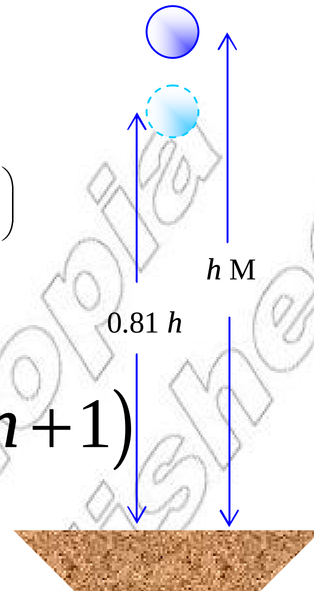
Unit 1

SEQUENCES AND SERIES

$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$

$$S_n = \frac{G_1(1-r^n)}{1-r}$$

$$\sum_{n=1}^{\infty} n(n+1)$$



Unit Outcomes:

After completing this unit, you should be able to:

- *revise the notions of sets and functions.*
- *grasp the concept of sequence and series.*
- *compute any terms of sequences from given rule.*
- *find out possible rules (formulas) from given terms.*
- *identify the types of sequences and series.*
- *compute partial and infinite sums of sequences.*
- *apply the knowledge of sequence and series to solve practical and real life problems.*

Main Contents

1.1 SEQUENCES

1.2 ARITHMETIC SEQUENCE AND GEOMETRIC SEQUENCE

1.3 THE SIGMA NOTATION AND PARTIAL SUMS

1.4 INFINITE SERIES

1.5 APPLICATIONS OF SEQUENCE AND SERIES

Key terms

Summary

Review Exercises

INTRODUCTION

MUCH OF THE MATHEMATICS WE ARE USING TODAY WAS DEVELOPED AS A RESULT OF M REAL WORLD SITUATIONS SUCH AS METEOROLOGY IN THE STUDY OF WEATHER PATTERNS, THE STUDY OF PATTERNS OF THE MOVEMENTS OF STARS AND GALAXIES AND NUMBER SEQ PATTERNS OF NUMBERS.

STUDYING ABOUT NUMBER SEQUENCES IS HELPFUL TO MAKE PREDICTIONS IN THE PATT NATURAL EVENTS.

FOR INSTANCE, FIBONACCI NUMBERS, A SERIES OF NUMBERS 1, 1, 2, 3, 5, 8, 13, 21, … WHERE EACH NUMBER IS THE SUM OF THE TWO PRECEDING NUMBERS, IS USED IN MODELLING TH RATES OF RABBITS.

IN SOME NUMBER SEQUENCE, IT IS POSSIBLE TO SEE THAT THE POSSIBILITY OF THE SU INFINITELY MANY NON-ZERO NUMBERS TO BE FINITE.

FOR EXAMPLE, IS IT POSSIBLE TO FIND THE FOLLOWING SUMS?

A $1+2+3+4+5+\cdots+n+\cdots$

B $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n}+\cdots$

C $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^{n-1}}+\cdots$

D $1+-1+1+-1+1+-1+\cdots+(-1)^{n-1}+\cdots$

THIS CONCEPT, WHICH MAY SEEM PARADOXICAL AT FIRST, PLAYS A CENTRAL ROLE IN SCI ENGINEERING AND HAS A VARIETY OF IMPORTANT APPLICATIONS.

ONE OF THE GOALS OF THIS UNIT IS TO EXAMINE THE THEORY AND APPLICATIONS OF INFIN WHICH WILL BE REFERRED TO AS INFINITE SERIES. WE WILL DEVELOP A METHOD WHICH M YOU TO DETERMINE WHETHER OR NOT SUCH AN INFINITE SERIES HAS A FINITE SUM.

OPENING PROBLEM

A FARMER HAS PLANTED CERTAIN TREES ON THE LAND IS IN THE FORM OF AN ISOSCELES TR WITH BASE 100 M AND HEIGHT 50 M. THE TREES UP IN DIFFERENT ROWS AS SHOWN IN FIGURE 1.1

IN EACH ROW, THE DISTANCE BETWEEN ANY TREES IS 5 M. THE DISTANCE BETWEEN ANY TW ROWS IS 5 M, TOO.

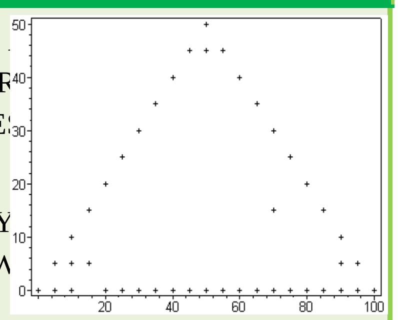


Figure 1.1

a HOW MANY ROWS OF TREES ARE THERE IN THE PIECE OF LAND?

b HOW LONG IS EACH ROW?

c HOW MANY TREES ARE THERE IN EACH ROW?

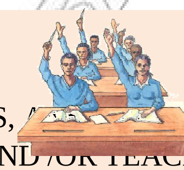
d WHAT IS THE TOTAL NUMBER OF TREES ON THE PIECE OF LAND?

TO SOLVE PROBLEMS LIKE THIS AND MANY OTHERS, A DETAILED STUDY OF SEQUENCES AND SUMS (CALLED *series*) IS REQUIRED.

1.1 SEQUENCES

1.1.1 Revision on Sets and Functions

ACTIVITY 1.1



FROM YOUR PREVIOUS KNOWLEDGE OF SETS AND FUNCTIONS, ANSWER THE FOLLOWING QUESTIONS, DISCUSSING EACH POINT WITH YOUR PARTNER AND/OR TEACHER.

1 DEFINE AND DISCUSS EACH OF THE FOLLOWING TERMS USING EXAMPLES.

a SET **b** FINITE SET **c** INFINITE SET

d EQUAL AND EQUIVALENT SETS **e** COUNTABLE SET

2 DEFINE A ONE-TO-ONE FUNCTION ON THE SET OF NATURAL NUMBERS ONTO EACH OF THE FOLLOWING SETS:

a THE SET OF WHOLE NUMBERS **b** THE SET OF INTEGERS

c THE SET OF EVEN INTEGERS

3 SHOW THAT THE SETS IN **2** **A**, **B** AND **C** ARE EQUIVALENT TO THE SET OF NATURAL NUMBERS.

4 LET S BE THE SET OF THE FIRST 10 PRIME NUMBERS. DEFINE THE FUNCTION

$f : S \to \mathbb{N}$ BY $f(n) = n^2 - 2$.

a WHAT IS THE DOMAIN OF f? **b** WHAT IS THE RANGE OF f?

c DRAW THE GRAPH OF f.

5 CONSIDER THE *Pascal's triangle*.

ROW 1				1			
ROW 2			1		1		
ROW 3		1		2		1	
ROW 4	1		3		3		1

LET $f(n)$ BE THE SUM OF THE NUMBERS IN THE n^{th} ROW.

FOR EXAMPLE, $f(1) = 1$, $f(2) = 1 + 1 = 2$, $f(3) = 1 + 2 + 1 = 4$.

a EVALUATE $f(10)$, $f(15)$ AND $f(n)$.

B PLOT THE POINTS WITH COORDINATES $(n, f(n))$ FOR n = 1, 2, 3, 4, 5, 6 ON A COORDINATE PLANE.

6 LET f BE A FUNCTION DEFINED ON ℕ BY $f(n) = \dfrac{n!}{2^n}$. EVALUATE

A $f(1)$ **B** $f(5)$ **C** $f(10)$

7 DEFINE q ON ℕ BY $q(n)$ = THE INTEGRAL FACTORS OF n.

PLOT THE POINTS WITH COORDINATES $(n, q(n))$ FOR n = 1, 2, 3, 4, 5, 6 ON A COORDINATE PLANE.

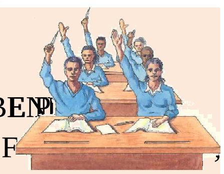
1.1.2 Number Sequences

THE WORD SEQUENCE, USED IN EVERYDAY CONVERSATION, USUALLY REFERS TO A LIST OCCURRING IN A SPECIFIC ORDER. IN MATHEMATICS, A SEQUENCE IS VIEWED AS A SET OF N ONE COMES AFTER ANOTHER IN A GIVEN RULE.

RECOGNIZING AND IDENTIFYING PATTERNS IS A FUNDAMENTAL IDEA THAT APPEARS IN SCIENCE AND MATHEMATICS. IT SERVES AS A FRUITFUL STARTING POINT FOR ANALYSING A WIDE V PROBLEMS.

SEQUENCES ARISE IN MANY DIFFERENT WAYS. FOR EXAMPLE, CONSIDER THE FOLLOWING AC AND TRY TO GET THE PATTERNS.

ACTIVITY 1.2



1 THE MONTHLY RENT OF A MACHINE, BIRR 200, IS TO BE PAID AT THE END OF EACH MONTH. IF IT IS NOT PAID AT THE END OF , THE AMOUNT DUE WILL INCREASE BIRR 3 PER DAY.

WHAT WILL BE THE AMOUNT TO BE PAID AFTER A DELAY OF

A 3 DAYS? **B** 10 DAYS? **C** n DAYS?

2 WHAT DO YOU UNDERSTAND ABOUT A SEQUENCE?

3 CONSIDER THE FUNCTION a GIVEN BY

$a(n) = 3n - 1$,

WHERE THE DOMAIN OF a IS THE SET OF NATURAL NUMBERS ℕ.

THEN $a(1) = 2$, $a(2) = 5$, $a(3) = 8$, …

THE FUNCTION a IS AN EXAMPLE OF A SEQUENCE.

INSTEAD OF THE STANDARD FUNCTION NOTATION, SEQUENCES ARE USUALLY DEFINED USIN NOTATION. THE VALUE $a(n)$ IS USUALLY SYMBOLIZED AS a_n.

THUS, WE HAVE $a_n = 3n - 1$.

THE ELEMENTS IN THE RANGE OF a_n ARE CALLED THE terms of the sequence; a_1 IS THE FIRST TERM, a_2 IS THE SECOND TERM, AND a_n IS the n^{TH} term, OR the general term of the sequence.

EVALUATE: a_{10}, a_{15} AND a_{25}

IN THIS SECTION, WE GIVE THE MATHEMATICAL DEFINITION OF A NUMBER SEQUENCE.

Definition 1.1

A SEQUENCE $\{a_n\}$ IS A FUNCTION WHOSE DOMAIN IS THE SET OF POSITIVE INTEGERS OR A SUBS
OF CONSECUTIVE POSITIVE INTEGERS STARTING WITH 1.

THE FUNCTIONAL VALUES $a_1, a_2, a_3, \ldots, a_n, \ldots$ ARE CALLED THE **terms of a sequence**, AND a_n IS CALLED THE **general term**, OR THE n^{th} TERM OF THE SEQUENCE.

WE USUALLY WRITE a_n INSTEAD OF THE FUNCTION $a(n)$ FOR THE VALUE OF THE FUNCTION AT THE NUMBER n, IF $n \in \mathbb{N}$.

NOTATION:

THE SEQUENCE $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ IS ALSO DENOTED BY $\{a_n\}$ OR $\{a_n\}_{n=1}^{\infty}$

SEQUENCES CAN BE DESCRIBED BY;

I LISTING THE TERMS.

II WRITING THE GENERAL TERM.

III DRAWING GRAPHS.

IV USING RECURRENCE RELATIONS.

Example 1

A BY ASSOCIATING EACH POSITIVE INTEGER n WITH ITS RECIPROCAL $\frac{1}{n}$, WE OBTAIN A SEQUENCE DENOTED BY $\left\{\frac{1}{n}\right\}$ WHICH REPRESENTS THE SEQUENCE OF NUMBERS

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$$

THE GENERAL TERM IS $a_n = \frac{1}{n}$.

B GIVEN THE GENERAL TERM $a_n = \left(\frac{1}{2}\right)^{n-1}$, WE OBTAIN

$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{4},\ a_4 = \frac{1}{8}$ AND SO ON. LIST UP TO THE 10TH TERM.

C GIVEN CERTAIN TERMS OF A SEQUENCE, SAY 2, 4, 6, 8, ..., WHICH ONE OF THE FOLLOWING IS THE POSSIBLE GENERAL TERM?

$a_n = 2n$ OR $a_n = (n-1)(n-2)(n-3)(n-4) + 2n$ FOR n A POSITIVE INTEGER.

BOTH GENERAL TERMS HAVE THE SAME FIRST FOUR TERMS; BUT THEY DIFFER BY FI
TRY TO FIND THE FIFTH AND SIXTH TERMS FOR BOTH GENERAL TERMS.

Example 2 WRITE A FORMULA FOR THE GENERAL TERM OF EACH OF THE SEQUENCES.

A 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .

B $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

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C 2, 0, 2, 0, 2, 0, . . . **D** 2, 7, 24, 77, . . . **E** 1, 3, 6, 10, …

Solution TO WRITE A FORMULA FOR THE n^{th} TERM OF A SEQUENCE, EXAMINE THE TERMS AND LOOK FOR THE PATTERN. EACH OF YOU MAY COME WITH SEVERAL FORMULAE FOLLOWING ARE FEW OF THEM.

A $a_n = n^2$ **B** $a_n = \left(-\frac{1}{2}\right)^n$ **C** $a_n = 1-(-1)^n$

D $a_n = 3^n - n$ **E** $a_n = \frac{n}{2}(n+1)$

SOME SEQUENCES ARE DEFINED BY GIVING A FORMULA FOR THE n^{th} TERM:

Example 3

A FOR THE SEQUENCE $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$, THE GENERAL TERM IS $a_n = \frac{n}{n+1}$ AND TERMS OF THE SEQUENCE CAN BE GIVEN AS $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

B FOR THE SEQUENCE $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$, THE GENERAL TERM IS $a_n = \frac{(-1)^n}{n}$ AND TERMS OF THE SEQUENCE CAN BE GIVEN AS $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^n}{n}, \ldots$

GRAPHICALLY, THIS SEQUENCE IS DESCRIBED IN THE FIGURE BELOW.

