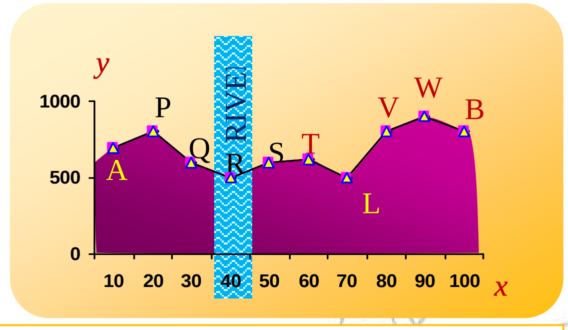
Unit 2



INTRODUCTION TO LIMITS AND CONTINUITY

Unit Outcomes:

After completing this unit, you should be able to:

- *understand the concept of "limit" intuitively.*
- *find out limits of sequences of numbers.*
- *determine the limit of a given function.*
- *determine continuity of a function over a given interval.*
- *apply the concept of limits to solve real life mathematical problems.*
- *develop a suitable ground for dealing with differential and integral calculus.*

Main Contents

2.1 LIMITS OF SEQUENCES OF NUMBERS

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INTRODUCTION

THIS UNIT DEALS WITH THE FUNDAMENTAL OBJECTS OF CALCULUS: LIMITS AND CONTINUIT

LIMITS ARE THEORETICAL IN NATURE BUT WE START WITH INTERPRETATIONS.

LIMIT CAN BE USED TO DESCRIBE HOW A FUNCTION BEHAVES AS THE INDEPENDENT VA APPROACHES A CERTAIN VALUE.

FOR EXAMPLE, CONSIDER THE FUNCTION $f(x)=\frac{x^2-1}{x-1}$. THEN $f(1)=\frac{0}{0}$ HAS NO MEANING. THE FORM $\frac{0}{0}$ IS SAID TO BE INDETERMINATE FORM BECAUSE IT IS NOT POSSIBLE TO ASSIGN A U VALUE TO IT.

THIS FUNCTION IS NOT DEFINED AT $x = 1$. HOWEVER, IT STILL MAKES SENSE TO ASK WHAT HAPPENS TO THE VALUES OF $f(x)$ AS THE VALUE OF x BECOMES CLOSER TO 1 WITHOUT ACTUALLY BEING EQUAL TO 1. YOU CAN VERIFY USING A CALCULATOR THAT $f(x)=\frac{x^2-1}{x-1}$ APPROACHES TO 2 WHENEVER YOU TAKE ANY VALUE VERY CLOSE TO 1 FOR x.

THIS MEANS THAT $f(x)$ HAS A WELL-DEFINED VALUE NEAR $x = 1$ ON EITHER SIDE OF 1.
LIMITS ARE USED IN SEVERAL AREAS OF MATHEMATICS, INCLUDING THE STUDY OF RATES O APPROXIMATIONS AND CALCULATIONS OF AREA.

FOR EXAMPLE, YOU KNOW HOW TO APPROXIMATE THE POPULATION OF YOUR KEBELE IN 2012 WHAT IS DIFFERENT IN LIMITS IS YOU WILL LEARN HOW TO KNOW THE RATE OF CHANGE OF IN YOUR KEBELE IN 2012.

OPENING PROBLEM

IMAGINE THAT A REGULAR POLYGON WITH n-SIDES IS INSCRIBED IN A CIRCLE.

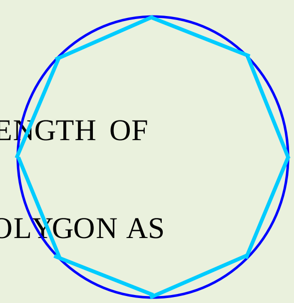


Figure 2.1

1 AS n GETS LARGE, WHAT HAPPENS TO THE LENGTH OF EACH SIDE OF THE POLYGON?

2 WHAT WILL BE THE LIMITING SHAPE OF THE POLYGON AS n GOES TO INFINITY?

3 WILL THE POLYGON EVER GET TO THE CIRCLE?

2.1 LIMITS OF SEQUENCES OF NUMBERS

ACTIVITY 2.1

1 FIND THE MAXIMUM AND MINIMUM ELEMENTS OF EACH OF THE FOLLOWING SETS.

A $\{1, 2, 3, \ldots, 10\}$ **B** $\{1, -1, 1, -1, \ldots\}$ **C** $\{x \in \mathbb{R}: -3 \le x < 5\}$

D $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ **E** $\{x \in \mathbb{R} : -1 \le x \le 2\}$ **F** $\{x \in \mathbb{R} : -5 < x \le 4\}$

G $\{x \in \mathbb{R} : |x| < 5\}$

2 FOR EACH OF THE FOLLOWING SEQUENCES $\{a_n\}$, FIND m AND k SUCH THAT

I $a_n \le m$, FOR ALL n **II** $a_n \ge k$, FOR ALL n

A $a_n = 2^n + 1$ **B** $a_n = \frac{1}{3^n}$ **C** $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$

D $a_n = \frac{n+1}{n}$ **E** $a_n = 7 + \frac{1}{n}$ **F** $a_n = \frac{10^n - 1}{10^n}$

2.1.1 Upper Bounds and Lower Bounds

THE NUMBERS m AND k IN ACTIVITY 2.1 ARE SAID TO BE AN **upper bound** AND A **lower bound** OF THE SEQUENCES, RESPECTIVELY.

Definition 2.1

LET $\{a_n\}$ BE A SEQUENCE AND $m, M \in \mathbb{R}$. THEN

I M IS SAID TO BE AN **upper bound** OF $\{a_n\}$, IF $M \ge a_i$ FOR ALL $a_i \in \{a_n\}$.

II m IS SAID TO BE A **lower bound** OF $\{a_n\}$, IF $m \le a_i$ FOR ALL $a_i \in \{a_n\}$

III A SEQUENCE IS SAID TO BE **bounded**, IF IT HAS AN UPPER BOUND (IS BOUNDED ABOVE) AND IF IT HAS A LOWER BOUND (IS BOUNDED BELOW).

Note:

✓ A SEQUENCE $\{a_n\}$ IS BOUNDED, IF AND ONLY IF THERE EXISTS $k > 0$ SUCH THAT $|a_n| \le k$ FOR ALL $n \in \mathbb{N}$.

Example 1 CONSIDER THE SEQUENCE $\left\{\frac{1}{n}\right\}$, WHERE THE TERMS ARE $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

CLEARLY, $0 < \frac{1}{n} \le 1$ FOR ALL $n \in \mathbb{N}$.

SOME UPPER BOUNDS ARE: $1, \sqrt{3}, 5, 2$, AND SOME LOWER BOUNDS ARE: $0, -2, -3, -5, -7$.

THUS, $\left\{\frac{1}{n}\right\}$ IS A BOUNDED SEQUENCE.

Example 2 SHOW THAT THE FOLLOWING SEQUENCES ARE BOUNDED.

A $\{(-1)^n\}$ **B** $\left\{\frac{4n-1}{2n}\right\}$

Solution

A THE SEQUENCE $\{(-1)^n\}$ IS BOUNDED BECAUSE $-1 \le (-1)^n \le 1$ FOR ALL $n \in \mathbb{N}$.

B CONSIDER THE GRAPH OF THE RATIONAL FUNCTION $y = \frac{4x-1}{2x}$. THE HORIZONTAL ASYMPTOTE, $y = 2$, IS THE LIMITING LINE OF THE CURVE.

IF WE MARK THE POINTS $\left(n, \frac{4n-1}{2n}\right)$ ON THE CURVE OF THE RATIONAL FUNCTION, IT GIVES THE GRAPH OF THE SEQUENCE. THE TERMS ARE INCREASING FROM $\frac{3}{2}$ TO 2.

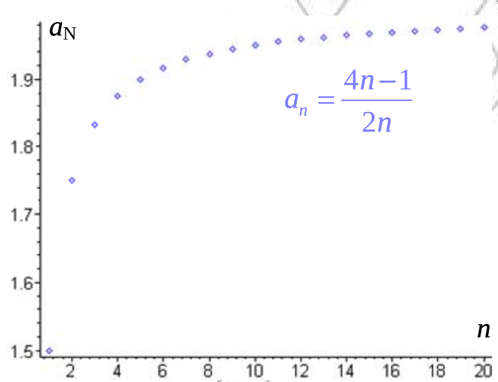


Figure 2.2

THUS, $\frac{3}{2} \le \frac{4n-1}{2n} < 2$ FOR ALL $n \in \mathbb{N}$. THIS SHOWS THAT $\left\{\frac{4n-1}{2n}\right\}$ IS BOUNDED.

Example 3 FOR EACH OF THE FOLLOWING SEQUENCES,

I FIND SOME UPPER BOUNDS AND SOME LOWER BOUNDS.

II DETERMINE THE GREATEST ELEMENT OF THE SET OF LOWER BOUNDS AND THE LEAST ELEMENT OF THE SET OF UPPER BOUNDS.

A $\left\{\frac{(-1)^n}{n}\right\}$ **B** $\{1 - n\}$ **C** $\{2^n\}$ **D** $\left\{\left(\frac{1}{n}\right)^n\right\}$

Solution ONE OF THE STRATEGIES IN FINDING UPPER BOUNDS AND LOWER BOUNDS OF A SEQUENCE IS TO LIST THE FIRST FEW TERMS AND OBSERVE ANY TREND.

A THE FIRST FEW TERMS OF $\left\{\frac{(-1)^n}{n}\right\}$ ARE:

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots,$

WHICH ARE CONSISTING OF NEGATIVE AND POSITIVE VALUES WITH – 1 THE MINIMUM TERM AND $\frac{1}{2}$ THE MAXIMUM TERM.

HENCE, $-1 \le \frac{(-1)^n}{n} \le \frac{1}{2}$ FOR ALL $n \in \mathbb{N}$.

THE SET OF LOWER BOUNDS IS THE INTERVAL $(-\infty, -1]$ WHOSE GREATEST ELEMENT IS -1.

THE SET OF UPPER BOUNDS IS THE INTERVAL $\left[\frac{1}{2}, \infty\right)$ WHOSE LEAST ELEMENT IS $\frac{1}{2}$.

B THE TERMS OF $\{1-n\}$ ARE:

0, –1, –2, –3, . . . ,

WHICH ARE DECREASING TO NEGATIVE INFINITY STARTING FROM 0. THIS SHOWS THAT THE SEQUENCE HAS NO LOWER BOUND (IS UNBOUNDED BELOW). THE SET OF UPPER BOUNDS IS $[0, \infty)$, WITH 0 THE LEAST ELEMENT OF ALL THE UPPER BOUNDS.

C WHEN WE CONSIDER $\{2^n\}$, THE TERMS ARE 2, 4, 8, 16, . . ., WHICH ARE INCREASING STARTING FROM 2 AND INDEFINITELY INCREASING. THUS, $\{2^n\}$ HAS NO UPPER BOUND, WHEREAS THE INTERVAL $(-\infty, 2]$ IS THE SET OF THE LOWER BOUNDS WITH 2 BEING THE GREATEST ELEMENT.

D THE TERMS OF $\left\{\left(\frac{1}{n}\right)^n\right\}$ ARE NON-NEGATIVE NUMBERS STARTING FROM 1 AND DECREASING TO 0 AT A FASTER RATE AS COMPARED TO $\left\{\frac{1}{n}\right\}$.

LOOK AT ITS TERMS: $1, \frac{1}{4}, \frac{1}{27}, \frac{1}{256}, \ldots$

CLEARLY, $0 < \left(\frac{1}{n}\right)^n \le 1$, FOR ALL $n \in \mathbb{N}$.

THUS THE SET OF LOWER BOUNDS IS $(-\infty, 0]$ WITH 0 BEING THE GREATEST ELEMENT AND THE SET OF UPPER BOUNDS IS $[1, \infty)$ WITH 1 THE LEAST ELEMENT.

THE FOLLOWING TABLE CONTAINS A FEW UPPER BOUNDS AND A FEW LOWER BOUNDS.

Sequence	Few upper bounds	Few lower bounds
$\left\{\frac{(-1)^n}{n}\right\}$	$\frac{1}{2}$, 1, 4, 10	–1, –2, –5, –7.5
$\{1-n\}$	0, 1, , 5	NONE
$\{2^n\}$	NONE	2, $\frac{1}{2}$, 0, $-\sqrt{10}$
$\left\{\left(\frac{1}{n}\right)^n\right\}$	1, 2, 3, 12	0, –1, –2, –

Least upper bound (lub) and greatest lower bound (glb)

IN EXAMPLE 3 ABOVE, YOU HAVE SEEN THE LEAST ELEMENT OF THE SET OF UPPER BOUNDS AND THE GREATEST ELEMENT OF THE SET OF LOWER BOUNDS. NOW, YOU CONSIDER SEQUENCES OF N GENERAL AND GIVE THE FOLLOWING FORMAL DEFINITION.

Definition 2.2

LET $\{a_n\}$ BE A SEQUENCE OF NUMBERS.

1 x IS SAID TO BE THE **least upper bound** (**lub**) OF $\{a_n\}$

- **I** IF x IS AN UPPER BOUND OF $\{a_n\}$, AND
- **II** WHENEVER y IS AN UPPER BOUND OF $\{a_n\}$, THEN $x \le y$.

2 x IS CALLED THE **greatest lower bound** (**glb**) OF $\{a_n\}$

- **I** IF x IS A LOWER BOUND OF $\{a_n\}$ AND
- **II** WHENEVER y IS A LOWER BOUND OF $\{a_n\}$, THEN $x \ge y$.

YOU MAY DETERMINE THE LUB OR GLB OF A SEQUENCE USING DIFFERENT TECHNIQUES OF D A SEQUENCES SUCH AS LISTING THE FIRST FEW TERMS OR PLOTTING POINTS.

IN THE FOLLOWING EXAMPLE, TO DETERMINE THE LUB AND GLB PLOTTING THE POINTS MIGH MUCH MORE HELPFUL THAN LISTING THE TERMS.

Example 4 FIND THE LUB AND GLB OF THE SEQUENCE $\left\{\frac{2n-3}{n+1}\right\}$

Solution IF THE GENERAL TERM OF A SEQUENCE HAS A RATIONAL EXPRESSION, THEN PLOTTING THE POINTS ON THE CURVE OF THE CORRESPONDING RATIONAL FUNCTION CAN BE HEL

CONSIDER THE GRAPH OF $y = \frac{2x-3}{x+1}$.

IF YOU HAVE VALUES FOR THE NATURAL NUMBERS, THEN IT GIVES THE GRAPH OF THE

THE SEQUENCE INCREASES FROM $-\frac{1}{2}$ TO 2.

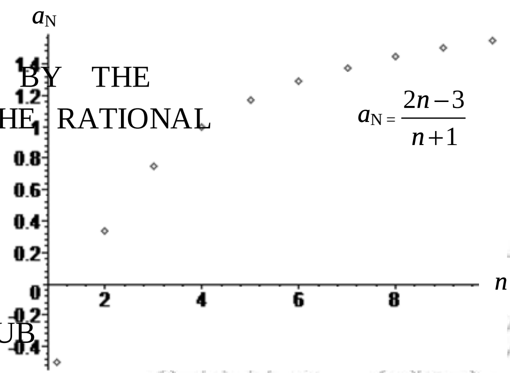


Figure 2.3

ITS ELEMENTS ARE LIMITED BY THE HORIZONTAL ASYMPTOTE OF THE RATIONAL FUNCTION.

HENCE, $-\frac{1}{2} \le \frac{2n-3}{n+1} \le 2$ FOR ALL $n \in \mathbb{N}$.

THEREFORE, THE GLB IS $-\frac{1}{2}$ AND THE LUB IS 2.

Example 5 FIND THE LUB AND GLB OF EACH OF THE FOLLOWING SEQUENCES.

A $\left\{\frac{1}{n}\right\}$ **B** $\left\{(-1)^n\right\}$ **C** $\left\{\frac{(-1)^n+1}{2}\right\}$

D $\left\{1-\frac{1}{n}\right\}$ **E** $\left\{1-\frac{(-1)^n}{n}\right\}$ **F** $\left\{\frac{2}{3^n}\right\}$

Solution IN THIS EXAMPLE, LISTING THE FIRST FEW TERMS IS SUFFICIENT TO DETERMINE THE LUB AND GLB.

LOOK AT THE FOLLOWING TABLE.

Sequence	First few terms	lub	glb
$\left\{\frac{1}{n}\right\}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ DECREASES	1	0
$\left\{(-1)^n\right\}$	$-1, 1, -1, 1, \ldots$ OSCILLATES	1	–1
$\left\{\frac{(-1)^n+1}{2}\right\}$	$0, 1, 0, 1, \ldots$ OSCILLATES	1	0
$\left\{1-\frac{1}{n}\right\}$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ INCREASES TO	1	0
$\left\{1-\frac{(-1)^n}{n}\right\}$	a_{2n-1}: DECREASE T $2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5}, \frac{5}{6}, \ldots$ a_{2n}: INCREASE T CONVERGES T	2	$\frac{1}{2}$
$\left\{\frac{2}{3^n}\right\}$	$\frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$ DECREASES TO	$\frac{2}{3}$	0

Example 6 FIND THE GLB AND LUB FOR EACH OF THE FOLLOWING SEQUENCES.

A $\left\{2^{\frac{1}{n}}\right\}$ **B** $\left\{(0.01)^{\frac{1}{n}}\right\}$

Solution THESE SEQUENCES NEED A CALCULATOR OR A COMPUTER TO LIST AS MANY TERMS AS POSSIBLE; ALTERNATIVELY PLOT THE CORRESPONDING FUNCTION GRAPH.

A THE LUB IS 2 AND THE GLB IS 1. **B** THE LUB IS 1 AND THE GLB IS 0.01.

Exercise 2.1

FOR EACH OF THE FOLLOWING SEQUENCES, FIND SOME UPPER BOUNDS AND LOWER BOU DETERMINE THE LUB AND GLB.

1 $\left\{\frac{(-1)^n}{n+3}\right\}$ **2** $\left\{\frac{n-1}{n+1}\right\}$ **3** $\left\{\frac{3n-2}{n}\right\}$ **4** $\left\{(-1)^n\left(1-\frac{1}{n}\right)\right\}$

5 $\left\{\frac{1-3n}{2n+5}\right\}$ **6** $\left\{2^{\frac{1}{n}}(-1)^n\right\}$ **7** $\left\{\frac{n+2}{3n-7}\right\}$ **8** $\left\{n^{\frac{1}{n}}\right\}$

9 $\left\{\frac{n!}{n^n}\right\}$ **10** $\left\{\frac{2^n}{n!}\right\}$

Monotonic sequences

Definition 2.3

LET $\{a_n\}$ BE A SEQUENCE OF NUMBERS. THEN,

I $\{a_n\}$ IS SAID TO BE AN INCREASING SEQUENCE, IF $a_n \le a_{n+1}$, FOR ALL $n \in \mathbb{N}$.

I.E. $\{a_n\}$ IS INCREASING, IF AND ONLY IF

$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le a_{n+1} \le \ldots$

II $\{a_n\}$ IS SAID TO BE STRICTLY INCREASING IF $a_n < a_{n+1}$, FOR ALL $n \in \mathbb{N}$

III $\{a_n\}$ IS SAID TO BE A DECREASING SEQUENCE, IF $a_n \ge a_{n+1}$, FOR ALL $n \in \mathbb{N}$. I.E., $\{a_n\}$ IS DECREASING, IF AND ONLY IF

$a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge a_{n+1} \ge \ldots$

IV $\{a_n\}$ IS SAID TO BE STRICTLY DECREASING IF $a_n > a_{n+1}$, FOR ALL $n \in \mathbb{N}$

Example 7 SHOW THAT THE SEQUENCE $\left\{3-\frac{1}{n}\right\}$ IS STRICTLY INCREASING.

Solution THIS CAN BE SEEN DIRECTLY FROM THE ORDER OF THE TERMS:

$$3-1<3-\frac{1}{2}<3-\frac{1}{3}<3-\frac{1}{4}$$

ALSO, $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1} \Rightarrow -\frac{1}{n} < -\frac{1}{n+1}$

$\Rightarrow 3-\frac{1}{n} < 3-\frac{1}{n+1}$, FOR ALL $n \in \mathbb{N} \Rightarrow \left\{3-\frac{1}{n}\right\}$ IS STRICTLY INCREASING.

Example 8 SHOW THAT $\left\{3+\frac{1}{n}\right\}$ IS STRICTLY DECREASING.

Solution NOTE THAT $3+1 > 3+\frac{1}{2} > 3+\frac{1}{3} > \ldots > 3+\frac{1}{n} > 3+\frac{1}{n+1} > \cdots$

$\Rightarrow \quad 3+\frac{1}{n} > 3+\frac{1}{n+1}, \ \forall n \in \mathbb{N}$

$\Rightarrow \left\{3+\frac{1}{n}\right\}$ IS STRICTLY DECR

Definition 2.4

A SEQUENCE $\{a_n\}$ IS SAID TO BE MONOTONIC OR A MONOTONE SEQUENCE, IF IT IS EITHER INCREASING OR DECREASING.

Example 9 SHOW THAT $\left\{\frac{(-1)^n}{n}\right\}$ IS NOT MONOTONIC.

Solution IT SUFFICES TO LIST THE FIRST FEW TERMS OF THE SEQUENCE.

THE TERMS $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$ ARE NEITHER IN AN INCREASING ORDER NOR IN A DECREASING ORDER. THUS, $\left\{\frac{(-1)^n}{n}\right\}$ IS NOT MONOTONIC.

Example 10 DECIDE WHETHER OR NOT EACH OF THE FOLLOWING SEQUENCES IS MONOTONIC.

