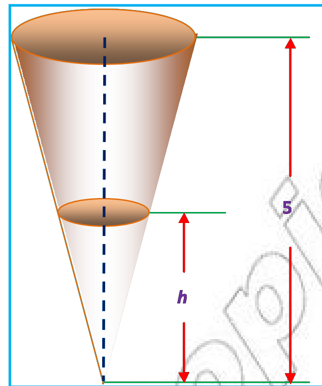
Unit 4



APPLICATIONS OF DIFFERENTIAL CALCULUS

Unit Outcomes:

After completing this unit, you should be able to:

- *find local maximum or local minimum value of a function on a given interval.*
- *find absolute maximum or absolute minimum value of a function on a given interval.*
- *apply the mean– value theorem.*
- *solve simple problems in which the studied theorems, formulae and procedures of differential calculus are applied.*
- *solve application problems.*

Main Contents

INTRODUCTION

IN UNIT 3 YOU HAVE STUDIED DERIVATIVES AND HAVE DEVELOPED METHODS TO FIND DERIVATIVES. DERIVATIVES CAN HAVE DIFFERENT INTERPRETATIONS IN EACH OF THE SCIENCES (NATURAL

FOR INSTANCE; THE VELOCITY OF A PARTICLE IS THE RATE OF CHANGE OF DISPLACEMENT W TO TIME. CHEMISTS WHO STUDY A CHEMICAL REACTION MAY BE INTERESTED IN THE RATE C IN THE CONCENTRATION OF A REACTANT WITH RESPECT TO TIME CALLED THE RATE OF REA MANUFACTURER IS INTERESTED IN THE RATE OF CHANGE OF THE COST OF PRODUCING x UNITS OF THE PER DAY WITH RESPECT TO x (CALLED THE MARGINAL COST). A BIOLOGIST IS INTERESTED IN THE RA OF CHANGE OF THE POPULATION OF A COLONY OF BACTERIA WITH RESPECT TO TIME. IN COMPUTATION OF RATES OF CHANGE IS IMPORTANT IN ALL OF THE NATURAL SCIENCES, IN E AND EVEN IN THE SOCIAL SCIENCES. ALL THESE RATES OF CHANGE CAN BE INTERPRETED A TANGENTS. THIS GIVES ADDED SIGNIFICANCE TO THE SOLUTION OF THE TANGENT P WHENEVER WE SOLVE A PROBLEM INVOLVING TANGENT LINES, WE ARE NOT JUST SOLVING IN GEOMETRY. WE ARE ALSO IMPLICITLY SOLVING A GREAT VARIETY OF PROBLEMS INVOL OF CHANGE IN SCIENCE AND ENGINEERING.

ONCE YOU HAVE DEVELOPED THE PROPERTIES OF THE MATHEMATICAL CONCEPT ONCE AND F CAN THEN TURN AROUND AND APPLY THESE RESULTS TO ALL OF THE SCIENCES. THIS IS M EFFICIENT THAN DEVELOPING PROPERTIES OF SPECIAL CONCEPTS IN EACH SEPARATE SCI FRENCH MATHEMATICIAN JOSEPH FOURIER (1768 – 1830) PUT IT BRIEFLY: "MATHEMATICS COMPARES THE MOST DIVERSE PHENOMENA AND DISCOVERS THE SECRET ANALOGIES T THEM."

YOU HAVE ALREADY INVESTIGATED SOME OF THE APPLICATIONS OF DERIVATIVES, BUT NOV KNOW THE DIFFERENTIATION RULES, YOU ARE IN A BETTER POSITION TO PURSUE THE APPL DIFFERENTIATION IN GREATER DEPTH. YOU WILL LEARN HOW DERIVATIVES AFFECT THE S GRAPH OF A FUNCTION AND, IN PARTICULAR, HOW THIS HELPS YOU LOCATE MAXIMU MINIMUM VALUES OF FUNCTIONS. MANY PRACTICAL PROBLEMS REQUIRE US TO MINIMIZE A MAXIMIZE AN AREA OR SOMEHOW FIND THE BEST POSSIBLE OUTCOME OF A SITUATION.

OPENING PROBLEM

A SQUARE SHEET OF CARDBOARD WHOSE AREA IS 12 M^2 IS USED TO MAKE AN OPEN BOX BY CUTTING SQUARES OF EQUAL SIZE FROM THE FOUR CORNERS AND FOLDING UP THE SIDES. WHAT SIZE SHOULD BE CUT TO OBTAIN A BOX WITH LARGEST POSSIBLE VOLUME?

4.1 EXTREME VALUES OF FUNCTIONS

4.1.1 Revision on Zeros of Functions

THE FUNDAMENTAL THEOREM OF ALGEBRA STATES THAT EVERY n^{th} DEGREE POLYNOMIAL HAS AT MOST n REAL ZEROS. THE PROBLEM OF FINDING ZEROS OF A POLYNOMIAL IS EQUIVALENT TO THE PR FACTORIZING THE POLYNOMIAL INTO LINEAR OR QUADRATIC FACTORS. IN THE EARLIER GRA STUDIED HOW TO FIND THE ZEROS OF A FUNCTION, TO REFRESH YOUR MEMORY, CONSI FOLLOWING REVISION QUESTIONS.