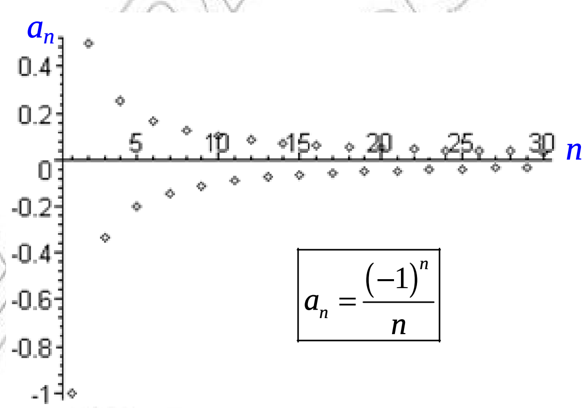

Figure 1.2

C FOR THE SEQUENCE $\left\{\text{SIN}\left(\frac{n}{3}\right)\right\}_{n=1}^{\infty}$ THE GENERAL TERM IS $a_n = \text{SIN}\left(\frac{n}{3}\right)$ AND TERMS OF THE SEQUENCE CAN BE GIVEN AS $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \ldots, \text{SIN}\left(\frac{n}{3}\right), \ldots$

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✍Note:

✓ A SEQUENCE THAT HAS A LAST TERM IS CALLED A **finite sequence**. A SEQUENCE THAT DOES NOT HAVE A LAST TERM IS CALLED AN **infinite sequence**. THE DOMAIN OF A FINITE SEQUENCE IS $\{1, 2, 3, ..., n\}$. THE DOMAIN OF AN INFINITE SEQUENCE IS

FOR INSTANCE, 1, 2, 3, . . ., 10 IS A FINITE SEQUENCE AND 1, 2, 3, . . . IS AN INFINITE SEQUENCE.

SOME SEQUENCES DO NOT HAVE A SIMPLE DEFINING FORMULA.

Example 4

A THE SEQUENCE $\{p_n\}$, WHERE p_n IS THE POPULATION OF ETHIOPIA AS OF MESKEREM 1 IN THE YEAR n.

B IF WE LET a_n TO BE THE DIGIT IN THE n^{th} DECIMAL PLACE OF THE NUMBER $\sqrt{2}$, THEN $\{a_n\}$ IS A WELL DEFINED SEQUENCE WHOSE FIRST FEW TERMS ARE 4, 1, 4, 2, 1, 3, 5, ...

Recursion Formula

A FORMULA THAT RELATES THE GENERAL TERM a_n OF A SEQUENCE TO ONE OR MORE OF THE TERMS THAT COME BEFORE IT IS CALLED A **recursion formula**. A SEQUENCE THAT IS SPECIFIED BY GIVING THE FIRST FEW TERMS TOGETHER WITH A RECURSION FORMULA IS SAID TO BE DEFINED RECURSI

Example 5 FIND THE FIRST SIX TERMS OF THE SEQUENCE $\{a_n\}$ DEFINED RECURSIVELY BY $a_1 = 2$ AND $a_n = \dfrac{a_{n-1}}{n}$ FOR $n \geq 2$.

Solution

$$a_1 = 2,\quad a_2 = \frac{a_1}{2} = \frac{2}{2} = 1,\quad a_3 = \frac{a_2}{3} = \frac{1}{3},\quad a_4 = \frac{a_3}{4} = \frac{\frac{1}{3}}{4} = \frac{1}{12}$$

$$a_5 = \frac{a_4}{5} = \frac{\frac{1}{12}}{5} = \frac{1}{60},\quad a_6 = \frac{a_5}{6} = \frac{\frac{1}{60}}{6} = \frac{1}{360}$$

THUS, THE FIRST SIX TERMS OF THE SEQUENCE $\{a_n\}$ ARE 2, 1, $\frac{1}{3}, \frac{1}{12}, \frac{1}{60}, \frac{1}{360}$.

✍Note:

✓ THE VALUES OF RECURSIVELY DEFINED FUNCTIONS ARE CALCULATED BY THE REPEATED APPLICATION OF THE FUNCTION TO ITS OWN VALUES.

Example 6 THE FIBONACCI SEQUENCE $\{f_n\}$ IS DEFINED RECURSIVELY BY THE CONDITIONS

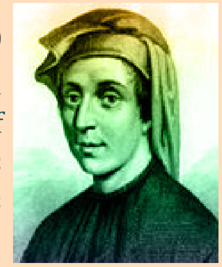
$f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$ FOR $n \geq 3$

Solution EACH TERM IS THE SUM OF THE TWO PRECEDING TERMS.

THE SEQUENCE DESCRIBED BY ITS FIRST FEW TERMS IS

{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . .}.

HISTORICAL NOTE



Leonardo Fibonacci (circa 1170, 1240)

Italian mathematician *Leonardo Fibonacci* made advances in number theory and algebra. Fibonacci, also called Leonardo of Pisa, produced numbers that have many interesting properties such as the birth rates of rabbits and the spiral growth of leaves on some trees.

He is especially known for his work on series of numbers, including the Fibonacci series. Each number in a Fibonacci series is equal to the sum of the two numbers that came before it. Fibonacci sequence arose when he was trying to solve a problem of the following kind concerning the breeding of rabbits.

"*Suppose that rabbits live forever and that every month each pair produces a new pair which become productive at the age of two months. If we start with one new born pair, how many pairs of rabbits will we have in the n^{th} month?*"

VERIFY THAT THE ANSWER TO THE ABOVE QUESTION IS THE FIBONACCI SEQUENCE DISCUSSED IN EXAMPLE 6 ABOVE.

Exercise 1.1

1 LIST THE FIRST FIVE TERMS OF EACH OF THE SEQUENCES WHOSE GENERAL TERMS ARE GIVEN BELOW, WHERE n IS A POSITIVE INTEGER.

A $a_n = 1-(0.2)^n$

B $a_n = \dfrac{n+1}{3n-1}$

C $a_n = \dfrac{3(-1)^n}{n}$

D $a_n = \cos\left(\dfrac{n}{2}\right)$

E $a_1 = 1,\quad a_{n+1} = \dfrac{1}{1+a_n}$

F $a_n = 2^n - 3n + 1$

G $a_n = (-1)^n + 1$

H $a_n = \dfrac{n^n}{n!}$

I p_n = THE n^{TH} PRIME NUMBER.

J q_n = THE SUM OF THE FIRST N NATURAL NUMBERS

K $a_1 = -1,\ a_2 = 2,\ a_{n+2} = na_1 + (n+1)a_2,\ n \geq 1$

L $a_1 = 1,\ a_{n+1} = \dfrac{1}{1+a_n^{\,2}}$ *for* $n \geq 1.$

2 FIND A FORMULA FOR THE GENERAL TERM a_n OF EACH OF THE FOLLOWING SEQUENCES, ASSUMING THAT THE PATTERN OF THE FIRST FEW GIVEN TERMS CONTINUES.

A $\{3, 6, 9, 12, 15, \dots\}$ **B** $\{2, 7, 12, 17, \dots\}$

C $\{0, 2, 0, 2, 0, 2, \dots\}$ **D** $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\}$

E $\{2, 3, 5, 8, 13, 21, \dots\}$ **F** $\{1, -1, 1, -1, \dots\}$

G $\{0, 1, 3, 7, 15, \dots\}$ **H** $\left\{1, 2, \frac{3}{2}, \frac{2}{3}, \frac{5}{24}, \dots\right\}$

I $\{0.2, 0.22, 0.222, 0.2222, \dots\}$ **J** $\left\{\frac{1}{2}, \frac{2}{3}, 1, \frac{8}{5}, \frac{8}{3}, \frac{32}{7}, 8, \dots\right\}$

1.2 ARITHMETIC SEQUENCES AND GEOMETRIC SEQUENCES

1.2.1 Arithmetic Sequences

OPENING PROBLEM

100 STUDENTS REGISTERED TO TAKE AN EXAM WERE GIVEN CARDS WITH NUMBERS RANGIN TO 100. THERE WERE FOUR EXAM ROOMS R_1, R_2, R_3 AND R_4.

STUDENTS WITH CARD NUMBERS

A 1, 4, 7, 10, . . . MUST BE IN R_1 **B** 2, 5, 8, 11, . . . IN R_2;

C 3, 9, 15, . . . IN R_3; **D** 6, 12, 18, 24, . . . IN R_4.

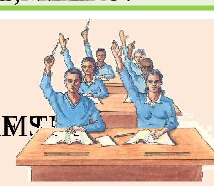
THE NUMBERS IN EACH ROOM CONTINUE WITH A CONSTANT DIFFERENCE.

1 FIND THE TOTAL NUMBER OF STUDENTS IN EACH ROOM.

2 ARE THERE ANY STUDENTS ASSIGNED TO DIFFERENT ROOMS SIMULTANEOUSLY? IF SO, WHICH CARD NUMBERS?

3 ARE THERE STUDENTS WHO ARE NOT ASSIGNED? IF SO, WHICH CARD NUMBERS?

ACTIVITY 1.3



1 FIND THE DIFFERENCE BETWEEN CONSECUTIVE TERMS FOR EACH OF THE SEQUENCES GIVEN BELOW.

A 1, 3, 5, 7, 9, … **B** 10, 15, 20, 25, …

C 10, 20, 30, 40, … **D** $1, -9, -19, -29, \dots$

2 CAN YOU SEE WAYS OF OBTAINING ANY TERM IN THE SEQUENCE, GIVEN THE FIRST TERM AND THE DIFFERENCE BETWEEN CONSECUTIVE TERMS?

NOW, ONE CAN OBSERVE FROM THE ACTIVITY 1.3 THAT THE DIFFERENCE BETWEEN EACH PAIR OF CONSECUTIVE TERMS IS A CONSTANT.

Definition 1.2

AN arithmetic sequence (OR arithmetic progression) IS ONE IN WHICH THE DIFFERENCE BETWEEN CONSECUTIVE TERMS IS A CONSTANT. THIS CONSTANT IS CALLED THE common difference. I.E., $\{A_n\}$ IS AN ARITHMETIC SEQUENCE WITH COMMON DIFFERENCE d, IF AND ONLY IF $A_{n+1} - A_n = d$ FOR ALL n.

FROM DEFINITION 1.2, WE OBSERVE THAT, IF $A_1, A_2, A_3, \ldots A_n, \ldots$ IS AN ARITHMETIC PROGRESSION, THEN, $A_2 - A_1 = A_3 - A_2 = A_4 - A_3 = \ldots = A_{n+1} - A_n = \ \ldots = d$.

EQUIVALENTLY, $A_2 = A_1 + d$, $A_3 = A_2 + d$, $A_4 = A_3 + d$, ..., $A_{n+1} = A_n + d$, ...

HENCE, $A_2 = A_1 + d$, $A_3 = A_1 + 2d$, $A_4 = A_1 + 3d$, ..., $A_{n+1} = A_1 + nd$, ...

THUS, WE HAVE PROVED THE FOLLOWING THEOREM FOR THE GENERAL TERM A_n.

Theorem 1.1

IF $\{A_n\}$ IS AN ARITHMETIC PROGRESSION WITH THE FIRST TERM A_1 AND COMMON DIFFERENCE d, THEN THE n^{th} TERM IS GIVEN BY

$$A_n = A_1 + (n-1)d.$$

Example 1 GIVEN AN ARITHMETIC SEQUENCE WITH FIRST TERM 5 AND COMMON DIFFERENCE 4, FIND THE FIRST FIVE TERMS AND THE TWENTIETH TERM.

Solution THE FIRST TERM OF THE ARITHMETIC SEQUENCE IS $A_1 = 5$, HENCE

$A_2 = A_1 + d = 5 + 4 = 9$ $\qquad$ $A_3 = A_1 + 2d = 5 + 2\times 4 = 13$

$A_4 = A_1 + 3d = 5 + 3\times 4 = 17$ $\qquad$ $A_5 = A_1 + 4d = 5 + 4\times 4 = 21$

THUS, THE FIRST FIVE TERMS ARE 5, 9, 13, 17, AND 21.

TO FIND THE TWENTIETH TERM, WE CAN USE THE FORMULA $A_n = A_1 + (n-1)d$

$$A_{20} = A_1 + 19d = 5 + 19\times 4 = 81.$$

Example 2 GIVEN AN ARITHMETIC SEQUENCE WHOSE FIRST TWO TERMS ARE −3 AND 7, FIND THE NEXT THREE TERMS AND THE FOURTEENTH TERM.

Solution SINCE THE FIRST TWO TERMS OF THE SEQUENCE ARE −3 AND 7, WE HAVE $A_1 = -3$ AND $A_2 = 7$. BECAUSE THE SEQUENCE IS ARITHMETIC

$d = A_2 - A_1 = 7 - (-3) = 10$

SINCE $A_n = A_1 + (n-1)d$, $\qquad$ $A_3 = A_1 + 2d = -3 + 2\times 10 = 17$

$A_4 = A_1 + 3d = -3 + 3\times 10 = 27$ $\qquad$ $A_5 = A_1 + 4d = -3 + 4\times 10 = 37$

THEREFORE, THE THREE TERMS FOLLOWING -3 AND 7 ARE 17, 27 AND 37.

THE FOURTEENTH TERM CAN BE FOUND BY USING THE FORMULA $A_n = A_1 + (n-1)d$

$$A_{14} = -3 + (14-1)10 = 127$$

Example 3 SHOW THAT THE SEQUENCE $\{2n-3\}$ IS AN ARITHMETIC SEQUENCE. DESCRIBE THE SEQUENCE GRAPHICALLY.