A $\left\{8-\frac{1}{n}\right\}$ **B** $\left\{8+\frac{1}{n}\right\}$ **C** $\left\{1-\frac{(-1)^n}{n}\right\}$

Solution

A IN $\left\{8-\frac{1}{n}\right\}$, SINCE $\left\{-\frac{1}{n}\right\}$ IS INCREASING TO 0, $\left\{8-\frac{1}{n}\right\}$ IS INCREASING TO 8.

HENCE, IT IS MONOTONIC.

B $\left\{\frac{1}{n}\right\}$ IS A DECREASING SEQUENCE; IT IS DECREASING TO 0. HENCE $\left\{8+\frac{1}{n}\right\}$ DECREASES TO 8.

HENCE, IT IS MONOTONIC.

C YOU CAN WRITE THE TERMS OF THE SEQUENCE AS:

$1+\frac{1}{n}$, IF n IS ODD; $\rightarrow 2 \searrow \frac{1}{2} \nearrow \frac{4}{3} \searrow \frac{3}{4} \nearrow \frac{6}{5} \searrow \frac{5}{6}$

$1-\frac{1}{n}$, IF n IS EVEN; $\rightarrow$

THIS SHOWS THAT $\left\{1+\frac{(-1)^n}{n}\right\}$ IS NOT MONOTONIC.

Exercise 2.2

1 SHOW THAT EACH OF THE FOLLOWING SEQUENCES IS MONOTONIC AND BOUNDED.

A $\left\{\frac{n+1}{2n-1}\right\}$ **B** $\left\{\frac{1}{n^2+4}\right\}$ **C** $\left\{3^{\frac{1}{n}}\right\}$

D $\sin\left(\frac{\quad}{2n}\right)$ **E** $\cos\left(\frac{1}{n}\right)$ **F** $\left\{\frac{2n+1}{n+5}\right\}$

2 GIVE EXAMPLES OF CONVERGENT SEQUENCES THAT ARE NOT MONOTONIC.

3 GIVE EXAMPLES OF BOUNDED SEQUENCES THAT ARE NOT CONVERGENT.

4 CAN YOU FIND A CONVERGENT SEQUENCE THAT IS NOT BOUNDED?

5 IN EACH OF THE FOLLOWING, DETERMINE WHETHER THE SEQUENCE IS BOUNDED OR NOT BOUNDED.

A $\left\{n+\frac{1}{n}\right\}$ **B** $\left\{7+\frac{2}{n}\right\}$ **C** $\left\{\frac{4}{n^2+1}\right\}$ **D** $\{\sin(n)\}$

E $\left\{7^{\frac{1}{n}}\right\}$ **F** $\left\{\left(\frac{1}{e}\right)^n\right\}$ **G** $\left\{\frac{\sqrt{n}-1}{\sqrt{n}+1}\right\}$ **H** $\left\{\ln\left(\frac{1}{n}\right)\right\}$

6 USE AN APPROPRIATE METHOD TO SHOW THAT EACH OF THE FOLLOWING SEQUENCES CONVERGES.

A $\left\{3+\frac{4}{n}\right\}$ **B** $\left\{\frac{2n-3}{3n+2}\right\}$ **C** $\left\{\frac{1}{n+1}-\frac{2}{n+3}\right\}$

D $\left\{\frac{1+3+5+\ldots+(2n-1)}{6n^2+1}\right\}$ **E** $\left\{\frac{2^{n+1}}{5^{n-4}}\right\}$ **F** $\left\{\frac{2n}{n^2+100}\right\}$

G $\left\{\sin\left(\frac{\quad}{n}\right)\right\}$ **H** $\left\{1+\frac{(-1)^n}{n}\right\}$

2.1.2 Limits of Sequences

OPENING PROBLEM

Consider the terms of the sequence $\left\{\frac{1}{n}\right\}$.

1 List terms of $\left\{\frac{1}{n}\right\}$ that satisfy the condition $0 < \frac{1}{n} < 10^{-2}$

2 Find the smallest natural number k such that $0 < \frac{1}{n} < 10^{-5}$ for all $n \geq k$.

Sequences are common examples in the study of limits. In particular, sequenc tend to a unique value when n increases indefinitely are important in the introductor part of limits of sequences of numbers.

ACTIVITY 2.2

Decide whether each of the following sequences tends to a unique real number as n increases.

1 $\left\{\frac{1}{n}\right\}$ **2** $\left\{\frac{(-1)^n}{n}\right\}$ **3** $\{4\}$ **4** $\{-10^{-n}\}$

5 $\left\{\left(\frac{2}{3}\right)^n\right\}$ **6** $\left\{\frac{n+5}{n}\right\}$ **7** $\left\{(-1)^n\right\}$ **8** $\{2^n\}$

In Activity 2.2, the terms of some of the sequences are tending to a unique real num as n gets larger and larger.

Consider the terms of the sequence $\left\{\frac{1}{n}\right\}$:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots$$

It is clear that as the value of n becomes larger and larger, the n^{th} term $\left(\frac{1}{n}\right)$ of the sequence becomes smaller in value and hence it becomes closer and closer to 0. For extremely large values of n, it will be very hard to distinguish the values of $\frac{1}{n}$ from 0.

In this case, 0 is said to be the limit of the sequence $\left\{\frac{1}{n}\right\}$ and you express this idea shortly by writing $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

READ $\lim_{n\to\infty}\frac{1}{n}=0$ AS "THE LIMIT OF $\frac{1}{n}$ AS n APPROACHES TO INFINITY IS 0."

ALSO, FOR THE SEQUENCE $\left\{\frac{1}{2^n}\right\}$, WHOSE TERMS ARE:

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\dots,\frac{1}{2^n},\frac{1}{2^{n+1}},\dots$$

YOU CAN SEE THAT $\lim_{n\to\infty}\left(\frac{1}{2^n}\right)=0$.

OBSERVE THAT THE TERMS OF THE $\left\{\frac{1}{2^n}\right\}$ ARE DECREASING TO 0 AT A RA THAN THAT OF $\left\{\frac{1}{n}\right\}$. FIGURE 2.4 SHOWS THIS COMPARISON.

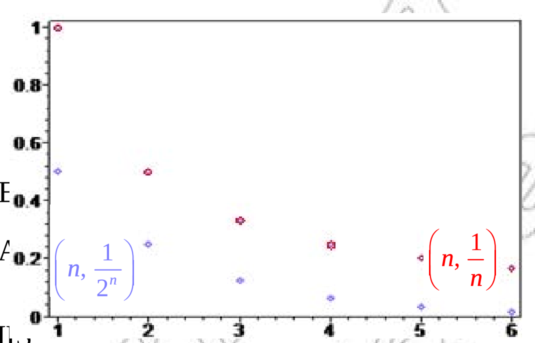


Figure 2.4

✍Note:

✓ IF A CONSTANT c IS ADDED TO nTH TERM OF THE SEQUENCE $\left\{\frac{1}{n}\right\}$, THEN YOU GET THE SEQUENCE $\left\{c+\frac{1}{n}\right\}$ WHICH CONVERGES TO c.

Example 11 CONSIDER THE SEQUENCE $\left\{5+\frac{1}{n}\right\}$, WHOSE TERMS ARE

$$5+1,\ 5+\frac{1}{2},\ 5+\frac{1}{3},\ 5+\frac{1}{4},\ \dots,\ 5+\frac{1}{n},\ \dots$$

AS n GETS LARGE, $\frac{1}{n}$ GETS CLOSE TO 0 SO THAT $5+\frac{1}{n}$ GETS CLOSE TO 5 + 0.

THEREFORE, $\lim_{n\to\infty}\left(5+\frac{1}{n}\right)=5$.

THIS CAN BE SEEN GRAPHICALLY, A SHIFTING THE GRAPH OF $a_n=\frac{1}{n}$ BY 5 UNITS THE POSITIVE y-DIRECTION GIVES THE G $a_n=5+\frac{1}{n}$, SO THAT AS n GETS LARGE ITS APPROACHES THE LINE WITH EQUATION $y = 5$ INSTEAD OF THE LINE WITH EQUATION $y = 0$.

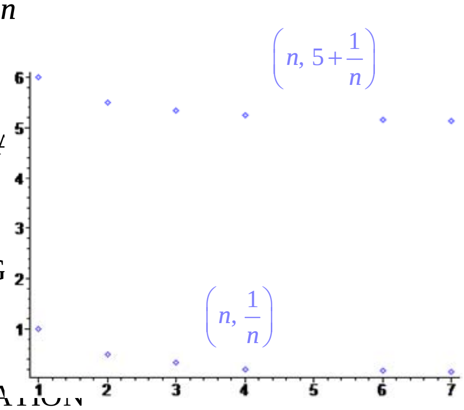


Figure 2.5

IN GENERAL, FOR A SEQUENCE $\{a_n\}$, IF THERE EXISTS A UNIQUE REAL NUMBER L SUCH THAT a_n BECOMES CLOSER AND CLOSER TO L AS n BECOMES INDEFINITELY LARGE, THEN L IS SAID TO BE THE LIMIT OF $\{a_n\}$ AS n APPROACHES INFINITY.

SYMBOLICALLY, THIS CONCEPT IS WRITTEN AS: $\lim_{n\to\infty} a_n = L$

IF SUCH A REAL NUMBER L EXISTS, THEN WE SAY THAT $\{a_n\}$ CONVERGES TO L. IF SUCH A NUMBER DOES NOT EXIST, WE SAY $\{a_n\}$ DIVERGES OR $\lim_{n\to\infty} a_n$ DOES NOT EXIST.

Example 12 SHOW THAT THE SEQUENCE $\{(-5)^n\}$ DIVERGES.

Solution THE TERMS OF THE SEQUENCE $\{(-5)^n\}$ ARE

$-5, 25, -125, 625, \ldots$

THUS, $\lim_{n\to\infty} (-5)^n$ DOES NOT APPROACH A UNIQUE NUMBER. THEREFORE, $\{(-5)^n\}$ DIVERGES.

Example 13 SHOW THAT THE SEQUENCE $\{2^n\}$ DIVERGES.

Solution THE TERMS OF THE SEQUENCE $\{2^n\}$ ARE: $2, 2^2, 2^3, 2^4, \ldots, 2^n, 2^{n+1}, \ldots$ WHICH ARE INDEFINITELY INCREASING AS n INCREASES TO INFINITY.

THUS, $\lim_{n\to\infty}\left(2^n\right) = \infty$. THIS SHOWS THAT $\{2^n\}$ DIVERGES.

Example 14 DECIDE WHETHER OR NOT THE SEQUENCE $\left\{\frac{5n-2}{3n}\right\}$ CONVERGES.

Solution FIRST WE NOTICE THAT $\frac{5n-2}{3n} = \frac{\left(\frac{5n-2}{n}\right)}{\left(\frac{3n}{n}\right)} = \frac{5-\frac{2}{n}}{3}$

TOGETHER WITH $\lim_{n\to\infty}\frac{1}{n} = 0$, WE HAVE $\lim_{x\to\infty}\left(\frac{5-\frac{2}{n}}{3}\right) = \frac{5}{3}$

HENCE, THE SEQUENCE $\left(\frac{5n-2}{3n}\right)$ CONVERGES TO $\frac{5}{3}$.

Example 15 SHOW THAT THE SEQUENCE {SIN (N)} IS DIVERGENT.

Solution YOU KNOW THAT $-1 \le \sin(n) \le 1$. AS n GETS LARGE, SIN (n) STILL OSCILLATES BETWEEN –1 AND 1. IT DOES NOT APPROACH A UNIQUE NUMBER.

THUS, {SIN (n)} DIVERGES.

Null sequence

Definition 2.5

A SEQUENCE $\{a_n\}$ IS SAID TO BE A NULL SEQUENCE, IF AND ONLY IF $\lim_{n\to\infty} a_n = 0$.

Example 16 EACH OF THE FOLLOWING SEQUENCES IS A NULL SEQUENCE.

A $\left\{\frac{1}{n}\right\}$ **B** $\left\{\frac{1}{10^n}\right\}$ **C** $\left\{\frac{1}{n^2+5}\right\}$ **D** $\left\{\frac{(-1)^n}{n}\right\}$

Example 17 SHOW THAT THE SEQUENCE $\left\{\frac{\cos(n)}{n}\right\}$ IS A NULL SEQUENCE.

Solution NOTICE THAT AS n APPROACHES TO INFINITY, $-1 \le \cos n \le 1$.

SO $\lim_{n\to\infty} \frac{\cos(n)}{n} = \frac{\text{FINITE QUANTITY}}{\text{INFINITE QUANTITY}} = 0$. THUS, $\left\{\frac{\cos(n)}{n}\right\}$ IS A NULL SEQUENCE.

Example 18 SHOW THAT THE SEQUENCE $\left\{\sin\left(\frac{1}{n}\right)\right\}$ IS A NULL SEQUENCE.

Solution THE TERMS OF THE SEQUENCE

$\sin(1)$, $\sin\left(\frac{1}{2}\right)$, $\sin\left(\frac{1}{3}\right)$, ... ARE DECREASING TO SIN 0.

THUS, $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$

THIS CAN BE SHOWN GRAPHICALLY:

AS n GOES TO INFINITY, $\frac{1}{n}$ TENDS TO 0. THUS,

$\sin\left(\frac{1}{n}\right)$ IS A NULL SEQUENCE

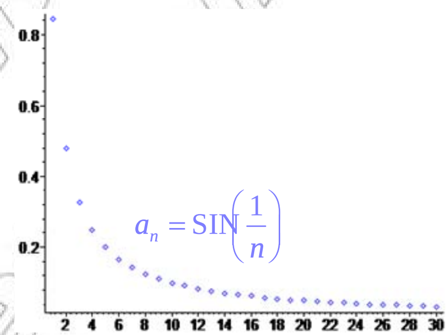


Figure 2.6

Exercise 2.3

1 FIND THE LIMIT OF EACH OF THE FOLLOWING SEQUENCES AS n TENDS TO INFINITY.

A $\left\{\frac{3}{n+1}\right\}$ **B** $\left\{\frac{(-1)^n}{n^2}\right\}$ **C** $\left\{\frac{1}{6^n}\right\}$ **D** $\left\{7^{\frac{1}{n}}\right\}$

E $\left\{(0.5)^{\frac{1}{n}}\right\}$ **F** $\left\{1-\frac{1}{n^2}\right\}$ **G** $\left\{\frac{\cos(n)}{n}\right\}$ **H** $\left\{\cos\left(\frac{1}{n}\right)\right\}$

I $\left\{n + \frac{1}{n}\right\}$ **J** $\left\{\frac{1+n}{2+n}\right\}$ **K** $\left\{1, 0, \frac{1}{3}, 0, \frac{5}{7}, 0, \frac{7}{9}, 0. . . .\right\}$

L $\left\{\frac{n+3}{1-2n}\right\}$ **M** $\left\{n-\frac{10}{n}\right\}$ **N** $\left\{\frac{(-1)^n(n-1)}{n+1}\right\}$

O { 0.6, 0.66, 0.666, . . . }

2 DECIDE WHETHER OR NOT EACH OF THE FOLLOWING SEQUENCES IS A NULL SEQUENCE.

A $\left\{-\frac{1}{n}\right\}$ **B** $\left\{1-\frac{2}{n+1}\right\}$ **C** $\left\{\frac{(-1)^n}{n^2+1}\right\}$

D $\left\{\frac{3}{n(n+1)}\right\}$ **E** $\left\{\left(\frac{7}{8}\right)^{n-2}\right\}$ **F** $\left\{2^n-2^{-n}\right\}$

G $\left\{\frac{4n-1}{n^2+1}\right\}$ **H** $\left\{\frac{2^n}{n^2+1}\right\}$ **I** $\left\{\frac{\sqrt{n}+1}{n}\right\}$

2.1.3 Convergence Properties of Sequences

ACTIVITY 2.3

1 GIVEN ON THE NEXT PAGE ARE GRAPHS OF SOME SEQUENCES. IDENTIFY THOSE GRAPHS WHICH ARE BOUNDED AND FIND THEIR LIMITS.

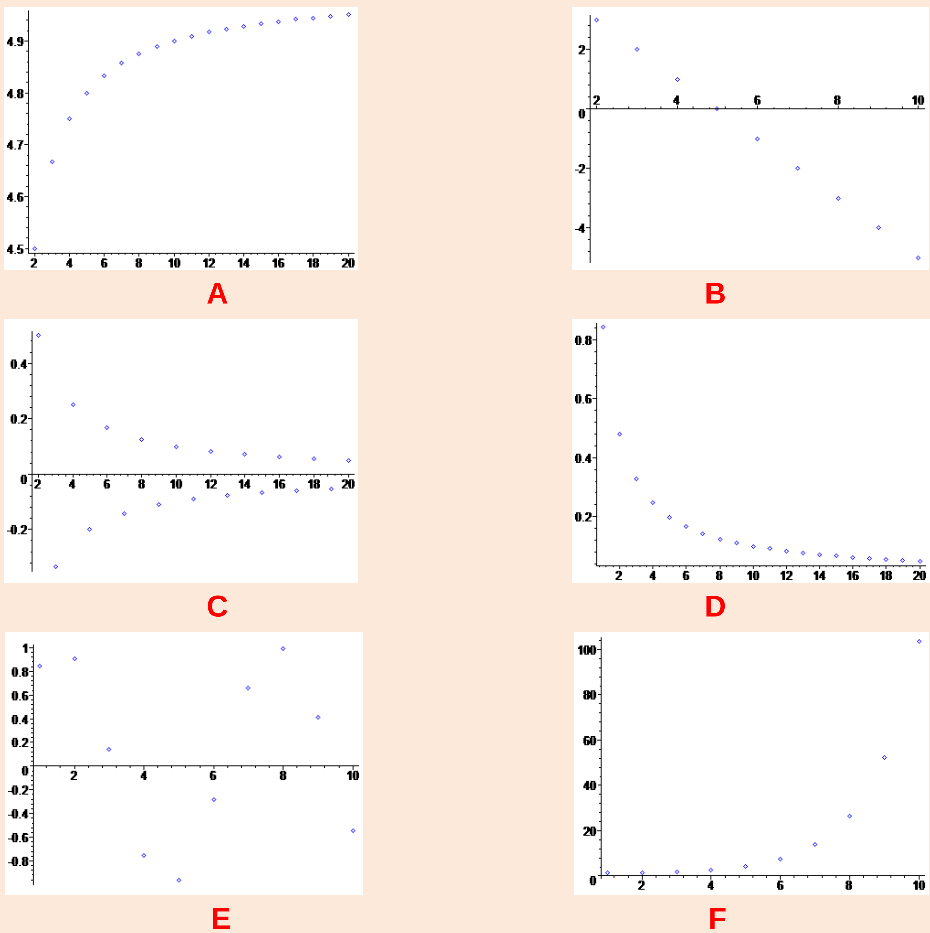


Figure 2.7

2 FOR EACH OF THE FOLLOWING SEQUENCES,

- **I** DECIDE WHETHER OR NOT IT IS BOUNDED AND/OR MONOTONIC
- **II** DETERMINE THE LIMITS IN TERMS OF THE GLB AND LUB.

A $\left\{1+\frac{1}{n}\right\}$ **B** $\left\{3-\frac{2}{n}\right\}$ **C** $\{4-n\}$

D $\{2^{1-n}\}$ **E** $\left\{\sin\left(\frac{1}{n}\right)\right\}$ **F** $\{-2^{n}\}$

FROM ACTIVITY 2.3, YOU HAVE THE FOLLOWING FACTS ABOUT MONOTONIC SEQUENCES.

1. IF A MONOTONIC SEQUENCE IS UNBOUNDED, THEN IT DIVERGES.
2. IF A MONOTONIC SEQUENCE IS BOUNDED, THEN IT CONVERGES.
   - **A** IF IT IS BOUNDED AND INCREASING, THEN IT CONVERGES TO ITS LEAST UPPER BOUND (LUB) OF THE SEQUENCE.
   - **B** IF IT IS BOUNDED AND DECREASING, THEN IT CONVERGES TO ITS GREATEST LOWER BOUND (GLB) OF THE SEQUENCE.

Example 19 SHOW THAT THE SEQUENCE $\left\{\frac{n+1}{2n+3}\right\}$ CONVERGES.

Solution OBSERVE THAT $\frac{n+1}{2n+3}=\frac{1}{2}-\frac{1}{2(2n+3)}$

THE SEQUENCE $-\frac{1}{2(2n+3)}$ IS INCREASING.

HENCE, $\frac{1}{2}-\frac{1}{2(2n+3)}$ IS INCREASING, WITH

$\frac{2}{5}\le\frac{n+1}{2n+3}<\frac{1}{2}$ FOR ALL $n\in\mathbb{N}$. *Explain!*

THEREFORE, $\left\{\frac{n+1}{2n+3}\right\}$ IS BOUNDED AND MONOTONIC AND HENCE IT CONVERGES.

ALSO, $\lim_{n\to\infty}\frac{n+1}{2n+3}=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2(2n+3)}\right)=\frac{1}{2}$. WHY?

THUS, $\left\{\frac{n+3}{2n+3}\right\}$ CONVERGES TO THE LEAST UPPER BOUND OF THE SEQUENCE.

SO FAR, THE LIMIT OF A SEQUENCE $\{a_n\}$ HAS BEEN DISCUSSED. YOUR NEXT TASK IS TO DETERMINE THE LIMITS OF THE SUM, DIFFERENCE, PRODUCT AND QUOTIENT OF TWO OR MORE SEQUENCES.

