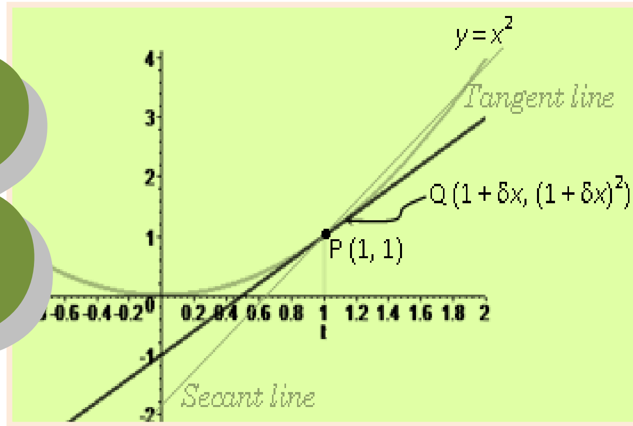
Unit 3



INTRODUCTION TO DIFFERENTIAL CALCULUS

Unit Outcomes:

After completing this unit, you should be able to:

- *describe the geometrical and mathematical meaning of derivative.*
- *determine the differentiability of a function at a point.*
- *find the derivatives of some selected functions over intervals.*
- *apply the sum, difference, product and quotient rule of differentiation of functions.*
- *find the derivatives of power functions, polynomial functions, rational functions. simple trigonometric functions, exponential and logarithmic functions.*

Main Contents

INTRODUCTION

IN EVERY ASPECT OF OUR LIFE, WE ENCOUNTER THINGS THAT CHANGE ACCORDING TO S
RECOGNIZABLE RULES OR FORMS. IN THE STUDY OF MANY PHYSICAL PHENOMENA, FOR EXA
ALWAYS SEE CHANGING QUANTITIES: THE SPEED OF A CAR, THE INFLATION OF PRICES OF C
NUMBER OF BACTERIA IN A CULTURE, THE SHOCK INTENSITY OF AN EARTHQUAKE, THE VOL
ELECTRIC SIGNAL AND SO ON.

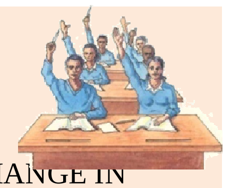
IN ORDER TO DEAL WITH QUANTITIES WHICH CHANGE AT VARIABLE RATE, YOU NEED THE
DIFFERENTIAL CALCULUS. MOREOVER, NOTIONS SUCH AS HOW FAST/SLOW THINGS ARE CHA
IS THE MOST SUITABLE QUANTITY TO BE CHOSEN FROM AMONG DIFFERENT ALTERNATIVES
IN THE DIFFERENTIAL CALCULUS.

IN THIS UNIT YOU ARE GOING TO STUDY THE MEANING AND METHODS OF DIFFERENTIATION
BEGINS BY CONSIDERING SLOPE AS A RATE OF CHANGE.

3.1 INTRODUCTION TO DERIVATIVES

3.1.1 Understanding Rates of Change

ACTIVITY 3.1



1 CONSIDER A CIRCLE OF RADIUS 3 CM.

- **A** IF THE RADIUS INCREASES BY 1 CM, FIND THE CHANGE IN CIRCUMFERENCE OF THE CIRCLE.
- **B** IF THE RADIUS INCREASES BY 1 CM/S FIND THE CIRCUMFERENCE OF THE CIRCLE WHEN t = 1 S, 2 S, 3 S
- **C** WHAT IS THE TIME RATE OF CHANGE OF THE CIRCUMFERENCE WHEN THE RADIUS INCREASES 1 CM/S?

THE FOLLOWING TABLE SHOWS THE TIME t, THE RADIUS r AND THE CIRCUMFERENCE c.

FROM THE FOLLOWING TABLE, WHAT IS THE RATE OF CHANGE OF THE CIRCUMFERENCE

t	1 S	2 S	3 S
r	4 CM	5 CM	6 CM
$c = 2\pi r$	8 CM	10 CM	12 CM

D IF Δr IS THE INCREASE IN THE RADIUS AND Δc IS THE INCREASE IN THE CIRCUMFERENCE, THEN $\Delta c = 2\pi(\Delta r + 3 \text{ CM}) - 2\pi(3 \text{ CM}) = 2\pi(\Delta r)$.

LET Δt BE THE INCREASE IN THE TIME. IF $\frac{\Delta r}{\Delta t} = 1$ CM/S, WHAT IS $\frac{\Delta c}{\Delta t}$?

2 *Average rate of change and instantaneous rates of change.*

SUPPOSE YOU DROVE 200 KM IN 4 HOURS, THEN THE AVERAGE SPEED AT WHICH YOU DROVE WAS 50 KM/HR. *This is the average rate of change.*

THE AVERAGE SPEED FOR THE WHOLE JOURNEY IS THE CONSTANT SPEED THAT WOULD BE REQUIRED TO COVER THE TOTAL DISTANCE IN THE SAME TIME.

SUPPOSE YOU DROVE AT 30 KM/HR FOR 2 KM AND THEN AT 120 KM/HR FOR 2 KM.

A WHAT IS YOUR AVERAGE SPEED?

B IS A PATROL OFFICER GOING TO STOP YOU FOR SPEEDING?

C IS THE OFFICER LIKELY TO CONSIDER YOUR AVERAGE SPEED AND NOT CHARGE YOU?

HERE WHAT IS CONSIDERED IS THE SPEED AT A PARTICULAR INSTANT.

3 SUPPOSE A PARTICLE MOVES ALONG A STRAIGHT LINE. LET O BE A FIXED POINT ON THAT LINE. THE FOLLOWING TABLE SHOWS THE DISTANCE OF THE PARTICLE FROM POINT O AT A GIVEN INSTANT OF TIME t.

t (s)	0	1	2	3	4	5
position (m)	4	4	4	4	4	4

A DRAW THE POSITION - TIME GRAPH.

B FIND THE GRADIENT (SLOPE) OF THE GRAPH.

C FIND THE SPEED OF THE PARTICLE IN THE INTERVAL OF TIME

$t = 0$ TO $t = 1$, $t = 1$ TO $t = 2$,…, $t = 4$ TO $t = 5$.

4 REPEAT PROBLEM 3 FOR THIS NEW DATA.

I

t (*s*)	0	1	2	3	4	5
position (m)	0	1	2	3	4	5

II

t (*s*)	0	1	2	3	4	5
position (m)	0	20	40	60	80	100

5 FROM THE POSITION - TIME GRAPHS, WHAT IS THE RELATION BETWEEN THE GRADIENT (OR SLOPE) AND THE SPEED IN THE GIVEN INTERVALS OF TIME?

IN ACTIVITY 3.1, THE POSITION-TIME GRAPHS ARE ALL STRAIGHT LINES AND THE SPEED IS REPRESENTED BY THE GRADIENT (OR THE SLOPE) OF THE LINE.

NOW, LET'S CONSIDER A POSITION-TIME GRAPH WHICH IS NOT A STRAIGHT LINE.

Example 1 SUPPOSE A PARTICLE MOVES ALONG A STRAIGHT LINE FROM POINT O ON THAT LINE. THE POSITION d OF THE PARTICLE FROM POINT O AT A GIVEN INSTANT OF TIME t ARE SHOWN BELOW.

t (seconds)	1	2	3	4	5
position (m)	1	4	9	16	25

A DRAW A POSITION - TIME GRAPH.

B LET A, B, C, D, E, AND F BE POINTS ON THE GRAPH WHEN $t = 0, 1, 2, 3, 4, 5$ RESPECTIVELY.

FIND THE GRADIENTS OF THE CHORDS AB, BC, CD, DE AND EF.

C FIND THE AVERAGE SPEEDS OVER THESE INTERVALS OF TIME

I $t = 0$ TO $t = 1$ **II** $t = 1$ TO $t = 2$ **III** $t = 2$ TO $t = 3$

V $t = 4$ TO $t = 5$ **IV** $t = 3$ TO $t = 4$

Solution FROM THE TABLE, IT IS OBSERVED THAT DIFFERENT DISTANCES ARE COVERED IN EQUAL INTERVALS OF TIME. THUS THE POSITION - TIME GRAPH IS NOT A STRAIGHT LINE.

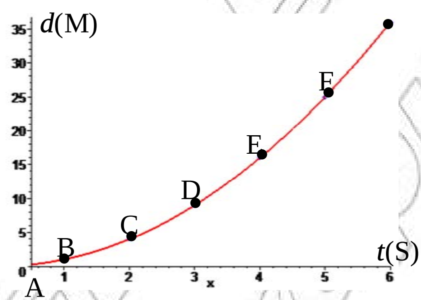


Figure 3.1

Chord	AB	BC	CD	DE	EF
Gradient	1	3	5	7	9
Average speed(m/s)	1	3	5	7	9

Example 2 A PARTICLE MOVES ALONG A STRAIGHT LINE FROM A FIXED POINT O ON THAT LINE. THE POSITION d IN METRES OF THE PARTICLE FROM POINT O AS A FUNCTION OF TIME t IN SECOND IS GIVEN BY $d = (t + 2)(4 - t)$.

A DRAW THE POSITION-TIME GRAPH FOR THE INTERVAL OF TIME $t = 0$ TO $t = 5$.

B USING THE GRAPH, FIND THE AVERAGE SPEED OF THE PARTICLE OVER THE INTERVALS

$t = 0$ TO $t = 1$, $t = 1$ TO $t = 2$, $t = 2$ TO $t = 3$, $t = 3$ TO $t = 4$, $t = 4$ TO $t = 5$.

C FIND THE TIME AT WHICH THE SPEED IS 0.

D APPROXIMATE THE GRADIENT OF THE GRAPH AT $t = 2$.

Solution

a

t (s)	0	1	2	3	4	5
$d = (t + 2)(4 - t)$	8	9	8	5	0	−7

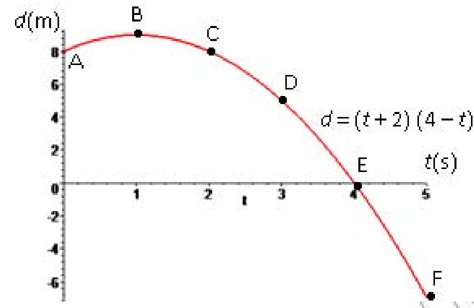


Figure 3.2

B

time interval $t = a$ to $t = b$	0 TO 1	1 TO 2	2 TO 3	3 TO 4	4 TO 5
chord	AB	BC	CD	DE	EF
gradient	$\frac{9-8}{1} = 1$	8 – 9 = −1	5 – 8 = −3	0 – 5 = −5	−7 – 0 = −7
Average speed	1 M/S	−1 M/S	−3 M/S	−5 M/S	−7 M/S

C FROM THE GRAPH ONE CAN SEE THAT THE GRADIENT IS ZERO AT B, WHEN $t = 1$, I.E., THE GRAPH HAS A HORIZONTAL TANGENT LINE AT $t = 1$.

D THERE WILL BE A VERY GOOD APPROXIMATION, IF WE CONSIDER VERY SMALL INTERVALS OF TIME.

LET Δt BE THE INCREASE IN TIME AND Δd BE THE INCREASE IN DISTANCE.

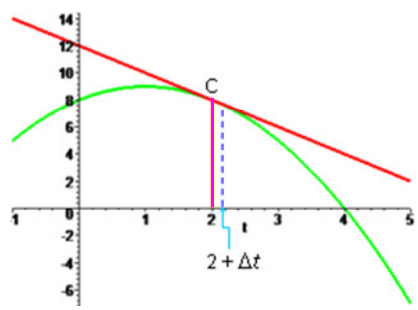
IF $t = 2 + \Delta t$, THEN

$\Delta d = (2 + \Delta t + 2)(4 - (2 + \Delta t)) = (4 + \Delta t)(2 - \Delta t)$. SINCE AT $t = 2$, $d = 8$,

THE GRADIENT AT $t = 2$ IS $\dfrac{(4+\Delta t)(2-\Delta t)-8}{2+\Delta t-2}$

$$= \frac{8 - 2\Delta t - \Delta t^2 - 8}{\Delta t}$$

$$= \frac{-(\Delta t)^2 - 2\Delta t}{\Delta t} = -\Delta t - 2$$



Figures 3.3

AS $\Delta t \to 0$, $-\Delta t - 2 \to -2$.

THE GRADIENT WHEN $t = 2$ IS −2.

BY A SIMILAR TECHNIQUE, WE CAN FIND THAT THE VELOCITY, $v(t) = 2 - 2t$.

Velocity-time graph

t (s)	0	1	2	3	4	5
Velocity m/s	2	0	−2	−4	−6	−8

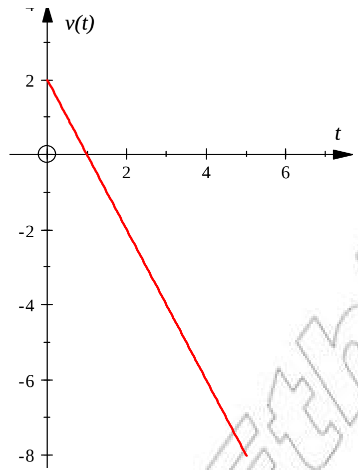


Figure 3.4

3.1.2 Graphical Definition of Derivative

The slope (gradient) of the graph of $y = f(x)$ at point P

NEWTON AND LEIBNIZ INVENTED CALCULUS AT ABOUT THE SAME TIME.

HISTORICAL NOTE

Sir Isaac Newton (1642-1727)

Isaac Newton's work represents one of the greatest contributions to science ever made by an individual. Most notably, Newton derived the law of universal gravitation, invented the branch of mathematics called calculus, and performed experiments investigating the nature of light and colour.

Gottfried Wilhelm Leibniz (1646-1716)

The 17th–century thinker **Gottfried Leibniz** made contributions to a variety of subjects, including theology, history, and physics, although he is best remembered as a mathematician and philosopher. According to Leibniz, the world is composed of monads—tiny units, each of which mirrors and perceives the other monads in the universe.

Definition 3.1 **Secant line and tangent line**

A LINE WHICH INTERSECTS A (CONTINUOUS) GRAPH IN EXACTLY TWO POINTS IS SAID TO BE A secant line.

A LINE WHICH TOUCHES A GRAPH AT EXACTLY ONE POINT IS SAID TO BE A tangent line AT THAT POINT. THE INTERSECTION POINT IS SAID TO BE THE point of tangency.

THE SLOPE OF THE GRAPH OF A FUNCTION AT A POINT P IS THE SLOPE OF THE TANGENT LINE A

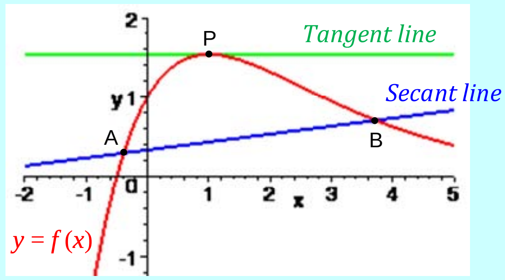


Figure 3.5

Example 3 CONSIDER THE GRAPH OF $y = x^2$.

A FIND THE SLOPE OF THE SECANT LINE PASSING THROUGH (1, 1) AND (2, 4).

B FIND THE SLOPE OF THE TANGENT LINE AT (1, 1).

Solution

A THE SLOPE OF THE SECANT LINE = $\frac{4-1}{2-1} = 3$.

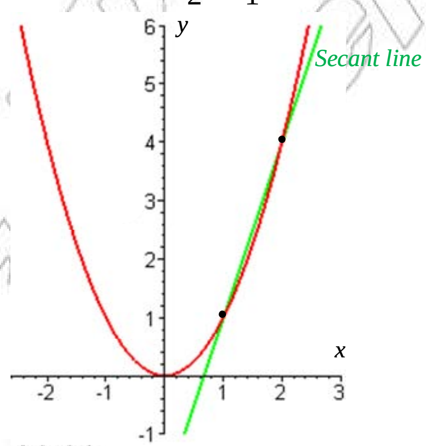


Figure 3.6

THIS IS THE AVERAGE RATE OF CHANGE OF y ON THE INTERVAL [1, 2].

IN GENERAL, THE AVERAGE RATE OF CHANGE OF A FUNCTION f ON $[x_1, x_2]$, $x_1 < x_2$, IS THE SLOPE (OR GRADIENT) OF THE SECANT LINE PASSING THROUGH THE TWO POINTS $(x_1, f(x_1))$ AND $(x_2, f(x_2))$.

SO AVERAGE RATE OF CHANGE = $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. (*See* FIGURE 3.7)

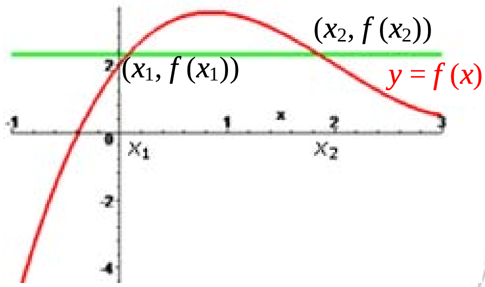


Figure3.7

b

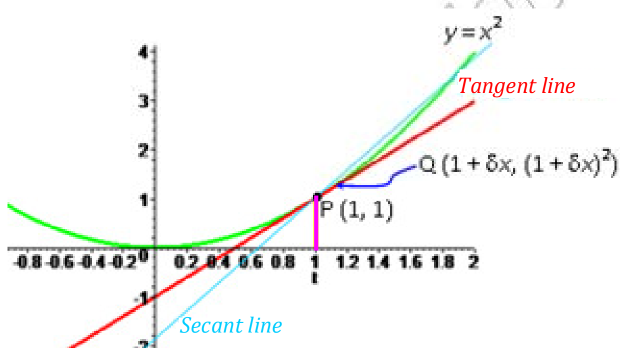


Figure 3.8

LET Q BE ANOTHER POINT ON THE GRAPH OF $y = x^2$ SUCH THAT THE INCREASE IN THE x-COORDINATE IN MOVING FROM P TO Q IS Δx.