Solution LET $A_n = 2n-3 \Rightarrow A_{n+1} = 2(n+1)-3 = 2n-1$

$\Rightarrow A_{n+1} - A_n = (2n-1)-(2n-3) = 2,$ A CONSTANT FOR ALL NATURAL NUMBERS

THUS $\{2n-3\}$ IS AN ARITHMETIC SEQUENCE.

IF WE PLOT THE SET OF POINTS WHOSE COORDINATES ARE $\{(n, 2n-3)\}$, WE GET THE GRAPH OF THE SEQUENCE.

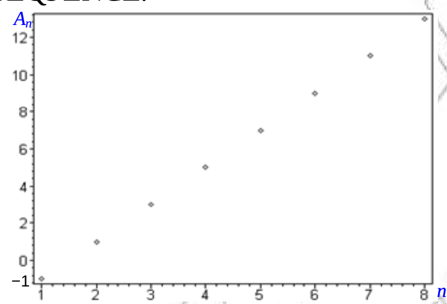


Figure 1.3

OBSERVE THAT THE GRAPH FOLLOWS THE PATTERN OF A LINEAR FUNCTION

Example 4 A MAN BOUGHT A MOTOR CAR FOR BIRR 80,000. IF THE VALUE OF THE CAR DEPRECIATES AT THE RATE OF BIRR 7000 PER YEAR, WHAT IS ITS VALUE AT TH OF THE 9TH YEAR?

Solution THE PRESENT VALUE OF THE CAR IS BIRR 80,000. THE RATE AT WHICH THE VALUE DEPRECIATES YEARLY IS 7,000. THUS, THE VALUE AT THE END OF THE FIRST YEA

BIRR 80,000 – BIRR 7,000 = BIRR 73,000.

THE VALUE AT THE END OF THE SECOND YEAR IS BIRR 73,000 – BIRR 7,000 = BIRR 66,000 AND AT THE END OF THE THIRD YEAR IT IS BIRR 66,000 – BIRR 7,000 = BIRR 59,000.

THUS, THE VALUES AT THE END OF CONSECUTIVE YEARS FORM AN ARITHMETIC SEQUEN

73,000, 66,000, 59,000,…

WITH $A_1 = 73{,}000$ AND $d = -7000$

$\Rightarrow A_n = 73{,}000 - 7{,}000\,(n-1)$

$\Rightarrow A_9 = 73{,}000 - 7{,}000 \times 8$

$= 17{,}000$

THEREFORE, THE VALUE OF THE MOTOR CAR AT THE END OF THE 9TH YEAR IS BIRR 17,000.00.

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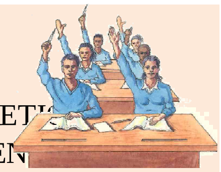
ACTIVITY 1.4



1 IF a, c, AND b; ARE THREE CONSECUTIVE TERMS OF AN ARITHMETIC SEQUENCE, THEN c IS CALLED THE ARITHMETIC MEAN BETWEEN a AND b.

- **A** EXPRESS c IN TERMS OF a AND b.
- **B** FIND THE ARITHMETIC MEAN BETWEEN 10 AND 15.

2 IF $\{a, m_1, m_2, m_3, \ldots, m_k, b\}$ IS AN ARITHMETIC SEQUENCE, THEN WE SAY THAT $m_1, m_2, m_3, \ldots, m_k$ ARE k-ARITHMETIC MEANS BETWEEN a AND b.
INSERT 5 ARITHMETIC MEANS BETWEEN 4 AND 13.

Exercise 1.2

1 DETERMINE WHETHER THE SEQUENCES WITH THE FOLLOWING nTH TERMS ARE ARITHMETIC.

- **A** $a_n = 4n - 7$ **B** $a_n = 4n$ **C** $a_n = 5n + 3$
- **D** $a_n = n^2 - n$ **E** $a_n = 5$ **F** $a_n = \dfrac{7 - 4n}{3}$

2 CONSIDER THE SEQUENCE 97, 93, 89, 85, …

- **A** SHOW THAT THE SEQUENCE CAN BE CONTINUED ARITHMETICALLY.
- **B** FIND A FORMULA FOR THE GENERAL TERM. **C** IS 60 A TERM IN THE SEQUENCE?

3 THE n^{TH} TERM OF A SEQUENCE IS GIVEN BY $3n - 7$.

- **A** SHOW THAT THE SEQUENCE IS ARITHMETIC. **B** FIND THE 75TH TERM.
- **C** WHAT IS THE LEAST TERM OF THE SEQUENCE GREATER THAN 528?

4 GIVEN AN ARITHMETIC SEQUENCE WITH $a_3 = 12$ AND $a_9 = 14$, FIND a_1 AND a_{30}.

5 GIVEN AN ARITHMETIC SEQUENCE WITH $a_4 = 18$ AND $a_8 = 10$, FIND a_1 AND THE COMMON DIFFERENCE d.

6 GIVEN AN ARITHMETIC SEQUENCE WITH $a_4 =$ AND D = 5 6, FIND a_1 AND a_9.

7 GIVEN AN ARITHMETIC SEQUENCE WITH $a_{61} = 102$ AND $d = -\dfrac{5}{3}$, FIND a_1 AND a_{30}.

8 IN AN ARITHMETIC SEQUENCE, IF p^{TH} TERM IS q AND THE q^{TH} TERM IS p, FIND THE $(p + q)^{TH}$ TERM.

9 FIND THE TOTAL NUMBER OF WHOLE NUMBERS THAT ARE LESS THAN 1000 AND DIVISIBLE BY 7.

10 IF n-ARITHMETIC MEANS ARE INSERTED BETWEEN a AND b, EXPRESS THE COMMON DIFFERENCE IN TERMS OF n, a AND b.

11 A MAN ACCEPTS A POSITION WITH AN INITIAL SALARY OF BIRR 18000.00 PER YEAR. IF IT IS KNOWN THAT HIS SALARY WILL INCREASE AT THE END OF EVERY YEAR BY BIRR 1500.00, WHAT WILL BE HIS ANNUAL SALARY AT THE BEGINNING OF THE 11TH YEAR?

1.2.2 Geometric Sequences

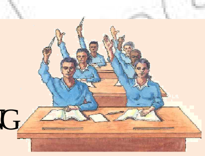
OPENING PROBLEM

THE POPULATION OF ETHIOPIA IN 2001 EC WAS APPROXIMATELY 75 MILLION. IF THE POPULATION IS INCREASING AT A RATE OF 2% EACH YEAR,

A WHAT WILL THE ETHIOPIAN POPULATION BE IN 2020?

B WHAT IS THE DOUBLING PERIOD OF THE POPULATION?

ACTIVITY 1.5



1 FIND THE RATIO BETWEEN CONSECUTIVE TERMS IN EACH OF THE FOLLOWING SEQUENCES.

A $2, 4, 8, 16, 32, \ldots$ **B** $100, 10, 1, \frac{1}{10}, \frac{1}{100}, \ldots$

C $-81, 27, -9, 3, -1, \frac{1}{3}, \ldots$ **D** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{32}, \ldots$

2 CAN YOU SEE HOW TO OBTAIN ANY TERM IN SUCH SEQUENCES, GIVEN THE FIRST TERM AND THE RATIO?

FROM ACTIVITY 1.5, YOU SHOULD HAVE SEEN THAT THE RATIO BETWEEN CONSECUTIVE TERMS IS A CONSTANT.

Definition 1.3

A geometric sequence (OR geometric progression) IS ONE IN WHICH THE RATIO BETWEEN CONSECUTIVE TERMS IS A NON ZERO CONSTANT. THIS CONSTANT IS CALLED THE common ratio. I.E., $\{G_n\}$ IS A GEOMETRIC SEQUENCE, IF AND ONLY IF

$G_{n+1} = rG_n$; $r \in \mathbb{R} \setminus \{0\}$. THE COMMON RATIO, $r = \frac{G_{n+1}}{G_n}$ FOR ALL n.

FROM THE DEFINITION, WE OBSERVE THAT, IF $G_1, G_2, G_3, \ldots G_n, \ldots$ IS A GEOMETRIC PROGRESSION, THEN

$$\frac{G_2}{G_1} = \frac{G_3}{G_2} = \frac{G_4}{G_3} = \frac{G_5}{G_4} = \ldots = \frac{G_n}{G_{n-1}} = \ldots = r$$

EQUIVALENTLY, $G_2 = rG_1$, $G_3 = rG_2$, $G_4 = rG_3$, $\ldots$, $G_{n+1} = rG_n$, $\ldots$

HENCE, $G_2 = rG_1$, $G_3 = r^2G_1$, $G_4 = r^3G_1$, $\ldots$, $G_{n+1} = r^nG_1$, $\ldots$

THUS, WE HAVE PROVED THE FOLLOWING THEOREM FOR THE GENERAL TERM G_n.

Theorem 1.2

IF $\{G_n\}$ IS A GEOMETRIC PROGRESSION WITH THE FIRST TERM G_1 AND COMMON RATIO r, THEN THE n^{th} TERM IS GIVEN BY $G_n = r^{n-1}G_1$

Example 5 GIVEN THE GEOMETRIC PROGRESSION 3, 6, 12, 24, …, FIND THE NEXT THREE TERMS AND THE SIXTEENTH TERM.

Solution SINCE WE ARE GIVEN A GEOMETRIC SEQUENCE, WE FIRST FIND THE COMMON RATIO, r, WHICH IS $\frac{6}{3} = 2$. NOTE THAT WE CAN USE ANY TWO CONSECUTIVE TERMS TO FIND r.

THEREFORE, THE TERM FOLLOWING 24 IS $2 \times 24 = 48$,

THE TERM FOLLOWING 48 IS $2 \times 48 = 96$

AND THE TERM FOLLOWING 96 IS $2 \times 96 = 192$.

THE SIXTEENTH TERM IS FOUND USING THE FORMULA

$$G_n = r^{n-1}G_1$$

$$G_{16} = 2^{16-1} \times 3 = 2^{15} \times 3 = 98{,}304$$

Example 6 FIND THE SEVENTH TERM OF A GEOMETRIC SEQUENCE WHOSE FIRST TERM IS 6 AND WHOSE FOURTH TERM IS $\frac{1}{36}$.

Solution FIRST, YOU NEED TO FIND THE COMMON RATIO r BY USING THE FORMULA

$$G_n = r^{n-1}G_1$$

$$G_4 = r^{4-1}G_1 \text{ WHERE } G_1 = 6, \text{ AND } G_4 = \frac{1}{36}$$

$$\frac{1}{36} = r^3 \times 6 \Rightarrow r^3 = \frac{1}{216}$$

THIS GIVES YOU $r = \frac{1}{6}$.

THUS, $G_7 = \left(\frac{1}{6}\right)^6 (6) = \frac{1}{7{,}776}$.

Example 7 A MACHINE DEPRECIATES BY $\left(\frac{1}{20}\right)^{TH}$ OF ITS PREVIOUS VALUE EVERY YEAR. IF ITS ORIGINAL COST IS BIRR 100,000.00, FIND THE VALUE OF THE MACHINE AT THE END OF THE 6^{TH} YEAR.

Solution THE VALUE OF THE MACHINE AT THE END OF THE FIRST YEAR IS

$$= \text{BIRR } 100{,}000.00 - \text{BIRR } \frac{100{,}000.00}{20} = \text{BIRR } 100{,}000\left(1 - \frac{1}{20}\right).$$

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THE VALUE AT THE END OF THE SECOND YEAR

$$= \text{BIRR } 100{,}000\left(1-\frac{1}{20}\right) - \text{BIRR } \frac{100{,}000}{20}\left(1-\frac{1}{20}\right)$$

$$= \text{BIRR } 100{,}000\left(1-\frac{1}{20}\right)^2.$$

SIMILARLY, THE VALUE AT THE END OF THE 3RD YEAR

$$= \text{BIRR } 100{,}000\left(1-\frac{1}{20}\right)^3.$$

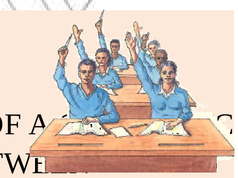
THE VALUES OF THE MACHINE AT THE END OF EVERY YEAR FORM A GEOMETRIC SE

$$100{,}000\left(1-\frac{1}{20}\right),\ 100{,}000\left(1-\frac{1}{20}\right)^2,\ 100{,}000\left(1-\frac{1}{20}\right)^3, \ldots$$

THUS, THE VALUE AT THE END OF THE 6TH YEAR

$$= \text{BIRR } 100{,}000\left(1-\frac{1}{20}\right)^6 = \text{BIRR } 73509.18906$$

ACTIVITY 1.6



1 IF a, c, b ARE THREE POSITIVE CONSECUTIVE TERMS OF A GEOMETRIC SEQUENCE, THEN c IS CALLED THE GEOMETRIC MEAN BETWEEN a AND b.

A FIND AN EXPRESSION FOR c IN TERMS OF a AND b.

B FIND THE GEOMETRIC MEAN BETWEEN 4 AND 8.

2 IF $\{a, m_1, m_2, m_3, \ldots, m_k, b\}$ IS A GEOMETRIC SEQUENCE, THEN WE SAY THAT $m_1, m_2, m_3, \ldots, m_k$ ARE k-GEOMETRIC MEANS BETWEEN a AND b.

INSERT THREE GEOMETRIC MEANS BETWEEN 0.4 AND 5.

Exercise 1.3

1 DECIDE WHETHER OR NOT EACH OF THE FOLLOWING SEQUENCES IS GEOMETRIC. FOR THOSE THAT ARE GEOMETRIC, DETERMINE THE n^{TH} TERM.