Theorem 2.1

LET $\{a_n\}$ AND $\{b_n\}$ BE CONVERGENT SEQUENCES WITH $\lim\limits_{n\to\infty} a_n = L$ AND $\lim\limits_{n\to\infty} b_n = M$. THEN THE SUM $\{a_n + b_n\}$, THE DIFFERENCE $\{a_n - b_n\}$, A CONSTANT MULTIPLE $\{c\,a_n\}$, THE PRODUCT $\{a_n b_n\}$, AND THE QUOTIENT $\left\{\frac{a_n}{b_n}\right\}$, PROVIDED THAT $M \neq 0$ AND $b_n \neq 0$ FOR EVERY n, ARE CONVERGENT WITH

1. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$
2. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = L - M$
3. $\lim\limits_{n\to\infty}(c\,a_n) = c\lim\limits_{n\to\infty} a_n = cL$ FOR A CONSTANT c
4. $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = LM$
5. $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{M}$
6. IF $a_n \geq 0$, AS $n \to \infty$, $\lim\limits_{n\to\infty}\sqrt{a_n} = \sqrt{\lim\limits_{n\to\infty} a_n} = \sqrt{L}$

Example 20 EVALUATE $\lim\limits_{n\to\infty}\left(8 + \frac{1}{n}\right)$

Solution USING PROPERTY 1,

$$\lim_{n\to\infty}\left(8 + \frac{1}{n}\right) = \lim_{n\to\infty} 8 + \lim_{n\to\infty}\frac{1}{n} = 8 + 0 = 8$$

Example 21 EVALUATE $\lim\limits_{n\to\infty}\frac{n+2}{3n-5}$

Solution FIRST, YOU DIVIDE THE NUMERATOR AND THE DENOMINATOR OF THE EXPRESSION BY n.

THEN, $\frac{n+2}{3n-5} = \frac{\left(\frac{n+2}{n}\right)}{\frac{3n-5}{n}} = \frac{1+\frac{2}{n}}{3-\frac{5}{n}}$

$$\Rightarrow \lim_{n\to\infty}\frac{n+2}{3n-5} = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{3-\frac{5}{n}}\right) = \frac{\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)}{\lim\limits_{n\to\infty}\left(3-\frac{5}{n}\right)} = \frac{\lim\limits_{n\to\infty}(1) + \lim\limits_{n\to\infty}\left(\frac{2}{n}\right)}{\lim\limits_{n\to\infty}(3) - \lim\limits_{n\to\infty}\left(\frac{5}{n}\right)}$$

$$= \frac{1 + 2\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)}{3 - 5\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{1+2\times 0}{3-5\times 0} = \frac{1}{3}$$

Example 22 FIND $\lim_{n\to\infty} \frac{1}{n(n+3)}$

Solution USING PARTIAL FRACTIONS

$$\frac{1}{n(n+3)} = \frac{a}{n} + \frac{b}{n+3}$$, FOR CONSTANTS a AND b.

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n(n+3)} = \lim_{n\to\infty} \frac{a}{n} + \lim_{n\to\infty} \frac{b}{n+3}$$

$$= a \lim_{n\to\infty} \frac{1}{n} + b \lim_{n\to\infty} \frac{1}{n+3} = a \times 0 + b \times 0 = 0$$

Example 23 FIND $\lim_{n\to\infty} \frac{3n^2+4n+1}{2n^2+7}$

Solution SINCE BOTH THE NUMERATOR AND THE DENOMINATOR HAVE THE SAME DEGREE, FIRST DIVIDE BOTH BY n^2

$$\lim_{n\to\infty} \frac{3n^2+4n+1}{2n^2+7} = \lim_{n\to\infty} \frac{\frac{3n^2+4n+1}{n^2}}{\frac{2n^2+7}{n^2}} = \lim_{n\to\infty} \left(\frac{3+\frac{4}{n}+\frac{1}{n^2}}{2+\frac{7}{n^2}} \right) = \frac{\lim_{n\to\infty}\left(3+\frac{4}{n}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(2+\frac{7}{n^2}\right)}$$

$$\frac{\lim_{x\to\infty} 3 + \lim_{n\to\infty} \frac{4}{n} + \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{7}{n^2}} = \frac{3+0+0}{2+0} = \frac{3}{2}$$

Example 24 EVALUATE $\lim_{n\to\infty} \left(\frac{2^{n+2}}{3^{n-3}} \right)$

Solution

$$\lim_{n\to\infty} \left(\frac{2^{n+2}}{3^{n-3}} \right) = \lim_{n\to\infty} \left(\frac{2^n \times 2^2}{3^n \times \frac{1}{27}} \right) = \lim_{n\to\infty} 108 \left(\frac{2}{3} \right)^n = 108 \times 0 = 0$$

Example 25 FIND THE LIMIT OF THE SEQUENCE WHOSE TERMS ARE:

0.3, 0.33, 0.333, 0.3333, . . .

Solution CLEARLY, THE SEQUENCE CONVERGES TO $0.\dot{3}$, IF THE TERMS CONTINUE BY A SERIES OF 3'S.

MOREOVER, THE n^{TH} TERM OF THE SEQUENCE CAN BE EXPRESSED IN TERMS OF n AS FOLLOWS

$$0.3 = \frac{3}{10} = 3\left(\frac{9}{9\times 10}\right) = 3\left(\frac{10-1}{9\times 10}\right)$$

$$0.33=\frac{33}{100}=\frac{3}{10^2}\left(\frac{99}{9}\right)=\frac{3}{10^2}\left(\frac{10^2-1}{9}\right)$$

Also, $0.333=\dfrac{3}{10^3}\left(\dfrac{10^3-1}{9}\right)$ so that

$$a_n=\frac{3}{10^n}\left(\frac{10^n-1}{9}\right)\text{ or } a_n=\frac{3}{9}\left(\frac{10^n-1}{10^n}\right)=\frac{1}{3}\left(1-\frac{1}{10^n}\right)$$

Thus, $\lim\limits_{n\to\infty}\frac{1}{3}\left(1-\frac{1}{10^n}\right)=\lim\limits_{n\to\infty}\left(\frac{1}{3}-\frac{1}{3}\times\frac{1}{10^n}\right)=\lim\limits_{n\to\infty}\frac{1}{3}-\frac{1}{3}\lim\limits_{n\to\infty}\frac{1}{10^n}=\frac{1}{3}-\frac{1}{3}\cdot 0=\frac{1}{3}$

Example 26 Evaluate $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+1}-1}{\sqrt{n^2+1}+1}$

Solution

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}-1}{\sqrt{n^2+1}+1}=\lim_{n\to\infty}\left(\frac{\dfrac{\sqrt{n^2+1}-1}{n}}{\dfrac{\sqrt{n^2+1}+1}{n}}\right)=\lim_{n\to\infty}\frac{\sqrt{\dfrac{n^2+1}{n^2}}-\dfrac{1}{n}}{\sqrt{\dfrac{n^2+1}{n^2}}+\dfrac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\dfrac{1}{n^2}}-\dfrac{1}{n}}{\sqrt{1+\dfrac{1}{n^2}}+\dfrac{1}{n}}$$

$$=\frac{\lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n^2}}-\lim\limits_{n\to\infty}\dfrac{1}{n}}{\lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n^2}}+\lim\limits_{n\to\infty}\dfrac{1}{n}}=\frac{\sqrt{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)}-0}{\sqrt{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)}+0}=1$$

Exercise 2.4

Evaluate each of the limits given in 1 – 12.

1. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}+\dfrac{3}{n+2}\right)$

2. $\lim\limits_{n\to\infty}\left(\dfrac{3^n+2^n}{6^n}\right)$

3. $\lim\limits_{n\to\infty}\left(\left(\sqrt{3}\right)^n\right)$

4. $\lim\limits_{n\to\infty}\left(\dfrac{25}{n+10}\right)$

5. $\lim\limits_{n\to\infty}\left(\dfrac{n^2+1}{30n+100}\right)$

6. $\lim\limits_{n\to\infty}\left(\dfrac{1+n+n^2}{n}\right)$

7. $\lim\limits_{n\to\infty}\left(-\dfrac{3}{5}\right)^n$

8. $\lim\limits_{n\to\infty}\left(20+\left(-\dfrac{1}{3}\right)^n\right)$

9. $\lim\limits_{n\to\infty}\left(\left(\dfrac{1}{3}\right)^n-n\right)$

10. $\lim\limits_{n\to\infty}\dfrac{(3n+1)^2}{2n^2+3n+1}$

11. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+5}}{n+1}$

12. $\lim\limits_{n\to\infty}\left(\dfrac{2n+3}{2n+5}\times\dfrac{5n-2}{6n+1}\right)$

13 $\lim_{n\to\infty}\left(\frac{1+2^2+3^2+...+n^2}{n^3}\right)$ **14** $\lim_{n\to\infty}\left(ne^{-n}\right)$ **15** $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

16 $\lim_{n\to\infty}\left(\frac{n+3}{1+\sqrt{n}}\right)$ **17** $\lim_{n\to\infty}\left(\frac{1}{2}\right)^{1-\frac{1}{2n}}$ **18** $\lim_{n\to\infty}\frac{\sqrt{n^2+1}-3}{n+2}$

19 GIVE EXAMPLES OF SEQUENCES $\{a_n\}$ AND $\{b_n\}$ SUCH THAT

A $\lim_{n\to\infty}(a_n+b_n)$ EXISTS BUT NEITHER $\lim_{n\to\infty}a_n$ NOR $\lim_{n\to\infty}b_n$ EXISTS.

B $\lim_{n\to\infty}(a_nb_n)$ EXISTS BUT NEITHER $\lim_{n\to\infty}a_n$ NOR $\lim_{n\to\infty}b_n$ EXISTS.

20 LET $a_n = 2^n$ AND $b_n = n$. EVALUATE $\lim_{n\to\infty}\frac{a_n}{b_n}$

2.2 LIMITS OF FUNCTIONS

IN THIS TOPIC, YOU WILL USE FUNCTIONS SUCH AS POLYNOMIAL, RATIONAL, EXPONENTIAL, LO ABSOLUTE VALUE, TRIGONOMETRIC AND OTHER PIECE-WISE DEFINED FUNCTIONS IN ORDER T THE CONCEPT "LIMIT OF A FUNCTION".

WE WILL SEE DIFFERENT TECHNIQUES OF FINDING THE LIMIT OF A FUNCTION AT A POINT SUCH A COMMON FACTORS IN RATIONAL EXPRESSIONS, LIKE $\frac{(x-2)(x+5)}{(x-2)(x+1)}$, FOR $x \neq 2$, RATIONALIZATION, LIKE $\frac{\left(\sqrt{x}-1\right)}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$, GRAPHS, TABLES OF VALUES AND OTHER PROPERTIES.

Limits of Functions at a Point

ACTIVITY 2.4

1 USE THE GRAPH TO ANSWER THE QUESTIONS BELOW IT.

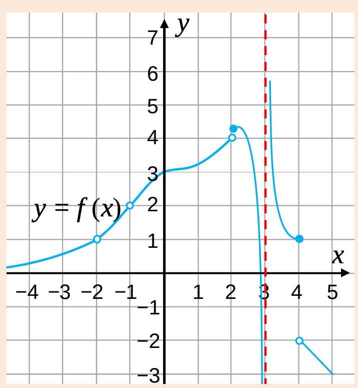


Figure 2.8

i WHAT IS THE DOMAIN OF f?

ii GIVE THE VALUES OF

a $f(-2)$ **b** $f(-1)$ **c** $f(2)$ **d** $f(3)$ **e** $f(4)$

iii WHAT NUMBER DOES $f(x)$ APPROACH TO AS x APPROACHES

a $-\infty$? **b** -2? **c** -1 FROM THE RIGHT?

d -1 FROM THE LEFT? **e** 0? **f** 2 FROM THE RIGHT?

g 2 FROM THE LEFT? **h** 4 FROM THE RIGHT?

i 4 FROM THE LEFT? **j** ∞ ?

2 EXPLAIN THE DIFFERENCE BETWEEN THE LIMITS $\lim_{n\to\infty} \frac{1}{n}$ AND $\lim_{x\to\infty} \frac{1}{x}$, WHERE $n \in \mathbb{N}$ AND $x \in \mathbb{R}$.

Definition 2.6 **The intuitive definition of the limit of a function at a point**

LET $y = f(x)$ BE A FUNCTION DEFINED ON AN INTERVAL SURROUNDING $x_0 \in \mathbb{R}$ (but f need not be defined at $x = x_0$). IF $f(x)$ GETS CLOSER AND CLOSER TO A SINGLE REAL NUMBER L AS x GETS CLOSER AND CLOSER TO (BUT NOT EQUAL TO) x_0, THEN WE SAY THAT THE LIMIT OF $f(x)$ AS x APPROACHES x_0 IS L.

Symbolically, this is written as

$$\lim_{x\to x_0} f(x) = L$$

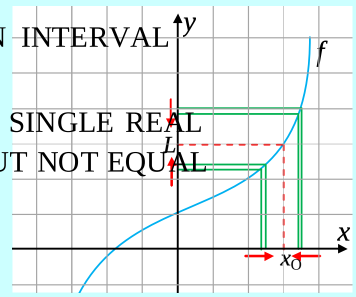


Figure 2.9

Example 1 LET $f(x) = x$. THEN $\lim_{x\to x_0} f(x) = x_o$

Example 2 LET $f(x) = \frac{x^2-4}{x-2}$. EVALUATE $\lim_{x\to 2} f(x)$

Solution LOOK AT THE GRAPH OF

$$f(x) = \frac{x^2-4}{x-2} = \begin{cases} x+2, & \text{IF } x \neq 2 \\ \nexists, & \text{IF } x = 2 \end{cases}$$

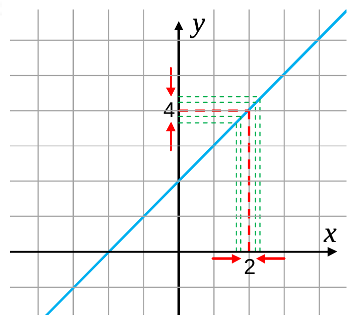


Figure 2.10

AS x GETS CLOSER AND CLOSER TO 2, $f(x)$ GETS CLOSER AND CLOSER TO 4.

$$\Rightarrow \lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4$$

Note:

✓ IF $f(x)$ APPROACHES TO DIFFERENT NUMBERS AS x APPROACHES x_0 FROM THE RIGHT AND FROM THE LEFT, THEN WE CONCLUDE THAT $\lim_{x\to x_o} f(x)$ DOES NOT EXIST.

ACTIVITY 2.5

1 EXPLAIN THE DIFFERENCE BETWEEN $\lim_{x \to a} f(x)$ AND $f(a)$.

2 WHAT HAPPENS TO $\lim_{x \to a} f(x)$, IF $f(x)$ APPROACHES TO DIFFERENT NUMBERS AS x APPROACHES TO a FROM THE RIGHT AND FROM THE LEFT? EXPLAIN THIS BY PRODUCING EXAMPLES.

3 THE LIMIT OF A FUNCTION $f(x)$ AS x APPROACHES a FROM THE RIGHT IS REPRESENTED BY THE SYMBOL $\lim_{x \to a^+} f(x)$ AND FROM THE LEFT BY $\lim_{x \to a^-} f(x)$.

ARE $\lim_{x \to a^+} f(x)$ AND $\lim_{x \to a^-} f(x)$ THE SAME FOR EVERY FUNCTION f?

WHAT CAN YOU SAY ABOUT $\lim_{x \to a} f(x)$, IF $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$?

4 CONSIDER THE FOLLOWING GRAPH OF A FUNCTION f.

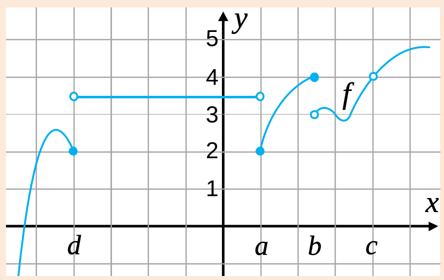


Figure 2.11

EVALUATE THE FOLLOWING LIMITS FROM THE GRAPH.

A $\lim_{x \to a^+} f(x)$ **B** $\lim_{x \to d^+} f(x)$ **C** $\lim_{x \to d} f(x)$ **D** $\lim_{x \to a^-} f(x)$

E $\lim_{x \to d^-} f(x)$ **F** $\lim_{x \to b^+} f(x)$ **G** $\lim_{x \to b^-} f(x)$ **H** $\lim_{x \to c^+} f(x)$

Example 3 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x \to 2}(2x-1)$ **B** $\lim_{x \to 0} \frac{|x|}{x}$ **C** $\lim_{x \to 3} \frac{x^2-5x+2}{x+4}$

D $\lim_{x \to -2} \frac{x^2+x-2}{x+2}$ **E** $\lim_{x \to 1} \frac{x}{x^2-1}$ **F** $\lim_{x \to \frac{\pi}{2}^+} \tan x$

Solution

A $\lim_{x \to 2}(2x-1) = 2(2) - 1 = 3$

B $\frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ \nexists, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$

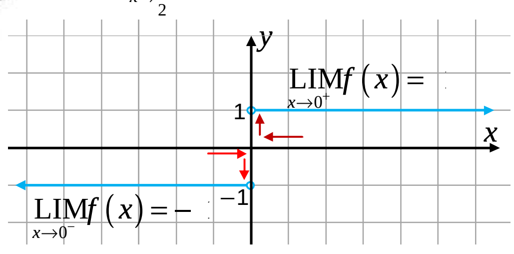


Figure 2.12

$\Rightarrow \lim_{x \to 0} f(x)$ DOESN'T EXIST.

C LOOK AT THE FOLLOWING TABLES OF VALUES (TAKEN UP TO 4 DECIMAL PLACES)

x	2.9	2.99	2.999	3.1	3.01	3.001	. . .	3
$\frac{x^2-5x+2}{x+4}$	−0.5927	−0.5736	−0.5717	−0.5479	−056917	−0.5712		

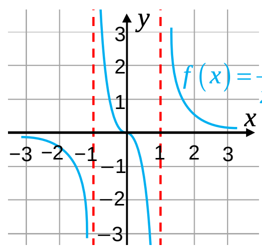
TO WHICH NUMBER DOES $f(x)$ APPROACH AS x APPROACHES TO 3?

$$\lim_{x\to 3}\frac{x^2-5x+2}{x+4}=-\frac{4}{7}=-0.5714$$

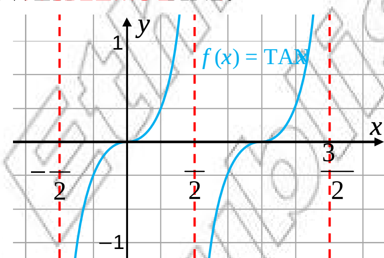
D $\frac{x^2+x-2}{x+2}=\frac{(x+2)(x-1)}{x+2}=x-1;\ x\neq -2.$

$$\lim_{x\to -2}\frac{x^2+x-2}{x+2}=\lim_{x\to -2}(x-1)=-3.$$

LOOK AT FIGURES 2.13 AND 2.14 TO ANSWER PROBLEMS E AND F.



Figures 2.13



Figures 2.14

E $\lim_{x\to 1^+}\frac{x}{x^2-1}=\infty$; $\lim_{x\to 1^-}\frac{x}{x^2-1}=-\infty \Rightarrow \lim_{x\to 1}\frac{x}{x^2-1}$ DOESN'T EXIST.

F $\lim_{x\to \frac{\pi}{2}^+}(\tan x)=-\infty$; $\lim_{x\to \frac{\pi}{2}^-}(\tan x)=+\infty \Rightarrow \lim_{x\to \frac{\pi}{2}}(\tan x)$ DOESN'T EXIST.

Example 4 THE LIMIT OF A CONSTANT FUNCTION AS $x \to a$ IS THE CONSTANT ITSELF.

TO VERIFY THIS:

LET $f(x) = c$. CLEARLY, $f(x)$ IS APPROACHING c AS x IS APPROACHING ANY NUMBER, SO THAT $\lim_{x\to a} c = c$.

Example 5 THE LIMIT OF THE IDENTITY FUNCTION AS $x \to a$ IS a. THAT IS, $\lim_{x\to a} x = a$.

Example 6 LET $f(x)=\begin{cases}0, & \text{IF } x\in\mathbb{Z}\\ 1, & \text{IF } x\notin\mathbb{Z}\end{cases}$. EVALUATE

A $\lim_{x\to -2} f(x)$ **B** $\lim_{x\to 0.3} f(x)$

Solution SKETCH THE GRAPH OF f (SEE FIGURE 2.15)

A $\lim_{x\to 2} f(x) = 1$ BUT $f(-2) = 0$.

B $\lim_{x\to 0.3} f(x) = 1$

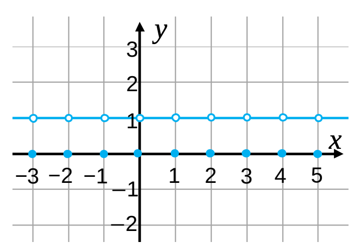


Figure 2.15

IS $\lim_{x\to c} f(x) = 1$ FOR ALL $c \in \mathbb{R}$?

What can you say about c if $\lim_{x\to c} f(x) = f(c) = 1$?

CLEARLY, c MUST NOT BE AN INTEGER.

Exercise 2.5

USE GRAPHS OR CALCULATORS TO DETERMINE THE LIMITS IN EXERCISES 1 – 15.

1 $\lim_{x\to 4}(5x+7)$	**2** $\lim_{x\to 0} \sin x$	**3** $\lim_{x\to \frac{1}{3}} \frac{1}{(3x-1)}$
4 $\lim_{x\to 0}(2^x)$	**5** $\lim_{x\to 0} \frac{1}{e^x - 1}$	**6** $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$
7 $\lim_{x\to 2} \frac{x-2}{x^2-x-2}$	**8** $\lim_{x\to -3} \frac{x^3+27}{x+3}$	**9** $\lim_{x\to 1} \frac{x^4-1}{x^6-1}$
10 $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$	**11** $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$	**12** $\lim_{x\to 0} \frac{x-4\lvert x\rvert}{x}$
13 $\lim_{x\to 5} \frac{5x-x^2}{x-5}$	**14** $\lim_{x\to 0} \frac{x^3}{\lvert x\rvert}$	**15** $\lim_{x\to -2} \frac{x^2-5x-14}{x^2-4}$

16 DISCUSS THE FOLLOWING POINT IN GROUPS. IS THE LIMIT OF THE SUM OF TWO FUNCTIONS AT A POINT THE SAME AS THE SUM OF THE LIMITS AT THE GIVEN POINT? JUSTIFY YOUR ANSWER BY PRODUCING SEVERAL EXAMPLES.