THEN Q HAS COORDINATES $(1 + \Delta x, (1 + \Delta x)^2)$.

HENCE, THE SLOPE OF THE SECANT LINE IS

$$\frac{(1 + \Delta x)^2 - 1}{1 + \Delta x - 1} = \frac{1 + 2\Delta x + (\Delta x)^2 - 1}{\Delta x} = 2 + \Delta x$$

NOTICE THAT THE TANGENT LINE IS THE LIMIT OF THE SECANT LINES THROUGH

$(1 + \Delta x, (1 + \Delta x)^2)$ AS $\Delta x \to 0$.

THUS, THE SLOPE OF THE TANGENT LINE AT (1, 1) IS

$$\lim_{\Delta x \to 0} (2 + \Delta x) = 2$$

IN GENERAL, THE INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT x_0 IS REPRESENTED BY THE SLOPE OF THE TANGENT LINE AT $(x_0, f(x_0))$.

Functional notation to find the slope of $y = f(x)$ at point P

GRADIENT OF THE SECANT LINE

$$= \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

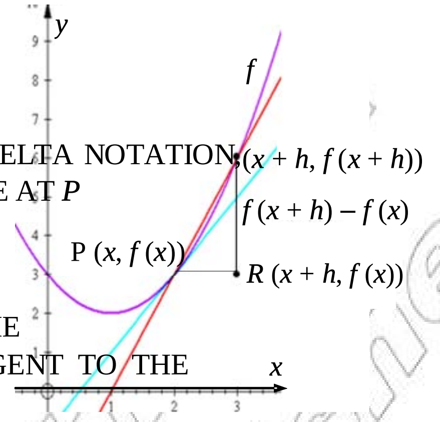


Figure 3.9

USING THE SAME METHOD AS WITH THE DELTA NOTATION, YOU HAVE GRADIENT OF THE TANGENT LINE AT P

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 4 LET $f(x) = 2x^2 - 5x + 1$. FIND THE GRADIENT OF THE LINE TANGENT TO THE GRAPH OF f AT $(2, -1)$.

Solution GRADIENT $= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$$= \lim_{h\to 0}\frac{2(2+h)^2-5(2+h)+1-(-1)}{h}$$

$$= \lim_{h\to 0}\frac{8+8h+2h^2-5h-8}{h}$$

$$= \lim_{h\to 0}\frac{3h+2h^2}{h} = \lim_{h\to 0}(3+2h) = 3.$$

Limit of the quotient difference

LET $f(x)$ BE A FUNCTION DEFINED IN A NEIGHBOURHOOD OF A POINT x_o.

THE RATIO $\frac{f(x)-f(x_o)}{x-x_o}$ IS CALLED THE **quotient difference** OF f AT $x = x_o$.

Example 5 FIND THE QUOTIENT-DIFFERENCE OF EACH OF THE FOLLOWING FUNCTIONS FOR THE GIVEN VALUES OF x_o.

A $f(x) = x + 1$; $x_o = 3$ **B** $f(x) = x^2 - 2x + 3$; $x_o = -1$

C $f(x) = x^3 - 4x + 1$; $x_o = 1$

Solution

A $\frac{f(x)-f(3)}{x-3} = \frac{(x+1)-4}{x-3} = \frac{x-3}{x-3} = 1;\ x \neq 3$

B $\frac{f(x)-f(-1)}{x-(-1)} = \frac{x^2-2x+3-(1+3-2(-1))}{x+1} = \frac{x^2-2x+3-6}{x+1}$

$$= \frac{x^2-2x-3}{x+1} = \frac{(x-3)(x+1)}{x+1} = (x-3);\ x \neq -1$$

C $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{x^3 - 4x + 1 - (1 - 4 + 1)}{x - 1} = \dfrac{x^3 - 4x + 3}{x - 1}$

$$= \frac{(x-1)\,(x^2 + x - 3)}{x - 1} = x^2 + x - 3; \quad x \neq 1$$

IF THE LIMIT OF THE QUOTIENT DIFFERENCE EXISTS AS $x \to x_0$, THEN IT IS SAID TO BE THE DERIVATIVE OF $f(x)$ AT $x = x_0$. IN THE ABOVE EXAMPLES,

A $\lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x - 3)}{x - 3} = 1.$

$\Rightarrow$ THE DERIVATIVE OF $f(x) = x + 1$ AT $x = 3$ IS 1.

B $\lim\limits_{x \to -1} \dfrac{f(x) - f(-1)}{x - (-1)} = \lim\limits_{x \to -1} \dfrac{(x-3)(x+1)}{x+1} = \lim\limits_{x \to -1} (x - 3) = -4$

$\Rightarrow$ THE DERIVATIVE OF $f(x) = 3 - 2x + x^2$ AT $x = -1$ IS -4.

C $\lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x^2 + x - 3)}{x - 1} = \lim\limits_{x \to 1} (x^2 + x - 3) = -1$

$\Rightarrow$ THE DERIVATIVE OF $f(x) = x^3 - 4x + 1$ AT $x = 1$ IS -1.

Exercise 3.1

1 LET $f(x) = x^2 - x + 3$. FIND

A $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

B $\lim\limits_{\Delta x \to 0} \dfrac{f(-1+\Delta x) - f(-1)}{\Delta x}$

C $\lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x - 4}$

2 FIND THE SLOPE OF EACH OF THE FOLLOWING FUNCTIONS AT THE GIVEN POINTS.

A $f(x) = x^2$; $(-3, 9)$

B $g(x) = 1 - 3x^2$; $(1, -2)$

C $h(x) = \dfrac{1}{x}$; $\left(\dfrac{1}{2}, 2\right)$

D $k(x) = \sqrt{x}$; $(9, 3)$

E $f(x) = \dfrac{3x - 1}{5x - 3}$; $\left(-3, \dfrac{5}{9}\right)$

F $f(x) = \sqrt{x}$; $(4, 2)$

G $f(x) = \begin{cases} x, & \text{IF } x < 0 \\ x^2, & \text{IF } x \geq 0 \end{cases}$; $(0, 0)$

H $f(x) = \begin{cases} x^2 + 2; & x \geq -2 \\ 4x - 2; & x < -2 \end{cases}$; $(-2, 6)$

I $f(x) = \begin{cases} \sqrt{1 - x}; & 0 \leq x \leq 1 \\ x^2 + 1; & x < 0 \end{cases}$; $(0, 1)$

3.1.3 Differentiation of a Function at a Point

Definition 3.2

LET f BE A FUNCTION AND x_o BE IN THE DOMAIN OF f.

IF $\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}$ EXISTS, WE SAY THAT THE GRAPH OF f HAS A TANGENT LINE AT $(x_o, f(x_o))$.

IN THAT CASE THE LINE TANGENT TO THE GRAPH OF f AT $(x_o, f(x_o))$ IS DEFINED TO BE THE LINE THROUGH $(x_o, f(x_o))$ WITH THIS LIMIT AS ITS SLOPE.

Example 6 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $f(x) = x^2 - 2$ AT $(1, -1)$.

Solution BY THE DEFINITION OF THE SLOPE, m OF THE TANGENT LINE,

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 2 - (-1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

$\Rightarrow$ THE SLOPE OF THE TANGENT LINE IS 2.

THE EQUATION OF THE TANGENT LINE IS

$$y - (-1) = 2(x - 1) \Rightarrow y = 2x - 3.$$

Example 7 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $y = \sin x$ AT $\left(\frac{}{2}, 1\right)$.

Solution THE SLOPE OF THE TANGENT LINE, $m = \lim_{x \to \frac{}{2}} \frac{\sin x - \sin \frac{}{2}}{x - \frac{}{2}}$

LET $z = x - \frac{}{2}$, THEN $x = z + \frac{}{2}$ AND AS $x \to \frac{}{2}$, $z \to 0$

THUS, $m = \lim_{z \to 0} \frac{\sin\left(z + \frac{}{2}\right) - \sin \frac{}{2}}{z} = \lim_{z \to 0} \frac{\sin z \cos \frac{}{2} + \cos z \sin \frac{}{2}}{z}$

$= \lim_{z \to 0} \frac{\cos z - 1}{z} = 0 \Rightarrow$ THE EQUATION OF THE TANGENT LINE IS $y = 1$.

Definition 3.3

LET f BE A CONTINUOUS FUNCTION AT x_o.

IF $\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = \infty$ OR $\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = -\infty$, THEN THE VERTICAL LINE $x = x_o$ IS TANGENT TO THE GRAPH OF f AT $(x_o, f(x_o))$.

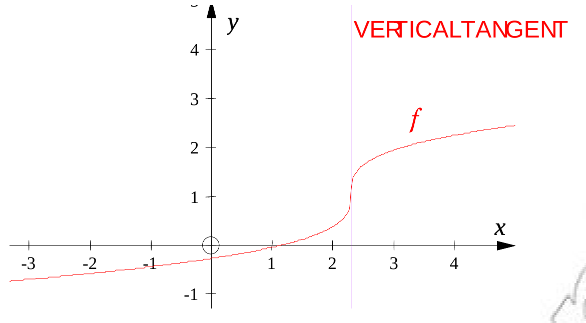


Figure 3.10

Example 8 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $f(x) = x^{\frac{1}{3}}$ AT $x = 0$.

Solution LOOK AT THE GRAPH OF f. IT IS CONTINUOUS ON $\mathbb{R}$. SO IT IS CONTINUOUS AT 0.

BUT, $\lim_{x\to 0} \dfrac{x^{\frac{1}{3}} - 0}{x - 0} = \infty$.

$\Rightarrow$ THE LINE $x = 0$ (THE Y-AXIS) IS TANGENT TO THE GRAPH OF f AT $(0, 0)$.

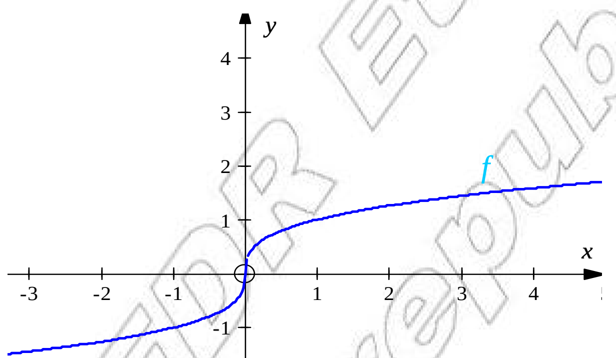


Figure 3.11

The derivative

Definition 3.4

LET x_o BE IN THE DOMAIN OF A FUNCTION f.

IF $\lim_{x\to x_o} \dfrac{f(x) - f(x_o)}{x - x_o}$ EXISTS, THEN WE CALL THIS LIMIT THE DERIVATIVE OF f AT x_o.

Notation:

THE DERIVATIVE OF f AT x_0 IS DENOTED BY $f'(x_0)$, WHICH IS READ AS 'f prime of x_o'.

IF $f'(x_0)$ EXISTS, THEN WE SAY THAT f HAS A DERIVATIVE AT x_0 OR f IS DIFFERENTIABLE AT x_0.

DIFFERENTIATION IS THE PROCESS OF FINDING THE DERIVATIVE OF A FUNCTION.

Example 9 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS AT THE GIVEN NUMBER.

A $f(x) = 4x + 5;\ x_o = 2$ **B** $f(x) = \frac{1}{4}x^2 + x;\ x_o = -1$

C $f(x) = x^3 - 9x;\ x_o = \frac{1}{3}$ **D** $f(x) = \sqrt{x};\ x_o = 4$

Solution USING THE DEFINITION,

$$f'(x_o) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}, \text{ YOU OBTAIN,}$$

A $$f'(2) = \lim_{x\to 2}\frac{(4x+5)-(4(2)+5)}{x-2} = \lim_{x\to 2}\frac{4x-8}{x-2} = \lim_{x\to 2}\frac{4(x-2)}{x-2} = \quad .$$

B $$f'(-1) = \lim_{x\to -1}\frac{\frac{1}{4}x^2 + x - \left(\frac{1}{4}(-1)^2 - 1\right)}{x-(-1)} = \lim_{x\to -1}\frac{\frac{1}{4}x^2 + x + \frac{3}{4}}{x+1}$$

$$= \lim_{x\to -1}\frac{\frac{1}{4}(x+3)(x+1)}{x+1} = \frac{1}{4}\lim_{x\to -1}(x+3) = \frac{1}{2}.$$

C $$f'\left(\frac{1}{3}\right) = \lim_{x\to \frac{1}{3}}\frac{x^3 - 9x - \left(\left(\frac{1}{3}\right)^3 - 9\left(\frac{1}{3}\right)\right)}{x - \frac{1}{3}} = \lim_{x\to \frac{1}{3}}\frac{\left(x^2 + \frac{1}{3}x - \frac{80}{9}\right)\left(x - \frac{1}{3}\right)}{x - \frac{1}{3}}$$

$$= \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \frac{1}{3} - \frac{80}{9} = \frac{-26}{3}.$$

D $$f'(x) = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

LET f BE A FUNCTION DEFINED AT a. IF $f'(a)$ EXISTS, THEN THE GRAPH OF f HAS A TANGENT LINE AT $(a, f(a))$ AND THE EQUATION OF THE TANGENT LINE IS

$$y - f(a) = f'(a)(x - a)$$

Example 10 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $f(x) = x^2$ AT

A $x = 1$, **B** $x = 0$, **C** $x = -5$

Solution $f(x) = x^2 \Rightarrow f'(x) = 2x$

A $f(1) = 1$ AND $f'(1) = 2$

$\Rightarrow$ THE EQUATION OF THE TANGENT LINE IS:

$y - f(1) = f'(1)(x - 1) \Rightarrow y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$

B $y - f(0) = f'(0)(x - 0) \Rightarrow y = 0$

C $y - f(-5) = f'(-5)(x - (-5))$

$\Rightarrow y - 25 = -10(x + 5) \Rightarrow y = -10x - 25$

Example 11 FIND THE EQUATIONS OF THE LINES TANGENT TO THE GRAPH OF $f(x) = x^3 + 1$ AT

A $x = -2$ **B** $x = -1$ **C** $x = 2$

Solution $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^3 + 1 - (a^3 + 1)}{x - a} = \lim_{x\to a} \frac{x^3 - a^3}{x - a}$

$= \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} = \lim_{x\to a} (x^2 + ax + a^2)$

$= a^2 + a^2 + a^2 = 3a^2$

THEREFORE,

A $f'(-2) = 3(-2)^2 = 12$

$\Rightarrow$ THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $x = -2$ IS

$y - f(-2) = 12(x - (-2))$

$\Rightarrow y - (-7) = 12(x + 2) \Rightarrow y = 12x + 17$

B $f'(-1) = 3(-1)^2 = 3$

$\Rightarrow$ THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $x = -1$ IS

$y - f(-1) = 3(x - (-1))$

$\Rightarrow y - 0 = 3x + 3 \Rightarrow y = 3x + 3$

C $f'(2) = 3(2)^2 = 12$

THE EQUATION OF THE TANGENT LINE AT $x = 2$ IS

$y - f(2) = 12(x - 2) \Rightarrow y - 9 = 12x - 24$

$\Rightarrow y = 12x - 15$

Example 12 LET $f(x) = \begin{cases} x^2, & \text{IF } x \geq 0 \\ x^3, & \text{IF } x < 0 \end{cases}$

DETERMINE THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $x = 0$.

Solution $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}$

HERE, WE CONSIDER THE ONE-SIDE LIMITS

$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2}{x} =$ AND $\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x^3}{x} =$

$\Rightarrow$ THE SLOPE OF THE GRAPH OF f AT 0 IS 0.

THE EQUATION OF THE TANGENT LINE IS

$y - f(0) = 0(x - 0) \Rightarrow y = 0$.

Exercise 3.2

1 FIND THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF THE FUNCTION AT THE INDI

A $f(x) = x^2$; (1, 1) **B** $f(x) = 4x^2 - 3x - 5$; (−2, 17)

C $f(x) = x^3 + 1$; (−1, 0) **D** $f(x) = (x-1)^{\frac{1}{3}}$; (1, 0)

E $f(x) = \frac{1}{\sqrt{x}}$; (1, 1)

2 LET $f(x) = \begin{cases} x, & \text{if } x > 3 \\ x^2 - 6, & \text{if } x \le 3 \end{cases}$. FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT EACH OF THE FOLLOWING POINTS.

A (0, −6) **B** (−2, −2) **C** (1, −5) **D** (3, 3) **E** (4, 4)

3 LET $f(x) = x^3 - 3x$, FIND THE VALUES OF x AT WHICH THE SLOPE OF THE TANGENT LINE IS 0.

4 LET $f(x) = x^3 + x^2 - x + 1$; FIND THE SET OF VALUES OF x SUCH THAT THE SLOPE IS POSITIVE.

5 IF THE GRAPHS OF THE FUNCTIONS f AND g GIVEN BY

$f(x) = 4 - x^2$ AND $g(x) = x^3 - 8x$ HAVE THE SAME SLOPE, DETERMINE THE VALUES OF x.

6 LET $f(x) = x^3 - x^2 - x + 1$; FIND THE EQUATION OF THE TANGENT LINE AT THE POINT WHERE T GRAPH CROSSES

A THE x-AXIS **B** THE y-AXIS **C** THE GRAPH OF $y = 1 - x^2$

The derivative as a function

ACTIVITY 3.2

FOR EACH OF THE FOLLOWING FUNCTIONS, FIND THE SET OF x_0 SUCH THAT $f'(x_0)$ EXISTS.