A $1, -2, 4, -8, \ldots$

B $5, 5, 5, 5, 5, \ldots$

C $-2, 0, 2, 4, \ldots$

D $9, -3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots$

E $0.9, 0.99, 0.999, 0.9999, \ldots$

F $2x,\ 4x^2,\ 8x^3,\ 16x^4, \ldots;\ x \neq 0.$

G $5+\sqrt{5},\ 1+\sqrt{5},\ \frac{5+\sqrt{5}}{5},\ \frac{1+\sqrt{5}}{5}, \ldots$

2 GIVEN THE GEOMETRIC SEQUENCE 5, 15, 45, ..., FIND THE NEXT THREE TERMS AND THE TENTH TERM.

3 FIND THE EIGHTH TERM OF THE GEOMETRIC SEQUENCE WHOSE FIRST TERM IS 5 AND WHOSE FOURTH TERM IS $\frac{1}{25}$.

4 FIND THE FIFTH TERM OF THE GEOMETRIC SEQUENCE WHOSE FIRST TERM IS 1 AND WHOSE FOURTH TERM IS 343.

5 IF x, $4x + 3$ AND $7x + 6$ ARE CONSECUTIVE TERMS OF A GEOMETRIC SEQUENCE, FIND THE VALUE(S) OF x.

Puzzle

A building company organizes a society to invest money starting from the first day of a month. If the society invests 1 cent for the first day, 2 cents for the second day, 4 cents for the third day and so on, with everyday investment being twice that of the previous day, how much will they invest on the 30th day of the month? Calculate the total amount invested in the entire month.

Exercise 1.4

1 DETERMINE WHETHER THE GIVEN SEQUENCE IS ARITHMETIC, GEOMETRIC OR NEITHER.

A 4, 7, 10, 13, … **B** 2, 6, 10, 14, 20, 26, … **C** $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{14}, \ldots$

D 1, 4, 9, 16, … **E** 2, -4, 8, -16, … **F** $\frac{4}{3}$, 8, 48, …

G $a_n = 5 - 2n$, WHERE n IS A POSITIVE INTEGER.

H $a_n = \frac{1}{n}$, WHERE n IS A POSITIVE INTEGER.

I $a_n = \frac{1}{n^2}$, WHERE n IS A POSITIVE INTEGER.

J $a_n = \frac{4^n}{7^{n+2}}$, WHERE n IS A POSITIVE INTEGER.

2 USE THE GIVEN INFORMATION ABOUT AN ARITHMETIC SEQUENCE TO FIND THE COMMON DIFFERENCE d AND THE GENERAL TERM A_n.

A $A_1 = 3$ AND $A_5 = 23$ **B** $A_6 = -8$ AND $A_{11} = 53$

C $A_4 = 8$ AND $A_8 = 10$

3 USE THE GIVEN INFORMATION ABOUT A GEOMETRIC SEQUENCE TO FIND THE INDICATED VALUES.

A $G_1 = 10$ AND $r = 2$, FIND G_4. **B** $G_1 = 4$ AND $r = -3$, FIND G_6.

C $G_3 = 1$ AND $G_6 = 216$, FIND G_1 AND r.

D $G_2 = \frac{1}{\sqrt{3}}$, $G_5 = -\frac{1}{9}$, FIND r, G_8 AND THE GENERAL TERM G_n.

4 FOR ANY PAIR OF NON-NEGATIVE INTEGERS a AND b, SHOW THAT THE ARITHMETIC MEAN BETWEEN a AND b IS GREATER THAN OR EQUAL TO THE GEOMETRIC MEAN BETWEEN THEM.

5 FIND THE FIRST TERM OF THE SEQUENCE 4, 12, 36, 108, ... WHICH EXCEEDS 20,000.

6 FIND THE FIRST TERM OF THE SEQUENCE 10, 5, 2.5, 1.25, ... WHICH IS LESS THAN 0.0001.

7 INSERT FOUR ARITHMETIC AND FIVE GEOMETRIC MEANS BETWEEN 2 AND 20.

8 IF x, 4, y ARE IN GEOMETRIC PROGRESSION AND x, 5, y ARE IN ARITHMETIC PROGRESSION, DETERMINE THE VALUE(S) OF x AND y.

9 IF $\{g_n\}$ IS A GEOMETRIC SEQUENCE WITH $g_n > 0$ FOR ALL $n \in \mathbb{N}$, THEN PROVE THAT $\{\ln g_n\}$ IS AN ARITHMETIC SEQUENCE.

1.3 THE SIGMA NOTATION AND PARTIAL SUMS

OPENING PROBLEM

AS WE KNOW, EACH OF US HAS PARENTS, GRANDPARENTS, GREAT GRANDPARENTS, GREA GRANDPARENTS AND SO ON. WHAT IS THE TOTAL NUMBER OF SUCH RELATIVES YOU HAVE PARENTS TO YOUR TENTH GRANDPARENTS?

IN THE PREVIOUS SECTION, YOU WERE INTERESTED IN THE INDIVIDUAL TERMS OF A SEQUENC SECTION, YOU DESCRIBE THE PROCESS OF ADDING THE TERMS OF A SEQUENCE. I.E., GIVEN A S $\{a_n\}$, YOU ARE INTERESTED IN FINDING THE SUM OF THE FIRST n TERMS. THIS IS CALLED THE partial sum, DENOTED BY S_n. THUS IF $a_1, a_2, a_3, ..., a_n, ...$ ARE THE TERMS OF THE SEQUENCE, THEN YOU PUT;

$S_1 = a_1$, S_1 IS THE FIRST TERM OF THE SEQUENCE.

$S_2 = a_1 + a_2$, S_2 IS THE SUM OF THE FIRST TWO TERMS OF THE SEQUENCE.

$S_3 = a_1 + a_2 + a_3$, S_3 IS THE SUM OF THE FIRST THREE TERMS OF THE SEQUENCE.

$S_4 = a_1 + a_2 + a_3 + a_4$, S_4 IS THE SUM OF THE FIRST FOUR TERMS OF THE SEQUENCE.

AND SO ON.

$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, S_n IS THE SUM OF THE FIRST n TERMS OF THE SEQUENCE.

Example 1 FIND THE SUM OF THE FIRST

A 5 NATURAL NUMBERS.

HENCE, $S_5 = 1+2+3+4+5 = 15$.

B 10 NATURAL NUMBERS THAT ARE MULTIPLES OF 3.

HENCE, $S_{10} = 3+6+9+12+15+18+21+24+27+30 = 165$.

Example 2 GIVEN THE GENERAL TERM $a_n = 3n - 5$, FIND

a THE SUM OF THE FIRST 6 TERMS. **b** THE SUM OF THE FIRST 10 TERMS.

Solution

a THE FIRST 6 TERMS OF THE SEQUENCE $a_n = 3n - 5$ ARE -2, 1, 4, 7, 10 AND 13.

HENCE, $S_6 = -2+1+4+7+10+13 = 33$.

b THE FIRST 10 TERMS OF THE SEQUENCE $a_n = 3n - 5$ ARE

-2, 1, 4, 7, 10, 13, 16, 19, 21 AND 24.

HENCE, $S_{10} = -2+1+4+7+10+13+16+19+21+24 = 113$

Example 3 GIVEN THE GENERAL TERM $a_n = \frac{1}{n} - \frac{1}{n+1}$, FIND THE SUM OF THE FIRST,

a 99 TERMS **b** n-TERMS

Solution

a $S_{99} = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots + (\frac{1}{99}-\frac{1}{100}) = 1 - \frac{1}{100} = 0.99$

b $S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

SO THAT $S_1 = \frac{1}{2}, S_2 = \frac{2}{3}, S_{10} = \frac{10}{11}, S_{99} = \frac{99}{100}, \ldots$ ETC.

✍Note:

✓ SUCH A SEQUENCE IS SAID TO BE TELESCOPING SEQUENCE.

WHEN YOU HAVE A FORMULA FOR THE GENERAL TERM OF A SEQUENCE, YOU CAN EXPRESS THE SUM OF THE FIRST n TERMS OF THE SEQUENCE IN A MORE COMPACT FORM USING A SPECIAL NOTATION FOR SUMS. THE GREEK (UPPER CASE) LETTER $\sum$ CALLED SIGMA, OR THE summation symbol, IS USED ALONG WITH THE GENERAL TERM OF THE SEQUENCE.

NOTATION:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

IN THIS NOTATION, i IS CALLED index of the summation OR SIMPLY THE INDEX. 1 IS THE lower limit AND n IS THE upper limit.

Definition 1.4

LET $\{a_n\}_{n=1}^{\infty}$ BE A SEQUENCE, THE SUM OF THE FIRST n TERMS OF THE SEQUENCE, DENOTED BY S_n, IS CALLED THE partial sum OF THE SEQUENCE.

NOTATION:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

SINCE SIGMA NOTATION IS MERELY A SHORTHAND WAY OF DENOTING A SUM, WE CAN REST OF THE REAL NUMBER PROPERTIES USING SIGMA NOTATION.

Properties of $\sum$

1 $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, WHERE c IS A CONSTANT. **2** $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

3 $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

4 $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$, WHERE $1 \le m < n$.

Example 4 EVALUATE EACH OF THE FOLLOWING.

A $\sum_{i=1}^{10} 3i$ **B** $\sum_{j=3}^{10}(5j-4)$ **C** $4\sum_{k=1}^{6} k^2 + 4\sum_{k=7}^{10} k^2$

Solution

A $\sum_{i=1}^{10} 3i = 3(1)+3(2)+3(3)+3(4)+3(5)+3(6)+3(7)+3(8)+3(9)+3(10)$

$= 3+6+9+12+15+18+21+24+27+30 = 165$

WHEREAS $3\sum_{i=1}^{10} i = 3(1+2+3+\ldots+10) = 3\times 55 = 165$

B $\sum_{j=3}^{10}(5j-4) = 5(3)-4+5(4)-4+5(5)-4+5(6)-4+5(7)-4+5(8)-4+5(9)-4+5(10)-4$

$= 11+16+21+26+31+36+41+46 = 228$

WHEREAS $5\sum_{j=3}^{10} j - \sum_{j=3}^{10} 4 = 5(3+4+5+\cdots+10) - \underbrace{(4+4+\cdots+4)}_{8-times} = 5\times 52 - 4\times 8 = 228$

C $\sum_{k=1}^{10} 4k^2 = 4(1^2)+4(2^2)+4(3^2)+\cdots+4(10^2)$

$= 4(1+4+9+\ldots+100) = 4(385) = 1540$

WHEREAS $4\sum_{k=1}^{6} k^2 + 4\sum_{k=7}^{10} k^2 = 4(91)+4(294) = 1540$

Example 5 GIVEN A SEQUENCE FOR WHICH $a_n = 2n^3$, EVALUATE

A S_4 **B** S_6

Solution

A $S_4 = \sum_{k=1}^{4} 2k^3 = 2\sum_{k=1}^{4} k^3 = 2(1^3+2^3+3^3+4^3) = 200$

B $S_6 = \sum_{k=1}^{6} 2k^3 = 2\sum_{k=1}^{6} k^3 = 2\sum_{k=1}^{4} k^3 + 2\sum_{k=5}^{6} k^3 = 2(1^3 + 2^3 + 3^3 + 4^3) + 2(5^3 + 6^3)$

$= 200 + 682 = 882.$

Exercise 1.5

1 USE THE GIVEN SEQUENCE OR GENERAL TERM TO FIND THE INDICATED SUM.

A $3, 7, 11, 15, \ldots; S_6$

B $-8, -3, 2, 7, \ldots; S_5$

C $-2, 0, 2, 4, 6, \ldots; S_6$

D $1, 2, 4, 8, 16, \ldots; S_6, S_{10}, S_{20}$

E $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots; S_6, S_{10}, S_{20}, S_{100}$. CAN YOU GUESS WHAT S_n IS?

F $a_n = 3n + 1$, WHERE n IS A POSITIVE INTEGER; $S_6, S_{10}, S_{20}, S_{100}$.

G $a_n = 2n - 1$, WHERE n IS A POSITIVE INTEGER; $S_6, S_{10}, S_{20}, S_{100}$.

H $a_n = \log\left(\frac{n}{n+1}\right)$, WHERE n IS A POSITIVE INTEGER; $S_6, S_{10}, S_{20}, S_{100}$.

I $a_n = \frac{n}{n+1} - \frac{n+1}{n+2}$, WHERE n IS A POSITIVE INTEGER; $S_6, S_{10}, S_{20}, S_{100}$.

CAN YOU GIVE A FORMULA FOR S_n?

2 REWRITE EACH SUM WITHOUT USING SIGMA NOTATION; THEN CALCULATE EACH SUM.

A $\sum_{n=1}^{5} n$ **B** $\sum_{k=1}^{4} 4(k+2)$ **C** $\sum_{k=1}^{6} 5(k-1)$ **D** $\sum_{k=1}^{8} 3k$

E $\sum_{k=2}^{6} k^2$ **F** $\sum_{k=3}^{5} k^3$ **G** $\sum_{k=1}^{5} 4$ **H** $\sum_{k=3}^{10} 7$

I $\sum_{m=1}^{10}\left(\frac{2}{m} - \frac{2}{m+1}\right)$ **J** $\sum_{n=1}^{8} \log_3\left(\frac{n+1}{n}\right)$ **K** $\sum_{k=1}^{6} \log_8 2^k$

3 USE THE SIGMA NOTATION TO REPRESENT THE SUM OF THE FIRST n TERMS OF THE GIVEN SEQUENCES.

A $4, 8, 12, \ldots, 4k, \ldots$ FOR $n = 5$.

B $2, 5, 8, \ldots, 3k-1, \ldots$ FOR $n = 8$.

C $2, 8, 18, \ldots, 2k^2, \ldots$ FOR $n = 7$.

D $7, 9, 11, \ldots, (2k+5), \ldots$ FOR $n = 10$.

