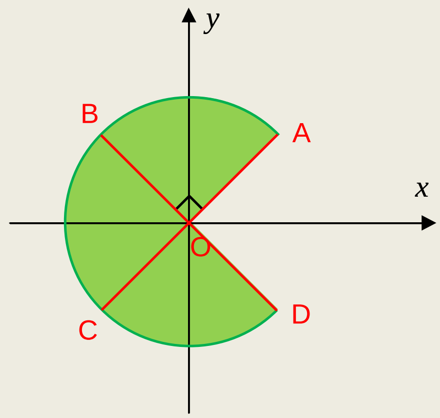


Figure 8.32

DISCUSS THE FOLLOWING QUESTIONS IN GROUPS.

A IF A = (2, 3) FIND THE COORDINATES OF B, C AND D.

B IF A = (x , y) EXPRESS THE COORDINATES OF B, C AND D IN TERMS OF x AND y.

2 LOOK AT THE FIGURE BELOW.

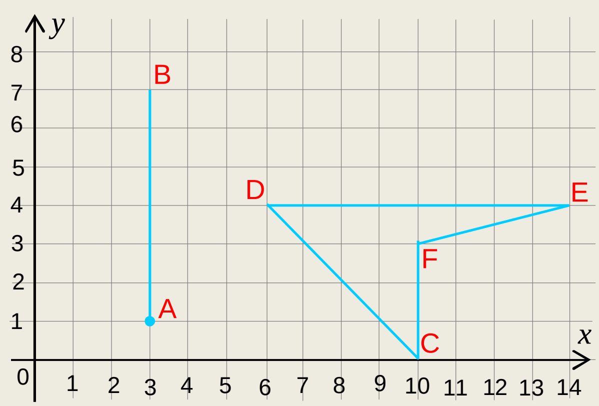


Figure 8.33

BY PLACING A PIECE OF TRANSPARENT PAPER ON THIS FIGURE, TRACE $\overline{AB}$ AND CDEF.

HOLD A PENCIL AT THE ORIGIN AND ROTATE THE PAPER 90° COUNTERCLOCKWISE. AFTER THIS ROTATION, WRITE THE IMAGES OF A, B, C, D, E AND F TO A', B', C', D', E' AND F', RESPECTIVELY, ON THE PAPER.

A FIND THE COORDINATES OF THOSE POINTS ON THE TRANSPARENT PAPER BY REFERRING THE x AND y COORDINATES OF THE ORIGINAL FIGURE.

B IS THERE A FIXED POINT IN THIS ROTATION?

C DISCUSS WHETHER OR NOT THIS TRANSFORMATION IS A RIGID MOTION.

D WHAT DO YOU THINK THE IMAGES OF THE x AND y AXES ARE?

3 DISCUSS WHAT YOU NEED TO DEFINE ROTATION.

IN THE GROUP WORK, YOU HAVE SEEN A THIRD TYPE OF TRANSFORMATION CALLED **rotation**. ROTATION IS FORMALLY DEFINED AS FOLLOWS.

Definition 8.13

A ROTATION R ABOUT A POINT O THROUGH AN ANGLE θ IS A TRANSFORMATION OF THE PLANE ONTO ITSELF WHICH CARRIES EVERY POINT P OF THE PLANE INTO THE POINT P' OF THE PLANE SUCH THAT $OP = OP'$ AND $m(\measuredangle POP') = \theta$. O IS CALLED THE **centre of rotation** AND θ IS CALLED THE **angle of rotation**.

Note:

I THE ROTATION IS IN THE **counter clockwise** DIRECTION IF $\theta > 0$ AND IN THE **clockwise** DIRECTION IF $\theta < 0$.

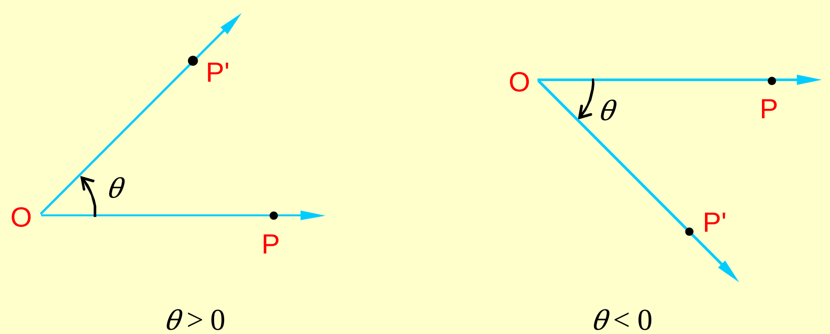


Figure 8.34

II ROTATION IS A RIGID MOTION.