NOTE THAT A NUMBER c IS A ZERO OF A FUNCTION f, IF AND ONLY IF $f(c) = 0$.

Revision Exercises

1 FIND THE REAL ZEROS OF EACH OF THE FOLLOWING FUNCTIONS.

A $f(x) = 3x - 2$ **B** $f(x) = x^3 - 8$ **C** $f(x) = x^3 + 8$

D $g(x) = \dfrac{1 - \sqrt{x}}{(x+1)^2}$ **E** $g(x) = \sqrt{x-1} + x - 1$

F $h(x) = 7x^2 - 51x + 14$ **G** $h(x) = \dfrac{x^2 - 8x + 7}{x^2 + 1}$

2 FIND x-INTERCEPT(S) OF THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS

A $y = 3 - 2x$ **B** $y = \dfrac{x-1}{3x+1}$ **C** $y = \sqrt{1-x}$

D $y = x^2 - 4$ **E** $y = \dfrac{x^2 + x - 6}{x^2 + 4}$ **F** $y = x^4 + 1$

G $y = x^2 + 1$

3 EXPLAIN WAYS OF FINDING ZEROS OF FUNCTIONS. CONSIDER PARTICULAR CASES SUCH AS LINEAR AND QUADRATIC FUNCTIONS AND OTHER POLYNOMIALS.

4.1.2 Critical Numbers and Critical Values

Maximum and minimum values of functions

ONE OF THE PRINCIPAL GOALS OF CALCULUS IS TO INVESTIGATE THE BEHAVIOR OF VARIOUS FUNCTIONS. AS PART OF THIS INVESTIGATION, YOU WILL BE LAYING THE GROUNDWORK FOR SOLVING A OF PROBLEMS THAT INVOLVE FINDING THE MAXIMUM OR MINIMUM VALUE OF A FUNCTIO EXISTS. SUCH PROBLEMS ARE OFTEN CALLED OPTIMIZATION PROBLEMS. YOU WILL BE INTI SOME USEFUL TERMINOLOGY, BUT BEFORE THAT DO THE FOLLOWING ACTIVITY

ACTIVITY 4.1

1 GIVEN A SET $S = \{0, 1, 2, 3, 4, 5\}$ AND $f(x) = 2x + 3$

A FIND $S' = \{f(x) \mid x \in S\}$ **B** WHAT IS THE LARGEST ELEMENT OF S'?

C WHAT IS THE SMALLEST ELEMENT OF S'?

2 GIVEN A SET S AS AN OPEN INTERVAL $S = (0, 5)$ AND $f(x) = 2x + 3$

A FIND $S' = \{f(x) \mid x \in S\}$ **B** CAN WE LIST ALL THE ELEMENTS OF S'?

C CAN YOU GUESS THE LARGEST ELEMENT OF S'?

D CAN YOU GUESS THE SMALLEST ELEMENT OF S'?

3 GIVEN A SET S AS A CLOSED INTERVAL $S = [0, 5]$ AND $f(x) = 2x + 3$

A FIND $S' = \{f(x) \mid x \in S\}$ **B** CAN WE LIST ALL THE ELEMENTS OF S'?

C CAN YOU GUESS THE LARGEST ELEMENT OF S'?

D CAN YOU GUESS THE SMALLEST ELEMENT OF S'?

4 LET $f(x) = \frac{1}{x}$. FIND THE LARGEST AND SMALLEST ELEMENTS, IF EACH ONE OF THEM EXISTS, ON THE FOLLOWING INTERVALS.

A $(0, 5)$ **B** $(0, 5]$

C $[0, 5)$ **D** $[-1, 5]$

NOW THE ABOVE DISCUSSION LEADS TO THE FOLLOWING DEFINITION.

Definition 4.1

LET f BE A FUNCTION DEFINED ON SET S.

IF FOR SOME c IN S

$f(c) \ge f(x)$ FOR EVERY x IN S, THEN $f(c)$ IS CALLED AN **absolute maximum** OF f ON S.

IF $f(c) \le f(x)$ FOR EVERY x IN S, THEN $f(c)$ IS CALLED AN **absolute minimum** OF f ON S.

THE ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM OF f ARE CALLED **extreme values** OR THE **absolute extreme values** OF f ON S.

SOMETIMES WE JUST USE THE TERMS MAXIMUM AND MINIMUM INSTEAD OF ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM, IF THE CONTEXT IS CLEAR.

NOTE THAT FROM DEFINITION 4.1 AND ACTIVITY 4.1, A FUNCTION DOES NOT NECESSARILY HAVE EXTREME VALUES ON A GIVEN SET.

FOR INSTANCE,

1 $f(x) = 2x + 3$ WHICH IS CONTINUOUS ON $(0, 5)$ HAS NO MAXIMUM VALUE AND MINIMUM VALUE (*See* ACTIVITY 4.1 *above*).