4 EXPRESS THE GIVEN SUMS USING SIGMA NOTATION.

A $2 + 6 + 10 + 14 + 18$

B $5 + 25 + 125 + 625$

C $1 + 3 + 5 + 7 + \ldots + 51$

D $4 + 7 + 10 + 13 + \ldots + 52$

E $\frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{99\times 100}$

F $\frac{2}{5} + \frac{4}{9} + \frac{6}{13} + \ldots + \frac{20}{41}$

1.3.1 Sum of Arithmetic Progressions

THE PARTICULAR STRUCTURE OF AN ARITHMETIC PROGRESSION ALLOWED YOU TO DEVEL FOR ITS GENERAL TERM A_n. THIS SAME STRUCTURE ALLOWS YOU TO DEVELOP FORMULAE FOR S_n, THE SUM OF THE FIRST n TERMS OF AN ARITHMETIC PROGRESSION.

YOU BEGIN BY EXAMINING A SPECIAL ARITHMETIC SEQUENCE, 1, 2, 3, 4, …, n, … AND ITS ASSOCIATED SUM S_n

$S_n = 1 + 2 + 3 + \ldots + n$, THE SUM OF THE FIRST n TERMS (NATURAL NUMBERS).

FOR $n = 100$, THAT IS, $S_{100} = 1 + 2 + 3 + \ldots + 98 + 99 + 100$

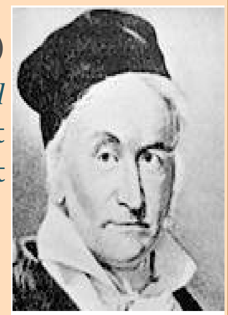
WRITE THE SUM IN REVERSE ORDER: $S_{100} = 100 + 99 + 98 + \ldots + 3 + 2 + 1$

ADDING THE TWO SUMS TOGETHER, WE GET: $2S_{100} = 101 + 101 + 101 + \ldots + 101 + 101 + 101$

I.E., $2S_{100} = 100 \times 101$.

THEREFORE, $S_{100} = \frac{1}{2}100(101) = 5050$.

Historical Note



Carl Friedrich Gauss (1777-1855)

A teacher of Gauss, at his elementary school, asked him *to add all the integers from 1 to 100.* When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{l} 1 + 2 + 3 + \ldots + 100 \\ 100 + 99 + 98 + \ldots + 1 \\ \hline 101 + 101 + 101 + \ldots + 101 \end{array}$$

$$\frac{100 \times 101}{2} = 5050$$

YOU CAN GENERALIZE THIS APPROACH AND DERIVE A FORMULA FOR S_n, THE SUM OF THE FIRST n NATURAL NUMBERS. YOU FOLLOW THE SAME STEPS AS YOU JUST DID FOR S_{100}.

$S_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$

WRITE THE SUM IN REVERSE ORDER: $S_n = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$

ADD THE TWO SUMS TOGETHER: $2S_n = (n+1) + (n+1) + \ldots + (n+1) + (n+1)$

THEREFORE, YOU HAVE $2S_n = n(n+1)$ AND SO, $S_n = \frac{n}{2}(n+1)$.

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THUS, YOU HAVE DERIVED THE FOLLOWING FORMULA.

THE SUM OF THE FIRST n POSITIVE INTEGERS IS GIVEN BY,

$$S_n = 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n+1).$$

Example 6 FIND THE SUM OF THE FIRST

A 30 NATURAL NUMBERS. **B** 150 NATURAL NUMBERS.

Solution

A USING FORMULA $S_n = \frac{n}{2}(n+1)$, $S_{30} = \frac{30}{2}(30+1) = 15(31) = 465$

B USING FORMULA $S_n = \frac{n}{2}(n+1)$ $S_{150} = \frac{150}{2}(150+1) = 75(151) = 11{,}325$.

YOU CAN NOW DERIVE THE GENERAL FORMULA FOR S_n, THE SUM OF THE FIRST n TERMS OF AN ARITHMETIC PROGRESSION.

THAT IS, $S_n = A_1 + A_2 + A_3 + \ldots + A_n$, WHERE $\{A_n\}_{n=1}^{\infty}$ IS AN ARITHMETIC SEQUENCE.

BUT THEN, $A_n = A_1 + (n-1)d$, WHERE D IS THE COMMON DIFFERENCE AND SO,

$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + (A_1 + 3d) + \ldots + (A_1 + (n-1)d)$$

BY COLLECTING ALL A_1 TERMS (THERE ARE n OF THEM) WE GET,

$$S_n = nA_1 + [d + 2d + 3d + \ldots + (n-1)d]$$

NOW FACTORING d FROM WITHIN THE BRACKETS,

$$S_n = nA_1 + d[1 + 2 + 3 + \ldots + (n-1)]$$

INSIDE THE BRACKETS, YOU HAVE THE SUM OF THE FIRST $(n-1)$ POSITIVE INTEGERS. THUS BY USING THE FORMULA $S_n = \frac{n}{2}(n+1)$, YOU GET

$$S_n = nA_1 + d\left(\frac{n-1}{2}\right)n = \frac{2nA_1 + n(n-1)d}{2} = \frac{n[2A_1 + (n-1)d]}{2}$$

HENCE, YOU HAVE PROVED THE FOLLOWING THEOREM.

Theorem 1.3

THE SUM S_n OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE WITH FIRST TERM A_1 AND COMMON DIFFERENCE d IS

$$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d].$$

THIS FORMULA CAN ALSO BE WRITTEN AS

$$S_n = \frac{n}{2}\left(A_1 + \left(A_1 + (n-1)d\right)\right) = \frac{n}{2}(A_1 + A_n) = n\left(\frac{A_1 + A_n}{2}\right),$$

WHERE A_n IS THE n^{TH} TERM. THIS ALTERNATIVE FORMULA IS USEFUL WHEN THE FIRST AND THE LAST TERMS ARE KNOWN.

Example 7 GIVEN THE ARITHMETIC SEQUENCE: 3, 7, 11, 15, …, FIND

A S_{20} **B** S_{80}

Solution

A SINCE THE GIVEN SEQUENCE IS AN ARITHMETIC SEQUENCE WITH $A_1 = 3$ AND COMMON DIFFERENCE $d = 4$, YOU CAN SUBSTITUTE THESE VALUES IN THE FORMULA

$$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d]$$

THUS, $S_{20} = \sum_{k=1}^{20} A_k = \frac{20}{2}\left(2(3) + (20-1)4\right) = 10\left(6 + 19(4)\right) = 10(82) = 820.$

B $$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d]$$

$$S_{80} = \sum_{k=1}^{80} A_k = \frac{80}{2}\left(2(3) + (80-1)4\right) = 40\left(6 + 79(4)\right) = 12{,}880.$$

Example 8 FIND THE SUM OF THE FIRST 35 TERMS OF THE SEQUENCE WHOSE GENERAL TERM IS $A_n = 5n$.

Solution FROM THE GENERAL TERM, WE GET $A_1 = 5$ AND $A_{35} = 5(35) = 175$.

SINCE WE CAN EASILY IDENTIFY THE FIRST AND n^{th} TERM, WE USE THE FORMULA,

$$S_n = \frac{n}{2}(A_1 + A_n) = n\left(\frac{A_1 + A_n}{2}\right)$$

THUS SUBSTITUTING $A_1 = 5$, AND $A_{35} = 175$, WE GET

$$S_{35} = \frac{35}{2}(5 + 175) = 35\left(\frac{5 + 175}{2}\right) = 35(90) = 3{,}150.$$

TRY TO FIND THE SUM OF THIS SEQUENCE USING THE OTHER FORMULA

$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d]$. WHICH FORMULA IS EASIER TO USE IN THIS EXAMPLE?

Example 9 IF THE n^{TH} PARTIAL SUM OF AN ARITHMETIC SEQUENCE $\{a_n\}$ IS $3n^2$, FIND a_n.

Solution NOTICE THAT $a_n = S_n - S_{n-1}$. *(Explain)*

$$\Rightarrow a_n = 3n^2 - 3(n-1)^2 = 6n - 3.$$

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Example 10 A WATER RESERVOIR IS BEING FILLED WITH WATER AT THE RATE OF 4,000 M³ FOR THE FIRST HOUR, 5,000 M³ FOR THE SECOND HOUR, 6,000 M³ FOR THE THIRD HOUR AND IT INCREASES BY 1,000 M³ AT THE END OF EVERY HOUR. IT IS COMPLETELY FILLED IN 8 HOURS. FIND THE CAPACITY OF THE RESERVOIR.

Solution OBSERVE THE SEQUENCE OF THE VOLUMES OF WATER FILLED AT THE END OF EVERY HOUR 4,000M³, 5,000M³, 6,000M³, …, FORM AN ARITHMETIC SEQUENCE WITH A_1 = 4,000 AND d = 1,000.

THE VOLUME OF WATER BEING FILLED IN 8 HOURS IS

$$S_8 = \frac{8}{2}(2 \times 4{,}000 + 7 \times 1{,}000)\ \text{M}^3 = 60{,}000\ \text{M}^3.$$

THUS, THE CAPACITY OF THE RESERVOIR IS 60,000 M³

1.3.2 Sum of Geometric Progressions

THE PARTICULAR STRUCTURE OF A GEOMETRIC PROGRESSION ALLOWED YOU TO DEVELO FOR ITS GENERAL TERM G_n. THIS SAME STRUCTURE ALLOWS YOU TO DEVELOP THE FORMULAE FOR S_n, THE SUM OF THE FIRST n TERMS OF A GEOMETRIC PROGRESSION, AS YOU DID FOR ARITHMETI PROGRESSIONS.

IF $\{G_n\}_{n=1}^{\infty}$ IS A GEOMETRIC SEQUENCE, THEN ITS ASSOCIATED GEOMETRIC SUM IS S_n

$$S_n = G_1 + G_2 + G_3 + \ldots + G_{n-1} + G_n$$

AS WITH THE CASE OF THE SUM OF ARITHMETIC SEQUENCE, WE CAN FIND A FORMULA TO DI GEOMETRIC SUM WHICH IS ASSOCIATED WITH A GEOMETRIC SEQUENCE.

LET $\{G_n\}_{n=1}^{\infty}$ BE A GEOMETRIC SEQUENCE WITH COMMON RATIO r. THEN $G_n = r^{n-1}G_1$ FOR EACH N.

THUS, $S_n = G_1 + G_2 + G_3 + \ldots + G_{n-1} + G_n$ IMPLIES THAT

$$S_n = G_1 + rG_1 + r^2G_1 + \ldots + r^{n-2}G_1 + r^{n-1}G_1$$

FACTORING OUT G_1, YOU GET

$$S_n = G_1(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1})$$

$$rS_n = G_1(r + r^2 + r^3 + \ldots + r^{n-1} + r^n) \qquad \textit{Multiplying both sides by } r$$

$$S_n - rS_n = G_1(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}) - G_1(r + r^2 + r^3 + \ldots + r^{n-1} + r^n) \quad \textit{Subtracting } rS_n \textit{ from } S_n$$

$(1-r)S_n = G_1(1-r^n)$, AND SO $S_n = \dfrac{G_1(1-r^n)}{1-r}$ *for* $r \neq 1$

THUS, YOU HAVE PROVED THE FOLLOWING THEOREM:

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Theorem 1.4

LET $\{G_n\}_{n=1}^{\infty}$ BE A GEOMETRIC SEQUENCE WITH COMMON RATIO r. THEN THE SUM OF THE FIRST n TERMS, S_n, IS GIVEN BY,

$$S_n = \begin{cases} nG_1, \text{ if } r = 1. \\ G_1 \dfrac{(1-r^n)}{1-r} = G_1 \dfrac{(r^n - 1)}{r-1}, \text{ IF } r \neq 1. \end{cases}$$

Example 11 GIVEN THE GEOMETRIC SEQUENCE: 1, 3, 9, 27, …, FIND

A S_5 **B** S_{10}

Solution

A FROM THE GIVEN SEQUENCE $G_1 = 1$ AND $r = 3$, THUS USING THE FORMULA

$S_n = \dfrac{G_1(1-r^n)}{1-r}$, YOU GET $S_5 = \dfrac{1(1-3^5)}{1-3} = \dfrac{-242}{-2} = 121.$

B BY USING THE SAME FORMULA, S_n AS IN **A**, $\dfrac{G_1(1-r^n)}{1-r}$, WE GET

$$S_{10} = \frac{1(1-3^{10})}{1-3} = \frac{-59048}{-2} = 29{,}524$$

Exercise 1.6

1 FIND THE SUM S_8 OF THE ARITHMETIC SEQUENCE WHOSE FIRST TERM IS 4 AND COMMON DIFFERENCE IS 5.

2 FIND THE SUM S_{10} OF THE ARITHMETIC SEQUENCE WHOSE FIRST TERM IS 8 AND THE COMMON DIFFERENCE IS −1.

3 FIND THE SUM S_7 OF THE ARITHMETIC SEQUENCE WHOSE FOURTH TERM IS 2 AND WHOSE SEVENTH TERM IS 17.

4 FIND THE SUMS S_8, S_{12}, S_{20} AND S_{100} OF THE GEOMETRIC SEQUENCE WHOSE FIRST TERM IS 4 WITH COMMON RATIO 5.

(WHAT HAPPENS TO THE SUM S_n AS n BECOMES "LARGER AND LARGER"?)

5 FIND THE SUM S_8, S_{12}, S_{20} AND S_{100} OF THE GEOMETRIC SEQUENCE WHOSE FIRST TERM IS 4 WITH COMMON RATIO $\dfrac{2}{3}$.

(WHAT HAPPENS TO THE SUM S_n AS n BECOMES "LARGER AND LARGER"?)