Example 15 FIND THE IMAGE OF POINT A (1, 0) WHEN IT IS ROTATED THROUGH 30^0 ABOUT THE ORIGIN.

Solution LET THE IMAGE OF A (1, 0) BE A'(a, b) AS SHOWN IN THE FIGURE.

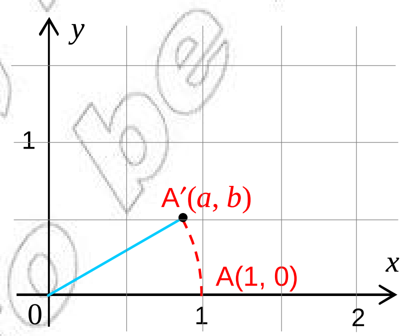


Figure 8.35

BUT FROM TRIGONOMETRY, $(a, b) = (r \cos\theta, r \sin\theta)$ WHERE $r = 1$ AND $\theta = 30^{\circ}$ IN THIS EXAMPLE. THEREFORE, THE IMAGE OF A $(1, 0)$ IS A' $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

NOTATION:

IF R IS ROTATION THROUGH AN ANGLE θ, THEN THE IMAGE OF (x, y) IS DENOTED BY $R_\theta(x, y)$. IN THE ABOVE EXAMPLE, $R_{30^\circ}(1, 0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

AT THIS LEVEL, WE DERIVE A FORMULA FOR A ROTATION R ABOUT O(0, 0) THROUGH AN ANG

Theorem 8.6

LET R BE A ROTATION THROUGH AN ANGLE θ ABOUT THE ORIGIN. IF $R_\theta(x, y) = (x', y')$,

THEN $x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

Proof

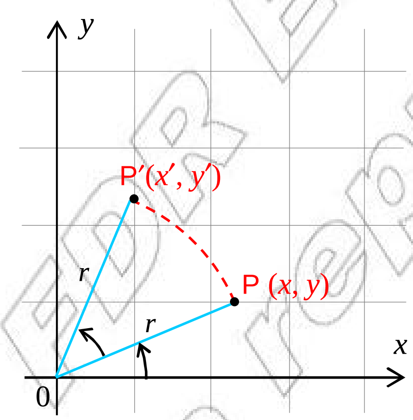


Figure 8.36

FROM TRIGONOMETRY WE HAVE,

$(x, y) = (r\cos\alpha, r\sin\alpha)$ AND $(x', y') = (r\cos(\alpha + \theta), r\sin(\alpha + \theta))$

$\Rightarrow r\cos(\alpha + \theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$= x\cos\theta - y\sin\theta$

$r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$= y\cos\theta + x\sin\theta$

$\therefore R_\theta(x, y) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta)$

Note:

LET R BE A COUNTER-CLOCKWISE ROTATION THROUGH THE ANGLE θ ABOUT THE ORIGIN. THEN

I $= \frac{}{2} \Rightarrow R(x, y) = (-y, x)$

II $= \quad \Rightarrow R(x, y) = (-x, -y)$

III $= \frac{3}{2} \Rightarrow R(x, y) = (y, -x)$

IV $= 2n$ FOR $n \in \mathbb{Z} \Rightarrow$ R IS THE IDENTITY TRANS

V EVERY CIRCLE WITH CENTRE AT THE CENTRE OF ROTATION IS FIXED.

Example 16 USING THE FORMULA, FIND THE IMAGES OF THE POINTS IN ROTATION ABOUT THE ORIGIN THROUGH THE INDICATED ANGLE.