2 $f(x) = \frac{1}{x}$ IS NOT CONTINUOUS ON $[-1, 5]$ AND HAS NO MAXIMUM AND MINIMUM VALUE.

3 $f(x) = 2x + 3$ HAS A MAXIMUM VALUE ON $(0, 5]$ WHICH IS 13 BUT HAS NO MINIMUM VALUE.

4 $f(x) = 2x + 3$ HAS A MINIMUM VALUE ON $[0, 5)$ WHICH IS 3 BUT HAS NO MAXIMUM VALUE ON $[0, 5)$.

AT THIS POINT ONE CAN ASK HOW ONE CAN BE SURE WHETHER A GIVEN FUNCTION f HAS MAXIMUM AND MINIMUM VALUES ON A GIVEN INTERVAL.

ACTUALLY, IF A FUNCTION f IS CONTINUOUS ON A CLOSED BOUNDED INTERVAL, IT CAN BE SHOWN T BOTH THE ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM MUST OCCUR. THIS RESULT, CA extreme value theorem, PLAYS AN IMPORTANT ROLE IN THE APPLICATION OF DERIVATIVES.

Extreme-value theorem

LET A FUNCTION f BE CONTINUOUS ON A CLOSED, BOUNDED INTERVAL $[a, b]$. THEN f HAS BOTH THE ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM VALUES ON $[a, b]$.

TO ILLUSTRATE THIS THEOREM, LET'S CONSIDER THE FOLLOWING GRAPH OF A FUNCTI INTERVAL $[a, b]$.

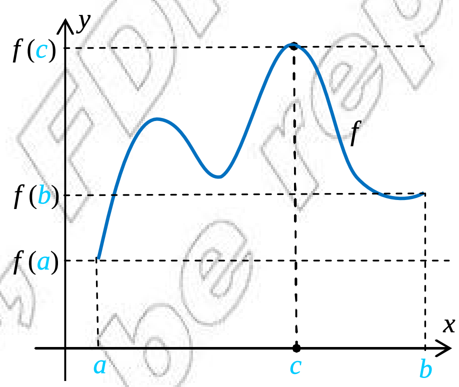


Figure 4.1

FROM THE GRAPH ONE CAN SEE THAT $f(a) \le f(x) \le f(c)$ FOR ALL x IN $[a, b]$

HENCE $f(a)$ IS THE ABSOLUTE MINIMUM AND $f(c)$ IS THE ABSOLUTE MAXIMUM VALUE OF f ON $[a, b]$.

NOTE THAT THIS THEOREM DOES NOT TELL US WHERE AND HOW TO FIND THE MAXIMU MINIMUM VALUES ON $[a, b]$; IT SIMPLY ASSERTS THAT A CONTINUOUS FUNCTION ON A CLOSED A BOUNDED INTERVAL HAS EXTREME VALUES.

IN THE NEXT SECTION, YOU WILL SEE HOW AND WHERE TO FIND THE MAXIMUM AND MINI
VALUE OF f ON $[a, b]$. TO THIS END, WE NEED TO DEFINE RELATIVE EXTREME VALUES AND CRITI
NUMBERS.

SOMETIMES THERE ARE EXTREME VALUES EVEN WHEN THE CONDITIONS OF THE THEOREM
SATISFIED, BUT IF THE CONDITIONS HOLD, THE EXISTENCE OF EXTREME VALUE IS GUARANTE

NOTE THAT THE MAXIMUM VALUE OF A FUNCTION OCCURS AT THE HIGHEST POINT ON ITS G
THE MINIMUM VALUE OCCURS AT THE LOWEST POINT.

Relative extreme values and critical numbers

CONSIDER THE FOLLOWING GRAPH OF A FUNCTION f AND ANSWER THE QUESTIONS IN ACTIVITY 4.2 BELOW.

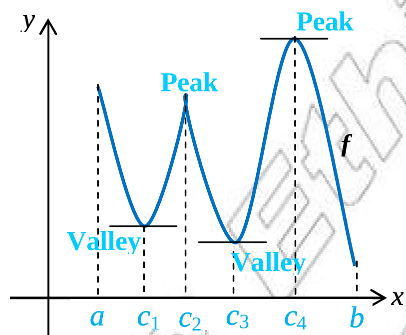


Figure 4.2

ACTIVITY 4.2

IDENTIFY THE NUMBERS AT WHICH THE MAXIMUM OR MINIM OCCUR ON THE GIVEN INTERVAL.

A $[a, c_2]$ **B** $[a, c_4]$ **C** $[c_2, c_4]$ **D** $[a, b]$

OBSERVE FROM THE ACTIVITY THAT, THE EXTREMA OF A CONTINUOUS FUNCTION OCCUR EITHE
AT END POINTS OF THE INTERVAL OR AT POINTS WHERE THE GRAPH HAS A "PEAK" OR A
(POINTS WHERE THE GRAPH IS HIGHER OR LOWER THAN ALL NEARBY POINTS).