Basic limit theorems

SUPPOSE $\lim_{x\to a} f(x)$ AND $\lim_{x\to a} g(x)$ EXIST AND $k \in \mathbb{R}$.

THEN $\lim_{x\to a}(f(x)+g(x))$, $\lim_{x\to a}(f(x)-g(x))$, $\lim_{x\to a} kf(x)$, $\lim_{x\to a}(fg)(x)$, $\lim_{x\to a}\left(\frac{f}{g}\right)(x)$,

PROVIDED THAT $\lim_{x\to a} g(x) \neq 0$, EXIST AND

1 $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2 $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3 $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$

4 $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5 $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

6 $\lim_{x\to a}\sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)}$, PROVIDED THAT $f(x) \geq 0$ FOR x NEAR a.

SEE HOW TO APPLY LIMIT THEOREMS IN THE FOLLOWING EXAMPLE.

Example 7 $\lim_{x\to 2}\left(x^3+4x^2-\frac{1}{x}+7x+11\right)$

$$= \lim_{x\to 2}x^3 + 4\lim_{x\to 2}x^2 - \lim_{x\to 2}\left(\frac{1}{x}\right) + 7\lim_{x\to 2}x + \lim_{x\to 2}(11)$$

$$= (2)^3 + 4\left(\lim_{x\to 2}x\right)^2 - \frac{\lim_{x\to 2}(1)}{\lim_{x\to 2}(x)} + 7(2) + 1$$

$$= 2^3 + 4\times 2^2 - \frac{1}{2} + 25 = 48.5$$

The limit of a polynomial function

SUPPOSE $p(x)$ IS A POLYNOMIAL, THEN $\lim_{x\to c} p(x) = p(c)$. *Explain!*

Example 8 $\lim_{x\to 3}(x^4-2x^3+5x^2+7x+1) = 3^4-2(3)^3+5(3)^2+7(3)+1 = 94$

Theorem 2.2

LET f AND g BE FUNCTIONS. SUPPOSE $\lim_{x\to a} f(x)$ AND $\lim_{x\to a} g(x)$ EXIST AND $f(x) = g(x)$, $\forall x \neq a$.

THEN $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Example 9 FIND $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

Solution $\frac{x^2-1}{x-1} = x+1$; FOR $x \neq 1$. LET $f(x) = \frac{x^2-1}{x-1}$ AND $g(x) = x+1$.

$f(x) = g(x)$, $\forall x \neq 1$. THEN $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) \Rightarrow \lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1)$

Example 10 FIND $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$.

Solution WHAT HAPPENS TO $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$ WHEN $x = 1$? IS THE RESULT DEFINED?

REWRITE THE EXPRESSION BY RATIONALIZING THE DENOMINATOR.

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)\left(\sqrt{x}+1\right)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}\left(\sqrt{x}+1\right) = 2$$

Example 11 EVALUATE $\lim_{x\to -3}\frac{x^3+3x^2-x-3}{4x^3+12x^2-x-3}$

Solution $x^3+3x^2-x-3 = x^2(x+3)-(x+3) = (x^2-1)(x+3)$

$$4x^3 + 12x^2 - x - 3 = 4x^2 (x + 3) - (x + 3) = (4x^2 - 1)(x + 3)$$

$$\Rightarrow \lim_{x \to -3} \frac{x^3 + 3x^2 - x - 3}{4x^3 + 12x^2 - x - 3} = \lim_{x \to -3} \frac{(x^2 - 1)(x + 3)}{(4x^2 - 1)(x + 3)} = \lim_{x \to -3} \frac{x^2 - 1}{4x^2 - 1} = \frac{8}{35}$$

Example 12 EVALUATE $\lim_{x \to 2} \dfrac{\frac{2}{x} - 1}{x^3 - 8}$.

Solution

$$\frac{\frac{2}{x} - 1}{x^3 - 8} = \frac{\left(\frac{2 - x}{x}\right)}{(x - 2)(x^2 + 2x + 4)} = -\frac{1}{x(x^2 + 2x + 4)};\ x \neq 0, 2$$

$$\Rightarrow \lim_{x \to 2} \frac{\frac{2}{x} - 1}{x^3 - 8} = -\lim_{x \to 2} \frac{1}{x(x^2 + 2x + 4)} = -\frac{1}{24}$$

Example 13 LET $f(x) = \sqrt{2 - x}$. SIMPLIFY THE EXPRESSION $\dfrac{f(x) - f(1)}{x - 1}$ AND EVALUATE $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$.

Solution

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\sqrt{2 - x} - 1}{x - 1} = \lim_{x \to 1} \frac{-1}{1 + \sqrt{2 - x}} = -\frac{1}{2}.$$

Example 14 IF $\lim_{x \to x_o} (f(x) + g(x))$ EXISTS, DO THE LIMITS $\lim_{x \to x_o} f(x)$ AND $\lim_{x \to x_o} g(x)$ EXIST?

Solution TAKE, FOR EXAMPLE, $f(x) = \dfrac{1}{x - 1}$ AND $g(x) = \dfrac{2}{1 - x^2}$.

DO $\lim_{x \to 1} f(x)$ AND $\lim_{x \to 1} g(x)$ EXIST? EVALUATE $\lim_{x \to 1} (f + g)(x)$.

$\lim_{x \to 1} f(x)$ AND $\lim_{x \to 1} g(x)$ BOTH DON'T EXIST. BUT

$$\lim_{x \to 1} (f(x) + g(x)) = \lim_{x \to 1} \left(\frac{1}{x - 1} + \frac{2}{1 - x^2}\right) = \lim_{x \to 1} \frac{1 - x}{1 - x^2} = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}$$

Example 15 FIND $\lim_{x \to 4} \dfrac{x - 4}{\sqrt{x} - 2}$

Solution

$$\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{(x - 4)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \to 4} (\sqrt{x} + 2) = 4$$

Example 16 LET $f(x) = \sqrt{x}$. FIND $\lim_{h \to 0} \dfrac{f(4 + h) - f(4)}{h}$.

Solution $\lim\limits_{h\to 0}\dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0}\dfrac{\sqrt{4+h}-2}{h} = \lim\limits_{h\to 0}\left[\dfrac{\sqrt{4+h}-2}{h}\cdot\dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right]$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{4+h}+2}\right) = \frac{1}{4}$$

Example 17 Evaluate $\lim\limits_{x\to 1}\sqrt{x^3+x^2-6x+5}$

Solution $x^3 + x^2 - 6x + 5 \ge 0$ for x near 1.

$$\Rightarrow \lim_{x\to 1}\sqrt{x^3+x^2-6x+5} = \sqrt{\lim_{x\to 1}\left(x^3+x^2-6x+5\right)} = \sqrt{1} = 1$$

Example 18 Find $\lim\limits_{x\to 0}\dfrac{\sqrt{5-x}-\sqrt{5}}{x}$

Solution $\lim\limits_{x\to 0}\dfrac{\sqrt{5-x}-\sqrt{5}}{x} = \lim\limits_{x\to 0}\dfrac{\left(\sqrt{5-x}-\sqrt{5}\right)\left(\sqrt{5-x}+\sqrt{5}\right)}{x\left(\sqrt{5-x}+\sqrt{5}\right)}$

$$= \lim_{x\to 0}\frac{5-x-5}{x\left(\sqrt{5-x}+\sqrt{5}\right)} = -\lim_{x\to 0}\frac{1}{\sqrt{5-x}+\sqrt{5}} = -\frac{1}{2\sqrt{5}}$$

Example 19 Find $\lim\limits_{x\to 7}\dfrac{\dfrac{1}{x+7}-\dfrac{1}{14}}{x-7}$.

Solution $\lim\limits_{x\to 7}\dfrac{\dfrac{1}{x+7}-\dfrac{1}{14}}{x-7} = \lim\limits_{x\to 7}\dfrac{14-(x+7)}{14(x+7)(x-7)} = \lim\limits_{x\to 7}\left(\dfrac{7-x}{x-7}\cdot\dfrac{1}{14(x+7)}\right)$

$$= -\lim_{x\to 7}\frac{1}{14(x+7)} = -\frac{1}{196}$$

Example 20 Evaluate $\lim\limits_{x\to -3}\dfrac{\sqrt{1+\sqrt{4+x}}-\sqrt{2}}{x+3}$.

Solution $\lim\limits_{x\to -3}\dfrac{\sqrt{1+\sqrt{4+x}}-\sqrt{2}}{x+3} = \dfrac{\sqrt{1+\sqrt{4+x}}-\sqrt{2}}{x+3}\cdot\dfrac{\sqrt{1+\sqrt{4+x}}+\sqrt{2}}{\sqrt{1+\sqrt{4+x}}+\sqrt{2}}$

$$= \frac{\sqrt{4+x}-1}{x+3}\cdot\frac{1}{\sqrt{1+\sqrt{4+x}}+\sqrt{2}}. \quad (\textit{Explain!})$$

$$= \frac{x+3}{x+3}\cdot\frac{1}{\left(\sqrt{4+x}+1\right)\left(\sqrt{1+\sqrt{4+x}}+2\right)} \quad (\textit{Explain!})$$

$$\Rightarrow \lim_{x\to -3}\frac{\sqrt{1+\sqrt{4+x}}-\sqrt{2}}{x+3} = \frac{\sqrt{2}}{8}. \quad (\textit{Explain!})$$

Exercise 2.6

1 USE THE FOLLOWING GRAPH OF THE FUNCTION f TO DETERMINE EACH OF THE LIMITS.

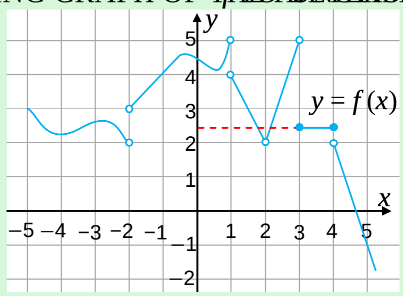


Figure 2.16

A $\lim\limits_{x\to 1^-} f(x)$ **B** $\lim\limits_{x\to 2} f(x)$ **C** $\lim\limits_{x\to -2} f(x)$

D $\lim\limits_{x\to 1^+} f(x)$ **E** $\lim\limits_{x\to 4^-} f(x)$ **F** $\lim\limits_{x\to 3} f(x)$

2 LET $f(x) = \begin{cases} 1-x^2, \text{if } -1 < x < 2 \\ -3 \text{ if } x = -1 \\ -x-1, \text{if } x < -1 \\ x-5, \text{ if } x \geq 2 \end{cases}$

SKETCH THE GRAPH OF f AND DETERMINE EACH OF THE FOLLOWING LIMITS.

A $\lim\limits_{x\to -1} f(x)$ **B** $\lim\limits_{x\to 2} f(x)$ **C** $\lim\limits_{x\to 5} f(x)$ **D** $\lim\limits_{x\to 3} f(x)$

3 SUPPOSE THAT f, g AND h ARE FUNCTIONS WITH $\lim\limits_{x\to 2} f(x) = 7$, $\lim\limits_{x\to 2} g(x) = -4$ AND $\lim\limits_{x\to 2} h(x) = \frac{3}{5}$, EVALUATE

A $\lim\limits_{x\to 2} \left(f(x) + g(x) \right)$ **B** $\lim\limits_{x\to 2} \left((fg)(x) - 3h(x) \right)$

C $\lim\limits_{x\to 2} \dfrac{f(x)g(x)h(x)}{f(x)+g(x)-5h(x)}$

4 DETERMINE EACH OF THE FOLLOWING LIMITS.

A $\lim\limits_{x\to 3} \dfrac{x-3}{\sqrt{x^2-6x+9}}$ **B** $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{x^2}$ **C** $\lim\limits_{x\to \frac{1}{3}} \dfrac{x+1}{3x-1}$

D $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2}$ **E** $\lim\limits_{x\to 0} \dfrac{x^3}{|x|+x}$ **F** $\lim\limits_{x\to -5} \dfrac{x^2+x-20}{x^2+4x-5}$

G $\lim\limits_{x\to 0} \dfrac{\sin x + 1}{x + \cos x}$ **H** $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ **I** $\lim\limits_{x\to 2} \dfrac{\sqrt{x-2\sqrt{x+1}-1}}{\sqrt{x-2}}$

J $\lim\limits_{x\to 1} \dfrac{\sqrt{x-1}+\sqrt{x}-1}{\sqrt{x^2-1}}$

Limits at infinity

Limits as x approaches ∞

ACTIVITY 2.6

1 USING THE CONCEPT OF LIMITS OF SEQUENCES OF NUM… ATE EACH OF THE FOLLOWING LIMITS AT INFINITY.

A $\lim\limits_{x\to\infty} \frac{1}{x}$ **B** $\lim\limits_{x\to-\infty} \frac{3x-1}{x+5}$ **C** $\lim\limits_{x\to\infty} \frac{x^2+1}{x-1}$

2 LET $f(x) = \frac{p(x)}{q(x)}$ BE A RATIONAL FUNCTION.

A IF DEGREE OF $p(x)$ = DEGREE OF $q(x)$, EVALUATE $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$ IN TERMS OF THE LEADING COEFFICIENTS OF $p(x)$ AND $q(x)$.

B IF DEGREE OF $p(x)$ < DEGREE OF $q(x)$, DISCUSS HOW TO EVALUATE $\lim\limits_{x\to\infty} \frac{p(x)}{q(x)}$.

C DO YOU SEE A RELATIONSHIP BETWEEN THESE LIMITS AND HORIZONTAL ASYMPTO… RATIONAL FUNCTIONS?

Definition 2.7

LET f BE A FUNCTION AND L BE A REAL NUMBER.

IF $f(x)$ GETS CLOSER TO L AS x INCREASES WITHOUT BOUND, THEN L IS SAID TO BE THE LIMIT OF $f(x)$ AS x APPROACHES TO INFINITY.

THIS STATEMENT IS EXPRESSED SYMBOLICALLY BY $\lim\limits_{x\to\infty} f(x) = L$

Example 21 EVALUATE $\lim\limits_{x\to\infty} \frac{3x^2-5x+4}{2x^2+4}$

Solution YOU APPLY THE TECHNIQUE WHICH ARE USED IN EVALUATING LIMITS OF NUM… SEQUENCES. I.E. DIVIDE THE NUMERATOR AND DENOMINATOR BY x^2 (THE HIGHEST POWER MONOMIAL).

$$\lim_{x\to\infty}\left(\frac{3x^2-5x+4}{2x^2+4}\right) = \lim_{x\to\infty}\left(\frac{\frac{3x^2-5x+4}{x^2}}{\frac{2x^2+4}{x^2}}\right) = \frac{\lim\limits_{x\to\infty}\left(3-\frac{5}{x}+\frac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2+\frac{4}{x^2}\right)} = \frac{3-0+0}{2+0} = \frac{3}{2}$$

Example 22 EVALUATE $\lim\limits_{x\to\infty}\left(\frac{1-3x}{6x+5}+\frac{2x+1}{x^2+7x+1}\right)$

Solution

$$\lim_{x\to\infty}\left(\frac{1-3x}{6x+5}+\frac{2x+1}{x^2+7x+1}\right)=\lim_{x\to\infty}\left(\frac{1-3x}{6x+5}\right)+\lim_{x\to\infty}\frac{2x+1}{x^2+7x+1}=\lim_{x\to\infty}\frac{\frac{1}{x}-3}{6+\frac{5}{x}}+0=-\frac{1}{2}.$$

Non-existence of limits

IN THE PREVIOUS TOPIC, YOU ALREADY SAW ONE CONDITION IN WHICH A LIMIT FAILS TO EXI

FOR EXAMPLE, $\lim_{x\to 0}\frac{|x|}{x}$, DOES NOT EXIST, AS THE LIMIT FROM THE LEFT AND THE RIGHT DO NOT AC

Do you see any other condition in which a limit fails to exist?

CONSIDER $f(x)=\text{SIN}\left(\frac{}{x}\right)$.

YOU KNOW THAT $y = \text{SIN } x$ HAS ONE COMPLETE CYCLE ON THE INTERVAL [2, 4]. AS $\frac{}{x}$ MOVES

FROM 2 TO 4 , x MOVES FROM $\frac{}{2}$ TO $\frac{}{4}$ WHICH IS $\frac{1}{2}$ TO $\frac{1}{4}$. THEREFORE, THE GRAPH OF f HAS

A COMPLETE CYCLE ON THE INTERVAL $\left[\frac{1}{4},\frac{1}{2}\right]$. SIMILARLY THERE IS A COMPLETE CYCLE ON

INTERVALS $\left[\frac{1}{6},\frac{1}{4}\right]$, $\left[\frac{1}{8},\frac{1}{6}\right]$, AND SO ON.

HENCE, THE GRAPH OF f GETS MORE AND MORE CROWDED AS x APPROACHES 0. I.E. CHANGES TOO FREQUENTLY BETWEEN –1 AND 1, AS x APPROACHES 0. THE GRAPH DOES NOT SETTLE DOWN. THAT IS, IT DOES NOT APPROACH A FIXED POINT. INSTEAD, IT OSCILLATES BETWEEN –1 AND 1. THE

$\lim_{x\to 0}\text{SIN}\left(\frac{}{x}\right)$ DOES NOT EXIST. THIS IS THE SECOND CONDITION IN WHICH A LIMIT FAILS TO EXIST

THE FOLLOWING IS THE GRAPH OF $f(x)=\text{SIN}\left(\frac{}{x}\right)$ SHOWING THE NON-EXISTENCE OF $\lim_{x\to 0}\text{SIN}\left(\frac{}{x}\right)$.

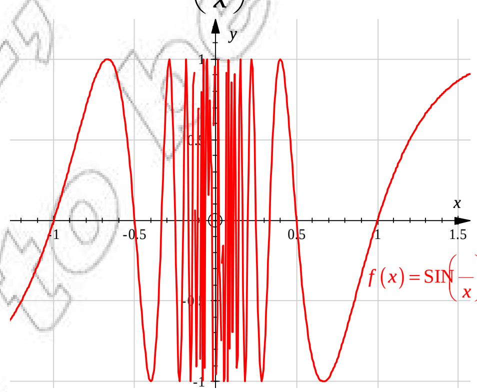


Figure 2.17

One side limits

ACTIVITY 2.7

1 SKETCH THE GRAPH OF $f(x)=\sqrt{x}$ AND $g(x)=\sqrt{-x}$

EVALUATE EACH OF THE FOLLOWING ONE-SIDED LIMITS BASED ON YOUR KNOWLEDGE OF A FUNCTION f AT A POINT $x = a$ AS x APPROACHES a FROM THE RIGHT, $\lim\limits_{x\to a^+} f(x)$ AND AS x APPROACHES a FROM THE LEFT, $\lim\limits_{x\to a^-} f(x)$.

A $\lim\limits_{x\to 0^+} f(x)$ **B** $\lim\limits_{x\to 0^-} f(x)$ **C** $\lim\limits_{x\to 0^+} g(x)$ **D** $\lim\limits_{x\to 0^-} g(x)$

2 USE THE FOLLOWING GRAPH OF A FUNCTION f TO EVALUATE THE ONE SIDE LIMIT.

A $\lim\limits_{x\to 1^+} f(x)$ **B** $\lim\limits_{x\to 1^-} f(x)$ **C** $\lim\limits_{x\to 3^+} f(x)$

D $\lim\limits_{x\to 3^-} f(x)$ **E** $\lim\limits_{x\to 4^+} f(x)$ **F** $\lim\limits_{x\to 4^-} f(x)$

G $\lim\limits_{x\to 2^-} f(x)$ **H** $\lim\limits_{x\to 2^+} f(x)$

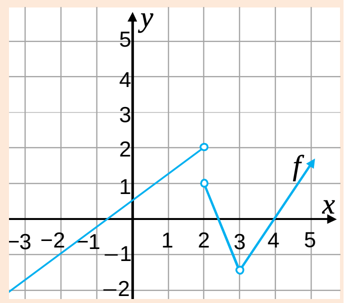
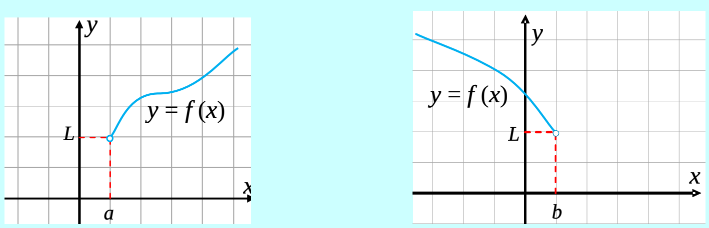


Figure 2.18

Definition 2.8

1 *Right Hand Limit*

LET f BE DEFINED ON SOME OPEN INTERVAL (a, c). SUPPOSE $f(x)$ APPROACHES A NUMBER L AS x APPROACHES a FROM THE RIGHT, THEN L IS SAID TO BE THE RIGHT HAND LIMIT OF f AT $x = a$. THIS IS ABBREVIATED AS: $\lim\limits_{x\to a^+} f(x) = L$



A **B**

Figure 2.19

2 *Left Hand Limit*

LET f BE DEFINED ON SOME OPEN INTERVAL (c, b). SUPPOSE $f(x)$ APPROACHES A NUMBER L AS x APPROACHES b FROM THE LEFT. THEN L IS SAID TO BE THE LEFT HAND LIMIT OF f AT $x = b$. THIS IS ABBREVIATED BY: $\lim\limits_{x\to b^-} f(x) = L$

Example 23 Let $f(x) = \sqrt{x-4}$.