6 GIVEN THE SUM $S_{10} = 165$ OF AN ARITHMETIC SEQUENCE AND $A_1 = 3$, FIND A_{10}.

7 GIVEN THE SUM $S_{20} = 910$ OF AN ARITHMETIC SEQUENCE AND $A_{20} = 95$, FIND A_1.

8 GIVEN THE SUM $S_{16} = 368$ OF AN ARITHMETIC SEQUENCE AND $A_1 = 1$, FIND A_8.

9 GIVEN THE SUM $S_n = 969$ OF AN ARITHMETIC SEQUENCE, $A_1 = 9$ AND COMMON DIFFERENCE $d = 6$ FIND n.

10 FIND THE SUM OF ALL 3-DIGIT WHOLE NUMBERS THAT ARE DIVISIBLE BY 13.

11 FIND THE SUM OF n-ARITHMETIC MEANS WHICH ARE INSERTED BETWEEN ANY TWO REA NUMBERS a AND b.

12 IN AN ARITHMETIC SEQUENCE, THE FOURTH TERM IS 84 AND THE TENTH TERM IS 60. FIND THE MAXIMUM POSSIBLE PARTIAL SUM.

13 IF A_1 AND A_2 ARE ARITHMETIC MEANS BETWEEN ANY TWO REAL NUMBERS a AND b, AND G_1 AND G_2 ARE GEOMETRIC MEANS BETWEEN a AND b, EXPRESS $\dfrac{A_1 + A_2}{G_1 G_2}$ IN TERMS OF a AND b.

14 EVALUATE EACH OF THE FOLLOWING SUMS.

A $\sum_{n=1}^{20}(5n+7)$ **B** $\sum_{n=1}^{6}(-1)^{n+1}\dfrac{n}{n+1}$ **C** $\sum_{n=2}^{5}\dfrac{3^n}{5^{n+1}}$

D $\sum_{k=0}^{7}\dfrac{2^k}{k!}$ **E** $\sum_{j=2}^{10}\dfrac{(-1)^{j-3}}{j}$ **F** $\sum_{k=1}^{20}k^2$

15 A WOMAN STARTED A BUSINESS BY BIRR 3000. SHE LOST BIRR 100 IN THE FIRST MONTH, BIRR 60 IN THE SECOND MONTH, BIRR 20 IN THE THIRD MONTH AND SO ON. ASSUMING T THIS IMPROVEMENT CONTINUED AT THE SAME RATE, DETERMINE HER TOTAL CAPITAL AND 7 MONTHS.

16 THE POPULATION OF A CERTAIN CITY INCREASES 3% AFTER EACH YEAR. IF THE PRESENT POPULATION OF THE CITY IS 400000, FIND THE POPULATION AFTER

A 4 YEARS **B** 10 YEARS

17 A PERSON INVESTED IN TWO DIFFERENT ORGANIZATIONS. HE INVESTED BIRR 10,000 IN A THAT INCREASES BIRR 300 PER YEAR AND BIRR 16,000 IN B THAT INCREASES BY 5% PER YEAR.

A DETERMINE THE AMOUNT IN EACH ORGANIZATION AFTER 10 YEARS.

B FIND A FORMULA FOR THE AMOUNT OF MONEY IN EACH ORGANIZATION AFTER n YEARS.

C DETERMINE THE NUMBER OF YEARS THAT THE AMOUNT IN A EXCEEDS THE AMOUNT IN B.

18 SUPPOSE YOU PAY 20% TAX WHEN YOU BUY A CERTAIN MACHINE. IF YOU BUY TH MACHINE FOR BIRR 20,000 AND SALE IT FOR BIRR 12,000, THE BUYER WILL PAY 20% TA AND SALE IT FOR BIRR 7,200. IF THIS PROCESS CONTINUES WITHOUT END, FIND THE TO THAT CAN BE COLLECTED.

1.4 INFINITE SERIES

OPENING PROBLEM

A BALL IS DROPPED FREELY FROM A HEIGHT OF 16 M. EACH TIME IT DROPS H METRE, IT REBOUNDS 0.81h METRE.

A WHAT IS THE TOTAL VERTICAL DISTANCE TRAVELLED BY THE BALL BEFORE IT COMES TO REST?

B THE BALL TAKES THE FOLLOWING TIMES FOR EACH FALL.

$$
\begin{array}{ll}
S_1 = -16t^2 + 16, & S_1 = 0 \text{ if } t = 1; \\
S_2 = -16t^2 + 16(0.81), & S_2 = 0 \text{ if } t = 0.9; \\
S_3 = -16t^2 + 16(0.81)^2, & S_3 = 0 \text{ if } t = (0.9)^2; \\
S_4 = -16t^2 + 16(0.81)^3, & S_4 = 0 \text{ if } t = (0.9)^3; \\
\quad\vdots & \quad\vdots \\
S_n = -16t^2 + 16(0.81)^{n-1}, & S_n = 0 \text{ if } t = (0.9)^{n-1};
\end{array}
$$

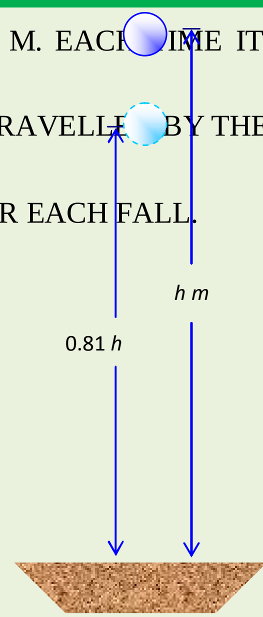


Figure 1.4

BEGINNING WITH S_2, THE BALL TAKES THE SAME AMOUNT OF TIME TO BOUNCE UP AS IT DOES TO

WHAT IS THE TOTAL TIME BEFORE IT COMES TO REST?

IF YOU TRY TO ADD THE TERMS OF AN INFINITE SEQUENCE $\{a_n\}_{n=1}^{\infty}$, YOU GET AN EXPRESSION OF THE FORM:

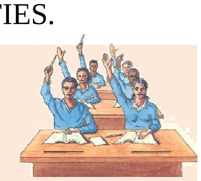
$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

WE CALL SUCH A SUM AN infinite series AND DENOTE IT BY THE SIGMA NOTATION AS

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$$

BUT DOES IT MAKE SENSE TO TALK ABOUT THE SUM OF INFINITELY MANY TERMS? WE MAY GET THE ANSWER AFTER THE FOLLOWING ACTIVITIES.

ACTIVITY 1.7



1 IS IT POSSIBLE TO FIND THE SUM OF THE FOLLOWING?

A $1 + 2 + 3 + 4 + 5 + \ldots + n + \ldots$ **B** $1 + 3 + 5 + 7 + \ldots + (2n-1) + \ldots$

C $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}+\ldots$ **D** $-1+1+-1+1+-1+\ldots+(-1)^n+\ldots$

E $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots+\frac{1}{3^n}+\ldots$ **F** $2+4+8+16+\ldots+2^n+\ldots$

2 FIND THE n^{th} PARTIAL SUM S_n FOR EACH OF THE ABOVE **A**–**F**.

3 WHAT HAPPENS TO THE PARTIAL SUM S_n AS n GETS "LARGER AND LARGER"?

LET US EXAMINE **A**, **C** AND **D**.

A $S_n=\frac{n}{2}(n+1)$ THE SUM OF THE FIRST n NATURAL NUMBERS.

AS N BECOMES "LARGER AND LARGER", S_n GETS "LARGER AND LARGER".

THAT IS AS n INCREASES INDEFINITELY, S_n ALSO INCREASES INDEFINITELY. OR AS n TENDS TO INFINITY, S_n TENDS TO INFINITY. SYMBOLICALLY, AS $n\to\infty$, $S_n\to\infty$

C $S_n=\frac{G_1(1-r^n)}{1-r}$; PARTIAL SUM OF A GEOMETRIC SERIES WITH $G_1=\frac{1}{2}$ AND $r=\frac{1}{2}$

THEREFORE $S_n=\dfrac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}=1-\left(\frac{1}{2}\right)^n$

AS $n\to\infty$, *the value of* $\left(\frac{1}{2}\right)^n$ *becomes almost zero. Hence,* $S_n\to 1$.

D $S_n=-1+1+-1+1+-1+\ldots+(-1)^n=\begin{cases}0 & \text{if } n \text{ is even}\\ -1 & \text{if } n \text{ is odd}\end{cases}$

AS $n\to\infty$, $S_n\to 0$ *or* -1, DEPENDING ON WHETHER n IS EVEN OR ODD.

THUS, AS $n\to\infty$, S_n *does not approach a unique number*. IN SUCH CASES, THE INFINITE SUM DOESN'T EXIST.

IN THE CASE OF **C** IN WHICH AS $n\to\infty$, $S_n\to 1$; WE DEFINE THE INFINITE SUM TO BE 1.

THAT IS, $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\left(\frac{1}{2}\right)^n+\ldots=1$

HOWEVER, IN THE CASES **A** AND **D** THE SUM DOES NOT EXIST; THE SUM IS NOT UNIQUE IN **D**, AND THE SUM IS NOT A FINITE NUMBER IN **A**.

NOW IN GENERAL, AS n TENDS TO INFINITY, IF THE PARTIAL SUM TENDS TO A UNIQUE FINITE NUMBER S, THEN WE SAY THE INFINITE SERIES **converges** AND THE INFINITE SUM IS EQUAL TO S; OTHERWISE, THE INFINITE SERIES IS SAID TO BE **divergent**.

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Definition 1.5

LET $\{a_n\}_{n=1}^{\infty}$ BE A SEQUENCE AND S_n BE THE n^{th} PARTIAL SUM SUCH THAT, AS $n \to \infty$, $S_n \to s$ WHERE s IS A FINITE REAL NUMBER, THEN WE SAY THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ **converges** AND IT IS WRITTEN AS $\sum_{n=1}^{\infty} a_n = s$.

HOWEVER, IF SUCH s DOES NOT EXIST OR IS INFINITE, WE SAY THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ **diverges**.

Example 1 DETERMINE WHETHER THE SERIES $\sum_{n=1}^{\infty} (3)^n$ CONVERGES OR DIVERGES.

Solution THE SERIES $\sum_{n=1}^{\infty} (3)^n = 3 + 9 + 27 + ... + 3^n + ...$ IS A GEOMETRIC SERIES WITH $G_1 = 3$ AND COMMON RATIO $r = 3$. HENCE, THE PARTIAL SUM IS GIVEN BY

$$S_n = \frac{G_1(1 - r^n)}{1 - r}$$

SUBSTITUTING THE VALUES, WE OBTAIN,

$$S_n = \frac{3[1-(3)^n]}{1-3} = -\frac{3}{2}[1-(3)^n] = \frac{-3}{2} + \frac{3}{2}(3)^n$$

THUS, AS $n \to \infty$, $S_n \to \infty$

THEREFORE, THE SERIES DIVERGES.

RECALL THAT, IF $\{G_n\}_{n=1}^{\infty}$ IS A GEOMETRIC SERIES WITH COMMON RATIO r, THEN

$$S_n = G_1 \frac{(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r}$$

IF $|r| < 1$, AS $n \to \infty$, $r^n \to 0$ SO THAT

$$S_n = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r} \to \frac{G_1}{1-r} \Rightarrow S_\infty = \frac{G_1}{1-r}$$

Example 2 EVALUATE

A $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ **B** $1 - \frac{3}{5} + \frac{9}{25} - \frac{27}{125} + \ldots$

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Solution

A THIS IS AN INFINITE GEOMETRIC SERIES WITH $G_1 = \frac{1}{2}$ AND $r = \frac{1}{2}$.

HENCE, $S_\infty = \dfrac{G_1}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$

B HERE, $G_1 = 1$, $r = -\frac{3}{5}$. HENCE, $S_\infty = \dfrac{1}{1-\left(-\frac{3}{5}\right)} = \dfrac{5}{8}.$

Example 3 WRITE $0.\dot{6}$ AS A RATIONAL NUMBER.

Solution $0.\dot{6} = 0.6 + 0.06 + 0.006 + \ldots = \dfrac{6}{10} + \dfrac{6}{100} + \dfrac{6}{1000} + \ldots$

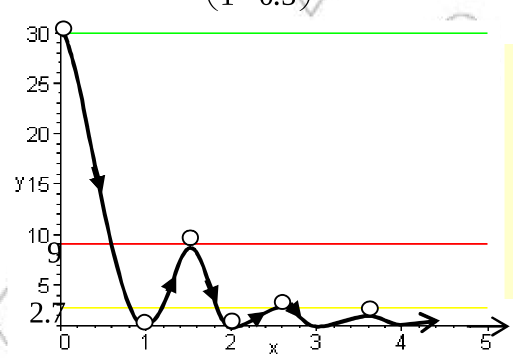
WHICH IS AN INFINITE GEOMETRIC SERIES WITH $r = \frac{1}{10}$ AND $G_1 = \frac{6}{10}$.

THUS, $S_\infty = \dfrac{G_1}{1-r} = \dfrac{\frac{6}{10}}{1-\frac{1}{10}} = \dfrac{\frac{6}{10}}{\frac{9}{10}} = \dfrac{2}{3}.$

Example 4 A BALL IS DROPPED FROM A HEIGHT OF 30M ABOVE A FLAT SURFACE. IN EACH BOUNCE, IT REBOUNDS TO 0.3 OF THE DISTANCE IT FELL. FIND THE MAXIMUM POSSIBLE VERTICAL DISTANCE THE BALL COULD TRAVEL.

Solution EACH TIME, THE BALL TRAVELS 0.3 TIMES THE DISTANCE IT FELL, AS SHOWN IN FIGURE 1.5. ASSUMING THAT THE BALL NEVER COMES TO REST, THE TOTAL DISTANCE IS

$$S = 30 + 2\left(\frac{9}{1-0.3}\right)m = 55.7m. \qquad (\textit{Explain!}).$$



Note that the ball is bouncing vertically and we assume no horizontal movement. The horizontal axis of this graph is only to indicate the bounces of the ball.