FOR EXAMPLE, THE FUNCTION f IN THE ABOVE FIGURE 4.2 HAS PEAKS AT $(c_2, f(c_2))$, $(c_4, f(c_4))$ AND VALLEYS AT $(c_1, f(c_1))$, $(c_3, f(c_3))$. PEAKS AND VALLEYS ARE WHAT YOU CALL RELATIVE EXTREMA.

Definition 4.2

A FUNCTION f IS SAID TO HAVE A RELATIVE

A MAXIMUM AT A NUMBER c IF IN AN OPEN INTERVAL I, $f(c) \geq f(x)$ FOR ALL x IN I.

B MINIMUM AT A NUMBER c IF IN AN OPEN INTERVAL I, $f(c) \leq f(x)$ FOR ALL x IN I.

THE RELATIVE MAXIMA AND RELATIVE MINIMA ARE CALLED **relative extrema**.

Example 1 AS SHOWN IN THE ABOVE ACTIVITY, THE VALLEYS AND PEAKS ARE RELATIVE MINIMUM AND RELATIVE MAXIMUM POINTS RESPECTIVELY;

$f(c_1)$ AND $f(c_3)$ ARE RELATIVE MINIMUM VALUES OBTAINED AT THE VALLEYS $(c_1, f(c_1))$ AND $(c_3, f(c_3))$, RESPECTIVELY.

$f(c_2)$ AND $f(c_4)$ ARE RELATIVE MAXIMUM VALUES OBTAINED AT THE PEAKS $(c_2, f(c_2))$ AND $(c_4, f(c_4))$ RESPECTIVELY.

Observe that:

1. AT $(c_1, f(c_1))$, $(c_3, f(c_3))$ AND $(c_4, f(c_4))$ THERE ARE HORIZONTAL TANGENT LINES, AND HENCE THE SLOPE OF THE TANGENT LINE IS ZERO THERE.

   THUS $f'(c_1) = 0$, $f'(c_3) = 0$ AND $f'(c_4) = 0$.

2. NO TANGENT LINE CAN BE DRAWN AT $(c_2, f(c_2))$ AND HENCE THE DERIVATIVE OF f DOES NOT EXIST AT c_2.

THEREFORE, FROM OBSERVATIONS 1 AND 2, ONE CAN CONCLUDE THAT RELATIVE EXTREMA OF A FUNCTION OCCUR EITHER WHERE THE DERIVATIVE IS ZERO (HORIZONTAL TANGENT) OR W DERIVATIVE DOES NOT EXIST (NO TANGENT). THIS NOTION LEADS TO THE FOLLOWING CONCL

Theorem 4.1

IF A CONTINUOUS FUNCTION f HAS A RELATIVE EXTREMUM AT c, THEN EITHER $f'(c) = 0$ OR f HAS NO DERIVATIVE AT c.

DOES THE CONVERSE HOLD TRUE? JUSTIFY BY AN EXAMPLE.

Definition 4.3

LET c BE IN THE DOMAIN OF f. THEN IF $f'(c) = 0$ OR f HAS NO DERIVATIVE AT c, THEN c IS SAI TO BE A **critical number** OF f.

Example 2 FIND THE CRITICAL NUMBERS OF THE GIVEN FUNCTIONS

1. $f(x) = 4x^3 - 5x^2 - 8x + 20$
2. $f(x) = 2\sqrt{x}\,(6 - x)$

Solution

1. $f'(x) = 12x^2 - 10x - 8$ IS DEFINED FOR ALL VALUES OF x.

   SOLVE $12x^2 - 10x - 8 = 0$

   $\Rightarrow 2(3x - 4)(2x + 1) = 0 \Rightarrow 3x - 4 = 0$ OR $2x + 1 = 0 \Rightarrow 3x = 4$ OR $2x = -1$

   $\Rightarrow x = \frac{4}{3}$ OR $x = \frac{-1}{2}$

   HENCE THE CRITICAL NUMBERS ARE $\frac{4}{3}$ AND $\frac{-1}{2}$.

2 $f'(x) = 6x^{-\frac{1}{2}} - 3x^{\frac{1}{2}}$

THE DERIVATIVE IS NOT DEFINED AT $x = 0$, BUT 0 IS IN THE DOMAIN OF f. HENCE, 0 IS A CRITICAL NUMBER.

TO FIND OTHER CRITICAL NUMBERS (IF THEY EXIST), SOLVE $f'(x) = 0$.

$$\Rightarrow 6x^{\left(-\frac{1}{2}\right)} - 3x^{\frac{1}{2}} = 0 \Rightarrow 3x^{\left(-\frac{1}{2}\right)}(2 - x) = 0 \Rightarrow 2 - x = 0 \Rightarrow x = 2$$

THEREFORE, THE CRITICAL NUMBERS ARE 0 AND 2.