Find $\lim_{x \to 4^+} f(x)$

Solution $\lim_{x \to 4^+} \sqrt{x-4} = 0$

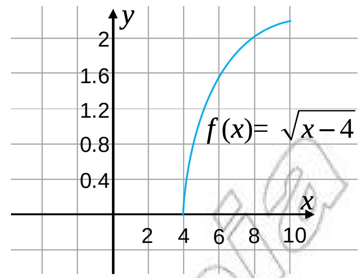


Figure 2.20

Example 24 Evaluate

A $\lim_{x \to 3^+} \sqrt{9-x^2}$ **B** $\lim_{x \to 3^-} \sqrt{9-x^2}$

C $\lim_{x \to -3^+} \sqrt{9-x^2}$ **D** $\lim_{x \to -3^-} \sqrt{9-x^2}$

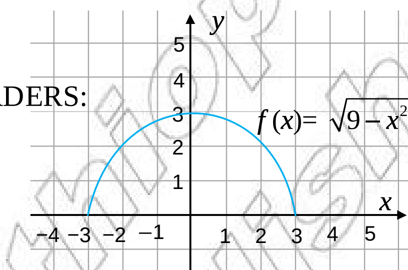


Figure 2.21

Solution Look at the following orders:

$3^- < 3 < 3^+$ and $-3^- < -3 < -3^+$

$(3^-)^2 = 9^-$ and $(3^+)^2 = 9^+$

$(-3^-)^2 = 9^+$ and $(-3^+)^2 = 9^-$

As $x \to 3^+$, $9 - x^2 \to 0^-$ and, as $x \to 3^-$, $9 - x^2 \to 0^+$

Therefore,

A $\lim_{x \to 3^+} \sqrt{9-x^2}$ doesn't exist **B** $\lim_{x \to 3^-} \sqrt{9-x^2} = 0$

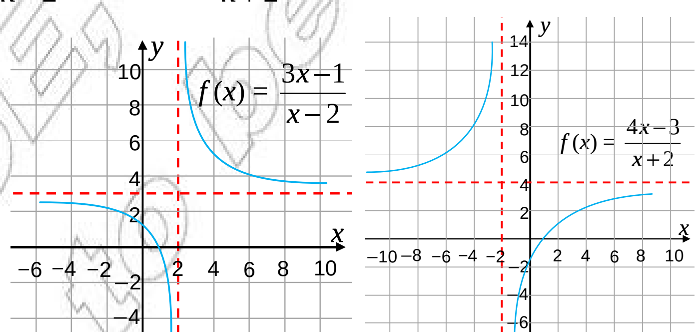
C $\lim_{x \to -3^+} \sqrt{9-x^2} = 0$ **D** $\lim_{x \to -3^-} \sqrt{9-x^2}$ doesn't exist.

Example 25 Evaluate

A $\lim_{x \to 2^-} \dfrac{3x-1}{x-2}$ **B** $\lim_{x \to -2^+} \dfrac{4x-3}{x+2}$

Solution Let us investigate these limits graphically.

Let $f(x) = \dfrac{3x-1}{x-2}$ and $g(x) = \dfrac{4x-3}{x+2}$.



A $\lim_{x \to 2^-} f(x) = -\infty$ **B** $\lim_{x \to -2^+} g(x) = -\infty$

Figure 2.22

ACTIVITY 2.8

1 USE THE ABOVE GRAPHS TO EVALUATE EACH OF THE FOLLOWING LIMITS.

I $\lim_{x\to 2^+} f(x)$ **II** $\lim_{x\to -2^-} g(x)$ **III** $\lim_{x\to \infty} f(x)$ **IV** $\lim_{x\to -\infty} g(x)$

2 DISCUSS THE EXISTENCE OF THE LIMIT OF A FUNCTION AT $x = a$, IF

I $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ **II** $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

WHAT CAN YOU SAY ABOUT $\lim_{x\to a^+} f(x)$ AND $\lim_{x\to a^-} f(x)$, IF $\lim_{x\to a} f(x) = L$?

Two side limits

Definition 2.9

LET f BE A FUNCTION DEFINED ON AN OPEN INTERVAL CONTAINING a, EXCEPT POSSIBLY AT a ITSELF. THEN, $\lim_{x\to a} f(x)$ EXISTS, IF BOTH $\lim_{x\to a^+} f(x)$ AND $\lim_{x\to a^-} f(x)$ EXIST AND ARE EQUAL: THAT IS,

$\lim_{x\to a} f(x)$ EXISTS, IF $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

IN THIS CASE, $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

Infinite limits

Example 26 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x\to 2^-} \frac{1}{4-x^2}$ **B** $\lim_{x\to 2^+} \frac{1}{4-x^2}$ **C** $\lim_{x\to -2^+} \frac{1}{4-x^2}$ **D** $\lim_{x\to -2^-} \frac{1}{4-x^2}$

Solution SKETCH THE GRAPH OF $f(x) = \frac{1}{4-x^2}$ IN ORDER TO DETERMINE EACH LIMIT AT THE SAME TIME.

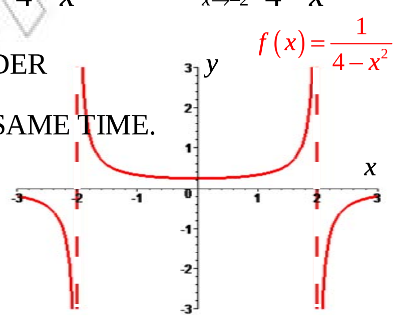


Figure 2.23

IF YOU TRY TO SUBSTITUTE $x = 2$, THE DENOMINATOR EQUALS 0.

A $\lim_{x\to 2^-} \frac{1}{4-x^2} = \infty$. THE GRAPH IS GOING UP INDEFINITELY TO ∞.

B $\lim_{x\to 2^+} \frac{1}{4-x^2} = -\infty$. THE GRAPH IS GOING INDEFINITELY DOWN TO $-\infty$.

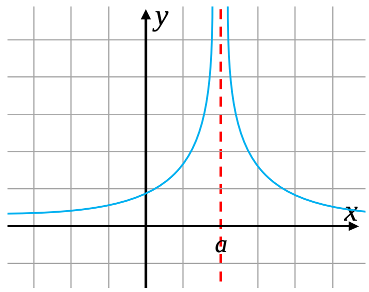
C $\lim_{x\to -2^+} \frac{1}{4-x^2} = \infty$ **D** $\lim_{x\to -2^-} \frac{1}{4-x^2} = -\infty$

RECALL THAT THE LINES $x = 2$ AND $x = -2$ ARE VERTICAL ASYMPTOTES OF THE RATIONAL FUNCTION $f(x) = \frac{1}{4-x^2}$.

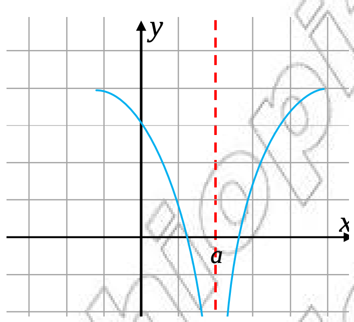
Vertical asymptotes

THE VERTICAL LINE $x = a$ IS A VERTICAL ASYMPTOTE TO THE GRAPH OF $y = f(x)$, IF ONE OF THE FOLLOWING IS TRUE.

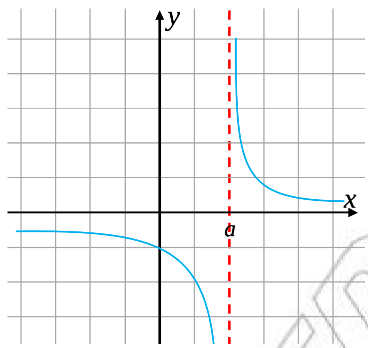
1 $\lim_{x \to a^-} f(x) = \infty$ **2** $\lim_{x \to a^+} f(x) = \infty$ **3** $\lim_{x \to a^-} f(x) = -\infty$ **4** $\lim_{x \to a^+} f(x) = -\infty$



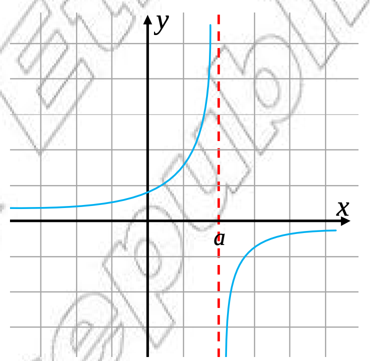
A $\lim_{x \to a} f(x) = \infty$



B $\lim_{x \to a} f(x) = -\infty$



C $\lim_{x \to a^+} f(x) = \infty$; $\lim_{x \to a^-} f(x) = -\infty$



D $\lim_{x \to a^+} f(x) = -\infty$; $\lim_{x \to a^-} f(x) = \infty$

Figure 2.24

Exercise 2.7

1 THE FOLLOWING TABLE DISPLAYS THE AMOUNT OF WHEAT PRODUCED IN QUINTALS PER

year	1995	1996	1997	1998	1999	2000	2001
Qutinal	33	43.6	49.5	53	55.8	57.5	59

BASED ON THIS DATA, THE ORGANIZATION THAT PRODUCES THE WHEAT PROJECTS THAT T

PRODUCT AT x^{TH} YEAR (TAKING 1995 AS THE FIRST YEAR) $p(x)$ WILL BE $\frac{140x+25}{2x+3}$

QUINTALS. APPROXIMATE THE YEARLY PRODUCT AFTER A LONG PERIOD OF TIME.

2 SUPPOSE THE UNEMPLOYMENT RATE OF A CERTAIN COUNTRY x-YEARS FROM NOW IS MODELLED BY $u(x)=\dfrac{45x+35}{9x+2}$ PERCENT. FIND THE LEVEL IT WILL REACH AS TIME GONE. BASED ON THE FORMULA, DISCUSS WHETHER THE UNEMPLOYMENT RATE INCREASES OR DECREAS

3 EVALUATE EACH OF THE FOLLOWING ONE-SIDE LIMITS.

A $\lim\limits_{x\to1^+}\sqrt{x-1}$	**B** $\lim\limits_{x\to1^-}\sqrt{x-1}$	**C** $\lim\limits_{x\to1^+}\sqrt{1-x^2}$
D $\lim\limits_{x\to-1^-}\sqrt{1-x^2}$	**E** $\lim\limits_{x\to-3^-}\sqrt{9-x^2}$	**F** $\lim\limits_{x\to-3^+}\sqrt{9-x^2}$
G $\lim\limits_{x\to5^+}\dfrac{1}{x-5}$	**H** $\lim\limits_{x\to5^-}\dfrac{1}{x-5}$	**I** $\lim\limits_{x\to0^+}\dfrac{1}{x^2}$
J $\lim\limits_{x\to0}\dfrac{1}{x^2}$	**K** $\lim\limits_{x\to0^+}\dfrac{4x+\lvert x\rvert}{4x-\lvert x\rvert}$	**L** $\lim\limits_{x\to5^+}\sqrt{4-\sqrt{x^2-9}}$

4 USE THE FOLLOWING GRAPH OF A FUNCTION f TO DETERMINE THE LIMITS BELOW.

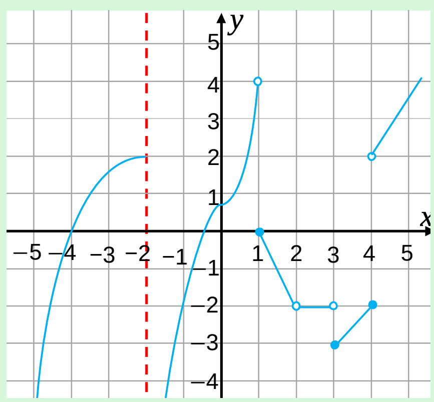


Figure 2.25

A $\lim\limits_{x\to-3^+}f(x)$	**B** $\lim\limits_{x\to-2^-}f(x)$	**C** $\lim\limits_{x\to-2^+}f(x)$
D $\lim\limits_{x\to1^-}f(x)$	**E** $\lim\limits_{x\to1^+}f(x)$	**F** $\lim\limits_{x\to2}f(x)$
G $\lim\limits_{x\to3^+}g(x)$	**H** $\lim\limits_{x\to3^-}f(x)$	**I** $\lim\limits_{x\to4^+}f(x)$

5 LET $f(x)=\begin{cases}e^x, & \text{IF } x\le 2\\ (e-1)x+3, & \text{IF } x>2\end{cases}$; $\quad g(x)=\begin{cases}x^2-x, & \text{IF } \lvert x\rvert\le 1\\ \dfrac{1}{x}, & \text{IF } \lvert x\rvert>1\end{cases}$

EVALUATE EACH OF THE FOLLOWING ONE SIDE LIMITS.

A $\lim\limits_{x\to2^+}\left(f(x)+g(x)\right)$	**B** $\lim\limits_{x\to2^-}\left(f(x)-g(x)\right)$
C $\lim\limits_{x\to1^+}f(x)\,g(x)$	**D** $\lim\limits_{x\to-1^+}\dfrac{f(x)-g(x)}{f(x)g(x)}$

6 IN EACH OF THE FOLLOWING FUNCTIONS, DETERMINE WHETHER THE GRAPH HAS A HO VERTICAL ASYMPTOTE AT THE GIVEN POINT. DETERMINE THE ONE SIDE LIMITS AT TH POINTS.

A $f(x)=\dfrac{x}{x+5}; x=-5$ **B** $f(x)=\dfrac{x^3+1}{x+1}; x=-1$

C $f(x)=\dfrac{|x^2-1|}{x-1}, x=1$ **D** $f(x)=\dfrac{(x-3)^3}{|x-3|}; x=3$

E $f(x)=\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}; x=0$ **F** $f(x)=\dfrac{x}{\text{SIN}\,x}; x=$

2.3 CONTINUITY OF A FUNCTION

THE TERM CONTINUOUS HAS THE SAME MEANING AS IT DOES IN OUR EVERYDAY ACTIVITY. FOR EXAMPLE, LOOK AT THE FOLLOWING TOPOGRAPHIC MAP BETWEEN TWO PLACES A AND B IN THE GRAPH. THE y-AXIS REPRESENTS HOW HIGH, IN METRES, ABOVE SEA LEVEL EACH POINT IS AND x-AXIS REPRESENTS DISTANCE IN KILOMETRES, BETWEEN POINTS.

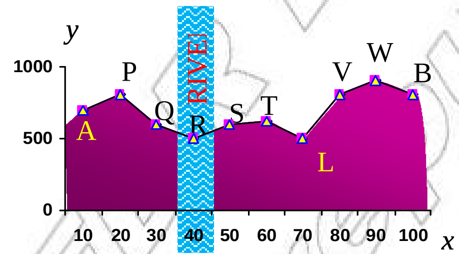


Figure 2.26

THIS CURVE IS DRAWN FROM A TO B WITHOUT LIFTING THE PENCIL FROM THE PAPER. THE GRAPH IS USEFUL FOR FINDING THE HEIGHT ABOVE SEA LEVEL OF ANY POINT BETWEEN A AND B.

THINK OF CONTINUITY AS DRAWING A CURVE WITHOUT TAKING THE PENCIL OFF OF THE PAPE

2.3.1 Continuity of a Function at a Point

ACTIVITY 2.9

LOOK AT THE FOLLOWING GRAPHS.

FROM EACH GRAPH EVALUATE $\lim_{x\to x_o} f(x)$ AND $f(x_o)$ AND DECIDE WHETHER THOSE VALUES ARE EQUAL OR UNEQUAL. DETERMINE WHETHER OR NOT EACH GRAPH HAS A HOLE, J GAP AT $x = x_O$.

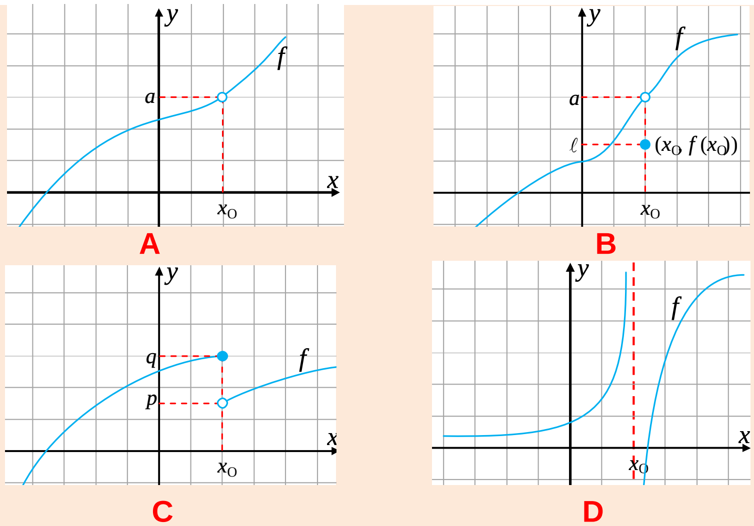


Figure 2.27

WHICH OF THE ABOVE GRAPHS ARE CONNECTED AT $x = x_0$?

Definition 2.10

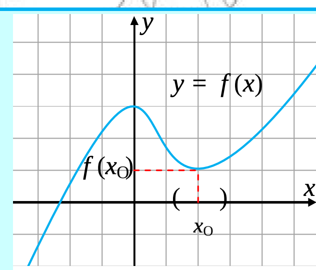
Continuous function at a point

A FUNCTION f IS SAID TO BE CONTINUOUS AT x_0, IF

I $\quad x_0 \in D_f$ (DOMAIN OF f)

II $\quad \lim_{x \to x_o} f(x)$ EXISTS AND

III $\quad \lim_{x \to x_o} f(x) = f(x_o)$



$\lim_{x \to x_o} f(x) = f(x_o)$

Figure 2.28

NOTICE THAT THE GRAPH HAS NO INTERRUPTION AT x_0.

IF ANY OF THESE THREE CONDITIONS IS NOT SATISFIED, THEN THE FUNCTION IS NOT CONTINUOUS AT $x = x_0$.

Definition 2.11

A FUNCTION f IS SAID TO BE **discontinuous at** x_0, IF f IS DEFINED ON AN **open interval** CONTAINING x_0 (EXCEPT POSSIBLY AT x_0) AND f IS NOT CONTINUOUS AT x_0.

Example 1 LET $f(x) = \frac{|x|}{x}$. IS f CONTINUOUS AT $x = -3$?, $x = 0$? AND $x = 1$?

Solution

$$f(x) = \frac{|x|}{x} \Rightarrow f(x) = \begin{cases} 1, & \text{IF } x > 0 \\ -1, & \text{IF } x < 0 \\ \nexists, & \text{IF } x = 0 \end{cases}$$

WHAT IS THE DOMAIN OF f? WHAT IS $\lim\limits_{x\to 0} f(x)$?

THE FUNCTION IS NOT CONTINUOUS AT $x = 0$.

$\lim\limits_{x\to 1} f(x) = f(1)$ AND $\lim\limits_{x\to -3} f(x) = f(-3)$

$\Rightarrow$ f IS CONTINUOUS AT $x = 1$ AND $x = -3$.

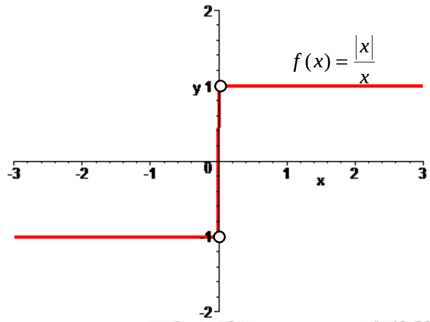


Figure 2.29

Suppose $c \neq 0$, then what is $\lim\limits_{x\to c} f(x)$?

What is the value of $f(c)$?

Is f continuous at $x = c$?

Example 2 LET $f(x) = \dfrac{x^2}{|x|}$. IS f CONTINUOUS AT $x = 0$?

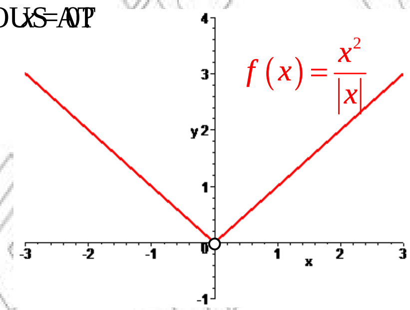


Figure 2.30

Solution

$$\frac{x^2}{|x|} = \begin{cases} x, & \text{IF } x > 0 \\ \nexists, & \text{IF } x = 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

$f(0)$ IS UNDEFINED. BUT $\lim\limits_{x\to 0} f(x) = 0$

$\Rightarrow$ f IS NOT CONTINUOUS AT $x = 0$

Example 3 FIND OUT THE CONDITION THAT MAKES $f(x) = \dfrac{1}{x-3}$ DISCONTINUOUS AT $x = 3$?

Solution f IS DISCONTINUOUS AT $x = 3$ BECAUSE

I $f(3)$ IS UNDEFINED

II $\lim\limits_{x\to 3^+} f(x) = \infty$

$\lim\limits_{x\to 3^-} f(x) = -\infty$

$\Rightarrow$ $\lim\limits_{x\to 3} f(x)$ DOESN'T EXIST.

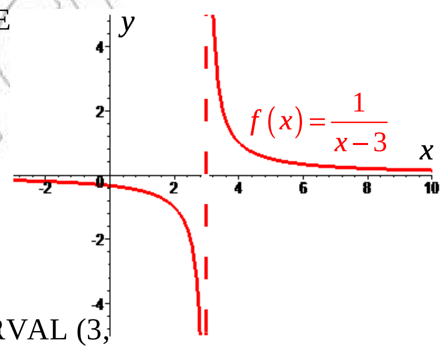


Figure 2.31

NOTE THAT f IS UNBROKEN ON THE INTERVAL $(3, \infty)$ AND ON $(-\infty, 3)$.