Figure 1.5

Example 5 IN FIGURE 1.6, AN INFINITE NUMBER OF RECTANGLES ARE CONSTRUCTED UNDER THE GRAPH OF $f(x) = 2^{-x}$. FIND THE SUM OF THE AREAS OF ALL THE RECTANGLES.

Solution THE n^{TH} RECTANGLE HAS UNIT LENGTH AND HEIGHT 2^{-n} UNITS. HENCE, THE AREA OF THE n^{TH} RECTANGLE IS $A_n = 2^{-n}$ SQUARE UNIT.

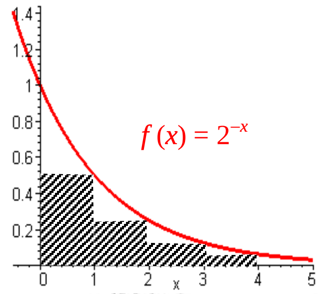


Figure 1.6

$$\Rightarrow \sum_{n=1}^{\infty} A_n = \frac{2^{-1}}{1-2^{-1}} = \frac{0.5}{1-0.5} \text{ SQUARE UNITS}$$

$$= 1 \text{ UNIT SQUARE.}$$

Exercise 1.7

1 FIND EACH OF THE FOLLOWING SUMS IF IT EXISTS, ASSUMING THE PATTERNS CONTINUE AS IN THE FIRST FEW TERMS.

A $2+1+\frac{1}{2}+\frac{1}{4}+...$ **B** $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+...$

C $\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+...$ **D** $\frac{1}{5}+\frac{-1}{10}+\frac{1}{20}+\frac{-1}{40}+...$

E $\frac{1}{5}+\frac{4}{15}+\frac{16}{45}+...$ **F** $7+\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+...$

G $\sum_{k=1}^{\infty} 4^{3-k}$ **H** $\sum_{k=1}^{\infty} 4^{k-3}$

I $\sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k-1}$ **J** $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2}{3}\right)^{n}$

K $\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-2}$

2 EXPRESS EACH OF THE FOLLOWING AS A FRACTION USING THE INFINITE SUM.

A $0.\overline{4}$ **B** $0.3\overline{7}$ **C** $3.2\overline{354}$ **D** $13.452\overline{981}$

3 FIND THE PRODUCT $5^{\frac{1}{2}}.5^{\frac{1}{4}}...5^{\frac{1}{2^n}}...$

4 IF $\sum_{k=1}^{\infty} 5^{kr} = \frac{1}{4}$, FIND THE VALUE OF r.

5 IF THE PRODUCT $3^r.3^{r^2}.3^{r^3}... = 3$, FIND r.

6 SUPPOSE A BALL IS DROPPED FROM A HEIGHT OF h AND ALWAYS REBOUNDS r OF THE HEIGHT FROM WHICH IT FALLS. SHOW THAT THE TOTAL VERTICAL DISTANCE THAT IS COVERED BY THE BALL IS $h\left(\frac{r+1}{1-r}\right)$. ASSUME THAT THE BALL WILL NEVER STOP BOUNCING.

1.5 APPLICATIONS OF ARITHMETIC PROGRESSIONS AND GEOMETRIC PROGRESSIONS

THIS SECTION IS DEVOTED TO THE APPLICATIONS OF ARITHMETIC AND GEOMETRIC PROGR GEOMETRIC SERIES (BINOMIAL SERIES) THAT ARE ASSOCIATED WITH REAL LIFE SITUATION SOME EXAMPLES FOLLOWED BY EXERCISES. THE EXAMPLES SHOWN HERE AND THE FOLL EXERCISES ILLUSTRATE SOME USEFUL APPLICATIONS.

Example 1 A JOB APPLICANT FINDS THAT A FIRM OFFERS A STARTING ANNUAL SALARY OF BIRR 32,500 WITH A GUARANTEED RAISE OF BIRR 1,400 PER YEAR.

A WHAT WOULD THE ANNUAL SALARY BE IN THE TENTH YEAR?

B OVER THE FIRST 10 YEARS, HOW MUCH WOULD BE EARNED AT THE FIRM?

Solution

A THE ANNUAL SALARY AT THE FIRM FORMS AN ARITHMETIC SEQUENCE;

32,500, 33,900, 35,300, … WITH FIRST TERM $A_1 = 32,500$

AND COMMON DIFFERENCE $d = 1,400$.

THUS, $A_n = A_1 + (n-1)d$, SUBSTITUTING THE VALUES WE OBTAIN;

$$A_{10} = 32,500 + (10-1)1,400 = \text{ BIRR } 45,10$$

B TO DETERMINE THE AMOUNT THAT WOULD BE EARNED OVER THE FIRST 10 YEARS, WE NEED TO ADD THE FIRST 10 ANNUAL SALARIES;

$$S_{10} = A_1 + A_2 + A_3 + ... + A_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$$

(It is 10 times the average of the first and the last term.)

$$S_{10} = \frac{10}{2}(32,500 + 45,100) = \text{ BIRR } 388,00$$

THEREFORE, OVER THE FIRST 10 YEARS A TOTAL OF BIRR 388,000 WOULD BE EARNED AT TH

Example 2 A WOMAN DEPOSITS BIRR 3,500 IN A BANK ACCOUNT PAYING ANNUAL INTEREST AT A RATE OF 6%. SHOW THAT THE AMOUNTS SHE HAS IN THE ACCOUNT AT TH OF EACH YEAR FORM A GEOMETRIC SEQUENCE.

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Solution LET $G_1 = 3{,}500$ THEN,

$$G_2 = G_1 + \frac{6}{100}G_1 = G_1(1+0.06) = 3{,}500(1.06) = 3{,}710.$$

$$G_3 = G_2 + \frac{6}{100}G_2 = G_2(1+0.06) = G_1(1.06)(1.06) = 3{,}710(1.06) = 3{,}932.6.$$

CONTINUING IN THIS WAY, YOU GET $G_n = (1.06)^{n-1} G_1$

SINCE THE RATIO OF ANY TWO CONSECUTIVE TERMS IS A CONSTANT, WHICH IS 1.06 SEQUENCE IS A GEOMETRIC SEQUENCE.

Example 3 SUPPOSE A SUBSTANCE LOSES HALF OF ITS MASS EACH YEAR. IF WE START WITH 100 GRAMS OF A RADIOACTIVE SUBSTANCE, HOW MUCH IS LEFT AFTER 10 YEA

Solution LET US RECORD THE AMOUNT OF THE RADIOACTIVE SUBSTANCE LEFT AFTER EACH YEAR STARTING WITH $G_0 = 100$. NOTE THAT EACH TERM IS HALF OF THE PREVIOUS TERM AND HENCE,

$$G_1 = \frac{1}{2}(100) = 50\text{G IS THE AMOUNT LEFT AT THE END OF YEAR 1.}$$

$$G_2 = \frac{1}{2}(50) = 25\text{G IS THE AMOUNT LEFT AT THE END OF YEAR 2.}$$

IF YOU CONTINUE IN THIS WAY, YOU SEE THAT THE RATIO OF ANY TWO CONSECUTIVE A CONSTANT, WHICH IS $\frac{1}{2}$, AND HENCE THIS SEQUENCE IS A GEOMETRIC SEQUENCE.

THEREFORE, AFTER TEN YEARS, THE AMOUNT OF THE SUBSTANCE LEFT IS GIVEN BY

$$G_{10} = \left(\frac{1}{2}\right)^{10} G_1 = \left(\frac{1}{2}\right)^{10}(100) = \frac{100}{1{,}024} = 0.09765625\text{G}.$$

Binomial series

YOU REMEMBER THAT THE BINOMIAL THEOREM STATES

$$(a+bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \ldots + (bx)^n$$

FOR ANY POSITIVE INTEGER n.

IN PARTICULAR, FOR $a = 1$ AND $b = 1$, YOU HAVE

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

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NOW, IF YOU CONSIDER THE INFINITE SERIES $1 - x + x^2 - x^3 + \ldots$, THEN IT IS A GEOMETRIC SERIES WITH COMMON RATIO $-x$. MOREOVER, FOR $|x| < 1$, IT CONVERGES TO $\frac{1}{1+x} = (1 + x)^{-1}$

IN GENERAL, FOR ANY VALUE OF n,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \quad (*)$$

AND THIS TYPE OF SERIES IS CALLED A **BINOMIAL SERIES.** THIS SERIES CONVERGES FOR $|x| < 1$. THE BINOMIAL SERIES GENERALIZES THE BINOMIAL THEOREM TO ANY REAL VALUES OF n. IF n IS A POSITIVE INTEGER THE BINOMIAL SERIES REDUCES TO BINOMIAL THEOREM.

Example 4 EXPAND EACH OF THE FOLLOWING EXPRESSIONS.

A $(1 + x)^{\frac{1}{2}}$ **B** $(1 - 3x)^{-5}$ **C** $(3x + 2)^{-4}$

Solution

A REPLACING n BY $\frac{1}{2}$ IN (*) GIVES YOU,

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots \quad \text{PROVIDED THAT } |x| < 1.$$

B REPLACING n BY -5 AND x BY $(-3x)$ IN (*) GIVES YOU.

$$(1 - 3x)^{-5} = 1 + (-5)(-3x) + \frac{(-5)(-5-1)(-3x)^2}{2!} + \frac{-5(-5-1)(-5-2)(-3x)^3}{3!} + \ldots$$

$$= 1 + 15x + 135x^2 + 945x^3 + 5670x^4 + \ldots \quad \text{PROVIDED THAT } |x| < \frac{1}{3}.$$

C OBSERVE THAT $(2x + 3)^{-4} = \left(3\left(\frac{2}{3}x + 1\right)\right)^{-4} = 3^{-4}\left(\frac{2}{3}x + 1\right)^{-4}$

HENCE,

$$(2x+3)^{-4} = 3^{-4}\left(1 + (-4)\left(\frac{2}{3}x\right) + \frac{(-4)(-4-1)\left(\frac{2}{3}x\right)^2}{2!} + \frac{(-4)(-4-1)(-4-2)\left(\frac{2}{3}x\right)^3}{3!} + \cdots\right)$$

$$= \frac{1}{81} - \frac{8}{243}x + \frac{40}{729}x^2 - \frac{160}{2187}x^3 + \frac{560}{6561}x^4 - \ldots$$

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THE BINOMIAL SERIES IS USEFUL FOR APPROXIMATIONS. WHEN YOU HAVE AN EXPRESSION FORM $(1+x)^n$ WHERE $|x| < 1$, YOU CAN TAKE $(1+x)^n$ TO BE EQUAL TO ONLY THE FIRST FEW TERMS OF THE SERIES.

Example 5 FIND THE APPROXIMATE VALUE OF $\sqrt[3]{6}$ CORRECT TO FOUR DECIMAL PLACES.

Solution: YOU KNOW THAT 6 IS NOT A PERFECT CUBE. BY USING THE PERFECT CUBE 8, REWRITE $\sqrt[3]{6}$ AS

$$\sqrt[3]{6} = \sqrt[3]{8-2} = \sqrt[3]{8\left(1-\frac{2}{8}\right)} = \sqrt[3]{8}\,\sqrt[3]{1-\frac{1}{4}} = 2\left(1-\frac{1}{4}\right)^{\frac{1}{3}}.$$

HENCE REPLACING n BY $\frac{1}{3}$ AND x BY $-\frac{1}{4}$ IN (*), YOU HAVE

$$\sqrt[3]{6} = 2\left(1-\frac{1}{4}\right)^{\frac{1}{3}} = 2\left(1+\frac{1}{3}\left(-\frac{1}{4}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(-\frac{1}{4}\right)^2}{2!}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(-\frac{1}{4}\right)^3}{3!}+\ldots\right)$$

$$= 2\left(1-\frac{1}{12}-\frac{1}{144}-\frac{5}{5184}-\ldots\right)$$

$$= 1.817515430988$$

$$\Rightarrow \sqrt[3]{6} = 1.8175 \text{ CORRECT TO FOUR DECIMAL PLACES.}$$

Exercise 1.8 (Application Problems)

1. A PERSON IS SCHEDULED TO GET A RAISE OF BIRR 250 EVERY 6 MONTHS DURING HIS/HER FIRST 5 YEARS ON THE JOB. IF HIS/HER STARTING SALARY IS BIRR 25,250 PER YEAR, WHAT HIS/HER ANNUAL SALARY BE AT THE END OF THE 3RD YEAR?
2. ROSA BEGINS A SAVING PROGRAM IN WHICH SHE WILL SAVE BIRR 1,000 THE FIRST YEAR, AND EACH SUBSEQUENT YEAR SHE WILL SAVE 200 MORE THAN SHE DID THE PREVIOUS YEAR MUCH WILL SHE SAVE DURING THE EIGHTH YEAR?
3. A CERTAIN ITEM LOSES ONE-TENTH OF ITS VALUE EACH YEAR. IF THE ITEM IS WORTH BIRR 28,000 TODAY, HOW MUCH WILL IT BE WORTH 4 YEARS FROM NOW?
4. A BOAT IS NOW WORTH BIRR 34,000 AND LOSES 12% OF ITS VALUE EACH YEAR. WHAT WILL IT BE WORTH AFTER 5 YEARS?
5. THE POPULATION OF A CERTAIN TOWN IS INCREASING AT A RATE OF 2.5% PER YEAR. IF THE POPULATION IS CURRENTLY 100,000, WHAT WILL THE POPULATION BE 10 YEARS FROM N

6 SOFIA DEPOSITS BIRR 3,500 IN A BANK ACCOUNT PAYING AN ANNUAL INTEREST RATE OF 6%. FIND THE AMOUNT SHE HAS AT THE END OF

A THE FIRST YEAR **B** THE SECOND YEAR **C** THE THIRD YEAR

D THE FOURTH YEAR **E** THE n^{th} YEAR

F DO THE AMOUNTS SHE HAS AT THE END OF EACH YEAR FORM A GEOMETRIC SEQUENCE? EXPLAIN.