A (4, 0) ; 60^{O} **B** (1, 1) ; $-\frac{\pi}{6}$ **C** (1, 2); 450^{O}

Solution

A $x' = x\text{COS} - y\ \text{SIN}$; $x = 4, y = 0, = 60^{o}$

$= 4\text{COS}\,60^{o} - 0 \times \text{SIN}\,60^{o} = 2$

$y' = x\text{SIN} + y\ \text{COS}$

$= 4\text{SIN}\,60 + 0 \times \text{COS}\,60^{o} = 2\sqrt{3}$

$\Rightarrow R_{60^{o}}(4, 0) = \left(2, 2\sqrt{3}\right)$

B $x' = 1 \times \text{COS}\left(-\frac{}{6}\right) - 1 \times \text{SIN}\left(-\frac{}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

$y' = x\ \text{SIN} + y\ \text{COS}$

$y' = 1 \times \text{SIN}\left(\frac{-}{6}\right) + 1 \times \text{COS}\left(\frac{-}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

$\Rightarrow R_{60^{o}}(1, 1) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2},\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

C $x' = 1 \times \text{COS}\frac{}{2} - 2 \times \text{SIN}\left(\frac{}{2}\right)$

$x' = 1 \times \text{COS}\frac{}{2} - 2 \times \text{SIN}\left(\frac{}{2}\right)$

$x' = -2(1) = -2$

$y' = 1 \times \text{SIN}\left(\frac{}{2}\right) + 2 \times \text{COS}\left(\frac{}{2}\right)$

$y' = 1 \times 1 + 2 \times 0$

$\Rightarrow y' = 1$

NOTICE THAT $450^{\circ} = (360^{\circ} + 90^{\circ})$

$\therefore R(x, y) = (-y, x)$

$\therefore R(1, 2) = (-2, 1)$

Rotation when the centre of rotation is (x_o, y_o)

SO FAR YOU HAVE SEEN ROTATION ABOUT THE ORIGIN. THE NEXT ACTIVITY INTRODUCES ROTATION ABOUT AN ARBITRARY POINT (x_0, y_0).

ACTIVITY 8.6

1 IN THE FOLLOWING FIGURE, A ROTATION R SENDS A T… O B'.

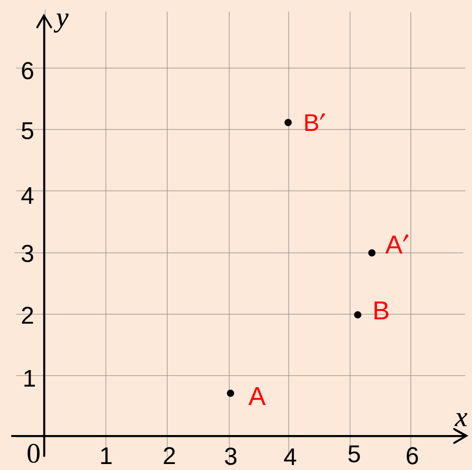


Figure 8.37

DISCUSS HOW TO DETERMINE THE CENTRE OF ROTATION.

2 IF R IS A ROTATION THROUGH $\frac{\pi}{4}$ ABOUT A(3, 2), DISCUSS HOW TO DETERMINE THE IMAGE OF A POINT P (2, 0).

THE ABOVE ACTIVITY LEADS TO THE FOLLOWING GENERALIZED FORMULA.

Corollary 8.4

IF P' (x', y') IS THE IMAGE OF P(x, y) AFTER IT HAS BEEN ROTATED THROUGH AN ANGLE θ ABOUT (x_0, y_0), THEN

$$x' = x_0 + (x - x_0)\text{COS}\,\theta - (y - y_0)\text{SIN}\,\theta$$
$$y' = y_0 + (x - x_0)\text{SIN}\,\theta + (y - y_0)\text{COS}\,\theta$$

Note:

As in the case of translation and reflection, to find the image of a circle under rotation we follow the following steps:

1. Find the centre and radius of the given circle.
2. Find the image of the centre of the given circle under the rotation.
3. Equation of the image circle will be the equation of the circle centred at the image of the centre of the given circle with radius the same as the radius of the given circle.

Example 17 Find the image of the circle $(x-3)^2 + (y+5)^2 = 1$ when it is rotated through $\frac{5\pi}{3}$ about (4, –3).