Example 4 CONSIDER THE PIECEWISE DEFINED FUNCTION $f(x) = \begin{cases} 0, & \text{IF } x \in \mathbb{Z} \\ 1, & \text{IF } x \notin \mathbb{Z} \end{cases}$

IS f CONTINUOUS AT $x = 1$? AT $x = \dfrac{1}{2}$?

DETERMINE THE SET OF NUMBERS AT WHICH f IS DISCONTINUOUS.

Solution

A $\lim_{x\to 1} f(x) = 1$ AND $f(1) = 0$

$\Rightarrow f$ IS DISCONTINUOUS AT $x = 1$

B $\lim_{x\to \frac{1}{2}} f(x) = 1$ AND $f\left(\frac{1}{2}\right) = 1$

$\Rightarrow f$ IS CONTINUOUS AT $x = \frac{1}{2}$.

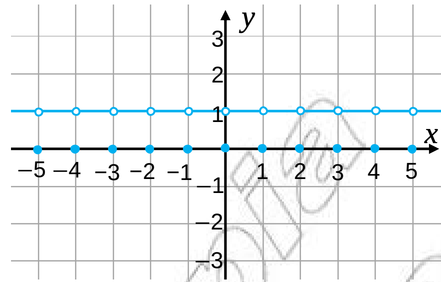


Figure 2.32

USE THE GRAPH IN FIGURE 2.32 TO EVALUATE $\lim_{x\to c} f(x)$ WHEN c IS AN INTEGER.

Do you see that f is discontinuous at every integer? Justify!

Example 5 SHOW THAT $f(x) = \frac{\sqrt{x^2 - 3x + 2}}{x - 5}$ IS CONTINUOUS AT $x = 3$.

Solution WHAT IS THE DOMAIN OF f? IS 3 IN THE DOMAIN OF f?

$$f(3) = \frac{\sqrt{3^2 - 3(3) + 2}}{3 - 5} \Rightarrow f(3) = -\frac{\sqrt{2}}{2}.$$

ALSO $\lim_{x\to 3} \frac{\sqrt{x^2 - 3x + 2}}{x - 5} = -\frac{\sqrt{2}}{2}$

$\Rightarrow f$ IS CONTINUOUS AT $x = 3$.

2.3.2 Continuity of a function on an Interval

CONSIDER THE FOLLOWING GRAPH OF A FUNCTION f.

DETERMINE THOSE INTERVALS ON WHICH THE GRAPH IS DRAWN WITHOUT TAKING THE PENCIL O

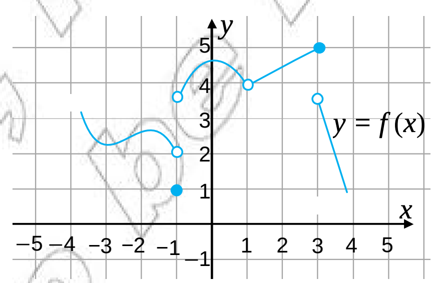


Figure 2.33

THE FUNCTION IS DISCONTINUOUS AT $x = -1$, $x = 1$ AND $x = 3$.

THE GRAPH IS CONTINUOUSLY DRAWN ON THE INTERVALS.

$(-\infty, -1)$, $(-1, 1)$, $(1, 3]$ AND $(3, \infty)$

Definition 2.12

(One side continuity)

A FUNCTION f IS CONTINUOUS FROM THE RIGHT AT $x = a$ PROVIDED THAT

$$\lim_{x\to a^+} f(x) = f(a).$$

A FUNCTION f IS CONTINUOUS FROM THE LEFT AT $x = b$ PROVIDED THAT

$$\lim_{x\to b^-} f(x) = f(b).$$

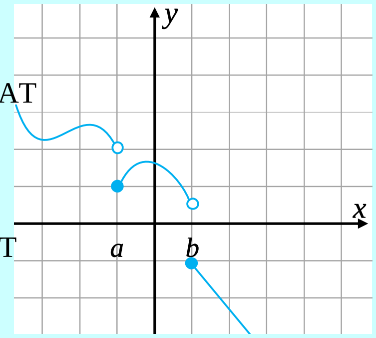


Figure 2.34

Example 6 LET $f(x) = \sqrt{1-x^2}$; SHOW THAT f IS CONTINUOUS FROM THE RIGHT AT $x = -1$ AND CONTINUOUS FROM THE LEFT AT $x = 1$.

Solution

A $\lim_{x\to -1^+} \sqrt{1-x^2} = 0$ AND $f(-1) = 0$

B $\lim_{x\to 1^-} \sqrt{1-x^2} = 0$ AND $f(1) = 0$

THE GRAPH OF f, SHOWN IN FIGURE 2.35, ALSO ILLUSTRATES THE ANSWERS.

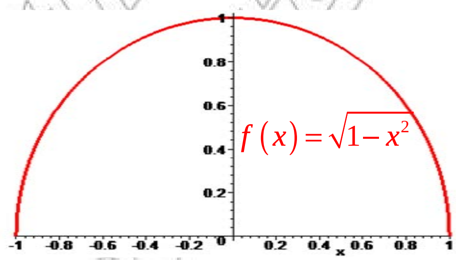


Figure 2.35

Example 7 SHOW THAT $g(x) = \sqrt{1-3x}$ IS CONTINUOUS FROM THE LEFT AT $x = \frac{1}{3}$.

Solution FROM THE GRAPH ONE CAN SEE THAT $\lim_{x\to \frac{1}{3}^-} g(x) = 0 = g\left(\frac{1}{3}\right)$

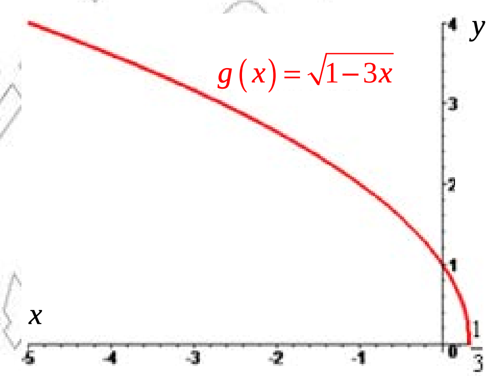


Figure 2.36

Example 8 LET $f(x) = \frac{x^2 - 9}{|x-3|}$. SHOW THAT f IS CONTINUOUS NEITHER FROM THE RIGHT NOR FROM THE LEFT AT $x = 3$.

Solution THE BASIC STRATEGY TO SOLVE SUCH A PROBLEM IS TO SKETCH THE GRAPH.

$$\frac{x^2-9}{|x-3|} \begin{cases} = x+3, \text{ IF } x > 3 \\ \nexists, \text{ IF } x = 3 \\ = -x-3, \text{ IF } x < 3 \end{cases}$$

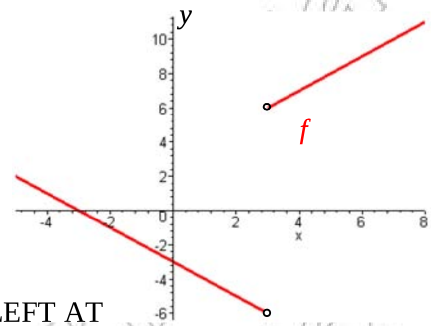


Figure 2.37

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-x-3) = -6$$

BUT $f(3)$ IS UNDEFINED

$\Rightarrow$ f IS NOT CONTINUOUS FROM THE LEFT AT $x = 3$.

SIMILARLY, f IS DISCONTINUOUS FROM THE RIGHT AT $x = 3$.

WE KNOW THAT THE POLYNOMIALS $x + 3$ AND $-x - 3$ ARE CONTINUOUS ON THE ENTIRE INTERVALS $(3, \infty)$ AND $(-\infty, -3)$, RESPECTIVELY.

Definition 2.13

Continuity of a function on an interval.

1 ***Open interval***

A FUNCTION f IS CONTINUOUS ON AN OPEN INTERVAL (a, b), IF

$$\lim_{x \to c} f(x) = f(c) \; \forall c \in (a, b).$$

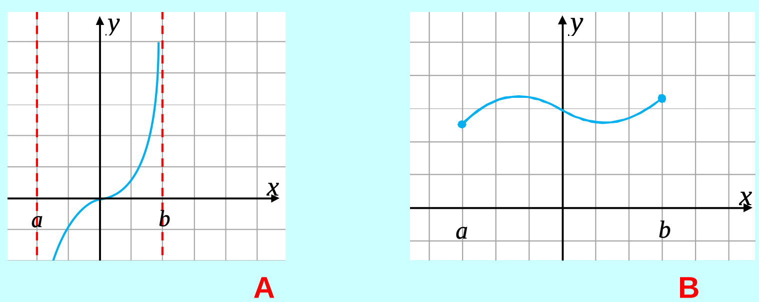


Figure 2.38

2 ***Closed interval***

A FUNCTION f IS CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$ PROVIDED THAT

I f IS CONTINUOUS ON (a, b)

II f IS CONTINUOUS FROM THE RIGHT AT a, AND

III f IS CONTINUOUS FROM THE LEFT AT b.

A FUNCTION f IS CONTINUOUS, IF IT IS CONTINUOUS OVER ITS DOMAIN.

Some continuous functions

- ✓ POLYNOMIAL FUNCTIONS
- ✓ ABSOLUTE VALUE OF CONTINUOUS FUNCTIONS
- ✓ THE SINE AND COSINE FUNCTIONS
- ✓ EXPONENTIAL FUNCTIONS
- ✓ LOGARITHMIC FUNCTIONS

Example 9 THE FOLLOWING IS THE GRAPH OF f. DETERMINE THE INTERVALS ON WHICH f IS CONTINUOUS.

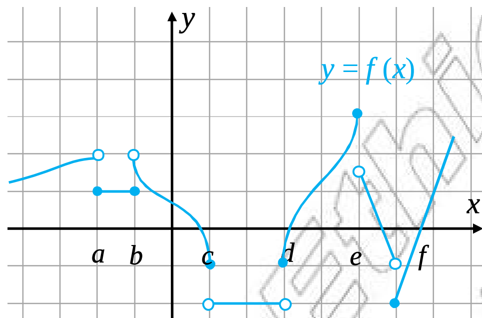


Figure 2.39

Solution IT IS CONTINUOUS ON $(-\infty, a)$, $[a, b]$, $(b, c]$, (c, d), $[d, e]$, (e, f), $[f, \infty)$.

Example 10 DETERMINE WHETHER OR NOT EACH OF THE FOLLOWING FUNCTIONS ARE CONTINUOUS ON THE GIVEN INTERVAL:

A $f(x) = \frac{1}{x}, (0, 5)$ **B** $f(x) = \frac{x^2 - 4}{x + 2}, (-3, 3)$

C $f(x) = 2x^3 - 5x^2 + 7x + 11, (-\infty, \infty)$.

Solution

A f IS A RATIONAL FUNCTION AND $x \neq 0$ FOR EACH $x \in (0, 5)$. HENCE, WE CONCLUDE THAT f IS CONTINUOUS ON $(0, 5)$.

B f IS UNDEFINED AT $x = -2$. HENCE f IS DISCONTINUOUS AT $x = -2$ BUT f IS CONTINUOUS AT ANY OTHER POINT ON $(-3, 3)$. THUS f IS NOT CONTINUOUS ON $(-3, 3)$.

C EVERY POLYNOMIAL FUNCTION IS CONTINUOUS ON $(-\infty, \infty)$. THUS f IS CONTINUOUS ON $(-\infty, \infty)$.

Example 11 LET $f(x) = \begin{cases} 4 - x^2, \text{ IF } x < 1 \\ 5, \text{ IF } 1 \leq x < 4 \\ -1, \text{ IF } x = 4 \\ x + 1, \text{ IF } x > 4 \end{cases}$

DETERMINE THE INTERVALS ON WHICH f IS CONTINUOUS.

Solution

FROM THE GRAPH (FIGURE 2.40) YOU MAY SEE THAT f IS CONTINUOUS ON $(-\infty, 1)$, $[1, 4)$ AND $(4, \infty)$. BUT IT IS DISCONTINUOUS AT $x = 1$ AND $x = 4$.

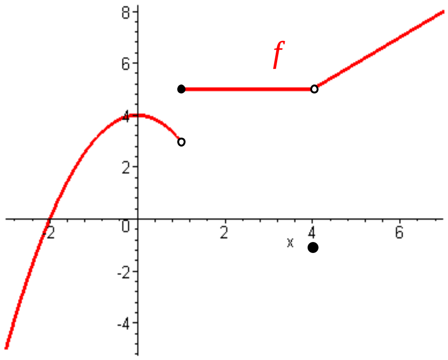


Figure 2.40

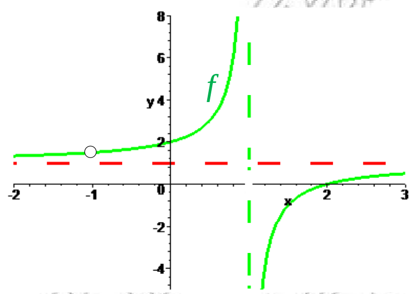


Figure 2.41

Example 12 LET $f(x) = \dfrac{x^2 - x - 2}{x^2 - 1}$. FIND THE INTERVALS WHERE f IS CONTINUOUS.

Solution $\dfrac{x^2 - x - 2}{x^2 - 1} = \dfrac{(x-2)(x+1)}{(x-1)(x+1)} = \dfrac{x-2}{x-1}$, *if* $x \neq -1, 1$

f IS DISCONTINUOUS AT $x = -1$ AND $x = 1$.

f IS CONTINUOUS ON $(-\infty, -1)$, $(-1, 1)$ AND $(1, \infty)$ AS IT IS SHOWN IN FIGURE 2.41

Example 13 LET $f(x) = \begin{cases} 2^{-x}, \text{ IF } x < -1 \\ 2x + 2, \text{ IF } -1 \leq x < 3 \\ 4 - x, \text{ IF } x \geq 3 \end{cases}$

DETERMINE THE INTERVALS ON WHICH f IS CONTINUOUS.

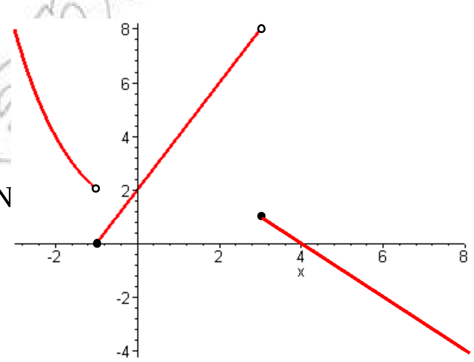


Figure 2.42

Solution LOOK AT THE GRAPH OF f.

ARE -1 AND 3 IN THE DOMAIN OF f?

f IS CONTINUOUS ON $(-\infty, -1)$, $[-1, 3)$, $[3, \infty)$

Example 14 DETERMINE THE INTERVAL ON WHICH $f(x) = \sqrt{x^2 - 1}$ IS CONTINUOUS.

Solution IN $f(x) = \sqrt{x^2 - 1}$, $x^2 - 1 \geq 0 \Rightarrow |x| \geq 1$

THE DOMAIN OF f IS $\{x : |x| \geq 1\}$

EXPLAIN WHY f IS DISCONTINUOUS ON $(-1, 1)$!

f IS CONTINUOUS ON $(-\infty, -1] \cup [1, \infty)$. *(Explain!)*

Example 15 Let $f(x)=\dfrac{1}{\sqrt{9-4x^2}}$. What is the largest interval on which f is continuous?

Solution First determine the domain of f and sketch the graph of f.

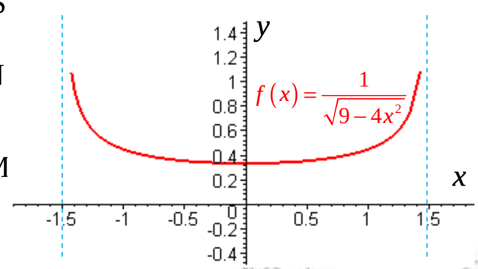


Figure 2.43

Explain why f is continuous on $\left(-\dfrac{3}{2},\dfrac{3}{2}\right)$.

Is there an interval larger than $\left(-\dfrac{3}{2},\dfrac{3}{2}\right)$ on which f is continuous?

Example 16 Determine the value of a so that the piecewise defined function

$$f(x)=\begin{cases} x+3, & \text{if } x>2 \\ ax-1, & \text{if } x\le 2 \end{cases}$$

is continuous on $(-\infty,\infty)$.

Solution If f is continuous on $(-\infty,\infty)$, then f must be continuous at $x = 2$.

$$\Rightarrow \lim_{x\to 2^+} f(x)=f(2) \Rightarrow \lim_{x\to 2^+}(x+3)=a(2)-1 \Rightarrow 5=2a-1 \Rightarrow a=3$$

$$\Rightarrow f(x)=\begin{cases} x+3, & \text{if } x>2 \\ 3x-1, & \text{if } x\le 2 \end{cases}$$

Example 17 Let $f(x)=\begin{cases} ax+b, & \text{if } x\le -2 \\ 2x+a, & \text{if } -2<x\le 3 \\ ax^2-bx+4, & \text{if } x>3 \end{cases}$

If f is a continuous function, find the values of a and b.

Solution f should be continuous at $x = -2$ and $x = 3$ because f is a continuous function.

I f is continuous at $x = -2$

$$\Rightarrow \lim_{x\to -2^+} f(x)=f(-2) \Rightarrow \lim_{x\to -2^+}(2\times(-2)+a)=(a(-2)+b) \Rightarrow -4+a=-2a+b$$

$\Rightarrow$ $3a - b = 4$ . . . . *equation* (1)

II f is continuous at $x = 3$

$$\Rightarrow \lim_{x\to 3^+} f(x)=f(3) \Rightarrow \lim_{x\to 3^+}(ax^2-bx+4)=2(3)+a$$

$\Rightarrow$ $9a - 3b + 4 = 6 + a$

$\Rightarrow$ $8a - 3b = 2$ . . . . *equation* (2)

Solving the system of equations

$\begin{cases} 3a-b=4 \\ 8a-3b=2 \end{cases}$ gives $a = 10$ and $b = 26$.

Example 18 DISCUSS THE CONTINUITY OF THE FUNCTION $f(x)=\sqrt{3-\sqrt{x^2-16}}$

Solution IN $\sqrt{x^2-16}$, $x^2-16\ge 0 \Rightarrow x^2\ge 16 \Rightarrow |x|\ge 4$

IN $\sqrt{3-\sqrt{x^2-16}}$, $3-\sqrt{x^2-16}\ge 0 \Rightarrow 3\ge \sqrt{x^2-16} \Rightarrow 25\ge x^2 \Rightarrow |x|\le 5$.

THUS, $|x|\ge 4$ AND $|x|\le 5$

$\Rightarrow$ f IS CONTINUOUS ON $[-5, -4]$ AND $[4, 5]$.

Example 19 A LIBRARY THAT RENTS BOOKS ALLOWS ITS CUSTOMERS TO KEEP A BOOK FOR 5 DAYS AT A FEE OF BIRR 10. CUSTOMERS WHO KEEP A BOOK MORE THAN 5 DAYS PAY BIRR 2 PENALTY PLUS BIRR 1.25 PER DAY FOR BEING LATE BEYOND THE FIRST 5 DAYS. IF $c(x)$ REPRESENTS THE COST OF KEEPING A BOOK FOR x DAYS, DISCUSS THE CONTINUITY OF c ON $[0, 20]$.

Solution WE FIRST DETERMINE A FORMULA FOR $c(x)$. FROM THE GIVEN INFORMATION, THE FEE FOR THE FIRST 5 DAYS IS BIRR 10.

$\Rightarrow c(x) = 10$, IF $0 < x \le 5$.

FOR $x > 5$, $c(x) = 10 + 2 + (x-5)(1.25)$. *Explain!*

$= 1.25x + 5.75$

$$\Rightarrow c(x)=\begin{cases}10, & \text{IF } 0< x\le 5\\ 1.25x+5.75, & \text{IF } x>5\end{cases}$$

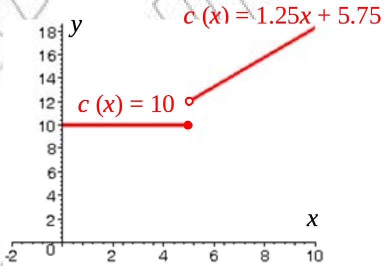


Figure 2.44

THE CONSTANT 10, AND THE POLYNOMIAL $1.25x + 5.75$ ARE CONTINUOUS ON (0, 5] AND (5, 20] RESPECTIVELY. THUS, C IS CONTINUOUS ON (0, 5] AND (5, 20].

BUT $\lim\limits_{x\to 5^+} c(x) = 1.25(5) + 5.75 = 12$

$\lim\limits_{x\to 5^-} c(x) = 10 \Rightarrow \lim\limits_{x\to 5} c(x)$ DOESN'T EXIST $\Rightarrow$ c IS NOT CONTINUOUS AT $x = 5$

Properties of continuous functions

SUPPOSE f AND g ARE CONTINUOUS AT $x = x_0$. DISCUSS THE CONTINUITY OF THE COMBINATIONS OF f AND g.

IS $f + g$ CONTINUOUS AT $x = x_0$?

$$\lim_{x\to x_o}(f+g)(x)=\lim_{x\to x_o}\left(f(x)+g(x)\right)=\lim_{x\to x_o}f(x)+\lim_{x\to x_o}g(x)$$ *Why?*

$= f(x_0) + g(x_0) = (f + g)(x_0)$

HENCE, $f + g$ IS CONTINUOUS AT $x = x_0$.

EXPLAIN THAT THE CONTINUITY OF THE COMBINATIONS OF f AND g IS AN IMMEDIATE CONSEQUENCE OF THE BASIC LIMIT THEOREMS.

Theorem 2.3 Properties of continuous functions

IF f AND g ARE CONTINUOUS AT $x = x_0$, THEN THE FOLLOWING FUNCTIONS ARE CONTINUOUS AT $x = x_0$.