7 A JOB APPLICANT FINDS THAT A FIRM A OFFERS A STARTING SALARY OF BIRR 31,100 WITH A GUARANTEED RAISE OF BIRR 1,200 PER YEAR, WHEREAS FIRM B OFFERS A HIGHER ST SALARY OF BIRR 35,100 BUT WILL GUARANTEE A YEARLY RAISE OF ONLY BIRR 900.

A WHAT WOULD THE ANNUAL SALARY BE IN THE 11^{th} YEAR AT FIRM A?

B WHAT WOULD THE ANNUAL SALARY BE IN THE 11^{th} YEAR AT FIRM B?

C OVER THE FIRST 11 YEARS, HOW MUCH WOULD BE EARNED AT FIRM A?

D OVER THE FIRST 11 YEARS, HOW MUCH WOULD BE EARNED AT FIRM B?

E COMPARE THE AMOUNT EARNED IN 11 YEARS IN FIRMS A AND B.

8 A THEATRE HALL HAS 38 ROWS OF SEATS. THE FIRST ROW HAS 17 SEATS, THE SECOND ROW HAS 20 SEATS, THE THIRD ROW HAS 23 SEATS AND SO ON. WHAT IS THE SEATING CAPACITY THEATRE HALL?

9 A CONTEST OFFERS A TOTAL OF 18 PRIZES. THE FIRST PRIZE IS WORTH BIRR 10,000, AND EACH CONSECUTIVE PRIZE IS WORTH BIRR 500 LESS THAN THE NEXT HIGHER PRIZE. FIND THE V OF THE EIGHTEENTH PRIZE AND THE TOTAL VALUE OF THE PRIZES.

10 A CONTEST OFFERS 10 PRIZES WITH A TOTAL VALUE OF BIRR 13,250. IF THE DIFFERENCE IN VALUE BETWEEN CONSECUTIVE PRIZES IS BIRR 250, WHAT IS THE VALUE OF THE FIRST P

11 A BALL BOUNCES UP ONE-HALF THE DISTANCE IT FALLS. HOW FAR DOES IT BOUNCE UP ON THE FIFTH REBOUND, IF THE BALL IS DROPPED FROM A HEIGHT OF 20 ME

12 A BALL IS DROPPED FROM A HEIGHT OF 60 METRES AND ALWAYS REBOUNDS ONE-THIRD THE DISTANCE FROM WHICH IT FALLS. FIND THE TOTAL DISTANCE THE BALL HAS TRAVELI HITS THE GROUND FOR THE FIFTH TIME AND THE TENTH TIME.

13 IN PROBLEM 12 ABOVE, IF YOU LET THE BALL BOUNCE "FOREVER", WHAT IS THE TOTAL DISTANCE IT WOULD TRAVEL UNTIL IT COMES TO REST?

14 EXPAND EACH OF THE FOLLOWING USING BINOMIAL SERIES.

A $(x-4)^{-7}$ **B** $(1+x)^{\frac{3}{2}}$ **C** $\dfrac{1}{\sqrt{4-x}}$

15 APPROXIMATE EACH OF THE FOLLOWING USING BINOMIAL SERIES.

A $\sqrt{5}$ **B** $\sqrt[3]{9}$ **C** $\sqrt[4]{17}$

Key Terms

arithmetic mean	finite sequence	recursion formula
arithmetic sequence	general term	sequence
common difference	geometric mean	series
common ratio	geometric sequence	sigma notation
convergent series	infinite sequence	telescoping sequence
divergent series	infinite series	terms of a sequence
Fibonacci sequence	partial sums	

Summary

1 *Sequence*

- ✓ A SEQUENCE $\{a_n\}$ IS A FUNCTION WHOSE DOMAIN IS THE SET OF POSITIVE INTEGERS OR SUBSET OF CONSECUTIVE INTEGERS STARTING WITH 1.
- ✓ THE SEQUENCE $\{a_1, a_2, a_3, \ldots\}$ IS DENOTED BY $\{a_n\}$ OR $\{a_n\}_{n=1}^{\infty}$.
- ✓ A SEQUENCE THAT HAS A LAST TERM IS CALLED A finite sequence. OTHERWISE IT IS CALLED infinite sequence.
- ✓ Recursion formula IS A FORMULA THAT RELATES THE GENERAL TERM a_n OF A SEQUENCE TO ONE OR MORE OF THE TERMS THAT COME BEFORE IT.

2 *Arithmetic and geometric progression*

I *Arithmetic progression*

- ✓ AN ARITHMETIC SEQUENCE IS ONE IN WHICH THE DIFFERENCE BETWEEN CONSECUTIVE TERMS IS A CONSTANT, AND THIS CONSTANT IS CALLED THE common difference.
- ✓ IF $\{A_n\}$ IS AN ARITHMETIC PROGRESSION WITH THE FIRST TERM A_1 AND THE COMMON DIFFERENCE d, THEN THE n^{TH} TERM IS GIVEN BY:

$$A_n = A_1 + (n - 1)\, d.$$

II *Geometric progression*

- ✓ A GEOMETRIC PROGRESSION IS ONE IN WHICH THE RATIO OF CONSECUTIVE TERMS IS A CONSTANT, AND THIS CONSTANT IS CALLED THE common ratio.
- ✓ IF $\{G_n\}$ IS A GEOMETRIC PROGRESSION WITH THE FIRST TERM G_1 AND COMMON RATIO r, THEN THE n^{TH} TERM IS GIVEN BY:

$$G_n = r^{n-1} G_1.$$

3 Partial sums

✓ THE SUM OF THE FIRST n TERMS OF THE SEQUENCE $\{a_n\}_{n=1}^{\infty}$, DENOTED BY S_n, IS CALLED THE partial sum of the sequence.

✓ THE SUM S_n OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE WITH FIRST TERM A_1 AND COMMON DIFFERENCE d IS:

$$S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d].$$

✓ IN A GEOMETRIC SEQUENCE $\{G_n\}_{n=1}^{\infty}$ WITH COMMON RATIO r, THE SUM OF THE FIRST n TERMS S_n IS GIVEN BY;

$$S_n = \begin{cases} nG_1, \text{ IF } r = 1 \\ \dfrac{G_1(r^n - 1)}{r-1}, \text{ IF } r \neq 1 \end{cases}$$

4 Convergent series and divergent series

✓ IN A SEQUENCE $\{a_n\}_{n=1}^{\infty}$, IF S_n IS THE n^{th} PARTIAL SUM SUCH THAT, AS $n \to \infty$, $S_n \to s$ WHERE s IS A FINITE REAL NUMBER, WE SAY THE INFINITE SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGES TO s, OTHERWISE THE SERIES DIVERGES.

Review Exercises on Unit 1

1 FIND THE FIRST FIVE TERMS OF THE SEQUENCE WITH THE GIVEN GENERAL TERM.

A $a_n = \dfrac{1}{2n+1}$ **B** $a_n = (n-1)^2$ **C** $a_n = (-1)^n n!$

D $a_n = \dfrac{3n-1}{3n+1}$ **E** $a_n = \dfrac{1}{n}\text{SIN}\left(\dfrac{n}{6}\right)$ **F** $a_n = \dfrac{n^2-3}{n^2+3}$

2 FIND THE FIRST FIVE TERMS OF THE RECURSIVELY DEFINED SEQUENCE.

A $a_1 = -2$ AND $a_n = \dfrac{1}{a_{n-1}}$ FOR $n \geq 2$

B $a_1 = 1$, $a_2 = 3$ AND $a_n = \dfrac{a_{n-1}}{a_{n-2}}$ FOR $n \geq 3$

C $a_1 = 1$, $a_n = (a_{n-1})^n$ FOR $n \geq 2$

D $a_1 = 0$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ FOR $n \geq 3$

3 FIND THE GENERAL TERM OF AN ARITHMETIC SEQUENCE THAT SATISFIES THE GIVEN CONDITIONS.

A THE FIRST TWO TERMS ARE 4 AND 7.

B THE FOURTH TERM IS 11 AND THE TENTH TERM IS 35.

C THE TENTH TERM IS 3 AND THE FIFTEENTH TERM IS 5.5.

D THE THIRD TERM IS $\sqrt{2}$ AND THE SIXTH TERM IS $4\sqrt{2}$.

4 FIND THE GENERAL TERM OF A GEOMETRIC SEQUENCE THAT SATISFIES THE GIVEN CONDITIONS.

A THE FIRST TWO TERMS ARE -1 AND $\frac{1}{4}$.

B THE THIRD TERM IS $\frac{2}{9}$ AND THE FIFTH TERM IS $\frac{2}{243}$.

C THE SECOND TERM IS $\frac{\sqrt{2}}{2}$ AND THE FOURTH TERM IS $\sqrt{2}$.

D THE FIRST TERM IS 0.15 AND THE THIRD TERM IS 0.0015.

5 EVALUATE EACH OF THE FOLLOWING SUMS.

A $\sum_{k=1}^{10} 3k-1$ **B** $\sum_{n=1}^{12}(4-5n)$ **C** $\sum_{k=0}^{5}\frac{2^k}{2k+1}$

D $\sum_{k=0}^{10}\left(3+(-1)^k\right)$ **E** $\sum_{k=2}^{12}(-1)^k\, 2^k$ **F** $\sum_{k=1}^{5}\frac{2^{k+5}}{3^{k-1}}$

G $\sum_{k=1}^{\infty}\left(\frac{3^k+2^k}{6^k}\right)^2$ **H** $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k-4}$ **I** $\sum_{k=1}^{20}2\left(\frac{3}{2}\right)^k$

J $\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^{k+5}$ **K** $\sum_{k=0}^{10}\frac{5^k}{4^k}$ **L** $\sum_{k=1}^{5}\left(-3\left(5^k\right)\right)$

6 FIND THE SUM OF WHOLE NUMBERS THAT ARE LESS THAN 100 AND LEAVE REMAINDER 2 WHEN DIVIDED BY 5.

7 EVALUATE EACH OF THE FOLLOWING INFINITE SERIES.

A $2-\sqrt{2}+1-\frac{1}{\sqrt{2}}-\frac{1}{2}+\dots$ **B** $9+3\sqrt{3}+3+\sqrt{3}+1+\frac{1}{\sqrt{3}}+\dots$

C $\frac{-3}{4}+\frac{9}{4}-\frac{27}{4}+\dots$ **D** $1+x+x^2+x^3+\dots$ (IN TERMS OF x)

E $0.1+0.11+0.111+0.1111+\dots$

8 FIND THE SUM OF ALL TWO-DIGIT WHOLE NUMBERS WHICH ARE DIVISIBLE BY 11

9 IN AN ARITHMETIC SEQUENCE, THE SUM OF THE FIRST 20 TERMS IS 950 AND THE SUM OF THE SECOND 20 TERMS IS 0. FIND THE GENERAL TERM OF THE SEQUENCE.

10 WHEN n ARITHMETIC MEANS ARE INSERTED BETWEEN 8 AND 44, THE SUM OF THE RESULTING TERMS IS 338. FIND THE VALUE OF n AND THE COMMON DIFFERENCE.

11 A CAR THAT IS BOUGHT FOR BIRR 125000.00 DEPRECIATES IN VALUE BY BIRR 4000 PER YEAR. HOW LONG WILL IT TAKE FOR THE CAR TO MAKE A LOSS OF 25% OF ITS VALUE? (TO THE N
YEAR).

12 A PERSON INVESTS BIRR 1,000,000.00 FOR THE FIRST YEAR. DURING EACH SUCCEEDING YEARS HE INVESTS BIRR 300,000.00 MORE THAN HE DID THE YEAR BEFORE. HOW MUCH W
HE INVEST OVER A PERIOD OF 20 YEARS?

13 A FACTORY THAT PRODUCES CEMENT HAD SALES OF BIRR 100,000.00 ON THE FIRST DAY AND SALES INCREASED BY BIRR 8,000.00 EVERY DAY DURING EACH SUCCESSIVE DAYS. FIND THE T
SALES OF THE FACTORY DURING THE FIRST 30 DAYS.

14 IF THE CONSTRUCTION OF A CERTAIN SCHOOL IS NOT COMPLETED BY THE AGREED UPON DATE, THE CONTRACTOR PAYS A PENALTY OF BIRR 1,000 FOR THE FIRST WEEK, BIRR 2,000 FO
SECOND WEEK, BIRR 3,000 FOR THE THIRD WEEK AND SO ON THAT IT IS OVERDUE.
CONSTRUCTION OF THE SCHOOL IS COMPLETED 13 WEEKS LATE, CALCULATE THE TOT
OF THE PENALTIES THAT THE CONTRACTOR MUST PAY.

15 SUPPOSE YOU EARN BIRR 10,000.00 PER MONTH AND PAY 20% TAX AND SPEND 60% OF IT. AGAIN THE RECIPIENT OF THE MONEY SPENT BY YOU PAYS 20% TAX AND SPENDS 60%
IT. IF THIS PROCESS CONTINUES WITHOUT END, FIND THE TOTAL AMOUNT THAT WILL B
THE TAX.

16 EXPAND EACH OF THE FOLLOWING EXPRESSIONS AS BINOMIAL SERIES AND DETERMINE THE VALUES OF x FOR WHICH IT CONVERGES.

A $(9+x)^{\frac{1}{2}}$ **B** $(1+5x)^{-\frac{5}{2}}$ **C** $(2-x)^{\frac{3}{2}}$

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