1 $f(x) = x^2$ **2** $f(x) = |x|$ **3** $f(x) = \begin{cases} x^2, & \text{if } x < 2 \\ 4x - 4, & \text{if } x \ge 2 \end{cases}$

FROM ACTIVITY 3.2, YOU OBSERVED THAT THERE ARE FUNCTIONS THAT ARE DIFFERENTIABLE NUMBERS IN THEIR DOMAIN AND THERE ARE FUNCTIONS THAT ARE NOT DIFFERENTIABL NUMBERS IN THEIR DOMAIN. AT THIS LEVEL WE GIVE THE DEFINITION OF THE FUNCTION f' AS FOLLOWS AND DETERMINE THE INTERVAL ON WHICH f' IS DEFINED.

Definition 3.6

f' IS THE FUNCTION WHOSE DOMAIN IS THE SET OF NUMBERS AT WHICH f IS DIFFERENTIABLE AND WHOSE VALUE AT ANY SUCH NUMBER x IS GIVEN BY

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

HERE WE SAY $f'(x)$ IS THE DERIVATIVE OF f WITH RESPECT TO x. WE CONSIDER t AS A VARIABLE AND x AS A CONSTANT.

Example 13 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x

A $f(x) = x^2$ **B** $f(x) = \sqrt{x};\ x > 0$ **C** $f(x) = \dfrac{2x-1}{x+4};\ x \neq -4$

Solution USING DEFINITION 3.6 OF $f'(x)$ WE HAVE,

a $f'(x) = \lim\limits_{t\to x}\dfrac{t^2 - x^2}{t - x} = \lim\limits_{t\to x}(t + x) = 2x$

b $f'(x) = \lim\limits_{t\to x}\dfrac{\sqrt{t}-\sqrt{x}}{t - x}\times\dfrac{\sqrt{t}+\sqrt{x}}{\sqrt{t}+\sqrt{x}} = \lim\limits_{t\to x}\dfrac{1}{\sqrt{t}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

c $f'(x) = \lim\limits_{t\to x}\dfrac{\dfrac{2t-1}{t+4} - \dfrac{(2x-1)}{x+4}}{t - x} = \lim\limits_{t\to x}\dfrac{2xt - x + 8t - 4 - (2tx - t + 8x - 4)}{(t-x)(t+4)(x+4)}.$

$= \lim\limits_{t\to x}\dfrac{9t - 9x}{(t-x)(t+4)(x+4)} = \lim\limits_{t\to x}\dfrac{9}{(t+4)(x+4)} = \dfrac{9}{(x+4)^2}.$

The different notations for the derivative

RECALL THE FUNCTIONAL NOTATION AND THE DEFINITION OF THE GRADIENT OF A GRAPH AT A POINT. THE FOLLOWING ARE SOME OTHER NOTATIONS FOR THE DERIVATIVES.

IF $y = f(x)$, THEN $f'(x) = \dfrac{dy}{dx},\ \dfrac{d}{dx}f(x),\ D(f(x))$

USING THESE NOTATIONS, WE HAVE

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{d}{dx}f(x)\right|_{x=x_0} = D(f(x_0))$$

Example 14 FIND THE DERIVATIVE OF $f(x) = \dfrac{1}{x}$.

Solution $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim\limits_{h\to 0}\dfrac{x - (x+h)}{xh(x+h)}$

$= \lim\limits_{h\to 0}\dfrac{-h}{xh(x+h)} = -\lim\limits_{h\to 0}\dfrac{1}{x(x+h)} = -\dfrac{1}{x^2}$

Example 15 LET $y = x^4$, THEN $\dfrac{dy}{dx} = \lim\limits_{t\to x}\dfrac{t^4 - x^4}{t - x} = \lim\limits_{t\to x}\dfrac{(t^2 - x^2)(t^2 + x^2)}{t - x}$

$= \lim\limits_{t\to x}(t + x)(t^2 + x^2)$

$= (x + x)(x^2 + x^2) = 2x(2x^2) = 4x^3$

Example 16 LET $f(x) = \dfrac{x}{x^2+1}$, THEN

$$D(f(x)) = \frac{d}{dx} f(x) = f'(x) = \lim_{t \to x} \frac{\frac{t}{t^2+1} - \frac{x}{x^2+1}}{t-x} = \lim_{t \to x} \frac{tx^2 + t - xt^2 - x}{(t^2+1)(x^2+1)(t-x)}$$

$$= \lim_{t \to x} \frac{tx(x-t) + (t-x)}{(t^2+1)(x^2+1)(t-x)} = \lim_{t \to x} \frac{-tx+1}{(t^2+1)(x^2+1)} = \frac{-x^2+1}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Exercise 3.3

USING DEFINITION 3.6, FIND THE DERIVATIVES OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x.

1 $f(x) = x$	**2** $f(x) = 2x - 5$	**3** $f(x) = x^2 + 4x - 5$
4 $f(x) = \dfrac{1}{x};\ x \neq 0$	**5** $f(x) = \sqrt{x}$	**6** $f(x) = x^3 - 3$
7 $f(x) = (3x-2)^2$	**8** $f(x) = x^2(2+x)$	**9** $f(x) = 8 - \sqrt[3]{x}$
10 $f(x) = \dfrac{x+2}{3-2x};\ x \neq \dfrac{3}{2}$	**11** $f(x) = \left(x - \dfrac{3}{x}\right)^2;\ x \neq 0$	
12 $f(x) = \dfrac{4x^2 - 5x^3}{x^2};\ x \neq 0$	**13** $f(x) = 2x - 5 + \dfrac{x^2}{7} + x^5$	
14 $f(x) = \left(x + \dfrac{1}{x^2}\right)^3;\ x \neq 0$	**15** $f(x) = \sqrt[3]{x} + x - \dfrac{1}{\sqrt{x}};\ x > 0$	

3.1.4 Differentiability on an Interval

Definition 3.7

1 IF I IS AN OPEN INTERVAL, THEN WE SAY THAT THE FUNCTION f IS DIFFERENTIABLE ON I, IF f IS DIFFERENTIABLE AT EACH POINT IN I.

2 IF I IS A CLOSED INTERVAL $[a, b]$ WITH $a < b$, THEN WE SAY f IS DIFFERENTIABLE ON I IF f IS DIFFERENTIABLE ON (a, b) AND IF THE ONE SIDE LIMITS $\lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a}$ AND $\lim_{x \to b^-} \dfrac{f(x) - f(b)}{x - b}$ BOTH EXIST.

Example 17 LET $f(x) = |x - 3|$.

A IS f DIFFERENTIABLE AT $x = 3$?

B FIND THE INTERVAL(S) ON WHICH f IS DIFFERENTIABLE.

Solution

A $f'(3) = \lim_{x\to 3} \frac{|x-3| - |3-3|}{x-3} = \lim_{x\to 3}\frac{|x-3|}{x-3}$

BUT $\lim_{x\to 3^+}\frac{|x-3|}{x-3} = 1$ AND $\lim_{x\to 3^-}\frac{|x-3|}{x-3} = -1 \Rightarrow f'(3)$ DOESN'T EXIST.

THEREFORE, f IS NOT DIFFERENTIABLE AT 3.

B SINCE BOTH $\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3}$ AND $\lim_{x\to 3^+}\frac{f(x)-f(3)}{x}$ EXIST, f IS DIFFERENTIABLE ON $(-\infty, 3]$ AND $[3, \infty)$.

$f'(x) = 1$ FOR ALL $x \geq 3$ AND $f'(x) = -1$ FOR ALL $x \leq 3$.

Example 18 LET $f(x) = \sqrt{1-x^2}$. FIND THE LARGEST INTERVAL ON WHICH f IS DIFFERENTIABLE.

Solution

$$f'(x) = \lim_{t\to x}\frac{\sqrt{1-t^2}-\sqrt{1-x^2}}{t-x} = \lim_{t\to x}\frac{1-t^2-(1-x^2)}{(t-x)\left(\sqrt{1-t^2}+\sqrt{1-x^2}\right)}.$$ *Why? Explain!*

$$= \lim_{t\to x}\frac{x^2-t^2}{(t-x)\left(\sqrt{1-t^2}+\sqrt{1-x^2}\right)} = \lim_{t\to x}\frac{(x-t)(x+t)}{-(x-t)\left(\sqrt{1-t^2}+\sqrt{1-x^2}\right)}$$

$$= -\lim_{t\to x}\frac{(x+t)}{\sqrt{1-t^2}+\sqrt{1-x^2}} = -\frac{2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

NOTICE THAT THE DOMAIN OF $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ IS $(-1, 1) \Rightarrow f$ IS DIFFERENTIABLE ON $(-1, 1)$.

Definition 3.8

1 A FUNCTION f IS DIFFERENTIABLE ON $[a, \infty)$ IF f IS DIFFERENTIABLE ON (a, ∞) AND THE ONE SIDE LIMIT

$$\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} \text{ EXISTS.}$$

2 A FUNCTION f IS DIFFERENTIABLE ON $(-\infty, a]$, IF f IS DIFFERENTIABLE ON $(-\infty, a)$ AND THE ONE SIDE LIMIT

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} \text{ EXISTS.}$$

Example 19 THE ABSOLUTE VALUE FUNCTION $f(x) = |x|$ IS DIFFERENTIABLE ON $(-\infty, 0]$ AND ON $[0, \infty)$.

Example 20 FIND THE LARGEST INTERVAL ON WHICH $f(x) = \dfrac{1}{3-2x}$ IS DIFFERENTIABLE.

Solution $f'(x) = \lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x} = \lim\limits_{t\to x}\dfrac{\dfrac{1}{3-2t}-\dfrac{1}{3-2x}}{(t-x)} = \lim\limits_{t\to x}\dfrac{3-2x-3+2t}{(t-x)(3-2t)(3-2x)}$

$$= \lim_{t\to x}\frac{2(t-x)}{(t-x)(3-2t)(3-2x)} = \lim_{t\to x}\frac{2}{(3-2t)(3-2x)} = \frac{2}{(3-2x)^2}$$

$\Rightarrow f$ IS DIFFERENTIABLE ON $\left(-\infty, \dfrac{3}{2}\right)$ AND ON $\left(\dfrac{3}{2}, \infty\right)$.

Exercise 3.4

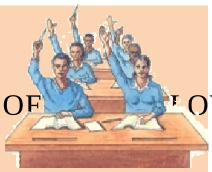
DETERMINE THE INTERVALS ON WHICH EACH OF THE FOLLOWING FUNCTIONS IS DIFFERENTIABLE.

1. $f(x) = 3x - 5$
2. $f(x) = x^2 + 7x + 6$
3. $f(x) = \dfrac{1}{x}$
4. $f(x) = \sqrt{x-2}$
5. $f(x) = \sqrt{9-4x^2}$
6. $f(x) = |x-5|$
7. $f(x) = |2x-3|$
8. $f(x) = |x| + |x-1|$
9. $f(x) = x|x|$
10. $f(x) = \begin{cases} x, & \text{IF } x > 1 \\ 2-x^2, & \text{IF } x \le 1 \end{cases}$
11. $f(x) = \dfrac{x-1}{3-x}$

3.2 DERIVATIVES OF SOME FUNCTIONS

Differentiation of power, simple trigonometric, exponential and logarithmic functions

ACTIVITY 3.3



1. USING YOUR KNOWLEDGE OF LIMITS, EVALUATE EACH OF THE FOLLOWING LIMITS.

   A $\lim\limits_{t\to x}\dfrac{t^3-x^3}{t-x}$ **B** $\lim\limits_{t\to x}\dfrac{t^{\frac{1}{3}}-x^{\frac{1}{3}}}{t-x}$ **C** $\lim\limits_{h\to 0}\dfrac{\cos h-1}{h}$

   D $\lim\limits_{h\to 0}\dfrac{e^h-1}{h}$ **E** $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}$, WHERE $a > 0$.

2. USING DEFINITION 3.6, FIND THE DERIVATIVES OF EACH OF THE FOLLOWING POWER FUNCTIONS.

   A $f(x) = x$ **B** $f(x) = x^2$ **C** $f(x) = x^4$ **D** $f(x) = x^{-1}$

   E $f(x) = x^{-5}$ **F** $f(x) = x^{\frac{1}{2}}$ **G** $f(x) = x^{\frac{-3}{2}}$ **H** $f(x) = x^{\frac{-1}{3}}$

THE DERIVATIVES OF THE POWER FUNCTIONS IN ACTIVITY CAN BE SUMMARIZED AS FOLLOWS:

Function $f(x)$	Derivative $f'(x)$
x	1
x^2	$2x$
x^4	$4x^3$
x^{-1}	$-x^{-2}$
x^{-5}	$-5x^{-6}$
$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
$x^{\frac{-3}{2}}$	$-\frac{3}{2}x^{-\frac{5}{2}}$
$x^{\frac{-1}{3}}$	$-\frac{1}{3}x^{-\frac{4}{3}}$

FROM THIS TABLE ONE CAN SEE THAT IF $f(x) = x^n$, THEN $f'(x) = nx^{n-1}$.

Derivative of a power function

HERE, WE CONSIDER THE DERIVATIVE OF $f(x) = x^n$ WITH RESPECT TO x, WHERE n IS A REAL NUMBER.

Theorem 3.1 Power rule for differentiation

LET $f(x) = x^n$, WHERE n IS A POSITIVE INTEGER. THEN $f'(x) = nx^{n-1}$

Proof:

LET $f(x) = x^n$. THEN, USING THE DEFINITION OF DERIVATIVE, WE OBTAIN,

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{t\to x}\frac{t^n-x^n}{t-x}$$

$$= \lim_{t\to x}\frac{(t-x)\left(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\ldots+x^{n-2}t+x^{n-1}\right)}{t-x}$$

$$= \lim_{t\to x}\left(t^{n-1}+xt^{n-2}+x^2t^{n-3}+\cdots+x^{n-2}t+x^{n-1}\right)$$

$$= x^{n-1}+x^{n-1}+x^{n-1}+\cdots+x^{n-1}+x^{n-1} = nx^{n-1}.$$

Example 1 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = x^4$ **B** $f(x) = x^{10}$ **C** $f(x) = x^{95}$ **D** $f(x) = x^{102}$

Solution USING THEOREM 3.1, WE HAVE

A $f'(x) = (x^4)' = 4x^3$ **B** $f'(x) = (x^{10})' = 10x^9$

C $f'(x) = 95x^{94}$ **D** $f'(x) = 102x^{101}$

Corollary 3.1

IF $f(x) = x^{-n}$, WHERE n IS A POSITIVE INTEGER, THEN $f'(x) = -n\,x^{-(n+1)}$.

Proof:

$$f(x)=\lim_{t\to x}\left(\frac{\frac{1}{t^n}-\frac{1}{x^n}}{t-x}\right)=\lim_{t\to x}\frac{x^n-t^n}{(t-x)t^nx^n}=\lim_{t\to x}\frac{x^n-t^n}{(t-x)}\times\lim_{t\to x}\frac{1}{t^nx^n}$$

$$=(-nx^{n-1})\left(\frac{1}{x^{2n}}\right).$$ *Why? Explain!*

$$=-nx^{-1-n}\quad=-nx^{-(n+1)}$$

Example 2 LET $f(x) = x^{-7}$, EVALUATE

A $f'(1)$ **B** $f'\left(-\frac{1}{2}\right)$ **C** $f'(c)$

Solution BY COROLLARY 3.1, $f'(x) = (x^{-7})' = -7(x^{-8})$. HENCE,

A $f'(1) = -7$

B $f'\left(-\frac{1}{2}\right)=-7\left(-\frac{1}{2}\right)^{-8}=-7\left((-2)^8\right)=-7\left((-2)^8\right)=-1792$

C $f'(c)=-7c^{-8}$.

Corollary 3.2

Let $f(x) = cx^n$, then $f'(x) = cnx^{n-1}$; where n is any non-zero integer.

Proof:

$$f(x)=cx^n\Rightarrow f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=\lim_{t\to x}\frac{ct^n-cx^n}{t-x}=\lim_{t\to x}\frac{c(t^n-x^n)}{t-x}$$

$$=c\lim_{t\to x}\frac{t^n-x^n}{t-x}=c\,f'(x)=cnx^{n-1}$$

Example 3 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTION

A $f(x) = 4x^7$ **B** $f(x) = -11x^6$ **C** $f(x)=\frac{6}{x^{10}}$ **D** $f(x)=\frac{-}{x^{13}}$

Solution BY COROLLARY 3.2, WE HAVE

A $f'(x) = 4(x^7)' = 4(7x^6) = 28x^6$

B $f'(x) = (-11x^6)' = -11\,(x^6)' = -11(6x^5) = -66x^5$

C $f'(x)=\left(\frac{6}{x^{10}}\right)'=6(x^{-10})'=6(-10x^{-11})=-60x^{-11}$

D $f'(x)=\left(\frac{-}{x^{13}}\right)'=-\quad\left(-13x^{-14}\right)=13\quad x^{-14}.$

Theorem 3.2 Derivatives of power functions

LET $f(x) = x^r$; WHERE r IS A REAL NUMBER. THEN, $f'(x) = rx^{r-1}$.

Example 4 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS

A $f(x) = x^{\frac{1}{2}}$ **B** $f(x) = x^{\frac{3}{5}}$ **C** $f(x) = x$

D $f(x) = x^{-4.05}$ **E** $f(x) = x^{\sqrt{2}}$ **F** $f(x) = x^{e-3}$

Solution BY USING THEOREM 3.2, WE OBTAIN,

A $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$ **B** $f'(x) = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5x^{\frac{2}{5}}}$

C $f'(x) = x^{-1}$ **D** $f'(x) = -4.05x^{-5.05}$

E $f'(x) = \sqrt{2}\,x^{\sqrt{2}-1}$ **F** $f'(x) = (e-3)\,x^{e-4}$

Exercise 3.5

1 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x

A $f(x) = x^3$ **B** $f(x) = x^5$ **C** $f(x) = x^{11}$

D $f(x) = x^{-7}$ **E** $f(x) = x^{-10}$ **F** $f(x) = x^{\frac{3}{4}}$

G $f(x) = x^{\frac{-5}{3}}$ **H** $f(x) = x^3\sqrt{x}$ **I** $f(x) = x^{\frac{3}{2}}\sqrt[3]{x^2}$

J $f(x) = x^{1-}$

2 LET $f(x) = \left(\sqrt[3]{x}\right)x^2$, FIND

A $f'(0)$ **B** $f'(1)$ **C** $f'(8)$

3 LET $f(x) = x^{-\frac{4}{5}}$

A IF $f'(a) = -\frac{4}{5}$, FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $(a, f(a))$.