Solution According to the note given above, we need to compute only the image of the centre of the circle. The centre is (3, –5) and its radius is 1 unit.

$$x' = x_o + (x - x_o)\cos\theta - (y - y_o)\sin\theta$$

where $(x, y) = (3, -5)$; $(x_o, y_o) = (4, -3)$; $\theta = \frac{5\pi}{3}$

$$x' = 4 + (3-4)\cos\frac{5\pi}{3} - (-5-(-3))\sin\frac{5\pi}{3} = 4 - \frac{1}{2} + \left(2\cdot\frac{-\sqrt{3}}{2}\right) = \frac{7}{2} - \sqrt{3}$$

$$y' = y_o + (x - x_o)\sin\theta + (y - y_o)\cos\theta$$

$$\Rightarrow y' = -3 + (3-4)\sin\frac{5\pi}{3} + (-5-(-3))\cos\frac{5\pi}{3} = -3 + \frac{\sqrt{3}}{2} - 1 = -4 + \frac{\sqrt{3}}{2}$$

Thus, the equation of the image of the circle is $\left(x - \frac{7}{2} + \sqrt{3}\right)^2 + \left(y + 4 - \frac{\sqrt{3}}{2}\right)^2 = 1$

Note:

One can also obtain the image of a line under a given rotation as follows:

✓ Choose two points on the line.

✓ Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

Example 18 Find the equation of the line $3x - 2y = 1$ after it has been rotated -135° about (–2, 3).

Solution According to the note, we choose any two arbitrary points, say R(1, 1) and (–1, –2). Together with $(x_o, y_o) = (-2, 3)$ and $\theta = -135^{\circ}$, we get

$$R(1, 1) = (-2 - 2.5\sqrt{2},\ 3 - 0.5\sqrt{2}) \text{ and } R(-1, -2) = (-2 - 3\sqrt{2},\ 3 + 2\sqrt{2})$$

$$\Rightarrow \text{THE SLOPE OF } \ell' = \frac{3+2\sqrt{2}-3+0.5\sqrt{2}}{-2-3\sqrt{2}+2+2.5\sqrt{2}} = -5$$

$$\Rightarrow \ell' : \frac{y-3-2\sqrt{2}}{x+2+3\sqrt{2}} = -5$$

$$\Rightarrow \ell' : y-3-2\sqrt{2} = -5x-10-15\sqrt{2}$$

$$\Rightarrow \ell' : y+5x+7+13\sqrt{2} = 0$$

Exercise 8.8

1. RECTANGLE ABCD HAS VERTICES A (1, 2), B (4, 2), C (4, –1) AND D (1, –1). FIND THE IMAGES OF THE VERTICES OF THE RECTANGLE WHEN THE AXES ARE ROTAT… THE ORIGIN THROUGH AN ANGLE θ =

2. FIND THE POINT INTO WHICH THE GIVEN POINT IS TRANSFORMED BY A ROTATION OF THE AXES THROUGH THE INDICATED ANGLES, ABOUT THE ORIGIN.

   A (–3, 4); 90° **B** (–2, 0); 60° **C** $(0, -1); \frac{\pi}{4}$ **D** (–1, 2); 30°

3. FIND AN EQUATION OF THE LINE INTO WHICH THE LINE WITH THE GIVEN EQUATION IS TRANSFORMED UNDER A ROTATION THROUGH THE INDICATED ANGLE.

   A $3x - 4y = 7$; ACUTE ANGLE θ SUCH THAT $\tan\theta = \frac{3}{4}$

   B $2x + y = 3$; $\theta = \frac{\pi}{3}$

4. FIND AN EQUATION OF THE CIRCLE INTO WHICH THE CIRCLE WITH THE GIVEN EQUATION IS TRANSFORMED UNDER A ROTATION THROUGH THE INDICATED ANGLE, ABOUT THE ORIG…

   A $x^2 + y^2 = 1$, $\theta = \frac{\pi}{3}$ **B** $(x+1)^2 + (y-2)^2 = 3^2$, $\theta = \frac{\pi}{4}$

5. FIND THE IMAGE OF (1, 0) AFTER IT HAS BEEN ROTATED –60° ABOUT (3, 2).

6. IF M IS A REFLECTION IN THE LINE $y = -x$ AND R IS A ROTATION ABOUT THE ORIGIN THROUGH 90°, FIND

   A M(R(3, 0)) **B** R(M(3, 0))