1 $f + g$ **2** $f - g$ **3** $kg,\ k \in \mathbb{R}$

4 fg **5** $\frac{f}{g}$, PROVIDED THAT $g(x_0) \neq 0$.

Example 20 LET $f(x) = x$, $g(x) = \text{SIN}\,x$. DISCUSS THE CONTINUITY OF THE COMBINATIONS OF f AND g AT $x = 0$.

Solution f AND g ARE CONTINUOUS AT $x = 0$. HENCE, $f + g$, $f - g$, kf AND fg ARE CONTINUOUS AT $x = 0$. $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{x}{\text{SIN}\,x} = 1$

BUT, $\frac{f}{g}(0)$ IS UNDEFINED. HENCE, $\frac{f}{g}$ IS NOT CONTINUOUS AT $x = 0$.

Example 21 DISCUSS THE CONTINUITY OF THE FUNCTION GIVEN BY

$$f(x) = \begin{cases} 4 - \sqrt{9 - x^2}, & \text{IF } |x| \leq 3 \\ 10 - 2x, & \text{IF } x > 3 \end{cases}$$

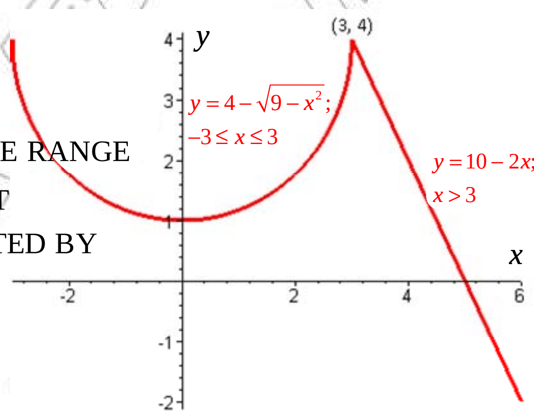


Figure 2.45

Solution CAN YOU DETERMINE THE RANGE OF VALUES OF $\sqrt{9 - x^2}$? WHAT IS THE CURVE REPRESENTED BY

$y = 4 - \sqrt{9 - x^2}$?

DO YOU SEE THAT

$1 \leq 4 - \sqrt{9 - x^2} \leq 4$?

THE FUNCTION IS CONTINUOUS ON

$[-3, \infty)$ AS IT IS SHOWN IN THE FIGURE.

SOME OF THE ABOVE EXAMPLES ARE THE COMPOSITIONS OF TWO OR MORE SIMPLE FUNCTIO

IN GENERAL, YOU HAVE THE FOLLOWING THEOREM ON THE CONTINUITY OF THE COMPOSITI FUNCTIONS.

Theorem 2.4 Continuity of compositions of functions

IF A FUNCTION f IS CONTINUOUS AT $x = x_0$ AND THE FUNCTION g IS CONTINUOUS AT $y = f(x_0)$, THEN THE COMPOSITION FUNCTION gof IS CONTINUOUS AT $x = x_0$.

I.E., $\lim\limits_{x\to x_o} g(f(x)) = \lim\limits_{y\to f(x_o)} g(y) = g(f(x_o)) = (gof)(x_o)$.

Example 22 LET $f(x) = x^2 - 3x + 2$ AND $g(x) = \sqrt{x}$.

SHOW THAT gof IS CONTINUOUS AT $x = -1$.

Solution $x_0 = -1$, f IS CONTINUOUS AT $x = -1$. *Explain!*

$f(x_0) = f(-1) = 6 \Rightarrow g$ IS CONTINUOUS AT $y = 6$.

IN SHORT, $\lim\limits_{x\to-1}(gof)(x) = \lim\limits_{x\to-1}\sqrt{x^2-3x+2} = \sqrt{\lim\limits_{x\to-1}(x^2-3x+2)}$

$= \sqrt{6}$

Maximum and minimum values

MAXIMUM AND MINIMUM ARE COMMON WORDS IN GENERAL REAL LIFE USE.

FOR EXAMPLE, DALOL DANAKIL DEPRESSION IN ETHIOPIA HAS THE MAXIMUM AVERAGE ANNUAL TEMPERATURE IN THE WORLD WHICH IS 35°C.

THE MINIMUM AVERAGE ANNUAL TEMPERATURE IN THE WORLD IS –57.2°C IN ANTARCTIC. DISCUSS OTHER MINIMUM AND MAXIMUM VALUES THAT EXIST IN REAL WORLD PHENOMENA.

Maximum and minimum values of a continuous function on a closed interval

Example 23 FIND THE MAXIMUM AND MINIMUM VALUES OF f ON THE CLOSED INTERVAL.

A $f(x) = 3x - 1$ ON $[-2, 3]$. **B** $f(x) = -x^2 + 3x - 4$ ON $[1, 5]$

Solution

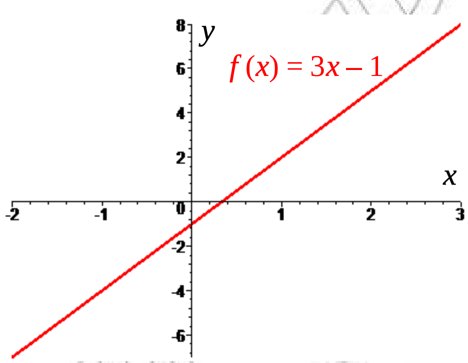


Figure 2.46

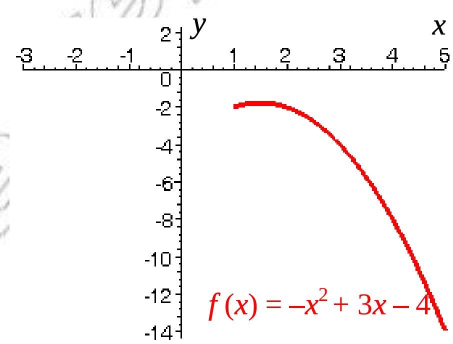


Figure 2.47

$-7 \le f(x) \le 8 \quad \forall x \in [-2, 3]$

THE MAXIMUM VALUE IS 8.

THE MINIMUM VALUE IS –7

$-14 \le f(x) \le -\frac{7}{4} \quad \forall x \in [1, 5]$

THE MAXIMUM VALUE IS $-1\frac{3}{4}$

THE MINIMUM VALUE IS –14.

The intermediate value theorem

Theorem 2.5 The intermediate value theorem

SUPPOSE f IS A CONTINUOUS FUNCTION ON THE CLOSED INTERVAL $[a, b]$ AND k IS ANY REAL NUMBER WITH EITHER $f(a) \le k \le f(b)$ OR $f(b) \le k \le f(a)$, THEN THERE EXISTS c IN $[a, b]$ SUCH THAT $f(c) = k$.

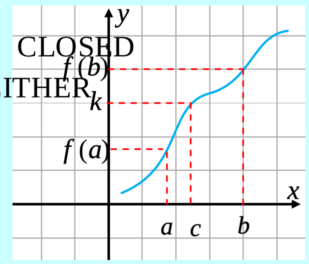


Figure 2.48

Example 24 SHOW THAT $f(x) = x^3 + x + 1$ HAS A ZERO BETWEEN $x = -1$ AND $x = 0$.

Solution USING THE INTERMEDIATE VALUE THEOREM, $k = 0$, $a = -1$, $b = 0$,

$f(-1) = (-1)^3 - 1 + 1 = -1 < 0$.

$f(0) = 0 + 0 + 1 = 1 \Rightarrow f(-1) < 0 < f(0)$

$\Rightarrow \exists c \in [-1, 0]$ SUCH THAT $f(c) = 0$.

Example 25 SHOW THAT THE GRAPH OF $f(x) = x^5 - 2x^3 + x - 7$ CROSSES THE LINE $y = 7.3$

Solution $f(1) = 1 - 2 + 1 - 7 = -7$

$f(2) = 32 - 16 + 2 - 7 = 11$

$\Rightarrow f(1) < 7.3 < f(2)$

$\Rightarrow$ THE GRAPH CROSSES THE LINE $y = 7.3$

Example 26 USE THE INTERMEDIATE VALUE THEOREM TO LOCATE THE ZEROS OF THE FUNCTION $f(x) = x^4 - x^3 - 5x^2 + 2x + 1$.

Solution EVERY POLYNOMIAL FUNCTION IS CONTINUOUS.

$f(0) = 1 > 0$

$f(1) = 1 - 1 - 5 + 2 + 1 = -2 < 0$

$\Rightarrow f$ HAS A ZERO BETWEEN $x = 0$ AND $x = 1$.

$f(2) = 16 - 8 - 20 + 4 + 1 = -7 < 0$ AND

$f(3) = 81 - 27 - 45 + 6 + 1 = 16 > 0$

$\Rightarrow f$ HAS A ZERO BETWEEN $x = 2$ AND $x = 3$

$f(-1) = 1 + 1 - 5 - 2 + 1 = -4 < 0$

$\Rightarrow f$ HAS A ZERO BETWEEN $x = 0$ AND $x = -1$

$f(-2) = 16 + 8 - 20 - 4 + 1 = 1 > 0$

$\Rightarrow f$ HAS A ZERO BETWEEN $x = -1$ AND $x = -2$

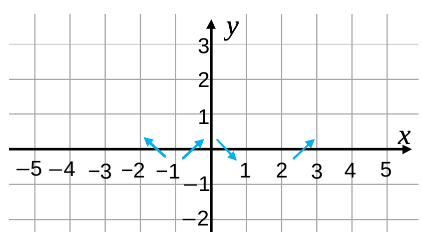


Figure 2.49

Note:

✓ DISCONTINUOUS FUNCTIONS MAY NOT POSSESS THE INTERMEDIATE VALUE PROPERTY. TO SEE THIS, CONSIDER $f(x) = \frac{1}{x}$ WHICH IS DISCONTINUOUS AT 0. $f(-1) < 0$ AND $f(1) > 0$ BUT THERE IS NO VALUE OF x IN $(-1, 1)$ SUCH THAT $f(x) = 0$

Approximating real zeros by bisection

LET f BE A CONTINUOUS FUNCTION ON THE CLOSED INTERVAL $[a, b]$. IF $f(a)$ AND $f(b)$ ARE OPPOSITE IN SIGN, THEN BY THE INTERMEDIATE VALUE THEOREM f HAS A ZERO IN (a, b). IN ORDER TO GET AN INTERVAL (a, b), IN WHICH f HAS ZERO, BISECT THE INTERVAL $[a, b]$ BY THE MIDPOINT $c = \frac{a+b}{2}$.

IF $f(c) = 0$, STOP SEARCHING A ZERO. IF $f(c) \neq 0$, THEN CHOOSE THE INTERVAL (a, c) OR (c, b) IN WHICH $f(c)$ HAS AN OPPOSITE SIGN AT THE END POINT.

REPEAT THIS BISECTION PROCESS UNTIL YOU GET THE DESIRED DECIMAL ACCURACY FOR THE APPROXIMATION.

Example 27 APPROXIMATE THE REAL ROOT OF $f(x) = x^3 + x - 1$ WITH AN ERROR LESS THAN $\frac{1}{16}$.

Solution USING A CALCULATOR, YOU CAN FILL IN THE FOLLOWING TABLE AND GET A NUMBER AS REQUIRED.

Opposite sign interval (a, b)	MID-POINT c	SIGN OF f		
		$f(a)$	$f(c)$	$f(b)$
(0, 1)	0.5	–	–	+
(0.5, 1)	0.75	–	+	+
(0.5, 0.75)	0.625	–	–	+
(0.625, 0.75)	0.6875	–	+	+

$f(0.6875) = 0.012451172 < 0.0625 = \frac{1}{16}$

$\Rightarrow$ 0.6875 IS A ROOT OF f WITH AN ERROR LESS THAN $\frac{1}{16}$

Example 28 USE THE BISECTION METHOD TO FIND AN APPROXIMATION OF $\sqrt[3]{7}$ WITH AN ERROR LESS THAN $\frac{1}{20}$.

Solution LET $x = \sqrt[3]{7}$, THEN $x^3 = 7 \Rightarrow x^3 - 7 = 0$. DEFINE A FUNCTION f BY

$f(x) = x^3 - 7$, $f(1) = -6 < 0$ AND $f(2) = 1 > 0$

$\Rightarrow f$ HAS A REAL ROOT IN (1, 2).

LOOK AT THE FOLLOWING TABLE.

Opposite sign interval (a, b)	MID-POINT	SING OF F		
		$f(a)$	$f(c)$	$f(b)$
(1, 2)	1.5	–	–	+
(1.5, 2)	1.75	–	–	+
(1.75, 2)	1.875	–	–	+
(1.875, 2)	1.9375	–	+	+
(1.875, 1.9375)	1.90625	–	–	+
(1.90625, 1.9375)	1.921875	–	+	+
(1.90625, 1.921875)	1.9140625	–	+	+

$$f(1.9140625) = 0.01242685 < 0.05 = \frac{1}{20}$$

$\Rightarrow \sqrt[3]{7} \approx 1.9140625$ WITH AN ERROR LESS THAN $\frac{1}{20}$

Theorem 2.6 Extreme value theorem

LET f BE A CONTINUOUS FUNCTION ON $[a, b]$. THEN THERE ARE TWO NUMBERS x_1 AND x_2 IN $[a, b]$ SUCH THAT $f(x_1) \le f(x) \le f(x_2)\ \forall x \in [a, b]$.

$f(x_2)$ IS THE MAXIMUM VALUE AND $f(x_1)$ IS THE MINIMUM VALUE.

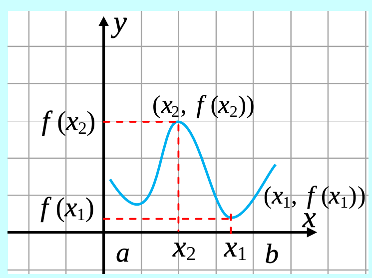


Figure 2.50

Group Work 2.1

1 DISCUSS THE FOLLOWING POINTS BY DRAWIN PRODUCING EXAMPLES.

ARE THERE MAXIMUM AND MINIMUM VALUES, IF

- **I** THE FUNCTION ON $[a, b]$ IS NOT CONTINUOUS?
- **II** THE FUNCTION IS CONTINUOUS ON (a, b)?
- **III** THE FUNCTION IS NOT CONTINUOUS BUT DEFINED ON AN OPEN INTERVAL?

2 LET f BE CONTINUOUS ON $[a, b]$. ANSWER THE FOLLOWING POINTS IN TERMS OF $f(a)$ AND $f(b)$. USE GRAPHS TO ILLUSTRATE YOUR ANSWERS.

i FIND THE MINIMUM AND THE MAXIMUM VALUES OF $f(x)$ WHEN f IS AN INCREASING FUNCTION.

ii FIND THE MINIMUM AND MAXIMUM VALUES OF $f(x)$ WHEN f IS DECREASING.

3 DISCUSS THE FOLLOWING STATEMENTS USING THE INTERMEDIATE VALUE THEOREM:

i AMONG ALL SQUARES WHOSE SIDES DO NOT EXCEED 10 CM, IS THERE A SQUARE WHOSE AREA IS $11\sqrt{5}$ CM², $11\sqrt{17}$ CM²?

ii AMONG ALL CIRCLES WHOSE RADII ARE BETWEEN 10 CM AND 20 CM, IS THERE A CIRCLE WHOSE AREA IS 628 CM²?

iii THERE WAS A YEAR WHEN YOU WERE HALF AS TALL AS YOU ARE TODAY.

Exercise 2.8

1 DETERMINE WHETHER OR NOT EACH OF THE FOLLOWING FUNCTIONS IS CONTINUOUS AT THE GIVEN NUMBER.

A $f(x) = 3,\ x = 5$

B $f(x) = 2x^2 - 5x + 3;\ x = 1$

C $f(x) = \dfrac{(x-3)^2}{|x-3|};\ x = 3$

D $f(x) = \dfrac{(x-4)}{x^2+1};\ x = -1$

E $f(x) = \begin{cases} \sin x, & x > 0 \\ 1, & x = 0 \\ \dfrac{1}{x}, & x < 0 \end{cases};\ x = 0$

F $f(x) = \begin{cases} |x| - 1, & \text{if } |x| > 1 \\ 0, & \text{if } x = \pm 1 \\ 1 - |x|, & \text{if } |x| < 1 \end{cases};\ x = \pm 1$

2 IF THE PIECEWISE DEFINED FUNCTIONS BELOW ARE CONTINUOUS, DETERMINE THE VALUES OF THE CONSTANTS.

A $f(x) = \begin{cases} ax - 3, & \text{if } x > 2 \\ 2x + 5, & \text{if } x \le 2 \end{cases}$

B $f(x) = \begin{cases} ax^2 + bx + 1, & \text{if } 2 \le x \le 3 \\ ax - b, & \text{if } x < 2 \\ bx + 4, & \text{if } x > 3 \end{cases}$

C $f(x) = \begin{cases} \sqrt{x^2 - 2x + a}, & \text{if } \dfrac{1}{2} \le x \le \dfrac{3}{2} \\ -\sqrt{-x^2 + 2x - \dfrac{3}{4}}, & \text{if } x < \dfrac{1}{2} \text{ or } x > \dfrac{3}{2} \end{cases}$

D $f(x) = \begin{cases} \dfrac{k(x-5)}{x^2 - 25}, & x \ne \pm 5 \\ 5 & \text{if } x = \pm 5 \end{cases}$

E $f(x) = \begin{cases} 2^{|x-c|}, & \text{if } x > 4 \\ 2x, & \text{if } x \le 4 \end{cases}$

3 FIND THE MAXIMUM POSSIBLE INTERVAL(S) ON WHICH THESE FUNCTIONS ARE CONTINU

A $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, \text{ IF } x\neq 2\\ 8, \text{ IF } x=2\end{cases}$

B $f(x)=e^{-x^2}$

C $f(x)=\begin{cases}4\dfrac{|x^2-1|}{x-1}, \text{ IF } x\neq 1\\ 5, \text{ IF } x=1\end{cases}$

D $f(x)=\sqrt{1-4x^2}$

E $f(x)=\dfrac{1}{\sqrt{9-4x^2}}$

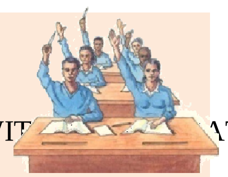
F $f(x)=\begin{cases}\dfrac{5(x^3+1)}{x+1}, \text{ IF } x\neq -1\\ 10, \text{ IF } x=-1\end{cases}$

G $F(X)=\sqrt{2-\sqrt{5-x^2}}$

4 THE MONTHLY BASE SALARY OF A SHOES SALES PERSON IS BIRR 900. SHE HAS A COMMIS OF 2% ON ALL SALES OVER BIRR 10,000 DURING THE MONTH. IF THE MONTHLY SALES AR 15,000 OR MORE, SHE RECEIVES BIRR 500 BONUS. IF x REPRESENTS THE MONTHLY SALES IN BIRR AND $f(x)$ REPRESENTS INCOME IN BIRR, EXPRESS $f(x)$ IN TERMS OF x AND DISCUSS THE CONTINUITY OF f ON [0, 25000].

2.4 EXERCISES ON APPLICATIONS OF LIMITS

ACTIVITY 2.10



1 LET x BE A REAL NUMBER. FILL IN THE TABLE BELOW WIT ATE VALUES.

x	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006
SIN x						
$\frac{\text{SIN } x}{x}$						

2 USE THE TABLE TO PREDICT $\lim_{x\to 0}\frac{\text{SIN } x}{x}$.

3 USE THE FOLLOWING GRAPH OF $f(x)=\frac{\text{SIN } x}{x}$ TO DETERMINE $\lim_{x\to 0}\frac{\text{SIN } x}{x}$.

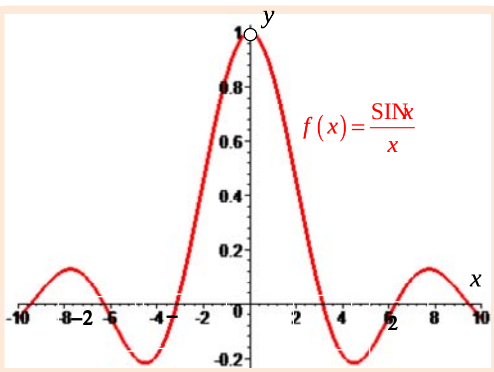


Figure 2.51

Theorem 2.7

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \text{ where } x \text{ is in radians.}$$

Example 1 Evaluate each of the following limits.

a $\lim_{x \to 0} \frac{\sin(3x)}{x}$ **b** $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$ **c** $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{x^2}\right)$

d $\lim_{x \to 0} \frac{x}{\sin x}$ **e** $\lim_{x \to 0} \frac{\tan x}{\sin x}$ **f** $\lim_{x \to 0} \frac{\sin(3x)}{\sin(4x)}$

g $\lim_{x \to 0} \frac{\sin^3 x}{x^3}$ **h** $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ **i** $\lim_{x \to 1} \frac{\sin(x-1)}{1 - x + x^2 - x^3}$

Solution

a $\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3\sin(3x)}{(3x)} = 3\lim_{x \to 0} \frac{\sin(3x)}{3x} = 3$

b $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y \to 0} \left(\frac{\sin y}{y}\right) = 1$, where $y = \frac{1}{x}$

c $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{x^2}\right) = \lim_{x \to \infty} \left(\frac{\sin \frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{y \to 0} \left(\frac{\sin(y)^2}{y^2}\right) = 1.$ *Why?*

d $\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \left(\frac{1}{\left(\frac{\sin x}{x}\right)}\right) = 1.$ *Why?*

E $\lim_{x\to 0}\frac{\tan x}{\sin x} = \lim_{x\to 0}\frac{\left(\frac{\tan x}{x}\right)}{\left(\frac{\sin x}{x}\right)} = \frac{\lim_{x\to 0}\frac{\tan x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1.$ *Why?*

F $\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)} = \lim_{x\to 0}\frac{\left(3\frac{\sin(3x)}{3x}\right)}{\left(4\frac{\sin(4x)}{4x}\right)} = \frac{3}{4}.$ *Why?*

G $\lim_{x\to 0}\frac{\sin^3 x}{x^3} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3 =$

H $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0}\frac{1}{(1+\cos x)} = 1\times\frac{1}{2} = \frac{1}{2}.$$

I $\lim_{x\to 1}\frac{\sin(x-1)}{1-x+x^2-x^3} = \lim_{x\to 1}\frac{-\sin(1-x)}{(1-x)+x^2(1-x)} = \lim_{x\to 1}\frac{-\sin(1-x)}{(1-x)(x^2+1)}$

$$= -\lim_{x\to 1}\frac{\sin(1-x)}{1-x}\cdot\lim_{x\to 1}\left(\frac{1}{x^2+1}\right) = -\frac{1}{2}$$

Example 2 The area A of a regular n-sided polygon inscribed in a circle of radius r is given by

$$A = nr^2\cos\frac{180^{\circ}}{n}\sin\frac{180^{\circ}}{n}$$

Using the fact that the circle is the limiting position of the polygon as $n\to\infty$, show that the area of the circle is $A = \pi r^2$.