B IF f HAS A VERTICAL TANGENT AT $(a, f(a))$, FIND THE VALUE OF a.

Derivatives of simple trigonometric functions

Theorem 3.3 Derivatives of sine and cosine functions

1 IF $f(x) = \sin x$, THEN $f'(x) = \cos x$. **2** IF $f(x) = \cos x$, THEN $f'(x) = -\sin x$.

Proof

1 $f(x) = \sin x \Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = \lim_{h\to 0}\left(\frac{\sin x(\cos h-1)}{h}+\frac{(\cos x)\sin h}{h}\right)$

$= \sin x \lim_{h\to 0}\frac{\cos h-1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} = 0 + (\cos x)\times 1 = \cos x.$

2 $f(x) = \cos x \Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = \lim_{h\to 0}\left(\frac{(\cos x)(\cos h-1)}{h}-\sin x\frac{\sin h}{h}\right)$

$= \cos x \lim_{h\to 0}\frac{\cos h-1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} = (\cos x)\times 0 - (\sin x)\times 1 = -\sin x$

Exercise 3.6

1 Find the derivatives of each of the following functions with respect to the appropriate variable.

a $f(x) = -\sin x$ **b** $g(\theta) = -\cos\theta$

c $f(x) = \sec x$ **d** $g(t) = \csc t$

2 Find the equation of the tangent line to the graph of f at the given point.

a $f(x) = \sin x;\ \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ **b** $g(x) = \cos x;\ \left(\frac{\pi}{2}, 0\right)$

c $h(x) = \tan x;\ \left(\frac{\pi}{4}, 1\right)$

3 If the line tangent to the graph of $f(x) = \sin x$ at $x = a$ has y-intercept $\frac{\sqrt{3}}{6} - \frac{\pi}{3}$, find the x-intercept of the line when $0 < a < \frac{\pi}{2}$.

Derivatives of exponential function

Theorem 3.4 Derivatives of exponential functions

If $f(x) = a^x;\ a > 0$, then $f'(x) = a^x \ln a$.

If $f(x) = e^x$, then $f'(x) = \lim_{t\to x}\frac{e^t - e^x}{t - x}$.

Let $h = t - x$. Then as $t \to x$, $h \to 0$.

THUS, $f'(x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right) = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x \ln e = e^x$

NOTICE ALSO THAT $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. YOU WILL USE THIS FACT FOR THE PROOF.

Proof:

LET $f(x) = a^x$.

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} = a^x \ln a$$

Example 5 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING EXPONENTIAL FUNCTIONS.

A $f(x) = 4^x$ **B** $f(x) = \sqrt{5^x}$ **C** $f(x) = {}^x$

D $f(x) = e^{x+3}$ **E** $f(x) = \sqrt[3]{e^x}$ **F** $f(x) = 2^{3x+5}$

Solution

A $f(x) = 4^x \Rightarrow f'(x) = 4^x \ln 4$

B $f(x) = \sqrt{5^x} \Rightarrow f'(x) = \sqrt{5^x}\ln\sqrt{5} = \frac{5^{\frac{1}{2}x}}{2}\ln$

C $f(x) = {}^x \Rightarrow f'(x) = {}^x \ln$

D $f(x) = e^{x+3} \Rightarrow f(x) = e^3.e^x \Rightarrow f'(x) = e^3 e^x = e^{x+3}$

E $f(x) = \sqrt[3]{e^x} \Rightarrow f'(x) = \sqrt[3]{e^x}\ln\sqrt[3]{e} = \frac{1}{3}\sqrt[3]{e^x}\ln e = \frac{1}{3}e^{\frac{x}{3}}$

F $f(x) = 2^{3x+5} \Rightarrow f'(x) = 2^5 \times 8^x \ln 8$

$\Rightarrow f'(x) = 2^5 \times 2^{3x}(3\ln 2) \Rightarrow f'(x) = 96(2^{3x})\ln 2.$

Derivatives of logarithmic functions

Theorem 3.5 **Derivatives of logarithmic functions**

IF $f(x) = \ln x$, $x > 0$, THEN $f'(x) = \frac{1}{x}$.

Proof:

LET $f(x) = \ln x$.

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)$$

$$\Rightarrow \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right) = \ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}$$

Corollary 3.3

If $f(x) = \log_a x$, $x > 0$, $a > 0$ and $a \neq 1$, then $f'(x) = \dfrac{1}{x \ln a}$.

Proof:

$$f(x) = \log_a x = \frac{\ln x}{\ln a} \Rightarrow f'(x) = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \times \frac{1}{x} = \frac{1}{x \ln a}.$$

Example 6 Find the derivatives of each of the following logarithmic functions.

a $f(x) = \log_2 x$ **b** $f(x) = \log x$ **c** $f(x) = \log_{\frac{1}{5}} x$

d $f(x) = \log(x^3)$ **e** $f(x) = \ln \sqrt[5]{x}$ **f** $f(x) = \log_5 \sqrt{x^3}$

Solution

a $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x \ln 2}$

b $f(x) = \log x \Rightarrow f'(x) = \dfrac{1}{x \ln 10}$

c $f(x) = \log_{\frac{1}{5}} x \Rightarrow f'(x) = \dfrac{1}{x \ln\left(\frac{1}{5}\right)} = -\dfrac{1}{x \ln 5}$

d $f(x) = \log(x^3) \Rightarrow f(x) = 3\log x \Rightarrow f'(x) = \dfrac{3}{x \ln 10}$

e $f(x) = \ln \sqrt[5]{x} \Rightarrow f(x) = \dfrac{1}{5} \ln x \Rightarrow f'(x) = \dfrac{1}{5x}$

f $f(x) = \log_5 \sqrt{x^3} \Rightarrow f(x) = \dfrac{3}{2} \log_5 x \Rightarrow f'(x) = \dfrac{3}{2x \ln 5}$

Exercise 3.7

1 Differentiate each of the following functions with respect to the appropriate variable.

a $f(x) = 3^x$ **b** $f(x) = \sqrt{3^x}$ **c** $f(x) = 49^x$

d $f(x) = (\pi + 1)^x$ **e** $f(x) = e^{4x}$ **f** $f(x) = \sqrt{e^{3x}}$

g $h(x) = 3^x \times 3^x \times 2^{2x}$ **h** $f(x) = \log_7 x$ **i** $h(x) = \ln(4x)$

j $f(x) = \log_{0.125}(6x)$ **k** $h(x) = \ln\left(x^{\frac{3}{5}}\right)$

2 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $y = e^x$ AT $(1, e)$.

3 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF $f(x) = \ln x$ AT $\left(\frac{1}{e^3}, -3\right)$.

4 SUPPOSE $f(x) = 2^x$. WHAT HAPPENS TO THE GRADIENT OF THE GRAPH OF $f(x)$ AS $x \to +\infty$ AND AS $x \to -\infty$?

5 LET $g(x) = \log_2 x$. DECIDE THE NATURE OF THE GRADIENT OF $g(x)$ AS $x \to 0^+$ AND AS $x \to \infty$.

3.3 DERIVATIVES OF COMBINATIONS AND COMPOSITIONS OF FUNCTIONS

ACTIVITY 3.4

1 FOR EACH OF THE FOLLOWING FUNCTIONS f AND g, EVALUATE

A $f(x) + g(x)$ **B** $f(x) - g(x)$ **C** $f(x)\, g(x)$ **D** $\dfrac{f(x)}{g(x)}$

- **I** $f(x) = 2x + 1$ AND $g(x) = 3x^2 + 5x + 1$
- **II** $f(x) = 4x^2 + 1$ AND $g(x) = \dfrac{2x-1}{4x+2}$
- **III** $f(x) = e^x$ AND $g(x) = \sin x$
- **IV** $f(x) = \log_2(x^2 + 1)$ AND $g(x) = \cos\left(\dfrac{1}{x}\right)$.
- **V** $f(x) = 3^{x^2+1}$ AND $g(x) = \tan x$.

2 USING THE DEFINITION OF THE DERIVATIVE OF A FUNCTION, DIFFERENTIATE EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = 4x + 5$ **B** $f(x) = 4x^2 + 3x + 1$

C $f(x) = \sqrt{x} + \dfrac{1}{x}$ **D** $f(x) = \dfrac{3x - 1}{4x + 2}$

3 GIVEN THE FUNCTIONS $f(x) = \dfrac{1}{x}$ AND $g(x) = \sqrt{x}$, DECIDE WHETHER OR NOT EACH OF THE FOLLOWING EQUALITIES IS CORRECT.

A $(f(x) + g(x))' = f'(x) + g'(x)$ **B** $(f(x) - g(x))' = f'(x) - g'(x)$

C $(f(x)\, g(x))' = f'(x)\, g'(x)$ **D** $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)}{g'(x)}$

4 GIVEN THE FUNCTIONS $f(x) = x^2 - 1$, $g(x) = 3^x$, $k(x) = \log_2 x$ AND $h(x) = \sin x$, EVALUATE

A $f'(x) + g'(x)$ **B** $h'(x) - k'(x)$

C $f'(x)\,g(x) + g'(x)\,f(x)$ **D** $\dfrac{h'(x)\,k(x) - h(x)\,k'(x)}{(k(x))^2}$

5 LET $f(x) = |x|$

A IS f CONTINUOUS AT 0? **B** IS f DIFFERENTIABLE AT 0?

6 LET $f(x) = \begin{cases} x^2, & \text{if } x \le 2 \\ 8 - 2x, & \text{if } x > 2 \end{cases}$. SKETCH THE GRAPH OF f AND DISCUSS THE CONTINUITY AND DIFFERENTIABILITY OF f AT

A 2 **B** 1 **C** 3

IN ACTIVITY 3.4 PROBLEMS 5 AND 6, YOU NOTICED THAT THERE ARE FUNCTIONS THAT ARE CONTINUOUS BUT NOT DIFFERENTIABLE AT A GIVEN POINT. THE FOLLOWING THEOREM SHOWS THAT THE CONDITION FOR A FUNCTION BEING DIFFERENTIABLE AT A GIVEN POINT IS STRONGER THAN THE CONDITION OF BEING CONTINUOUS AT THAT POINT.

Theorem 3.6

IF f IS DIFFERENTIABLE AT a, THEN f IS CONTINUOUS AT a.

Proof:

SUPPOSE f IS DIFFERENTIABLE AT a. THEN, $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ EXISTS.

OBSERVE THAT, FOR $x \ne a$, $f(x) - f(a) = \left(\dfrac{f(x) - f(a)}{x - a}\right)(x - a)$

HENCE, $\lim_{x\to a}(f(x) - f(a)) = \lim_{x\to a}\left(\dfrac{f(x)-f(a)}{x-a}\right)(x-a) = \lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)$

$$= f'(a) \times 0 = 0$$

$$\Rightarrow \lim_{x\to a}(f(x) - f(a)) = 0 \Rightarrow \lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a) \Rightarrow f \text{ IS CONTINUOUS AT } a.$$

THE FOLLOWING EXAMPLE SHOWS THAT THE CONVERSE OF THIS THEOREM IS NOT TRUE.

Example 1 SHOW THAT EACH OF THE FOLLOWING FUNCTIONS IS CONTINUOUS BUT NOT DIFFERENTIABLE AT THE INDICATED NUMBERS.

A $f(x) = |x|;\ x = 0$ **B** $f(x) = |3x - 1|;\ x = \dfrac{1}{3}$

C $f(x) = \begin{cases} \sin x, & \text{IF } x > 0 \\ -x, & \text{IF } x \le 0 \end{cases}$ AT $x = 0$

Solution

A $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}|x| = 0 = f(0)$ $(\Rightarrow f$ IS CONTINUOUS AT (0, 0).

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$$

HERE, WE HAVE $\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x}$. BUT $\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0}\frac{-x}{x} = -$

$$\Rightarrow \lim_{x\to 0}\frac{f(x) - f(0)}{x-0} \text{ DOESN'T EXIST.}$$

HENCE, f IS NOT DIFFERENTIABLE AT $x = 0$.

B $f(x) = |3x - 1|$

$$\lim_{x\to\frac{1}{3}} f(x) = \lim_{x\to\frac{1}{3}}|3x-1| = 0 = f\left(\frac{1}{3}\right) \Rightarrow f \text{ IS CONTINUOUS AT } x = \frac{1}{3}$$

$$f'\left(\frac{1}{3}\right) = \lim_{x\to\frac{1}{3}}\frac{f(x) - f\left(\frac{1}{3}\right)}{x-\frac{1}{3}} = \lim_{x\to\frac{1}{3}}\frac{|3x-1|-0}{x-\frac{1}{3}} \text{ HERE, } \lim_{x\to\left(\frac{1}{3}\right)^+}\frac{|3x-1|}{x-\frac{1}{3}} = 3$$

$$\text{BUT } \lim_{x\to\left(\frac{1}{3}\right)^-}\frac{|3x-1|}{x-\frac{1}{3}} = -3 \Rightarrow f \text{ IS NOT DIFFERENTIABLE AT } x = \frac{1}{3}$$

C $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = 0$. ALSO, $f(0) = 0$

THUS, f IS CONTINUOUS AT $x = 0$.

$$f(x) = \begin{cases} \cos x, & \text{IF } x > 0 \\ -1, & \text{IF } x < 0 \end{cases}. \text{ BUT } f'(0) \text{ DOESN'T EXIST.}$$

$\Rightarrow f$ IS NOT DIFFERENTIABLE AT $x = 0$.

What conclusion can you make about differentiability at a point where a graph has a sharp point?

3.3.1 Derivative of a Sum or Difference of Two Functions

Theorem 3.7 Derivative of a sum or difference of two functions

IF f AND G ARE DIFFERENTIABLE AT x_o, THEN $f + g$ AND $f - g$ ARE ALSO DIFFERENTIABLE AT x_o, AND THEIR DERIVATIVES ARE GIVEN AS FOLLOWS:

1. $(f + g)'(x_o) = f'(x_o) + g'(x_o)$ . . . . . . *The sum rule.*
2. $(f - g)'(x_o) = f'(x_o) - g'(x_o)$ . . . . . . *The difference rule.*

Proof:

1 $$(f+g)'(x_o)=\lim_{x\to x_o}\frac{(f+g)(x)-(f+g)(x_o)}{x-x_o}$$
$$=\lim_{x\to x_o}\frac{f(x)+g(x)-f(x_o)-g(x_o)}{x-x_o}$$
$$=\lim_{x\to x_o}\left(\frac{f(x)-f(x_o)}{x-x_o}+\frac{g(x)-g(x_o)}{x-x_o}\right)$$
$$=\lim_{x\to x_o}\left(\frac{f(x)-f(x_o)}{x-x_o}\right)+\lim_{x\to x_o}\left(\frac{g(x)-g(x_o)}{x-x_o}\right)$$
$$=f'(x_o)-g'(x_o)$$

2 THE PROOF FOLLOWS A SIMILAR ARGUMENT TO 1 ABOVE.

Example 2 LET $f(x) = 4x^3 + \text{SIN}\,x$. EVALUATE

A $f'(0)$ **B** $f'\left(\frac{\ }{4}\right)$

Solution FROM THE ABOVE THEOREM WE HAVE,

$f'(x) = (4x^3 + \text{SIN}\,x)' = (4x^3)' + (\text{SIN}\,x)' = 12x^2 + \text{COS}\,x$

A $f'(0) = 12(0^2) + \text{COS}\,0 = 1$

B $f'\left(\frac{\ }{4}\right)=12\left(\frac{\ }{4}\right)^2+\text{COS}\left(\frac{\ }{4}\right) = 12\frac{\ ^2}{16}+\frac{\sqrt{2}}{2}=\frac{3\ ^2}{4}+\frac{\sqrt{2}}{2}=\frac{3\ ^2+2\sqrt{2}}{4}$

Example 3 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = \sqrt{x} + 3^x$ **B** $h(x) = x^{\frac{1}{3}} + \text{LOG}_2 x$ **C** $k(x) = e^x - \text{COS}\,x$

Solution

A $f'(x) = \left(\sqrt{x}\right)' + \left(3^x\right)' = \frac{1}{2\sqrt{x}} + 3^x\ \text{LN}\ 3$

B $h'(x) = \left(x^{\frac{1}{3}}\right)' + (\text{LOG}_2 x)' = \frac{1}{3}x^{\frac{-2}{3}} + \frac{1}{x\,\text{LN}\,2} = \frac{1}{3x^{\frac{2}{3}}} + \frac{1}{x\,\text{LN}\,2}$.

C $k'(x) = (e^x)' - (\text{COS}\,x)' = e^x - (-\text{SIN}\,x) = e^x + \text{SIN}\,x$.

Example 4 DIFFERENTIATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x.