SUPPOSE YOU ARE LOOKING FOR THE ABSOLUTE EXTREME OF A CONTINUOUS FUNCTION f ON THE CLOSED AND BOUNDED INTERVAL $[a, b]$. THE EXTREME VALUE THEOREM TELLS YOU THAT THESE EXTREMA EXIST AND THEOREM 4.1 ENABLES YOU TO NARROW THE LIST OF "CANDIDATES" FOR POINTS WHERE EXTREMA CAN OCCUR FROM THE ENTIRE INTERVAL $[a, b]$ TO JUST THE END POINTS, AND THE CRITICAL NUMBERS BETWEEN a AND b. THIS SUGGESTS THE FOLLOWING PROCEDURES:

To find the absolute extrema of a continuous function *f* on [*a*, *b*]:

Step 1 COMPUTE $f'(x)$ AND FIND CRITICAL NUMBERS OF f ON (a, b).

Step 2 EVALUATE f AT THE ENDPOINTS a, b AND AT EACH CRITICAL NUMBER.

Step 3 COMPARE THE VALUES IN STEP 2.

THUS BY COMPARING THE VALUES OF f IN STEP 3 YOU HAVE:

- ✓ THE LARGEST VALUE OF f IS THE ABSOLUTE MAXIMUM OF f ON $[a, b]$
- ✓ THE SMALLEST VALUE OF f IS THE ABSOLUTE MINIMUM OF f ON $[a, b]$

Example 3 GIVEN $f(x) = x^2 - x^3$, FIND THE ABSOLUTE EXTREMUM VALUE OF f ON

A $[-1, 2]$ **B** $\left[-\frac{1}{2}, \frac{3}{2}\right]$ **C** $[0, 1]$

Solution $f'(x) = 2x - 3x^2$, $f'(x) = 0 \Rightarrow x(2 - 3x) = 0 \Rightarrow x = 0$ OR $x = \frac{2}{3}$

A BOTH 0 AND $\frac{2}{3}$ ARE CRITICAL NUMBERS ON $[-1, 2]$

HENCE THE FOLLOWING ARE THE CANDIDATES FOR EXTREME VALUES.

$$f(0) = 0,\quad f\left(\frac{2}{3}\right) = \frac{4}{27},\quad f(-1) = 2,\quad f(2) = -4$$

COMPARING THE VALUES, THE MAXIMUM VALUE IS 2 AND THE MINIMUM VALUE IS –4.

B BOTH 0 AND $\frac{2}{3}$ ARE CRITICAL NUMBERS ON $\left[-\frac{1}{2}, \frac{3}{2}\right]$. HENCE $f(0)$, $f\left(\frac{2}{3}\right)$, $f\left(-\frac{1}{2}\right)$ AND $f\left(\frac{3}{2}\right)$ ARE CANDIDATES FOR EXTREME VALUES.

$$f(0) = 0\ ,\ f\left(\frac{2}{3}\right) = \frac{4}{27}\ ;\ f\left(-\frac{1}{2}\right) = \frac{3}{8}\ ;\ f\left(\frac{3}{2}\right) = \frac{-9}{8}$$

COMPARING THE VALUES, $\frac{3}{8}$ IS THE MAXIMUM VALUE AND $-\frac{9}{8}$ IS THE MINIMUM VALUE.

C $\frac{2}{3}$ IS THE ONLY CRITICAL NUMBER IN $[0, 1]$. HENCE $f(0)$, $f\left(\frac{2}{3}\right)$ AND $f(1)$ ARE THE CANDIDATES FOR EXTREME VALUES.

$$f(0) = 0,\quad f\left(\frac{2}{3}\right) = \frac{4}{27} \text{ AND } f(1) = 0$$

COMPARING THE VALUES 0 IS THE MINIMUM VALUE AND $\frac{4}{27}$ IS THE MAXIMUM VALUE.

Example 4 FIND THE ABSOLUTE MAXIMUM AND MINIMUM VALUE OF $f(x) = x - x^{\frac{2}{3}}$ ON $[-1, 2]$.

Solution $f'(x) = 1 - \frac{2}{3}x^{-\frac{1}{3}} = \dfrac{3x^{\frac{1}{3}} - 2}{3x^{\frac{1}{3}}}$ BUT $f'(0)$ DOES NOT EXIST.

HENCE 0 IS ONE OF THE CRITICAL NUMBERS.

$$f'(x) = 0 \Rightarrow \frac{3}{2}x^{\frac{1}{3}} - 1 = 0 \Rightarrow x = x = \left(\frac{3}{2}\right)^3 = \frac{8}{27} \Rightarrow x = 0 \text{ AND } x = \frac{8}{27}$$

ARE CRITICAL NUMBERS.

HENCE THE FOLLOWING ARE THE CANDIDATES FOR EXTREME VALUES:

$$f(-1) = -2,\ f(2) = 2 - \sqrt[3]{4} > 0\ ,\ f(0) = 0,\ f\left(\frac{8}{27}\right) = \frac{-4}{27}$$

THEREFORE –2 IS THE MINIMUM VALUE AND $2 - \sqrt[3]{4}$ IS THE MAXIMUM VALUE ON [–1, 2].