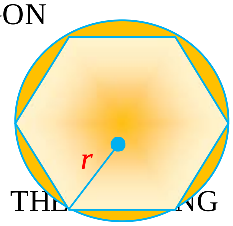


Figure 2.52

Proof:

$$A = \lim_{n\to\infty} nr^2\cos\frac{180^{\circ}}{n}\sin\frac{180^{\circ}}{n} = r^2\lim_{n\to\infty} n\cos\frac{\pi}{n}\sin\frac{\pi}{n}$$

$$= r^2\lim_{n\to\infty}\cos\frac{\pi}{n}\cdot\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} = r^2\times 1\times\pi = \pi r^2$$

Computation of *e* using the limit of a sequence

HISTORICAL NOTE

Leonhard Euler (1707-1783)

Swiss mathematician, whose major work was done in the field of pure mathematics. Euler was born in Basel and studied at the University of Basel under the Swiss mathematician Johann Bernoulli, obtaining his master's degree at the age of 16.

In his Introduction to Analysis of the Infinite (1748), Euler gave the first full analytical treatment of algebra, the theory of equations, trigonometry, and analytical geometry. In this work he treated the series expansion of functions and formulated the rule that only convergent infinite series can properly be evaluated.

He computed e to 23 decimal places using $\left(1+\frac{1}{k}\right)^k$.

IN GRADE 11, YOU HAVE USED THE IRRATIONAL NUMBER e IN EXPRESSIONS AND FORMULAE THAT MODEL REAL WORLD PHENOMENA.

ACTIVITY 2.11

1 CONSIDER THE SEQUENCE $\left\{\left(1+\frac{1}{k}\right)^k\right\}_{k\geq 1}$

A IS THE SEQUENCE MONOTONE?

JUSTIFY YOUR ANSWER BY FILLING UP THE VALUES IN THE FOLLOWING TABLE.

k	1	2	3	4	5	10	100	1000	10000
$\left(1+\frac{1}{k}\right)^k$									

B FIND THE SMALLER POSITIVE INTEGER k SUCH THAT $\left(1+\frac{1}{k}\right)^k$ IS GREATER THAN 2.5, 2.7, 2.8.

C WHAT DO YOU SEE FROM THE TABLE AS k INCREASES?

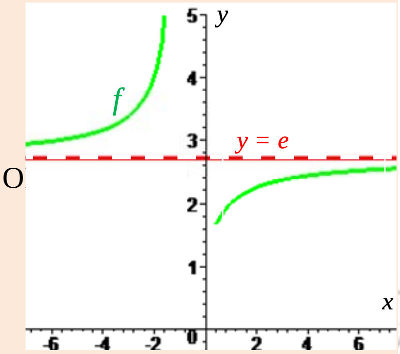
D FIND A POSITIVE INTEGER n SUCH THAT $n < \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k < n+1$

2 LET $f(x) = \left(1 + \frac{1}{x}\right)^x$

A WHAT IS THE DOMAIN OF f?

B LOOK AT THE GRAPH OF f. IS f CONTINUOUS ON (−1, 0]? WHY?

C USE THE GRAPH TO EVALUATE $\lim_{x\to\infty} f(x)$ AND $\lim_{x\to-\infty} f(x)$



Figures 2.53

Theorem 2.8

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \quad \text{AND} \quad \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

Example 3 EVALUATE $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+100}$

Solution

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+100} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \cdot \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{100} = e\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right)^{100} = e. \textit{ Why?}$$

IN GENERAL, YOU CAN SHOW THAT $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+c} = e$ FOR $c \in \mathbb{R}$

Example 4 EVALUATE $\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^x$

Solution LET $\frac{1}{y} = \frac{9}{x}$, THEN $x = 9y$.

THUS, $\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^x = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{9y} = \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^9 = e^9$. WHY?

IN GENERAL, WE CAN SHOW THAT $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = e^c$ FOR $c \in \mathbb{R}$

Example 5 EVALUATE $\lim_{x\to\infty}\left(\frac{x}{3-x}\right)^x$.

Solution

$$\lim_{x\to\infty}\left(\frac{x}{x-3}\right)^x = \lim_{x\to\infty}\left(\frac{1}{\left(1-\frac{3}{x}\right)^x}\right) = \frac{1}{e^{-3}} = e^3$$

Example 6 EVALUATE $\lim\limits_{x\to-\infty}\left(\dfrac{5x+1}{5x-3}\right)^{1-4x}$

Solution $\lim\limits_{x\to-\infty}\left(\dfrac{5x+1}{5x-3}\right)^{1-4x}=\lim\limits_{x\to-\infty}\left(\dfrac{5x-3}{5x+1}\right)^{4x-1}=\left(\dfrac{1+\frac{-3}{5x}}{1+\frac{1}{5x}}\right)^{4x-1}=\left(\dfrac{e^{\frac{-3}{5}}}{e^{\frac{1}{5}}}\right)^{4}=e^{-3.2}$

(Explain!)

Exercise 2.9

1 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim\limits_{x\to0}\dfrac{\tan(4x)}{\tan(3x)}$

B $\lim\limits_{x\to-2}\dfrac{\sin(x+2)}{x^3+2x^2+x+2}$

C $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{x-\frac{\pi}{2}}{\cos x}$

D $\lim\limits_{x\to0}\dfrac{\sin x}{x^3-x}$

E $\lim\limits_{x\to\pi^-}\dfrac{\sin x}{1-\cos x}$

F $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$

G $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x+5}$

H $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+3}\right)^{8-5x}$

I $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x-1}\right)^{3x-1}$

J $\lim\limits_{x\to\infty}\left(\dfrac{2x+5}{2x-11}\right)^{x+1}$

K $\lim\limits_{x\to0^+}(5x+1)^{\frac{1}{x}}$

L $\lim\limits_{X\to0}\sin\left(\dfrac{1}{X}\right)$

M $\lim\limits_{x\to\infty}\tan\left(\dfrac{1}{x}\right)$

2 *Continuous compounding formula*

CONSIDER THE COMPOUND INTEREST FORMULA $A=P\left(1+\dfrac{r}{100n}\right)^{nt}$.

IF THE LENGTH OF TIME PERIOD FOR COMPOUNDING OF THE INTEREST DECREASES FROM SEMI ANNUALLY, QUARTERLY, MONTHLY, DAILY, HOURLY, ETC., THEN A INCREASES THE AMOUNT

BUT THE INTEREST RATE FOR THE PERIOD DECREASES. THAT IS, AS $n\to\infty$, $\dfrac{r}{100n}\to0$. IN THIS

CASE, THE INTEREST IS SAID TO BE COMPOUNDED CONTINUOUSLY. FIND A FORMULA AMOUNT A OBTAINED WHEN THE INTEREST IS COMPOUNDED CONTINUOUSLY.

3 IF BIRR 4500 IS DEPOSITED IN AN ACCOUNT PAYING 3% ANNUAL INTEREST COMPOUNDED CONTINUOUSLY, THEN HOW MUCH IS IN THE ACCOUNT AFTER 10 YEARS AND 3 MONTHS

Key Terms

continuity	function	lower bound	null sequence
convergence	glb	lub	one side limit
decreasing	increasing	maximum	sequence
discontinuity	infinity	minimum	upper bound
divergence	limit	monotonic	

Summary

1 *Upper bound and lower bound*

I A NUMBER m IS CALLED AN upper bound OF A SEQUENCE $\{a_n\}$, IF AND ONLY IF $m \geq a_i\ \forall a_i \in \{a_n\}$

II A NUMBER k IS CALLED A lower bound OF A SEQUENCE $\{a_n\}$, IF AND ONLY IF $k \leq a_i\ \forall a_i \in \{a_n\}$

2 *Least upper bound (lub) and greatest lower bound (glb).*

I A NUMBER ℓ IS SAID TO BE THE least upper bound (LUB), IF AND ONLY IF ℓ IS AN UPPER BOUND AND IF y IS AN upper bound, THEN $\ell \leq y$.

II A NUMBER g IS SAID TO BE THE greatest lower bound (GLB), IF AND ONLY IF g IS A LOWER BOUND AND IF x IS A LOWER BOUND, THEN $g \geq x$.

3 A SEQUENCE $\{a_n\}$ IS SAID TO BE monotonic, IF IT IS EITHER INCREASING OR DECREASING.

4 A SEQUENCE $\{a_n\}$ IS SAID TO BE null sequence, IF AND ONLY IF $\lim_{n\to\infty} a_n = 0$.

5 *Convergence properties of sequences*

IF $\lim_{n\to\infty} a_n = L$ AND $\lim_{n\to\infty} b_n = M$, THEN

I $\lim_{n\to\infty}(a_n \pm b_n) = L \pm M$

II $\lim_{n\to\infty} ca_n = cL$; WHERE c IS A CONSTANT.

III $\lim_{n\to\infty}(a_n b_n) = LM$

IV $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{M}$, PROVIDED THAT $M \neq 0$, AND $b_n \neq 0$ FOR ANY n.

6 *Limit of a function*

✓ A NUMBER L IS THE LIMIT OF A FUNCTION f AT $x = a$, IF AND ONLY IF $f(x)$ APPROACHES L TO AS x - APPROACHES a BUT f NEED NOT BE DEFINED AT a. THIS IS EXPRESSED BY

$$\lim_{x\to a} f(x) = L$$

7 *One side limits*

I A NUMBER L IS SAID TO BE THE RIGHT SIDE LIMIT OF A FUNCTION f AT $x = a$, IF AND ONLY IF $f(x)$ APPROACHES L AS x APPROACHES a FROM THE RIGHT. THIS IS EXPRESSED AS: $\lim_{x\to a^+} f(x) = L$

II LIKEWISE, WE CAN DEFINE THE left hand side limit AND EXPRESS IT AS:

$$\lim_{x\to a^-} f(x) = L$$

III $\lim_{x\to a} f(x) = L$, IF AND ONLY IF $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

8 *Basic limit theorems*

IF $\lim_{x\to a} f(x) = L$ AND $\lim_{x\to a} g(x) = M$ THEN

I $\lim_{x\to a}\left(f(x) \pm g(x)\right) = L \pm M$

II $\lim_{x\to a} cf(x) = cL$ FOR A CONSTANT c

III $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = LM$

IV $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ PROVIDED THAT $M \neq 0$.

9 *Continuity*

I A FUNCTION f IS SAID TO BE CONTINUOUS AT $x = x_0$, IF THE FOLLOWING THREE CONDITIONS ARE MET.

A $f(x_0)$ IS DEFINED **B** $\lim_{x\to x_0} f(x)$ EXIST **C** $\lim_{x\to x_o} f(x) = f(x_o)$

II A FUNCTION IS CONTINUOUS ON AN OPEN INTERVAL (a, b), IF IT IS CONTINUOUS AT EACH NUMBER IN THE INTERVAL.

III A FUNCTION f IS CONTINUOUS ON A CLOSED INTERVAL $[a, b]$, IF IT IS CONTINUOUS ON (a, b) AND $\lim_{x\to a^+} f(x) = f(a)$ AND $\lim_{x\to b^-} f(x) = f(b)$.

IV A FUNCTION f IS SAID TO BE CONTINUOUS, IF IT IS CONTINUOUS ON ITS ENTIRE DOMAIN

10 *Properties of continuous functions*

IF f AND g ARE FUNCTIONS THAT ARE CONTINUOUS AT $x = x_o$ AND c IS A REAL NUMBER, THEN THE FOLLOWING FUNCTIONS ARE CONTINUOUS AT x_o.

I SCALAR MULTIPLE: cf

II SUM AND DIFFERENCE $f \pm g$

III PRODUCT: fg

IV QUOTIENT: $\dfrac{f}{g}$ PROVIDED THAT $g(x_0) \neq 0$.

11 *Continuity of composite functions*

IF g IS CONTINUOUS AT $x = x_0$ AND f IS CONTINUOUS AT $y = g(x_0)$, THEN THE COMPOSITE FUNCTION GIVEN BY $(f \circ g)(x) = f(g(x))$ IS CONTINUOUS AT $x = x_0$.

12 *Intermediate value theorem*

IF f IS CONTINUOUS ON $[a, b]$ AND k IS ANY REAL NUMBER BETWEEN $f(a)$ AND $f(b)$, THEN THERE IS AT LEAST ONE NUMBER c BETWEEN a AND b SUCH THAT $f(c) = k$.

13 *Extreme value theorem*

LET f BE A CONTINUOUS FUNCTION ON THE CLOSED INTERVAL $[a, b]$. THEN THERE EXIST TWO REAL NUMBERS x_1 AND x_2 IN $[a, b]$ SUCH THAT $f(x_2) \le f(x) \le f(x_1)$ FOR ALL $x \in [a, b]$. IN THIS CASE $f(x_2)$ IS THE MINIMUM VALUE OF THE FUNCTION f ON $[a, b]$ AND $f(x_1)$ IS THE MAXIMUM VALUE OF f ON $[a, b]$.

14 *Two important limits*

I $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **II** $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$

Review Exercises on Unit 2

1 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x\to 0}(2x - 1)$ **B** $\lim_{x\to -1}\frac{x+1}{x^2+7x+6}$ **C** $\lim_{x\to 9}\frac{\sqrt{x}-3}{x^2-81}$

D $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$ **E** $\lim_{x\to 0}\frac{\cos x}{x}$

2 LET $f(x) = \frac{x\,|x-5|}{x^2-25}$, EVALUATE

A $\lim_{x\to 5^+} f(x)$ **B** $\lim_{x\to -5^+} f(x)$ **C** $\lim_{x\to 5^-} f(x)$ **D** $\lim_{x\to -5^-} f(x)$

3 LET $f(x) = \begin{cases} 3, \text{ IF } x = -5 \\ -0.6, \text{ IF } -5 < x \le -2 \\ x^2 - 4, \text{ IF } -2 < x < 3 \\ x + 2, \text{ IF } x \ge 3 \end{cases}$

SKETCH THE GRAPH OF f AND EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x\to -5} f(x)$ **B** $\lim_{x\to -2} f(x)$ **C** $\lim_{x\to 3} f(x)$

4 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x\to 3}\left(x^3 - 4x^2 + 5x - 11\right)$ **B** $\lim_{x\to 2}\sqrt{x^2 - 5x}$

C $\lim_{x\to -7} \dfrac{x^2-49}{x^2+6x-7}$ **D** $\lim_{x\to 0} \dfrac{3x-4|x|}{5x}$

E $\lim_{x\to -5} \dfrac{x^3+125}{x+5}$ **F** $\lim_{x\to 1} \dfrac{\sin(x-1)+x^2-1}{x-1}$

G $\lim_{x\to \infty} \sin\left(\dfrac{\;}{x}\right)$ **H** $\lim_{x\to \infty} \cos x$

I $\lim_{x\to 0} \dfrac{\sin x}{x\cos x}$ **J** $\lim_{x\to 0} \dfrac{\sin^3(5x)}{\sin(4x^3)}$

5 TEST WHETHER OR NOT EACH OF THE GIVEN FUNCTIONS IS CONTINUOUS AT THE INDICATED NUMBER.

A $f(x)=\begin{cases} x^2-x, & \text{if } x \geq 1 \\ x+1, & \text{if } x < 1 \end{cases};\ x=1$ **B** $f(x)=\dfrac{x^2|9-x^2|}{3-x};\ x=3$

C $f(x)=\begin{cases} \dfrac{\sin x}{x}, & \text{IF } x \neq 0 \\ 1, & \text{IF } x=0 \end{cases};\ x=0$ **D** $f(x)=\begin{cases} \dfrac{1}{4}, & \text{IF } x \notin \mathbb{Z} \\ 4^x, & \text{IF } x \in \mathbb{Z} \end{cases};\ x=\dfrac{1}{2}$

E $f(x)=\begin{cases} \dfrac{\cos x}{e^x}, & \text{IF } x > 0 \\ e^x, & \text{IF } x \leq 0 \end{cases};\ x=0$

6 DETERMINE THE VALUES OF THE CONSTANTS SO THAT EACH OF THE FUNCTIONS IS CONTINUOUS.

A $f(x)=\begin{cases} ax-1, & \text{IF } x \leq 2 \\ x^2+3x, & \text{IF } x > 2 \end{cases}$ **B** $f(x)=\begin{cases} \dfrac{x^2-ax}{x-a}, & \text{if } x \neq a \\ 2, & \text{if } x = a \end{cases}$

C $f(x)=f(x)=\begin{cases} \sin(kx), & \text{IF } x \leq \\ 1, & \text{IF } x > 0 \end{cases}$ **D** $f(x)=\begin{cases} x^2+1, & \text{if } x < a \\ 15-5x, & \text{if } a \leq x \leq b \\ 5x-25, & \text{if } x > b \end{cases}$

7 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim_{x\to \infty} \dfrac{3x^3+5x^2-11}{2x^3-1}$ **B** $\lim_{x\to \infty} \dfrac{\sqrt{x^2+1}-10}{\sqrt{x^2+1}+9}$

8 EVALUATE EACH OF THE FOLLOWING ONE SIDE LIMITS.

A $\lim_{x\to 0^+} |x|-3x$ **B** $\lim_{x\to 3^+} \sqrt{3-x}$ **C** $\lim_{x\to 3^-} \sqrt{3x-9}$

D $\lim_{x\to 0^+} \ln x$ **E** $\lim_{x\to 5^+} \dfrac{x}{(x-5)^3}$ **F** $\lim_{x\to 2^+} \sqrt{1+\sqrt{x-1}}$

G $\lim\limits_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$ **H** $\lim\limits_{x\to -5^-} \sqrt{25 - x^2}$ **I** $\lim\limits_{x\to 7^-} \frac{x^2\left|x^2-49\right|}{x-7}$

9 DETERMINE THE LARGEST INTERVAL ON WHICH EACH OF THE GIVEN FUNCTIONS IS CONTINUOUS.

A $f(x) = \sqrt{\frac{1-x}{x}}$ **B** $f(x) = \sqrt{\ln\sqrt{x}}$

C $f(x) = \ln\left(\frac{x}{e^x - 1}\right)$ **D** $f(x) = \sqrt{\frac{4x-3}{x-4}}$

10 DETERMINE THE MAXIMUM AND MINIMUM VALUES OF EACH OF THE FUNCTIONS DEFINED ON THE INDICATED CLOSED INTERVAL.

A $f(x) = 3x + 5$; $[-3, 2]$ **B** $g(x) = 1 - x^2$; $[-2, 3]$

C $h(x) = x^4 - x^2$; $[-2, 2]$ **D** $f(x) = \frac{1}{x}$; $[-2, 2]$

E $h(x) = 4x^2 - 5x + 1$; $[-1.5, 1.5]$ **F** $f(x) = \begin{cases} x^2, & \text{IF } |x| \le 1 \\ 2 - |x|, & \text{IF } |x| > 1 \end{cases}$; $[-3, 2]$

11 LOCATE THE ZEROS OF EACH OF THE FOLLOWING FUNCTIONS USING THE INTERMEDIATE VALUE THEOREM.

A $f(x) = x^2 - x - 1$ **B** $g(x) = x^3 + 2x^2 - 5$

C $h(x) = x^3 - x + 2$ **D** $f(x) = x^4 - 2x^3 - x^2 + 3x - 2$

E $g(x) = x^4 - 9x^2 + 14$

12 EVALUATE EACH OF THE FOLLOWING LIMITS.

A $\lim\limits_{x\to 0} \frac{\sin\left(\frac{x}{\ }\right)}{\tan x}$ **B** $\lim\limits_{x\to 0} \frac{\sin(x^3)}{x^3}$ **C** $\lim\limits_{x\to\infty} x \tan\left(\frac{1}{x}\right)$

D $\lim\limits_{x\to 0} \frac{x - \tan x}{x}$ **E** $\lim\limits_{x\to\infty} \left(1 + \frac{3}{x+11}\right)^{x+6}$

13 IN A CERTAIN COUNTRY, THE LIFE EXPECTANCY OF MALES x YEARS FROM NOW IS GIVEN BY THE FORMULA $f(x) = \frac{210x + 116}{3x + 4}$ YEARS. WHAT WILL BE THE LIFE EXPECTANCY OF MALES IN THIS COUNTRY AS TIME PASSES? DISCUSS WHETHER OR NOT THE LIFE EXPECTANCY I
COUNTRY IS INCREASING.

14 A GIRL ENROLLING IN TYPING CLASS TYPES $\frac{60(x+1)}{x+9}$ WORDS PER MINUTE AFTER x WEEKS OF LESSONS. DETERMINE THE MAXIMUM POSSIBLE NUMBER OF WORDS THE GIRL CAN TYPE
TIME PASSES.