A $y = 2x^4 - 5x^2 + 7x - 11$ **B** $f(x)=\sqrt{x}+\text{LOG}\,x-{}^{x}4+\frac{1}{x^2}$

Solution USING THE DERIVATIVE OF A SUM AND DIFFERENCE

A $f'(x) = (2x^4 - 5x^2 + 7x - 11)' = (2x^4 - 5x^2)' + (7x - 11)'$

$= (2x^4)' - (5x^2)' + (7x)' - (11)' = 8x^3 - 10x + 7.$

B $f'(x) = \left(\sqrt{x} + \text{LOG}\, x - 4^x + \frac{1}{x^2}\right)' = \left(\sqrt{x} + \text{LOG}\, x\right)' - \left(4^x - \frac{1}{x^2}\right)'$

$= \left(\sqrt{x}\right)' + (\text{LOG}\, x)' - \left[\left(4^x\right)' - \left(\frac{1}{x^2}\right)'\right] = \frac{1}{2\sqrt{x}} + \frac{1}{x\ \text{LN}\ 10} - \left(4^x\ \text{LN}\ 4 + \frac{2}{x^3}\right)$

$= \frac{1}{2\sqrt{x}} + \frac{1}{x\ \text{LN}10} - 4^x\ \text{LN}\ 4 - \frac{2}{x^3}$

Corollary 3.4

IF $f_1, f_2, f_3, \ldots, f_n$ ARE DIFFERENTIABLE AT x_o, THEN $\sum_{i=1}^{n} f_i$ IS DIFFERENTIABLE AT x_o AND

$$\left(\sum_{i=1}^{n} f_i\right)'(x_o) = \sum_{i=1}^{n} f_i{}'(x_o).$$

Example 5 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS

A $f(x) = 4x^3 + 5x^2 - 11x + 12$ **B** $g(x) = 16x^9 - 12x^8 - 9x^5 + 23$

Solution

A

$$\frac{d}{dx} f(x) = \frac{d}{dx}(4x^3 + 5x^2 - 11x + 12) = \frac{d}{dx}(4x^3) + \frac{d}{dx}(5x^2) - \frac{d}{dx}(11x) + \frac{d}{dx}(12)$$

$= 12x^2 + 10x - 11 + 0$

$= 12x^2 + 10x - 11.$

B $\frac{d}{dx} g(x) = \frac{d}{dx}\left(16x^9 - 12x^8 - 9x^5 + 23\right)$

$= \frac{d}{dx}\left(16x^9\right) - \frac{d}{dx}\left(12x^8\right) - \frac{d}{dx}\left(9x^5\right) + \frac{d}{dx}(23)$

$= 144x^8 - 96x^7 - 45x^4.$

The derivative of a polynomial function

LET $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, THEN,

$f'(x) = na_n x^{n-1} + (n-1)\, a_{n-1} x^{n-2} + \ldots + 2a_2 x + a_1$

3.3.2 Derivatives of Product and Quotient of Functions

Derivatives of product of functions

ACTIVITY 3.5

1 Evaluate each of the following limits.

a $\lim_{h \to 0} \frac{(x+h)\,e^{x+h} - xe^x}{h}$ b $\lim_{h \to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$

2 Using the definition of derivative, differentiate each of the following functions with respect to x.

a $f(x) = xe^x$ b $f(x) = x \sin x$

Theorem 3.8.1 The product rule

If f and g are differentiable functions at x_o, then the product fg is differentiable at x_o and its derivative is given as follows:

$(fg)'(x_o) = f'(x_o)\,g(x_o) + g'(x_o)\,f(x_o)$.

Proof:

$$(fg)'(x_o) = \lim_{x \to x_o} \frac{(fg)(x) - (fg)(x_o)}{x - x_o} = \lim_{x \to x_o} \frac{f(x)g(x) - f(x_o)g(x_o)}{x - x_o}$$

$$= \lim_{x \to x_o} \frac{f(x)g(x) - f(x_o)g(x) + f(x_o)g(x) - f(x_o)g(x_o)}{x - x_o}$$

$$= \lim_{x \to x_o} \frac{g(x)\left(f(x) - f(x_o)\right) + f(x_o)\left(g(x) - g(x_o)\right)}{x - x_o}$$

$$= \lim_{x \to x_o} g(x)\left(\frac{f(x) - f(x_o)}{x - x_o}\right) + \lim_{x \to x_o} f(x_o)\left(\frac{g(x) - g(x_o)}{x - x_o}\right)$$

$$= \lim_{x \to x_o} g(x) \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} + f(x_o) \lim_{x \to x_o} \frac{g(x) - g(x_o)}{x - x_o}$$

$$= g(x_o)\,f'(x_o) + f(x_o)\,g'(x_o)$$

If $y = f(x) \cdot g(x)$, then $\left.\frac{dy}{dx}\right|_{x=x_o} = g(x_o)\frac{d}{dx}\left(f(x)\right)\Big|_{x=x_o} + f(x_o)\frac{d}{dx}\left(g(x)\right)\Big|_{x=x_o}$

Example 6 Let $h(x) = (x + 5)(x^2 + 1)$. Evaluate $h'(3)$.

Solution Let $f(x) = x + 5$ and $g(x) = x^2 + 1$.

Then, using the product rule you obtain, $h'(3) = f'(3)\,g(3) + g'(3)\,f(3)$.

BUT $f'(x) = 1$ SO THAT $f'(3) = 1$ AND $g'(x) = 2x$, SO THAT $g'(3) = 6$.

THEREFORE, $k'(3) = 1 \times 10 + 6 \times 8 = 58$.

Example 7 LET $f(x) = x\,e^{x+1}$, EVALUATE $f'(-1)$.

Solution LET $h(x) = x$ AND $k(x) = e^{x+1}$, THEN $f(x) = h(x)\,k(x)$.

THEN $h'(x) = 1$ AND $k'(x) = e^{x+1}$.

$\Rightarrow f'(-1) = h'(-1)\,k(-1) + k'(-1)\,h(-1) = 1 \times e^0 + e^0 \times (-1) = 0$

Example 8 LET $y = e^x \sin x$, EVALUATE $\left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{3}}$

Solution

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \left.\left(\sin x \frac{d}{dx}\left(e^x\right)\right)\right|_{x=\frac{\pi}{3}} + \left.\left(e^x \frac{d}{dx}(\sin x)\right)\right|_{x=\frac{\pi}{3}}$$

$$= \sin x\, e^x\Big|_{x=\frac{\pi}{3}} + e^x \cos x\Big|_{x=\frac{\pi}{3}} = \sin\frac{\pi}{3}\, e^{\frac{\pi}{3}} + e^{\frac{\pi}{3}} \cos\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2} e^{\frac{\pi}{3}} + e^{\frac{\pi}{3}} \times \frac{1}{2} = \frac{e^{\frac{\pi}{3}}}{2}\left(\sqrt{3}+1\right)$$

Theorem 3.8.2 The product rule

$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, FOR ALL x AT WHICH BOTH f AND g ARE DIFFERENTIABLE.

Note:

✓ IF $y = (fg)(x)$, THEN

$$\frac{dy}{dx} = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

Example 9 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING USING THE PRODUCT RULE.

A $f(x) = x \sin x$ **B** $f(x) = x^2 \cos x$

C $f(x) = \left(x^2 - 5x + 1\right)e^x$ **D** $f(x) = \sqrt{x}\,\log_2 x$

Solution

A $f'(x) = (x \sin x)' = (x)' \sin x + x(\sin x)' = 1 \times \sin x + x(\cos x)$

$= \sin x + x \cos x$

B $f'(x) = (x^2 \cos x)' = (x^2)' \cos x + x^2 (\cos x)' = 2x \cos x + x^2(-\sin x)$

$= 2x \cos x - x^2 \sin x$

C $f'(x) = \left((x^2 - 5x + 1)\, e^x\right)' = (x^2 - 5x + 1)'\, e^x + (e^x)'(x^2 - 5x + 1)$

$= (2x - 5)\, e^x + e^x (x^2 - 5x + 1) = (x^2 - 3x - 4)e^x.$

D $f'(x) = \left(\sqrt{x}\log_2 x\right)' = \dfrac{1}{2\sqrt{x}}\log_2 x + \dfrac{\sqrt{x}}{x \ln 2}$

Example 10 LET $y = 3^x \cos x$, FIND $\dfrac{dy}{dx}$

Solution $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(3^x \cos x\right) = \cos x \dfrac{d}{dx}(3^x) + 3^x \dfrac{d}{dx}(\cos x)$

$= 3^x \ln 3 \cos x - 3^x \sin x = 3^x (\ln 3 \cos x - \sin x)$

Example 11 FIND THE DERIVATIVE OF $f(x) = (x^2 + 1)(\ln x)(\sin x)$

Solution

$$f'(x) = \frac{d}{dx}\left((x^2+1)(\ln x)(\sin x)\right) = (\ln x)(\sin x)\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(\ln x \sin x)$$

$$= (\ln x \sin x)\, 2x + (x^2+1)\left(\frac{1}{x}\sin x + \ln x \cos x\right) \text{ (Explain!)}$$

$$= 2x \ln x \sin x + (x^2 + 1)\left(\frac{1}{x}\sin x + \ln x \cos x\right)$$

ONE OF THE PURPOSES OF THIS EXAMPLE IS TO EXTEND THE PRODUCT RULE FOR FINDING DERIVATIVES OF THE PRODUCTS OF THREE OR MORE FUNCTIONS SUCH AS

$(fgh)'(x) = (fg)'(x)\, h(x) + h'(x)\,(fg)(x)$

$= \left(f'(x)g(x) + g'(x)f(x)\right)h(x) + h'(x)(fg)(x)$

$= f'(x)\, g(x)\, h(x) + f(x)\, g'(x)\, h(x) + f(x)\, g(x)\, h'(x).$

Example 12 FIND THE DERIVATIVE OF $y = x^3 \sin x\, (3^x)$

Solution $\dfrac{dy}{dx} = (x^3)' \sin x\,(3^x) + x^3 (\sin x)' \times 3^x + x^3 \sin x\,(3^x)'$

$3x^2 \sin x \times 3^x + x^3 \cos x \times 3^x + x^3 \sin x \times 3^x \ln 3.$

Derivative of a quotient of functions

ACTIVITY 3.6

1 LET $f(x) = e^x$ AND $g(x) = x$. EVALUATE

A $f'(x)\, g(x)$ **B** $g'(x)\, f(x)$

C $\dfrac{f'(x)}{g'(x)}$ **D** $\dfrac{f'(x)\, g(x) - g'(x)\, f(x)}{(g(x))^2}$

LET f BE A DIFFERENTIABLE FUNCTION SUCH THAT $f(x) \neq 0$. THEN,

$$\left(\frac{1}{f(x)}\right)' = \lim_{h\to 0}\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \lim_{h\to 0}\frac{f(x)-f(x+h)}{hf(x+h)f(x)}$$

$$= \lim_{h\to 0}\frac{f(x)-f(x+h)}{h} \times \lim_{h\to 0}\frac{1}{f(x)f(x+h)}$$

$$= -\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \times \frac{1}{f(x)f(x+0)}$$

$$= -f'(x) \times \frac{1}{(f(x))^2} = \frac{-f'(x)}{(f(x))^2}$$

2 USING $\left(\frac{1}{f(x)}\right)' = \frac{-f'(x)}{(f(x))^2}$, FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = \frac{1}{x}$ **B** $f(x) = \frac{1}{\sqrt{x}}$ **C** $f(x) = \frac{1}{3x+1}$

D $f(x) = \frac{1}{\sqrt{x}+1}$ **E** $f(x) = \frac{1}{e^x+1}$

FROM THE ABOVE ACTIVITY, YOU OBSERVED THAT

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \times \frac{1}{g(x)}\right)' = \frac{1}{g(x)} \times f'(x) + f(x) \times \left(\frac{1}{g(x)}\right)' \quad \ldots \textit{By the product rule}$$

$$= \frac{f'(x)}{g(x)} + f(x)\left(\frac{-g'(x)}{(g(x))^2}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Theorem 3.9 The quotient rule

IF f AND G ARE DIFFERENTIABLE FUNCTIONS AND $g(x) \neq 0$, THEN $\frac{f}{g}$ IS DIFFERENTIABLE FOR ALL x AT WHICH f AND g ARE DIFFERENTIABLE WITH

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

✍Note:

IF $y = \left(\frac{f}{g}\right)(x)$, THEN $\frac{dy}{dx} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

Example 13 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS AT THE GIVEN NUMBER.

A $f(x) = \dfrac{x}{x+5}$ AT $x =$

B $f(x) = \tan x$ AT $x = \dfrac{\ }{3}$

C $f(x) = \dfrac{\ln x}{x}$ AT $x = e$

Solution USING THE QUOTIENT RULE WE OBTAIN,

A $$f'(x) = \frac{(x+5)(x)' - x(x+5)'}{(x+5)^2} = \frac{x+5-x}{(x+5)^2} = \frac{5}{(x+5)^2}$$

$$\Rightarrow f'(1) = \frac{5}{(1+5)^2} = \frac{5}{36}$$

B $$f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \Rightarrow f'\left(\frac{\ }{3}\right) = \sec^2\left(\frac{\ }{3}\right) =$$

> **Note:**
>
> ✓ $\dfrac{d}{dx}(\tan x) = \sec^2 x$

C $$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{x(\ln x)' - \ln x(x)'}{x^2} = \frac{x \times \frac{1}{x} - \ln x \times}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\Rightarrow f'(e) = \frac{1 - \ln e}{e^2} = 0$$

Example 14 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS USING THE QUOTIENT RULE.

A $f(x) = \dfrac{1}{\ln x}$ **B** $f(x) = \dfrac{1}{x^2 - 2}$ **C** $f(x) = \dfrac{4x^2 - 5x + 7}{x^2 - 3x + 1}$

D $f(x) = \dfrac{4^x}{x \ln x}$ **E** $f(x) = \dfrac{x \sin x}{x^2 + 1}$ **F** $f(x) = \dfrac{x \tan x}{e^x + \log_2 x}$

Solution

A $$f(x) = \frac{1}{\ln x} \Rightarrow f'(x) = \frac{-(\ln x)'}{(\ln x)^2} = -\frac{1}{x \ln^2 x}$$

B $$f(x) = \frac{1}{x^2 - 2} \Rightarrow f'(x) = -\frac{(x^2 - 2)'}{(x^2 - 2)} = \frac{-2x}{(x^2 - 2)^2}$$

C $f'(x)=\dfrac{(4x^2-5x+7)'(x^2-3x+1)-(x^2-3x+1)'(4x^2-5x+7)}{(x^2-3x+1)^2}$

$=\dfrac{(8x-5)(x^2-3x+1)-(2x-3)(4x^2-5x+7)}{(x^2-3x+1)^2}=\dfrac{-7x^2-6x+16}{(x^2-3x+1)^2}$

D $f'(x)=\dfrac{(4^x)'x\ln x-4^x(x\ln x)'}{(x\ln x)^2}=\dfrac{(4^x\ln 4)x\ln x-4^x(1+\ln x)}{(x\ln x)^2}$

$=\dfrac{4^x(x\ln 4\ln x-(1+\ln x))}{(x\ln x)^2}$

E $f(x)=\dfrac{x\sin x}{x^2+1}\Rightarrow f'(x)=\dfrac{(x\sin x)'(x^2+1)-(x^2+1)'(x\sin x)}{(x^2+1)^2}$

$=\dfrac{(\sin x+x\cos x)(x^2+1)-2x^2\sin x}{(x^2+1)^2}.$

F $f'(x)=\left(\dfrac{x\tan x}{e^x+\log_2 x}\right)'=\dfrac{(x\tan x)'(e^x+\log_2 x)-x\tan x(e^x+\log_2 x)'}{(e^x+\log_2 x)^2}$

$=\dfrac{(\tan x+x\sec^2 x)(e^x+\log_2 x)-x\tan x\left(e^x+\dfrac{1}{x\ln 2}\right)}{(e^x+\log_2 x)^2}$

Example 15 In each of the following, find $\dfrac{dy}{dx}$

A $y=\dfrac{x^2}{3x+1}$ **B** $y=\dfrac{x^2+1}{x^3+x-1}$ **C** $y=\dfrac{x^2+4}{\cos x}$

D $y=\dfrac{x+e^x}{2x+1}$ **E** $y=\dfrac{\cos x}{1-\sin x}$

Solution Applying the quotient rule,

A $y=\dfrac{x^2}{3x+1}\Rightarrow\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{x^2}{3x+1}\right)=\dfrac{(3x+1)\dfrac{d}{dx}(x^2)-x^2\dfrac{d}{dx}(3x+1)}{(3x+1)^2}$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{x^2}{3x+1}\right)=\dfrac{(3x+1)\dfrac{d}{dx}(x^2)-x^2\dfrac{d}{dx}(3x+1)}{(3x+1)^2}$

$$= \frac{(3x+1)(2x) - x^2(3)}{(3x+1)^2} = \frac{6x^2+2x-3x^2}{(3x+1)^2} = \frac{3x^2+2x}{(3x+1)^2}$$

B $y = \dfrac{x^2+1}{x^3+x-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^3+x-1}\right) = \frac{(x^3+x-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^3+x-1)}{(x^3+x-1)^2}$$

$$= \frac{(x^3+x-1)(2x)-(x^2+1)(3x^2+1)}{(x^3+x-1)^2} = \frac{2x^4+2x^2-2x-(3x^4+4x^2+1)}{(x^3+x-1)^2}$$

$$= -\frac{(x^4+2x^2+2x+1)}{(x^3+x-1)^2}$$

C $$\frac{d}{dx}\left(\frac{x^2+4}{\cos x}\right) = \frac{\cos x\frac{d}{dx}(x^2+4)-(x^2+4)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{2x\cos x+(x^2+4)\sin x}{\cos^2 x}$$

D $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+e^x}{2x+1}\right) = \frac{(2x+1)\frac{d}{dx}(x+e^x)-(x+e^x)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)(1+e^x)-(x+e^x)(2)}{(2x+1)^2} = \frac{2x+1+2xe^x+e^x-2x-2e^x}{(2x+1)^2}$$

$$= \frac{2xe^x-e^x+1}{(2x+1)^2}.$$

E $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right) = \frac{(1-\sin x)\frac{d}{dx}\cos x-\cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2} = \frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}.$$