Exercise 4.1

IDENTIFY CRITICAL NUMBERS AND FIND THE ABSOLUTE MAXIMUM VALUE AND ABSOLUTE M
VALUE FOR EACH OF THE GIVEN FUNCTIONS ON THE GIVEN INTERVAL.

1. $f(x) = x^3$; $[-2, 1]$
2. $f(x) = x^4 - 2x^2 + 3$; $[-1, 2]$
3. $f(x) = x^{\frac{2}{3}}(5 - 2x)$; $[-1, 2]$
4. $f(x) = 2\cos x + x$; $[0, 2\pi]$
5. $f(x) = x^3 - 3x^2$; $[-1, 3]$
6. $f(x) = 3x^5 - 20x^3$; $[-2, 2]$

Rolle's theorem and the mean-value theorem

YOU WILL SEE THAT MANY OF THE RESULTS OF THIS UNIT DEPEND ON ONE CENTRAL FACT CALLED THE mean-value theorem. BUT TO ARRIVE AT THE MEAN VALUE THEOREM YOU BEGIN WITH A SPECIAL CASE OF THE MEAN VALUE THEOREM CALLED Rolle's theorem, NAMED AFTER THE SEVENTEENTH-CENTURY FRENCH MATHEMATICIAN Michael Rolle. THIS RESULT IMPLIES THAT IF f IS CONTINUOUS ON $[a, b]$ AND $f(a) = f(b)$ THEN THERE ALWAYS EXISTS AT LEAST ONE CRITICAL NUMBER OF f IN (a, b).

ACTIVITY 4.3

LOOK AT THE FOLLOWING GRAPHS AND ANSWER THE QUESTIONS BELOW:

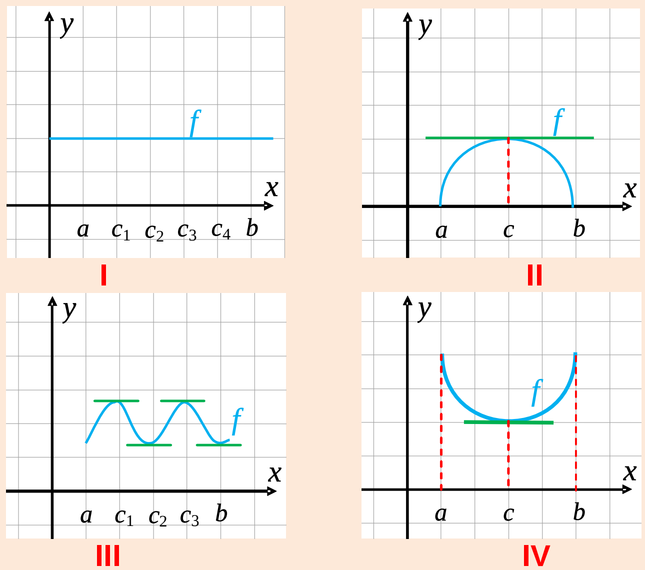


Figure 4.3

IN ALL CASES $f(a) = f(b)$

1 FIND THE COORDINATES OF POINTS ON EACH GRAPH AT WHICH HORIZONTAL TANGENT LINES OCCUR.

2 WHAT IS THE SLOPE OF A HORIZONTAL LINE?

3 HOW DO YOU RELATE SLOPES OF TANGENT LINES TO DERIVATIVES?

Rolle's theorem

LET f BE FUNCTION THAT SATISFIES THE FOLLOWING THREE CONDITIONS:

A f IS CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$

B f IS DIFFERENTIABLE ON THE OPEN INTERVAL (a, b)

C $f(a) = f(b)$

THEN, THERE IS A NUMBER c IN (a, b) SUCH THAT $f'(c) = 0$

Proof: THERE ARE THREE CASES:

Case 1 $f(x) = k$, A CONSTANT (AS IN FIGURE 4.3I IN THE ABOVE ACTIVITY)

THEN $f'(x) = 0$, SO THE NUMBER c CAN BE ANY NUMBER IN (a, b).

Case 2 $f(x) > f(a)$ FOR SOME x IN (a, b), (AS IN FIGURE 4.3II AND FIGURE 4.3III IN THE ABOVE ACTIVITY)

BY THE EXTREME VALUE THEOREM, f HAS A MAXIMUM VALUE SOMEWHERE IN $[a, b]$. SINCE $f(a) = f(b)$, IT MUST ATTAIN THIS MAXIMUM VALUE AT A NUMBER c IN (a, b). THEN f HAS A LOCAL MAXIMUM AT c AND, SINCE f IS DIFFERENTIABLE AT c, THIS IMPLIES THAT $f'(c) = 0$

Case 3 $f(x) < f(a)$ FOR SOME x IN (a, b) (AS IN FIGURE 4.3 IV AND FIGURE 4.3 V IN THE ABOVE ACTIVITY)

BY THE EXTREME VALUE THEOREM, f HAS A MINIMUM VALUE IN $[a, b]$ AND, SINCE $f(a) = f(b)$, IT ATTAINS THIS MINIMUM VALUE AT A NUMBER c IN (a, b) WHERE AGAIN $f'(c) = 0$.