7. IN A ROTATION R, THE IMAGE OF A(6, 5) IS A′(3, 2) AND THE IMAGE OF B(7, 3) IS B′(2, 6). FIND THE IMAGE OF (0, 0).

8. IN FIGURE 8.38, POINT B IS THE IMAGE OF POINT A IN A REFLECTION IN THE LINE ℓ_1 AND POINT C IS THE IMAGE OF POINT B in a reflection ABOUT THE LINE ℓ_2. PROVE THAT THERE IS A ROTATION ABOUT O THROUGH AN ANGLE 2θ THAT WILL MAP C TO A.

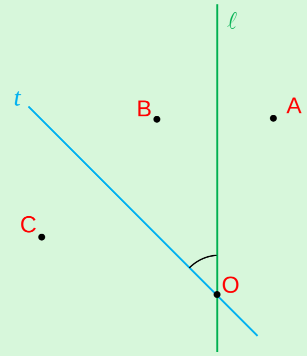


Figure 8.38

Key Terms

coordinate form of a vector
identity transformation
initial point
non-rigid motion
parallel vectors
perpendicular (orthogonal) vectors
reflection
resolution of vectors
rigid motion
rotation
scalar quantity
standard position
standard unit vector
terminal point
transformation
translation
unit vector
vector quantity
zero vector

Summary

1 *Vector*

I A QUANTITY WHICH CAN BE COMPLETELY DESCRIBED BY ITS MAGNITUDE EXPRESS SOME PARTICULAR UNIT IS CALLED A **scalar quantity**.

II A QUANTITY WHICH CAN BE COMPLETELY DESCRIBED BY STATING BOTH ITS MAGN EXPRESSED IN SOME PARTICULAR UNIT AND ITS DIRECTION IS CALLED A **vector quantity**.

III TWO VECTORS ARE SAID TO BE **equal**, IF THEY HAVE THE SAME MAGNITUDE AND DIRECTION.

IV A **zero vector** OR **null vector** IS A VECTOR WHOSE MAGNITUDE IS ZERO AND WHOSE DIRECTION IS INDETERMINATE.

V A **unit vector** IS A VECTOR WHOSE MAGNITUDE IS ONE.

2 ***Addition of vectors***

LET $\mathbf{u}$ AND $\mathbf{v}$ BE VECTORS, THEN THE SUM $\mathbf{u} + \mathbf{v}$ IS A VECTOR GIVEN BY THE PARALLELOGRAM LAW OR TRIANGLE LAW SATISFYING THE FOLLOWING PROPERTIES.

I VECTOR ADDITION IS COMMUTATIVE. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

II VECTOR ADDITION IS ASSOCIATIVE. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

III $\mathbf{u} + \mathbf{0} = \mathbf{u}$

IV $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

V $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$

3 ***Multiplication of a vector by a scalar***

LET $\mathbf{u}$ BE A VECTOR AND BE A SCALAR, THEN $\mathbf{u}$ IS A VECTOR SATISFYING THE FOLLOWING PROPERTIES.

I $| \ \mathbf{u}| = | \ | \, |\mathbf{u}|$

II IF IS A SCALAR, THEN $(\ + \ )\mathbf{u} = \ \mathbf{u} + \ \mathbf{u}$

III IF $\mathbf{v}$ IS A VECTOR, THEN $(\mathbf{u} + \mathbf{v}) = \ \mathbf{u} + \ \mathbf{v}$.

4 ***Scalar product or dot product***

THE DOT PRODUCT OF TWO VECTORS $\mathbf{u}$ AND $\mathbf{v}$, AND IS AN ANGLE BETWEEN THEM IS DEFINED AS: $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos$ SATISFYING THE FOLLOWING PROPERTIES.

I THE SCALAR PRODUCT OF VECTORS IS COMMUTATIVE. $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$

II IF $\mathbf{u} = \mathbf{0}$ OR $\mathbf{v} = \mathbf{0}$, THEN $\mathbf{u}.\mathbf{v} = 0$.