Exercise 3.8

1 DIFFERENTIATE EACH OF THE FOLLOWING FUNCTIONS USING THE APPROPRIATE RULES.

A $f(x) = 1 - x - x^2 + x^3$ **B** $g(x) = 7\sqrt{x} + e^x - \sin x$

C $h(x) = \dfrac{x}{x+5}$ **D** $l(x) = x + \sin x - e^x$

E $k(x) = \dfrac{x \sin x}{x - e^x}$ **F** $f(x) = \dfrac{\sqrt{x}}{x \cos x}$

G $g(x) = \csc x \sec x$ **H** $h(x) = \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{\sec x}{x^2}$

I $k(x) = \dfrac{4x+5}{x^2+1}$ **J** $f(x) = x^2 \ln x$

2 FOR EACH OF THE FOLLOWING FUNCTIONS, FIND $\dfrac{dy}{dx}$.

A $y = \ln x + e^x$ **B** $y = (x^2 - 2x - 3)\, e^x$ **C** $y = \dfrac{1 - \ln x}{x^2}$

D $y = \dfrac{x^2+1}{\cos x}$ **E** $y = \dfrac{e^x + x - 1}{x+1}$ **F** $y = \dfrac{\sin x}{1 - \cos x}$

G $y = \dfrac{1 - \sin x}{x + \cos x}$ **H** $y = \dfrac{e^x \sin x}{e^x + 1}$ **I** $y = \dfrac{x^2}{x + \ln x}$

J $y = e^x(1 + x^2)\tan x$ **K** $y = \dfrac{\left(1 + \dfrac{1}{x^2}\right)}{1 - \dfrac{1}{x^2}}$ **L** $y = \left(e^x - \sqrt{x}\right)^3$

3 IN EACH OF THE FOLLOWING, FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $(a, f(a))$.

A $f(x) = \dfrac{x-1}{x+1};\ a = 0$ **B** $f(x) = \dfrac{3x+1}{4-x^2};\ a = 1$

C $f(x) = e^x \sin x;\ a = 0$ **D** $f(x) = \dfrac{x^2 - 4x}{e^{-x}+1};\ a = 0$

3.3.3 The Chain Rule

SUPPOSE YOU INVEST BIRR 100 IN A BANK THAT PAYS r PERCENT ANNUAL INTEREST COMPOUNDED MONTHLY. THEN AT THE END OF 5 YEARS THE ACCOUNT BALANCE (IN BIRR) WILL BE

$$A(r) = 100\left(1 + \frac{r}{1200}\right)^{60}$$

THIS IS THE COMPOSITION OF THE TWO FUNCTIONS

$$f(r) = 1 + \frac{r}{1200} \text{ AND } g(x) = 100x^{60}.$$

$$g(f(r)) = 100(f(r))^{60} \text{ I.E., } A(r) = g(f(r)).$$

IN THIS SECTION, YOU WILL SEE HOW TO DETERMINE THE DERIVATIVE OF A COMPOSITION FUNC
$A(r)$ USING THE DERIVATIVES OF THE COMPONENT FUNCTIONS LIKE f AND g.

ACTIVITY 3.7

1 LOOK AT THE FOLLOWING TABLE.

Function $y = f(x)$	Expanded form	$\frac{dy}{dx}$	The derivative in factorized form
$2x^3 + 1$	$2x^3 + 1$	$6x^2$	$1 \times \frac{dy}{dx}$
$(2x^3 + 1)^2$	$4x^6 + 4x^3 + 1$	$24x^5 + 12x^2$	$2(2x^3+1)\frac{d}{dx}(2x^3+1)$
$(2x^3 + 1)^3$	$8x^9 + 12x^6 + 6x^3 + 1$	$72x^8 + 72x^5 + 18x^2$ $= 18x^2(4x^6 + 4x^3 + 1)$ $= 18x^2(2x^3 + 1)^2$	$3(2x^3+1)^2\frac{d}{dx}(2x^3+1)$
$(2x^3 + 1)^4$	$16x^{12} + 32x^9 + 24x^6 + 8x^3 + 1$	$192x^{11} + 288x^8 +$ $144x^5 + 24x^2$	$4(2x^3+1)^3\frac{d}{dx}(2x^3+1)$

FROM THE ABOVE TABLE YOU MIGHT HAVE NOTICED THAT THE DERIVATIVE IS THE PRODUCT O
THE EXPONENT, THE EXPRESSION WITH EXPONENT REDUCED BY 1 AND THE DERIVATIV
$2x^3 + 1$.

2 FIND THE DERIVATIVES OF EACH OF THE FOLLOWING FUNCTIONS WITHOUT EXPANDI
POWER.

A $(2x^3 + 1)^4$ **B** $(2x^3 + 1)^{11}$ **C** $(2x^3 + 1)^n$

3 LET $f(x) = 3x + 1$, $g(x) = \cos x$ AND $h(x) = \frac{3x - 1}{x^2 + 1}$. EVALUATE EACH OF THE FOLLOWING FUNCTIONS.

A $f(g(x))$ **B** $f(h(x))$ **C** $h(g(x))$

D $f'(g(x))$ **E** $f'(g(x)).g'(x)$ **F** $h'(g(x)).g'(x)$

AT THIS STAGE WE CAN GIVE THE DERIVATIVE OF COMPOSITIONS OF FUNCTIONS AT A GIVEN P

Theorem 3.10 The chain rule

LET g BE DIFFERENTIABLE AT x_o AND f BE DIFFERENTIABLE AT $g(x_o)$. THEN $f \circ g$ IS DIFFERENTIABLE AT x_o AND $(f \circ g)'(x_o) = f'(g(x_o)).g'(x_o)$

Proof:

FOR $g(x) - g(x_o) \neq 0$, WE HAVE,

$$\frac{f(g(x)) - f(g(x_o))}{x - x_o} = \frac{f(g(x)) - f(g(x_o))}{x - x_o} \times \frac{g(x) - g(x_o)}{g(x) - g(x_o)}$$

THUS,
$$(f \circ g)'(x_o) = \lim_{x \to x_o} \frac{(f \circ g)(x) - (f \circ g)(x_o)}{x - x_o}$$
$$= \lim_{x \to x_o} \frac{f(g(x)) - f(g(x_o))}{g(x) - g(x_o)} \times \frac{g(x) - g(x_o)}{x - x_o}$$
$$= \lim_{g(x) \to g(x_o)} \frac{f(g(x)) - f(g(x_o))}{g(x) - g(x_o)} \times \lim_{x \to x_0} \frac{g(x) - g(x_o)}{x - x_o}$$
$$= f'(g(x_O)).\, g'(x_o)$$

Example 16 LET $h(x) = \sin(3x + 1)$. EVALUATE $h'\left(\frac{-2}{6}\right)$.

Solution h IS THE COMPOSITION OF THE TWO SIMPLE FUNCTIONS $f(x) = \sin x$ AND $g(x) = 3x + 1$. I.e., $h(x) = f(g(x))$.

BY THE CHAIN RULE, $h'(x_o) = f'(g(x_o)).\, g'(x_O)$.

BUT $f'(x) = \cos x$, $g'(x) = 3$ AND $x_o = \frac{-2}{6}$

THUS $h'\left(\frac{-2}{6}\right) = f'\left(g\left(\frac{-2}{6}\right)\right) \times g'\left(\frac{-2}{6}\right) = f'\left(3\left(\frac{-2}{6}\right) + 1\right) \times 3$

$$= 3f'\left(\frac{\;}{2}\right) = 3\cos\left(\frac{\;}{2}\right) = 0.$$

Example 17 FIND THE DERIVATIVE OF $f(x) = \sqrt{1 + x^2}$ AT $x = 2$.

Solution $f(x)$ IS THE COMPOSITION OF THE FUNCTIONS $g(x) = \sqrt{x}$ AND $h(x) = 1 + x^2$. I.E.,

$f(x) = g(h(x)) \Rightarrow f'(2) = g'(h(2)) \times h'(2) = g'(5) \times h'(2)$

BUT $g'(x) = \frac{1}{2\sqrt{x}}$ AND $h'(x) = 2x$

THUS $f'(2) = \frac{1}{2\sqrt{5}} \times 4 = \frac{2\sqrt{5}}{5}$.

3.3.4 Derivatives of Composite Functions

IF g IS DIFFERENTIABLE AT x AND f IS DIFFERENTIABLE AT $g(x)$, THEN $f \circ g$ IS DIFFERENTIABLE AT x WITH

$$(f \circ g)'(x) = f'(g(x)).g'(x)$$

Example 18 FIND THE DERIVATIVE OF $f(x) = e^{x^2+x+3}$.

Solution LET $g(x) = e^x$ AND $h(x) = x^2 + x + 3$, THEN $f(x) = g(h(x))$, $g'(x) = e^x$ AND $h'(x) = 2x + 1$. BUT $f'(x) = g'(h(x)).\, h'(x)$

$$\Rightarrow f'(x) = g'(x^2 + x + 3) \times (2x + 1) = e^{x^2+x+3}(2x + 1) = f(x) \times (2x + 1).$$

$f'(x)$ CAN BE FOUND AS FOLLOWS.

$$f'(x) = \left(e^{x^2+x+3}\right)' = e^{x^2+x+3} \times (x^2 + x + 3)' = e^{x^2+x+3} \times (2x + 1)$$

Example 19 LOOK AT EACH OF THE FOLLOWING DERIVATIVES.

A $\left((x+5)^4\right)' = 4(x+5)^3\,(x+5)' = 4\,(x+5)^3$

WHERE $(x^4)' = 4x^3$ *derivative of the inner function*

B $\left((5x-2)^{10}\right)' = 10(5x-2)^9\,(5x-2)' = 10(5x-2)^9 \times 5 = 50\,(5x-2)^9$

$(x^{10})' = 10x^9$ *derivative of the inner function*

C $\left((3x^2+5x+2)^8\right)' = 8(3x^2+5x+2)^7(3x^2+5x+2)' = 8(3x^2+5x+2)^7(6x+5)$

D $(\cos(x^2+x+7))' = -\sin(x^2+x+7)(x^2+x+7)' = -\sin(x^2+x+7)(2x+1)$

$= -(2x+1)\sin(x^2+x+7)$

E $\left(\sin\sqrt{x^2+4x+1}\right)' = \cos\sqrt{x^2+4x+1}\left(\sqrt{x^2+4x+1}\right)'$

$$= \cos\sqrt{x^2+4x+1} \times \frac{1}{2\sqrt{x^2+4x+1}}\left(x^2+4x+1\right)'$$

$$= \cos\sqrt{x^2+4x+1} \times \frac{2x+4}{2\sqrt{x^2+4x+1}}$$

$$= \frac{x+2}{\sqrt{x^2+4x+1}}\cos\sqrt{x^2+4x+$$

THIS IS THE DERIVATIVE OF THE COMPOSITION OF THREE FUNCTIONS. THEREFORE, YOU I

Corollary 3.5

$$\left(f(g(h(x)))\right)' = f'(g(h(x))).g'(h(x)).h'(x).$$

Proof:-

$$\left(f\left(g\left(h(x)\right)\right)\right)' = f'\left(g\left(h(x)\right)\right).\left(g\left(h(x)\right)\right)' \text{. WHY?}$$

$$= f'\left(g\left(h(x)\right)\right).g'\left(h(x)\right).h'(x)$$

Example 20 FIND THE DERIVATIVE OF $k(x) = \cos^5(x^2 + 1)$.

Solution NOTICE THAT k IS THE COMPOSITION OF THE THREE SIMPLE FUNCTIONS,

$f(x) = x^5$, $g(x) = \cos x$ AND $h(x) = x^2 + 1$. I.E., $k(x) = f(g(h(x)))$

ALSO, $f'(x) = 5x^4$, $g'(x) = -\sin x$, $h'(x) = 2x$ AND

$k'(x) = f'(g(h(x))).\ g'(h(x)).\ h'(x) = f'(g(x^2 + 1))\ g'(x^2 + 1).\ (2x)$

$= f'(\cos(x^2 + 1)).\ (-\sin(x^2 + 1))\ (2x) = 5\cos^4(x^2 + 1)\ (-\sin(x^2 + 1))\ (2x)$

$= -10x \sin(x^2 + 1) \cos^4(x^2 + 1)$

IN SHORT,

$$(\cos^5(x^2 + 1))' = 5\cos^4(x^2 + 1).(-\sin(x^2 + 1))(2x) = -10x \sin(x^2 + 1) \cos^4(x^2 + 1)$$

The chain rule using the notation $\frac{dy}{dx}$

LET $y = f(u)$ AND $u = g(x)$. THEN,

$$y = f(g(x)),\ \frac{dy}{du} = f'(u) \text{ AND } \frac{du}{dx} = g'(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right) = f'\left(g(x)\right).\ g'(x) = f'(u).\ \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

THEREFORE, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 21 FIND THE DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x.

A $y = (3x + 4)^6$ **B** $y = \cos^6 x$

C $y = \left(x^3 + 1\right)^{\frac{3}{5}}$ **D** $y = \sqrt{3x^5 - 2x + 4}$

Solution

A $y = (3x + 4)^6$

LET $u = 3x + 4$, THEN $y = u^6 \Rightarrow \frac{du}{dx} = 3$ AND $\frac{dy}{du} = 6u^5$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 6u^5 \times 3 = 18u^5 = 18(3x+4)^5.$$

IN SHORT, $\frac{dy}{dx} = \frac{d}{dx}(3x+4)^6 = 6\,(3x+4)^5.\frac{d}{dx}(3x+4) = 18\,(3x+4)^5$

B $y = \cos^6 x$

LET $u = \cos x$, THEN $y = u^6$, $\dfrac{dy}{du} = 6u^5$ AND $\dfrac{du}{dx} = -\sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 6u^5(-\sin x) = -6\sin x \cos^5 x$$

OBSERVE THAT $\dfrac{d}{dx}\cos^6 x = 6\cos^5 x\ \dfrac{d}{dx}\cos x = -6\sin x\cos^5 x$

C $y = (x^3+1)^{\frac{3}{5}}$

LET $u = x^3 + 1$, THEN $y = u^{\frac{3}{5}}$ SO THAT $\dfrac{du}{dx} = 3x^2$ AND $\dfrac{dy}{du} = \dfrac{3}{5}u^{-\frac{2}{5}}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{3}{5}u^{-\frac{2}{5}}.3x^2 = \frac{3}{5}\left(x^3+1\right)^{-\frac{2}{5}}.3x^2 = \frac{9}{5}x^2\left(x^3+1\right)^{-\frac{2}{5}} = \frac{9x^2}{5\left(x^3+1\right)^{\frac{2}{5}}}$$

D $y = \sqrt{3x^5 - 2x + 4}$

LET $u = 3x^5 - 2x + 4$, THEN $y = \sqrt{u}$

HENCE, $\dfrac{du}{dx} = 15x^4 - 2$ AND $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot\left(15x^4 - 2\right) = \frac{1}{2\sqrt{3x^5 - 2x + 4}}\cdot\left(15x^4 - 2\right)$$

Example 22 DIFFERENTIATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO x.

A $y = \dfrac{x}{\sqrt{x^2+4}}$ **B** $y = \sqrt{\sin\left(x^2+1\right)}$ **C** $y = e^{\sqrt{x^2+5x+4}}$

D $y = \dfrac{\log\sqrt{x^2+1}}{x + \sin x}$ **E** $y = \cos\sqrt{\log\left(\sqrt{x^2+1} + x\right)}$

Solution IN THIS EXAMPLE, YOU DIFFERENTIATE EACH FUNCTION BY REWRITING IT AS THE COMPOSITION OF SIMPLE FUNCTIONS.

A $y = \dfrac{x}{\sqrt{x^2+4}}$. HERE YOU APPLY THE QUOTIENT RULE AND THE CHAIN RULE

$$\frac{dy}{dx} = \frac{(x)'\sqrt{x^2+4} - x\left(\sqrt{x^2+4}\right)'}{\left(\sqrt{x^2+4}\right)^2} \quad \textit{Quotient rule}$$

$$= \frac{\sqrt{x^2+4} - x\left(\dfrac{1}{2\sqrt{x^2+4}}\right)\left(x^2+4\right)'}{\left(x^2+4\right)} \quad \textit{Chain rule}$$

$$= \frac{\sqrt{x^2+4} - \dfrac{x}{2\sqrt{x^2+4}}(2x)}{(x^2+4)} = \frac{\left(\sqrt{x^2+4}\right)^2 - x^2}{(x^2+4)\sqrt{x^2+4}}$$

$$= \frac{x^2+4-x^2}{(x^2+4)\sqrt{x^2+4}} = \frac{4}{(x^2+4)\sqrt{x^2+4}}$$

B $y = \left(\sqrt{\sin(x^2+1)}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin(x^2+1)}}.\left(\sin(x^2+1)\right)' \quad \text{because } \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos(x^2+1)}{2\sqrt{\sin(x^2+1)}}(2x) = \frac{x\cos(x^2+1)}{\sqrt{\sin(x^2+1)}}$$

C $y = e^{\sqrt{x^2+5x+4}}$

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{x^2+5x+4}}\left(\sqrt{x^2+5x+4}\right)' \quad \text{becuase } \left(e^x\right)' = e^x$$

$$= e^{\sqrt{x^2+5x+4}} \times \frac{1}{2\sqrt{x^2+5x+4}}\left(x^2+5x+4\right)'$$

$$= \frac{e^{\sqrt{x^2+5x+4}}}{2\sqrt{x^2+5x+4}}(2x+5)$$

D $y = \dfrac{\log\sqrt{x^2+1}}{x+\sin x} = \dfrac{\frac{1}{2}\log(x^2+1)}{x+\sin x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\,\frac{(x+\sin x)\left(\log(x^2+1)\right)' - \log(x^2+1)(x+\sin x)'}{(x+\sin x)^2}$$

$$= \frac{1}{2}\,\frac{(x+\sin x)\left(\dfrac{1}{(x^2+1)\ln 10}(x^2+1)'\right) - \log(x^2+1)(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{1}{2(x+\sin x)^2}\left(\frac{(x+\sin x)(2x)}{(x^2+1)\ln 10} - \log(x^2+1)(1+\cos x)\right)$$

E $y = \cos\sqrt{\log\left(\sqrt{x^2+1}+x\right)}$. This is the composition of several functions.

$$\Rightarrow \frac{dy}{dx} = -\sin\sqrt{\log\left(\sqrt{x^2+1}+x\right)} \times \frac{1}{2\sqrt{\log\left(\sqrt{x^2+1}+x\right)}} \times \frac{1}{\left(\sqrt{x^2+1}+x\right)\ln 10} \times \left(\frac{x}{\sqrt{x^2+1}}+1\right)$$

$$-\frac{\sin\sqrt{\log\left(\sqrt{x^2+1}+x\right)}}{2\sqrt{\log\left(\sqrt{x^2+1}+x\right)}\left(\sqrt{x^2+1}+x\right)\ln 10}\left(\frac{x}{\sqrt{x^2+1}}+1\right)$$

Example 23 Find the equation of the line tangent to the graph of $y = \ln\left(\frac{x^2}{x^2+2x}\right)$ at $x = 1$.