Example 5 LET'S APPLY ROLLE'S THEOREM TO THE POSITION FUNCTION $s = f(t)$ OF A MOVING OBJECT. IF THE OBJECT IS IN THE SAME PLACE AT TWO DIFFERENT INSTANTS $t = a$ AND $t = b$, THEN $f(a) = f(b)$. ROLLE'S THEOREM SAYS THAT THERE IS SOME INSTANT OF TIME $t = c$ BETWEEN a AND b WHEN $f'(c) = 0$; THAT IS, THE VELOCITY IS 0. (IN PARTICULAR YOU CAN SEE THAT THIS IS TRUE WHEN A BALL IS THROWN DI UPWARD).

Example 6 PROVE THAT THE EQUATION $x^3 + x - 1 = 0$ HAS EXACTLY ONE REAL ROOT.

Solution FIRST YOU USE THE INTERMEDIATE VALUE THEOREM TO SHOW THAT A ROOT EXISTS.

LET $f(x) = x^3 + x - 1$ $\quad f(0) = -1 < 0$ AND $f(1) = 1 > 0$

SINCE f IS A POLYNOMIAL, IT IS CONTINUOUS, SO THE INTERMEDIATE VALUE THEOREM THAT THERE IS A NUMBER c BETWEEN 0 AND 1 SUCH THAT $f(c) = 0$. THUS THE GIVEN EQUATION HAS A ROOT.

TO SHOW THAT THE EQUATION HAS NO OTHER REAL ROOT, WE USE ROLLE'S THEOREM AND ARGUE BY CONTRADICTION.

SUPPOSE THAT IT HAD TWO ROOTS a AND b. THEN $f(a) = f(b) = 0$ AND, SINCE f IS A POLYNOMIAL, IT IS DIFFERENTIABLE ON (a, b) AND CONTINUOUS ON $[a, b]$. THUS BY ROLLE'S THEOREM, THERE IS A NUMBER c BETWEEN a AND b SUCH THAT $f'(c) = 0$.

BUT $f'(x) = 3x^2 + 1 \geq 1 \ \forall x$ (SINCE $x^2 \geq 0$)

SO $f'(x) \neq 0$. THIS LEADS TO A CONTRADICTION.

THEREFORE, THE EQUATION CANNOT HAVE TWO REAL ROOTS.

OUR MAIN USE OF ROLLE'S THEOREM IS IN PROVING ANOTHER IMPORTANT THEOREM, WHICH WAS FIRST STATED BY FRENCH MATHEMATICIAN, Joseph – Louis Lagrange.

ACTIVITY 4.4

CONSIDER THE FOLLOWING GRAPHS.

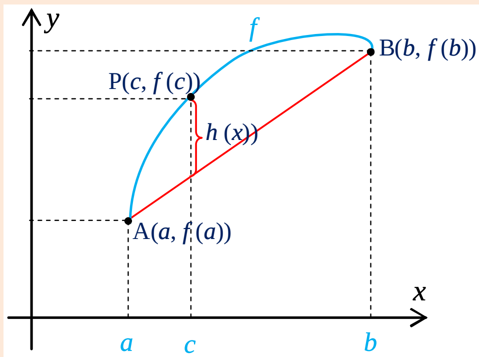


Figure 4.4

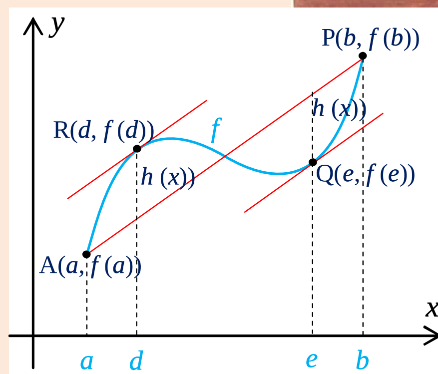


Figure 4.5

IN BOTH CASES *AB* AND *AP* ARE THE SECANT LINES OF THE GRAPH OF *f*.

A FIND THE SLOPE OF THE SECANT LINE *AB*.

B FIND THE EQUATION OF THE SECANT LINE.

C CAN YOU DRAW A LINE PARALLEL TO *AB* AND PASSING THROUGH *P*? WHY?

D CAN YOU DRAW A LINE PARALLEL TO *AB* AND PASSING THROUGH *R*? *Q*? WHY?

E RECALL THAT THE LINES THROUGH *P*, *R*, *Q* ARE TANGENT LINES. CAN YOU FIND THE SLOPE OF THESE TANGENT LINES, IF *f* IS DIFFERENTIABLE FUNCTION? EXPLAIN.

F FOR THE TANGENTS PARALLEL TO *AB*, COMPARE THE SLOPE WITH THAT OF THE SLOPE OF *AB*. ARE THESE EQUAL? WHY?

G CAN YOU FIND THE VERTICAL DISTANCE $h(x)$ AS IN FIGURE 4.4 AND FIGURE 4.5?