III TWO VECTORS $\mathbf{u}$ AND $\mathbf{v}$ ARE ORTHOGONAL IF $\mathbf{u}.\mathbf{v} = 0$.

5 ***Transformation of the plane***

I TRANSFORMATION CAN BE CLASSIFIED AS **rigid** MOTION AND NON-RIGID MOTION.

II **Rigid motion** IS A MOTION THAT PRESERVES DISTANCE. OTHERWISE IT IS NON-RIGID.

III **Identity transformation** IS A TRANSFORMATION THAT IMAGE OF EVERY POINT IS ITSELF.

6 ***Translation***

TRANSLATION IS A TRANSFORMATION IN WHICH EVERY POINT OF A FIGURE IS MOVED ALONG THE SAME DIRECTION THROUGH THE SAME DISTANCE.

I **Translation vector:** IF POINT P IS TRANSLATED TO P', THEN $\overrightarrow{PP'}$ IS SAID TO BE THE TRANSLATION VECTOR.

II IF $\mathbf{u} = (h, k)$ IS A TRANSLATION VECTOR, THEN $T(x, y) = (x + h, y + k)$

7 ***Reflection***

A REFLECTION M ABOUT A FIXED LINE L IS A TRANSFORMATION OF THE PLANE ONTO WHICH MAPS EACH POINT P OF THE PLANE INTO THE POINT P' OF THE PLANE SUCH THA THE PERPENDICULAR BISECTOR OF PP'.

- **I** REFLECTION IN THE x-AXIS, $M(x, y) = (x, -y)$
- **II** REFLECTION IN THE y-AXIS, $M(x, y) = (-x, y)$
- **III** REFLECTION IN THE LINE $y = x$, $M(x, y) = (y, x)$
- **IV** REFLECTION IN THE LINE $y = -x$, $M(x, y) = (-y, -x)$
- **V** REFLECTION IN THE LINE $y = mx$, $M(x, y) = (x', y')$

  $x' = x$ COS 2 + y SIN 2 $y' = x$ SIN 2 – y COS 2

  m = TAN

8 ***Rotation***

A ROTATION R ABOUT A POINT O THROUGH AN ANGLE IS A TRANSFORMATION OF THE PLANE ONTO ITSELF WHICH MAPS EVERY POINT P OF THE PLANE INTO THE POINT P' OF THE PLANE SUCH THAT $OP = OP'$ AND M($\angle POP'$) =

ROTATION FORMULAE

$x' = x$ COS – y SIN $y' = x$ SIN + y COS

Review Exercises on Unit 8

1 GIVEN VECTORS $\mathbf{u} = (2, 5)$, $\mathbf{v} = (-3, 3)$ AND $\mathbf{w} = (5, 3)$

A FIND $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$ AND $|\mathbf{u} - \mathbf{v} + 2\mathbf{w}|$ **B** FIND $2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$ AND $|2\mathbf{u} + 3\mathbf{v} - \mathbf{w}|$

C FIND THE UNIT VECTOR IN THE DIRECTION OF $\mathbf{u}$.

D FIND $\mathbf{z}$ IF $\mathbf{z} + \mathbf{u} = \mathbf{v} - \mathbf{w}$ **E** FIND $\mathbf{z}$ IF $\mathbf{u} + 2\mathbf{z} = 3\mathbf{v}$

2 TWO FORCES $\mathbf{F}_1$ AND $\mathbf{F}_2$ WITH $|\mathbf{F}_1| = 30$N AND $|\mathbf{F}_2| = 40$N ACT ON A POINT, IF THE ANGLE BETWEEN $\mathbf{F}_1$ AND $\mathbf{F}_2$ IS 30^o, THEN FIND THE MAGNITUDE OF THE RESULTANT FORCE.

3 A ROTATION R TAKES A (1, –3) TO A' (3, 5) AND B (0, 0) TO B' (4, –6). FIND THE CENTRE OF ROTATION.

4 IF $\mathbf{a}$ AND $\mathbf{b}$ ARE NON-ZERO VECTORS WITH $|\mathbf{a}| = |\mathbf{b}|$, SHOW THAT $\mathbf{a} + \mathbf{b}$ AND $\mathbf{a} - \mathbf{b}$ ARE ORTHOGONAL.