Solution $y = \ln\left(\frac{x^2}{x^2+2x}\right) \Rightarrow y = \ln(x^2) - \ln(x^2+2x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2}(2x) - \frac{1}{(x^2+2x)}.(2x+2) = \frac{2}{x} - \frac{2x+2}{x^2+2x}$$

$$\Rightarrow \text{The gradient is } \left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{1} - \frac{2(1)+2}{1^2+2(1)} = \frac{2}{3}$$

$\Rightarrow$ The equation of the tangent line is

$$y - \ln\left(\frac{1}{1+2}\right) = \frac{2}{3}(x-1) \Rightarrow y = \frac{2}{3}x - \frac{2}{3} - \ln .$$

Exercise 3.9

1 Use the chain rule and any other appropriate rule to differentiate each of the following functions.

A $f(x) = e^{x+6}$ **B** $f(x) = (x+5)^{10}$ **C** $f(x) = (4x+5)^{12}$

D $f(x) = \sin(3x)$ **E** $f(x) = \cos((x^2+1))$ **F** $f(x) = \frac{e^{(x+2)}}{xe^x - 1}$

G $f(x) = e^{-5x}\sin(4x^2+5x+1)$ **H** $f(x) = \sqrt{x^2+2x+3}$

I $f(x) = \log_3(x^2+4)$ **J** $f(x) = \frac{x^2}{x+\ln(x^2+9)}$

K $f(x) = \frac{\sin x}{\sqrt{2x+1}}$ **L** $f(x) = \sin(x^2) + \cos(x^2)$

M $f(x) = \ln\left(\frac{1}{x^2+1}\right)$ **N** $f(x) = \ln\sqrt{x^2+1}$

O $f(x) = \sin\sqrt{\ln(x^2+7)}$ **P** $\log_a x$

Q $f(x)=e^{-\sqrt{x^2+1}}\sin\left(\sqrt{x^2+1}\right)$ **R** $f(x)=\ln\sqrt{\cos(x^2+3)}$

2 FIND THE EQUATION OF THE LINE TANGENT TO THE GRAPH OF f AT $(a, f(a))$, IF

A $f(x)=xe^{-\sqrt{x+1}}$ AT $(0, 0)$ **B** $f(x)=e^{2-x^2}$ AT $(1, e)$

C $f(x)=\ln\left(\frac{x+1}{\cos x}\right)$ AT $(0, 0)$ **D** $f(x)=\frac{e^{3x+2}}{1-2x}$ AT $\left(-1, \frac{1}{3e}\right)$

E $f(x)=(8-x^3)\sqrt{2-x}$ AT $(-2, 32)$

3 FIND $\frac{dy}{dx}$.

A $y=\sqrt{1+x^6}$ **B** $y=\sqrt{1+3x^2}\,e^x$ **C** $y=\frac{2x^3}{\sqrt{1+x^4}}$

D $y=\sqrt{\frac{x^2}{x^3+1}}$ **E** $y=\left(\frac{2x-1}{3-4x}\right)^9$ **F** $y=\cos(\ln\sqrt{e^x})$

G $y=(ax+b)^r$; WHERE r IS A REAL NUMBER.

3.3.5 Higher Order Derivatives of a Function

YOU HAVE SEEN THAT FOR A FUNCTION f, f' IS THE FIRST DERIVATIVE OR SIMPLY THE DERIVATIVE OF f. f' IS A FUNCTION WHICH ASSIGNS x TO $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$.

FOR INSTANCE, IF $f(x)=x^2+1$, THEN $f'(x)=2x$ WHICH IS A FUNCTION, TOO. THEREFORE, YOU CAN COMPUTE THE DERIVATIVE OF f' WHICH IS $(f')'(x)=(2x)'=2$.

ACTIVITY 3.8

1 LET $f(x)=x^3+4x+5$.

A FIND $f'(x)$ **B** DIFFERENTIATE $f'(x)$ WITH RESPECT TO x.

2 IF $f(x)=\begin{cases} x^2, & \text{IF } x<3, \\ 6x-9, & \text{IF } x\geq 3. \end{cases}$

A FIND $f'(x)$ **B** SKETCH THE GRAPH OF $f'(x)$

C FIND THE DERIVATIVE OF $f'(x)$ AT $x=3$.

3 LET $f(x)=x^3+1$. SKETCH THE GRAPH OF $f'(x)$ AND THE DERIVATIVE OF $f'(x)$ USING THE SAME COORDINATE SYSTEM.

4 LET $f(x)=|x|\,x$.

A FIND $f'(x)$ **B** FIND $(f'(x))'(0)$

THE SECOND DERIVATIVE OF A FUNCTION f, DENOTED BY $f''(x)$ IS THE DERIVATIVE OF THE FIRST DERIVATIVE.

I.E., $f''(x) = (f'(x))'$.

YOU SAY THAT f IS TWICE DIFFERENTIABLE OR THE SECOND DERIVATIVE OF f EXISTS AT x IF f' IS DIFFERENTIABLE PROVIDED THAT $f''(x) = \lim_{t \to x} \frac{f'(t) - f'(x)}{t - x}$ EXISTS.

Example 24 FIND THE SECOND DERIVATIVES FOR EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = x^2$ **B** $f(x) = x^3$ **C** $f(x) = \sin x$

D $f(x) = e^x$ **E** $f(x) = xe^x$ **F** $f(x) = x \sin x$

G $f(x) = e^x \sin x$

Solution

A $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2.$

B $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$

C $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x$

D $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x$

E $f(x) = xe^x \Rightarrow f'(x) = (x)'e^x + x(e^x)'$ BY THE PRODUCT RULE

$= e^x + xe^x = e^x(1 + x) \quad \Rightarrow f''(x) = (e^x)'(1 + x) + e^x(1 + x)'$

$= e^x(1 + x) + e^x = e^x(2 + x)$

F $f(x) = x \sin x \Rightarrow f'(x) = \sin x + x \cos x$

$\Rightarrow f''(x) = \cos x + \cos x - x \sin x = 2 \cos x - x \sin x$

G $f(x) = e^x \sin x \Rightarrow f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

$\Rightarrow f''(x) = (e^x)'(\sin x + \cos x) + e^x(\sin x + \cos x)'$

$= e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= e^x(\sin x + \cos x + \cos x - \sin x) = 2e^x \cos x.$

Note:

IF $y = f(x)$, THEN $\frac{dy}{dx} = f'(x)$ SO THAT $f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ IS DENOTED BY $\frac{d^2y}{dx^2}$. I.E., $\frac{d^2y}{dx^2} = f''(x)$ ALSO, $\frac{d^2}{dx^2} f(x) = f''(x)$.

Example 25 FOR EACH OF THE FOLLOWING, FIND $\frac{d^2y}{dx^2}$

A $y = x^4$	**B** $y = \sqrt{x}$	**C** $y = \frac{x}{x+1}$
D $y = \frac{3}{\sqrt{x-1}}$	**E** $y = \sqrt{e^{4x+3}}$	**F** $y = e^{-x^2+2x+1}$
G $y = \text{LN}\,x$	**H** $y = \frac{x+1}{x^2+1}$	**I** $y = \frac{\text{SIN}\,x}{\sqrt{x}}$
J $y = \frac{x}{\sqrt{x^2+1}}$	**K** $y = \frac{x^2}{x+\text{LN}\,x}$	**L** $y = \text{LN}\left(\frac{x}{\text{COS}\,x}\right); 0 < x < \frac{}{2}$

Solution

A $y = x^4 \quad \Rightarrow \frac{d}{dx}\left(x^4\right) = 4x^3 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(4x^3\right) = 12x^2.$

B $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$

. $\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = -\frac{1}{4}x^{\frac{-3}{2}} = -\frac{1}{4x\sqrt{x}}.$

C $y = \frac{x}{x+1} \Rightarrow \frac{dy}{dx} = \frac{(x)'(x+1) - x(x+1)'}{(x+1)^2}.$ *Quotient Rule*

$= \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}((x+1)^{-2}) = -2(x+1)^{-3} = -\frac{2}{(x+1)^3}$

D $y = \frac{3}{\sqrt{x-1}} = 3(x-1)^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{1}{2}\right)(x-1)^{-\frac{3}{2}} = -\frac{3}{2}(x-1)^{-\frac{3}{2}}$

$\Rightarrow \frac{d^2Y}{dx^2} = \frac{d}{dx}\left(-\frac{3}{2}(x-1)^{-\frac{3}{2}}\right) = -\frac{3}{2}\left(-\frac{3}{2}(x-1)^{-\frac{5}{2}}\right)$

$= \frac{9}{4}(x-1)^{-\frac{5}{2}} = \frac{9}{4(x-1)^2\sqrt{x-1}}.$

E $y = \sqrt{e^{4x+3}} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{e^{4x+3}}} \times e^{4x+3} \times 4 = \frac{2e^{4x+3}}{\sqrt{e^{4x+3}}} = 2e^{4x+3}\frac{\sqrt{e^{4x+3}}}{e^{4x+3}} = 2\sqrt{e^{4x+3}}$

AlSO, $y = \sqrt{e^{4x+3}} = e^{\frac{4x+3}{2}}$

$\Rightarrow \frac{dy}{dx} = e^{\frac{4x+3}{2}} \times \frac{d}{dx}\left(\frac{4x+3}{2}\right) = e^{\frac{4x+3}{2}} \times 2 = 2\sqrt{e^{4x+3}}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(2e^{\frac{4x+3}{2}}\right) = 2e^{\frac{4x+3}{2}} \times \frac{d}{dx}\left(\frac{4x+3}{2}\right)$$

$$= 2e^{\frac{4x+3}{2}} \times 2 = 4e^{\frac{4x+3}{2}} = 4\sqrt{e^{4x+3}}$$

F $$y = e^{-x^2+2x+1} \Rightarrow \frac{dy}{dx} = e^{-x^2+2x+1}\frac{d}{dx}\left(-x^2+2x+1\right)$$

$$= e^{-x^2+2x+1}\left(-2x+2\right)$$

$$\frac{d^2y}{dx^2} = \left(e^{-x^2+2x+1}\left(-2x+2\right)\right)' = \left(e^{-x^2+2x+1}\right)'\left(-2x+2\right) + e^{-x^2+2x+1}\left(-2x+2\right)'$$

$$= e^{-x^2+2x+1}\left(-2x+2\right)^2 + e^{-x^2+2x+1}\left(-2\right) = e^{-x^2+2x+1}\left(4x^2-8x+2\right)$$

$$= e^{-x^2+2x+1}\left(\left(2-2x\right)^2-2\right)$$

G $$y = \mathrm{LN}\,x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

H $$y = \frac{x+1}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{x^2+1-(x+1)(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{1-x^2-2x}{(x^2+1)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$$

I $$y = \frac{\mathrm{SIN}\,x}{\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}\,(\mathrm{SIN}\,x)' - \mathrm{SIN}\,x\left(\sqrt{x}\right)'}{\left(\sqrt{x}\right)^2} = \frac{\sqrt{x}\,\mathrm{COS}\,x - \mathrm{SIN}\,x\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x}{2x\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = \left(\frac{2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x}{2x\sqrt{x}}\right)' = \frac{1}{2}\left(\frac{2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x}{x^{\frac{3}{2}}}\right)'$$

$$= \frac{(2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x)'\,x^{\frac{3}{2}} - \left(x^{\frac{3}{2}}\right)'(2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x)}{2\left(x^{\frac{3}{2}}\right)^2}$$

$$= \frac{(2\,\mathrm{COS}\,x - 2x\,\mathrm{SIN}\,x - \mathrm{COS}\,x)\,x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}(2x\,\mathrm{COS}\,x - \mathrm{SIN}\,x)}{2x^3}$$

$$= \frac{(\cos x - 2x\sin x)x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}\,x(2\cos x - x\sin x)}{2x^3}$$

J $\quad y = \dfrac{x}{\sqrt{x^2+1}} \Rightarrow \dfrac{dy}{dx} = \dfrac{(x)'\sqrt{x^2+1} - x\left(\sqrt{x^2+1}\right)'}{\left(\sqrt{x^2+1}\right)^2} = \dfrac{\sqrt{x^2+1} - x\dfrac{1}{2\sqrt{x^2+1}}\times 2x}{x^2+1}$

$$\frac{(x^2+1)-x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}} = (x^2+1)^{-\frac{5}{2}}$$

$$\frac{d^2y}{dx^2} = \left((x^2+1)^{-\frac{3}{2}}\right)' = -\frac{3}{2}(x^2+1)^{-\frac{5}{2}}\times 2x$$

$$= \frac{-3x}{(x^2+1)^{\frac{5}{2}}} = \frac{-3x}{(x^2+1)^2\sqrt{x^2+1}}$$

K $\quad y = \dfrac{x^2}{x+\ln x} \Rightarrow \dfrac{dy}{dx} = \dfrac{(x^2)'(x+\ln x) - x^2(x+\ln x)'}{(x+\ln x)^2} = \dfrac{2x(x+\ln x) - x^2\left(1+\dfrac{1}{x}\right)}{(x+\ln x)^2}$

$$= \frac{2x^2+2x\ln x - x^2 - x}{(x+\ln x)^2} = \frac{x^2+2x\ln x - x}{(x+\ln x)^2}$$

$$\frac{d^2y}{dx^2} = \left(\frac{x^2+2x\ln x - x}{(x+\ln x)^2}\right)'$$

$$= \frac{(x^2+2x\ln x - x)'(x+\ln x)^2 - \left((x+\ln x)^2\right)'(x^2+2x\ln x - x)}{(x+\ln x)^4}$$

$$= \frac{\left(2x+2\ln x+2x\left(\frac{1}{x}\right)-1\right)(x+\ln x)^2 - 2(x+\ln x)\left(1+\frac{1}{x}\right)(x^2+2x\ln x - x)}{(x+\ln x)^4}$$

$$= \frac{(2x+2\ln x+1)(x+\ln x) - \left(2+\frac{1}{x}\right)(x^2+2x\ln x - x)}{(x+\ln x)^3}$$

L $\quad y = \ln\left(\dfrac{x}{\cos x}\right) = \ln(x) - \ln(\cos x)$ because $0 < x < \dfrac{\pi}{2}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\ln x - \ln(\cos x)) = \frac{1}{x} - \frac{1}{\cos x}\times(-\sin x) = \frac{1}{x} + \tan x$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} + \sec^2 x$$

SIMILARLY, WE DEFINE THE THIRD, FOURTH, ETC. DERIVATIVES OF A FUNCTION f AS FOLLOWS:

THE THIRD DERIVATIVE OF A FUNCTION f IS THE DERIVATIVE OF THE SECOND DERIVATIVE. I.E.,

$$f'''(x) = ((f''(x))'$$

ALSO, THE FOURTH DERIVATIVE OF A FUNCTION f IS THE DERIVATIVE OF THE THIRD DERIVATIVE.

IN GENERAL, FOR $n \geq 2$, THE n^{TH} DERIVATIVE OF f, DENOTED BY $f^{(n)}(x)$ IS DEFINED AS

$$f^{(n)}(x) = \lim_{t \to x} \frac{f^{(n-1)}(t) - f^{(n-1)}(x)}{t - x}$$

IF THIS LIMIT EXISTS, THEN WE SAY THAT f IS n-TIMES DIFFERENTIABLE IF THE n^{th} DERIVATIVE OF $f(x)$ EXISTS.

Example 26 FIND THE FOURTH DERIVATIVE OF

A $f(x) = x^4 - 5x^3 + 6x^2 + 7x + 1$ **B** $f(x) = \sin x$

Solution

A $f(x) = x^4 - 5x^3 + 6x^2 + 7x + 1$

$\Rightarrow f'(x) = (x^4 - 5x^3 + 6x^2 + 7x + 1)' = 4x^3 - 15x^2 + 12x + 7$

$\Rightarrow f''(x) = (4x^3 - 15x^2 + 12x + 7)' = 12x^2 - 30x + 12$

$\Rightarrow f^{(3)}(x) = (12x^2 - 3x + 12)' = 24x - 30$

$\Rightarrow f^{(4)}(x) = 24$

NOTE THAT, FOR $n \geq 5$, $f^{(n)}(x) = 0$

B $f(x) = \sin x$,

$\Rightarrow f'(x) = \cos x \quad \Rightarrow f''(x) = -\sin x$

$\Rightarrow f'''(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x$.

NOTATION:

IF $y = f(x)$, THEN, WE WRITE $f^{(n)}(x) = \dfrac{d^n y}{dx^n} = \dfrac{d^n}{dx^n} f(x) = D^n f(x)$

Example 27 LET $y = xe^x$. FIND $\dfrac{d^n y}{dx^n}$.

Solution $y = xe^x \Rightarrow \dfrac{dy}{dx} = (x)' e^x + x(e^x)' = e^x + xe^x = e^x(1+x)$

$$\frac{d^2y}{dx^2} = (e^x(1+x))' = (e^x)'(1+x) + e^x(1+x)' = e^x(1+x) + e^x = e^x(2+x)$$

$$\frac{d^3y}{dx^3} = (e^x(2+x))' = (e^x)'(2+x) + e^x(2+x)' = e^x(2+x) + e^x = e^x(3+x)$$

FROM THIS PATTERN WE CONCLUDE THAT, $\dfrac{d^n y}{dx^n} = e^x(n+x)$

Example 28 LET f BE AN n-TIMES DIFFERENTIABLE FUNCTION. IF $g(x) = f(3x + 1)$, FIND $g^{n}(x)$.

Solution $g(x) = f(3x + 1)$

$$\Rightarrow g'(x) = f'(3x + 1) \cdot (3x + 1)' \quad \textit{by chain rule.}$$

$$= 3f'(3x + 1)$$

$$\Rightarrow G''(x) = 3f''(3x + 1)(3x + 1)' = 3f''(3x + 1) \times 3 = 3^2 f''(3x + 1)$$

$$g^{(3)}(x) = 3^2 f^{(3)}(3x + 1)(3x+1)' = 3^2 f^{(3)}(3x + 1) \times 3$$

$$= 3^3 f^{(3)}(3x + 1)$$

FROM THIS PATTERN ONE CAN SEE THAT $g^{(n)}(x) = 3^n f^{(n)}(3x + 1)$.

Example 29 LET $f(x) = |x|x^2$. FIND THE THIRD DERIVATIVE OF f.

Solution

$$f(x) = |x|\,x^2 = \begin{cases} x^3, \text{ IF } x \geq 0 \\ -x^3, \text{ IF } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2, \text{ IF } x \geq 0 \\ -3x^2, \text{ IF } x < 0 \end{cases}$$

$$\Rightarrow f''(x) = \begin{cases} 6x, \text{ IF } x \geq 0 \\ -6x, \text{ IF } x < 0 \end{cases} \Rightarrow f^{(3)}(x) = \begin{cases} 6, \text{ IF } x > 0 \\ \nexists, \text{ IF } x = 0 \\ -6, \text{ IF } x < 0 \end{cases}$$

$$f^{(3)}(0) = \lim_{x \to 0} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x \to 0} \frac{f''(x)}{x}$$

BUT, $\lim_{x \to 0^+} \frac{f''(x)}{x} = \lim_{x \to 0^+} \frac{6x}{x} =$ AND $\lim_{x \to 0^-} \frac{f''(x)}{x} = \lim_{x \to 0^-} \frac{6x}{x} = -$

$$\Rightarrow f^{(3)}(0) = \text{DOESN'T EXIST.}$$

THIS IS AN EXAMPLE OF A FUNCTION WHICH IS TWICE DIFFERENTIABLE AT 0 BUT IT IS NO TIMES DIFFERENTIABLE AT 0.

Example 30 LET $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

BE A POLYNOMIAL FUNCTION OF DEGREE n. SHOW THAT $f^{(k)}(x) = 0$ FOR ALL $k > n$.

Solution $f'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x$

$$f''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + 2a_2$$

$$f^{(3)}(x) = n(n-1)(n-2)a_n x^{n-3} + (n-1)(n-2)(n-3)x^{n-3} + \ldots + 6a_3$$

.
.
.

$$f^{(n-1)}(x) = n(n-1)(n-2)\ldots(n-1)a_n x$$

$$f^{(n)}(x) = n!\, a_n$$

$$\Rightarrow f^{(n+1)}(x) = 0.$$

Exercise 3.10

1 FIND THE SECOND DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = 3x - 9$ **B** $f(x) = 4x^3 - 6x^2 + 7x + 1$

C $f(x) = \sqrt{x} + \sin x$ **D** $f(x) = x\sqrt{x} + \sin x$

E $f(x) = \dfrac{\sin x}{x+1}$ **F** $f(x) = \ln(x^2 + 1)$

G $f(x) = \dfrac{x^2 - 4}{x+1}$ **H** $f(x) = \sec x$

I $f(x) = \dfrac{x^2}{\sqrt{4-x^2}}$

2 FOR EACH OF THE FOLLOWING, FIND $\dfrac{d^2y}{dx^2}$

A $y = e^{3x+2}$ **B** $y = \log_3\left(\sqrt{x+1}\right)$ **C** $y = \ln\left(\dfrac{1}{x^2+1}\right)$

D $y = \cos^2(2x+1)$ **E** $y = (\ln x)^3$ **F** $y = \ln(1 - x^3)$

G $y = \ln\left(\dfrac{x}{\sqrt{x+2}}\right)$ **H** $y = e^{-\sqrt{x}} \sin\sqrt{x}$ **I** $y = \sin(2x \cos x)$

J $y = \ln(\ln x)$ **K** $y = (x+1)\sqrt{x^2+1}$

3 FIND A FORMULA FOR THE n^{TH} DERIVATIVE OF EACH OF THE FOLLOWING FUNCTIONS FOR THE GIVEN VALUES OF n.

A $f(x) = e^{(3x+1)}$; $n \in \mathbb{N}$ **B** $f(x) = e^{x^2}$; $n = 6$

C $f(x) = \ln\left(\dfrac{1}{x^2+1}\right)$; $n = 4$ **D** $f(x) = e^{-x^2+7x-3}$; $n = 4$

Key Terms

chain rule	gradient	rate of change	slope
derivative	product rule	rules	tangent
differentiation	quotient rule	secant	work

Summary

1 The slope (gradient) of the graph of $y = f(x)$ at point P.

I A LINE WHICH TOUCHES A (CONTINUOUS) GRAPH AT EXACTLY ONE POINT IS SAID TO BE A TANGENT LINE AT THAT POINT, CALLED THE POINT OF TANGENCY.

II THE SLOPE OF THE GRAPH OF $y = f(x)$ AT A POINT IS THE SLOPE OF THE TANGENT LINE AT THE POINT OF TANGENCY.

III THE SLOPE OF $y = f(x)$ AT $(a, f(a))$ IS $m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

IV THE EQUATION OF THE TANGENT LINE AT $(a, f(a))$ IS $y - f(a) = m_a(x - a)$.

2 Differentiation of a function at a point

I *The Derivative*

LET x_o BE IN THE DOMAIN OF A FUNCTION f. THEN $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

ALSO, $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

II *Other notation*

SOME OF THE OTHER NOTATIONS FOR DERIVATIVES ARE

$$\frac{dy}{dx}, \ \frac{d}{dx} f(x), D(f(x))$$

IF $y = f(x)$, THEN THE DERIVATIVE AT POINT a, $f'(a)$ IS ALSO DENOTED BY

$$\left.\frac{dy}{dx}\right|_{x=a}$$

3 Differentiability on an interval

A FUNCTION f IS DIFFERENTIABLE ON AN OPEN INTERVAL (a, b), IF f IS DIFFERENTIABLE AT EACH POINT ON (a, b). f IS DIFFERENTIABLE ON THE CLOSED INTERVAL $[a, b]$, IF IT IS DIFFERENTIABLE ON (a, b) AND THE ONE SIDE LIMIT

$\lim_{x \to a^+} f(x)$ AND $\lim_{x \to b^-} f(x)$ BOTH EXIST.

4 The Derivatives of some functions

I *The Derivative of a constant function is 0.*

$$\frac{d}{dx}(c) = 0$$

II *The Derivative of the power function*

IF $f(x) = x^r$, THEN $f'(x) = rx^{r-1}$

III ***The Derivative of simple trigonometric functions***

- ✓ IF $f(x) = \sin x$, THEN $f'(x) = \cos x$
- ✓ IF $f(x) = \cos x$, THEN $f'(x) = -\sin x$

IV ***The Derivatives of exponential functions***

- ✓ IF $f(x) = e^x$, THEN $f'(x) = e^x$
- ✓ IF $f(x) = a^x;\ a > 0$, THEN $f'(x) = a^x \ln a$

V ***The Derivatives of logarithmic functions***

- ✓ IF $f(x) = \ln x$, THEN $f'(x) = \dfrac{1}{x}$
- ✓ IF $f(x) = \log_a x$; $a > 0$ AND $a \neq 1$, THEN $f'(x) = \dfrac{1}{x \ln a}$.

5 Derivatives of combinations of functions

LET f AND g BE DIFFERENTIABLE FUNCTIONS.

I THE DERIVATIVES OF A SUM OR A DIFFERENCE.

- ✓ ***The sum rule***

  $(f + g)'(x) = f'(x) + g'(x)$

- ✓ ***The difference rule***

  $(f - g)'(x) = f'(x) - g'(x)$

II THE DERIVATIVES OF PRODUCTS AND QUOTIENTS.

- ✓ ***The product rule***

  $(fg)'(x) = f'(x)\, g(x) + g'(x)\, f(x)$

- ✓ ***The quotient rule***

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

6 Differentiation of compositions of functions

The Chain Rule

LET f AND g BE DIFFERENTIABLE FUNCTIONS. THEN,

$(f \circ g)'(x) = f'(g(x)).\, g'(x)$.

IF u IS A FUNCTION OF x, $y = f(u)$, $u = g(x)$, THEN

$$\frac{dy}{dx} = (f \circ g)'(x) = \frac{dy}{du}.\frac{du}{dx}$$

I $\dfrac{d}{dx} u^n = nu^{n-1}\dfrac{du}{dx}$ **II** $\dfrac{d}{dx} \sin u = \cos u \dfrac{du}{dx}$

III $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$ **IV** $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

V $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$ **VI** $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

VII $\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$

7 *Higher Derivatives*

I ***The second Derivative***

$$f''(x) = \lim_{t\to x}\frac{f'(t)-f'(x)}{t-x}$$

II ***The nth Derivatives; n ≥ 3***

$$f^{(n)}(x) = \lim_{t\to x}\frac{f^{(n-1)}(t)-f^{(n-1)}(x)}{t-x}$$

If $y = f(x)$, then, $\frac{d^2y}{dx^2} = f''(x)$; $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

$$\frac{d^ny}{dx^n} = f^{(n)}(x).$$

Review Exercises on Unit 3

In Exercises 1 – 8, find the difference quotient of f at a.

1 $f(x) = 4x + 3;\ a = -2$ **2** $f(x) = 2x^2 + 1;\ a = -1$ **3** $f(x) = \frac{x+1}{x-2};\ a = -2$

4 $f(x) = \frac{x+1}{x^2};\ a = -\frac{1}{2}$ **5** $f(x) = |x+4|;\ a = -4$ **6** $f(x) = \sqrt{x} + 5;\ a = \frac{9}{4}$

7 $f(x) = 2^x;\ a = 0$ **8** $f(x) = \sqrt{1-3x^2};\ a = \frac{\sqrt{3}}{4}$

In Exercises 9 – 53, find the derivative of the expression with respect to x.

9 9 **10** $^2 + \frac{1}{\sqrt{2}}$ **11** $x^2 - 3x + 1$

12 $4x^2 - 8x$ **13** $x^4 - 7x^3 + 2$ **14** $(x-5)(3x+4)$

15 $(x-3)^2$ **16** $(5x+1)(5x-1)(x-5)$ **17** $4x^3 - x^{\frac{1}{3}} + \sqrt{x} + 1$

18 $3^{(x-2)} + \sqrt{x} + 5x^2 - \frac{1}{x}$ **19** $e^{-x} + e^x$ **20** $\sin(4x)$

21 $\cos(x^2 + 4)$ **22** $\tan(6x - 1)$ **23** $\ln(7x + 3)$

24 $\frac{x^2+4}{x}$ **25** $x - 2(x+1)^2$ **26** $\frac{x^3-5x+3}{x^4}$

27 $5x(x+1)$ **28** $1 + x^{-1} + x^{-2} + x^{-3}$ **29** $\frac{x-1}{x\sqrt{x}}$

30 $\sqrt{1-3x^2}$ **31** $\frac{x^2+1}{\log_2 x}$ **32** e^{4-x^2}

33 $\frac{1}{x^{-\frac{1}{3}}} + \sqrt[3]{x^2} + \sqrt[4]{x^3}$ **34** $x\,e^{1-x}$ **35** $x^{-2}(e^x+1)$

36 $(\ln x)(x^2+1)$ **37** $(2x+1)^4$ **38** $\frac{(x-1)^3}{\sqrt{x}}$

39 $x\ln x - x$ **40** $\log\sqrt{x^2+2}$ **41** $\sqrt{\ln x} + \sqrt{e^x} + 2^x$

42 $\sqrt{\log\sqrt{x}}$ **43** $e^x\cos x$ **44** $\left(\frac{1}{x\ \sin\ x}\right)^{\frac{5}{3}}$

45 $\cos\sqrt{\ln(x^2 +}$ **46** $\tan\left(\frac{x^2-1}{x}\right)$ **47** $\sec^2(x+3)$

48 $x^{-2}(\sin(x^2))$ **49** $\frac{x^3-4x+5}{x^2+1}$ **50** $\frac{e^x\sin x}{\ln x}$

51 $x\sqrt{1-(2+x)^{\frac{3}{2}}}$ **52** $e^{\sin\sqrt{x+3}}$ **53** $e^x\sin x$

54 FOR EACH OF THE FOLLOWING, FIND $f'(x)$.

A $f(x) = \begin{cases} x^3, & \text{IF } x \geq 0 \\ x^2, & \text{IF } x < 0 \end{cases}$

B $f(x) = \begin{cases} \frac{1}{2^x+1}, & \text{IF } x \leq 1 \\ \frac{1}{x+2}, & \text{IF } x > 1 \end{cases}$

C $f(x) = \begin{cases} \log_2 \frac{1}{x^2+1}, & \text{IF } x < \\ \log \frac{1}{x+3}, & \text{IF } x \geq 3 \end{cases}$

FIND THE GRADIENT (SLOPE) OF THE GIVEN CURVE AT THE GIVEN POINT IN EXERCISES 55 – 67 BELOW.

55 $f(x) = x^2 - 5x + 1;\ x = 1$ **56** $f(x) = x\sqrt{x+1}\ ;\ x = 0$

57 $f(x) = \frac{x}{x+2};\ x = 1$ **58** $f(x) = (x^2-1)\sqrt{x}\ ;\ x = 2$

59 $f(x) = x^2 + 5x + 4;\ x = -2.5$ **60** $f(x) = x^3 - 3x + 1;\ x = 2$

61 $f(x) = \frac{3x-1}{(x-1)^2};\ x = \frac{1}{2}$ **62** $f(x) = x^2 + \frac{2}{x^2}\ ;\ x = \sqrt{2}$

63 $f(x) = e^{\sqrt{x^2+1}}\ ;\ x = \sqrt{3}$ **64** $f(x) = \ln\left(x+\sqrt{x^2+1}\right);\ x = 1$

65 $f(x) = \cos(4x + 1)$; $x = \frac{-1}{4}$

66 $f(x) = |3x - 2|$; $x = \frac{2}{3}$

67 $f(x) = \begin{cases} x^3, \text{ if } x \le -1 \\ 3x + 2, \text{ if } x > -1 \end{cases}$; $x = -1$

For **Exercises 68 – 79**, find the equation of the line tangent to the curve at the given point.

68 $f(x) = x^2 - 2x + 3$; $x = -1$

69 $f(x) = x\sqrt{x}$; $x = 1$

70 $f(x) = \frac{x}{x^2 + 1}$; $x = 2$

71 $f(x) = \sqrt{4 - 3x}$; $x = -4$

72 $f(x) = \sin x$; $x = \frac{\pi}{3}$

73 $f(x) = 3 - |x - 1|$; $x = 1$

74 $f(x) = \sqrt{9 - x^2}$; $x = 2$

75 $f(x) = \log(x + 3)$; $x = 7$

76 $f(x) = e^{x+1}$; $x = -2$

77 $f(x) = \frac{\ln x}{x}$; $x = e$

78 $f(x) = \frac{1}{(x-2)^2}$; $x = 1$

79 $f(x) = \frac{e^x \sin x}{e^x + 1}$; $x = 0$

80 Find the equation of the line tangent to the graph of $y = x^2 - 5x + 1$ at the point where the curve crosses the line $y = 7$.

81 Find the equation of the tangent to the curve $y = \frac{1}{x} + x^2$ which has a slope of -3.

82 Find the value of k so that the line $y = -8x + k$ is tangent to the curve $y = 3x^2 - 4x + 1$.

In **Exercises 83 – 96**, find $\frac{d^2y}{dx^2}$.

83 $y = x^2$

84 $y = x^9$

85 $y = (x^2 + 5)^7$

86 $y = e^{1-x}$

87 $y = \frac{1}{4}x^7 - 2x^3 + x^2 - 1$

88 $y = \frac{1}{\sqrt{x}}$

89 $y = \ln(x^2 + 1)$

90 $y = \sin(x - \cos(x))$

91 $y = e^x \cos x$

92 $y = e^{-2x} \cos x$

93 $y = \frac{2x - 3}{2x + 3}$

94 $y = \frac{x^2 + 8}{x + 1}$

95 $y = (\sqrt{x + 3} + 5)^{10}$

96 $y = (x^2 + 1)\sin(4x + 5)$

97 If $y = x^3 e^{-x}$, find $\frac{d^3y}{dx^3}$

98 If $f(x) = e^x \ln x$, evaluate $f'''(1)$.