5 A PERSON PULLS A BODY 50 M ON A HORIZONTAL GROUND BY A ROPE INCLINED AT 30^o TO THE GROUND. FIND THE WORK DONE BY THE HORIZONTAL COMPONENT OF THE TENSION I ROPE, IF THE MAGNITUDE OF THE TENSION IS 10 N.

6 USING VECTOR METHODS, FIND THE EQUATION OF THE LINE TANGENT TO THE CIRCLE $x^2 + y^2 - x + y = 6$ AT

A A(1, –3) **B** B(1, 2).

7 IF A TRANSLATION T CARRIES THE POINT (7, –12) TO (9,–10), FIND THE IMAGES OF TH FOLLOWING LINES AND CIRCLES.

A $y = 2x - 5$ **B** $2y - 5x = 4$ **C** $x + y = 10$

D $x^2 + y^2 = 3$ **E** $x^2 + y^2 - 2x + 5y = 0$

8 IN A REFLECTION, THE IMAGE OF THE POINT P (3, 10) IS P' (7, 2). FIND THE EQUATION OF TH LINE OF REFLECTION.

9 IF THE PLANE IS ROTATED 30^O ABOUT (1, 4) FIND THE IMAGE OF

A THE POINT (–3, 2) **B** $x^2 + y^2 - 2x - 8y = 10$

C $x^2 + y^2 - 3y = 0$ **D** $y = x + 4$

10 PROVE THAT THE SUM OF ALL VECTORS FROM THE CENTRE OF A REGULAR POLYGON TO IS **0**.

11 USING A VECTOR METHOD, PROVE THAT AN ANGLE INSCRIBED IN A SEMI-CIRCLE MEASURE

12 FIND THE RESULTANT OF TWO VECTORS OF MAGNITUDES 6 UNITS AND 10 UNITS, IF TH BETWEEN THEM IS:

A 30^O **B** 120^O **C** 150^O

13 FOUR FORCES ACTING ON A PARTICLE ARE REPRESENTED BY **i** + 4**j**, 3**i** – 5**j**, 5**i** + 4**j** AND 2**i** + **j**. FIND THE RESULTANT FORCE **F**

14 A BALLOON IS RISING 4 METERS PER SECOND. IF A WIND IS BLOWING HORIZONTALLY SPEED OF 2.5 METER PER SECOND, FIND THE VELOCITY OF THE BALLOON RELATIVE GROUND.

15 THREE TOWNS A, B AND C ARE JOINED BY STRAIGHT RAILWAYS. TOWN B IS 600KM EA AND 1200KM NORTH OF TOWN A. TOWN C IS 800 KM EAST AND 900 KM SOUTH OF TOWN B. BY CONSIDERING TOWN A AS THE ORIGIN,

A FIND THE POSITION VECTORS OF B AND C USING THE UNIT VECTORS **i** AND **j**

B IF T IS A TRAIN STATION TWO THIRDS OF THE WAY ALONG THE RAIL WAY FORM TO TO TOWN B, PROVE THAT T IS THE CLOSEST STATION TO TOWN C ON THE RAIL WAY TOWN A TO TOWN B.

16 TWO VILLAGES A AND B ARE 2 KM AND 4 KM FAR AWAY FROM A STRAIGHT ROA RESPECTIVELY AS SHOWN IN FIGURE 8.39.

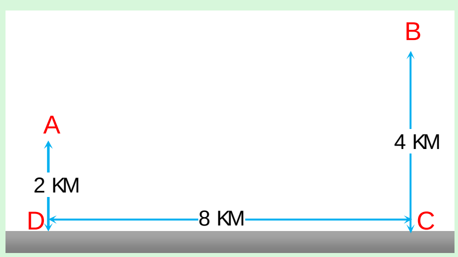


Figure 8.39

THE DISTANCE BETWEEN C AND D IS 8 KM. INDICATE THE POSITION OF A COMMON POW SUPPLIER THAT IS CLOSEST TO BOTH VILLAGES. DETERMINE THE SUM OF THE MIN DISTANCES FROM THE POWER SUPPLIER TO BOTH VILLAGES.