The mean-value theorem

LET *f* BE A FUNCTION THAT SATISFIES THE FOLLOWING CONDITIONS:

1 *f* IS CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$

2 *f* IS DIFFERENTIABLE ON THE OPEN INTERVAL (a, b)

THEN THERE IS A NUMBER c IN (a, b) SUCH THAT

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ OR EQUIVALENTLY } f(b) - f(a) = f'(c)(b - a)$$

Proof: USING ACTIVITY 4.4

$$h(x) = f(x) - y, \text{ WHERE } y = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)$$

$$h(a) = 0 = h(b) \text{ AND } \frac{dy}{dx} = \frac{f(b) - f(a)}{b - a}$$

OBSERVE THAT h IS CONTINUOUS ON $[a, b]$ AND DIFFERENTIABLE ON (a, b). THEN BY ROLLE'S THEOREM THERE IS A NUMBER c IN (a, b) SUCH THAT $h'(c) = 0$

$$\Rightarrow f'(c) - \left.\frac{dy}{dx}\right|_{x=c} = 0 \quad \Rightarrow f'(c) = \left.\frac{dy}{dx}\right|_{x=c} \quad \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 7 TO ILLUSTRATE THE MEAN-VALUE THEOREM WITH A SPECIFIC FUNCTION, CONSIDER $f(x) = x^3 - x$, $a = 0$, $b = 2$. SINCE f IS A POLYNOMIAL, IT IS CONTINUOUS AND DIFFERENTIABLE FOR ALL $x \in \mathbb{R}$, SO IT IS CERTAINLY CONTINUOUS ON [0, 2] AND DIFFERENTIABLE ON (0, 2). THEREFORE, BY THE MEAN-VALUE THEOREM, THERE IS A NUMBER c IN (0, 2) SUCH THAT

$f(2) - f(0) = f'(c)(2 - 0)$, $\quad f(2) = 6$, $f(0) = 0$

$f'(x) = 3x^2 - 1$

$f'(c) = 3c^2 - 1$

$$\Rightarrow 6 = (3c^2 - 1)(2) = 6c^2 - 2$$

$$\Rightarrow c^2 = \frac{4}{3} \quad \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

BUT c MUST LIE IN (0, 2), SO $c = \dfrac{2}{\sqrt{3}}$

2 IF AN OBJECT MOVES IN A STRAIGHT LINE WITH POSITION FUNCTION $s = f(t)$, THEN THE AVERAGE VELOCITY BETWEEN $t = a$ AND $t = b$ IS $\dfrac{f(b) - f(a)}{b - a}$ AND THE VELOCITY AT $t = c$ IS $f'(c)$.

THUS, THE MEAN-VALUE THEOREM TELLS US THAT AT SOME TIME $t = c$ BETWEEN a AND b THE INSTANTANEOUS VELOCITY $f'(c)$ IS EQUAL TO THAT OF THE AVERAGE VELOCITY. FOR INSTANC IF A CAR TRAVELLED 180 KM IN 2 HRS, THEN THE SPEEDOMETER MUST HAVE READ 90 K AT LEAST ONCE.

THE MEAN-VALUE THEOREM CAN BE USED TO ESTABLISH SOME OF THE BASIC FACTS OF DIFFER CALCULUS.

Theorem 4.2

IF $f'(x) = 0$ FOR ALL x IN AN INTERVAL I, THEN f IS CONSTANT ON I.

Proof: LET f BE A DIFFERENTIABLE FUNCTION ON AN INTERVAL I AND LET $f'(x) = 0$ FOR ALL x ON INTERVAL I

IF $x_1, x_2 \in$ I AND $x_1 < x_2$ WITH $f'(x) = 0$ $\forall x \in$ I

THE FUNCTION SATISFIES THE CONDITIONS OF MEAN-VALUE THEOREM ON $[x_1, x_2]$. *Why?*

THUS WE APPLY MEAN-VALUE THEOREM ON $[x_1, x_2]$; SO, THAT

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0. \text{ Why?}$$

THIS IMPLIES THAT $f(x_1) = f(x_2)\ \forall x_1, x_2 \in I$

THEREFORE; WE CONCLUDE THAT f IS A CONSTANT ON I.

Corollary 4.1

IF $f'(x) = g'(x)$ FOR ALL x ON AN INTERVAL I, THEN $f - g$ IS A CONSTANT OR

$f(x) = g(x) + c$, (c IS ARBITRARY CONSTANT.)

Proof: EXERCISE

(HINT: CONSIDER $(f - g)'(x) = 0\ \forall x$ AND APPLY THE ABOVE THEOREM)

Exercise 4.2

1 VERIFY THAT EACH OF THE FOLLOWING FUNCTIONS SATISFIES THE CONDITIONS OF ROLLE'S THEOREM ON THE GIVEN INTERVAL. THEN, FIND ALL VALUES OF c THAT SATISFY THE CONCLUSION OF ROLLE'S THEOREM.

- **A** $f(x) = x^2 - 4x + 1$ ON $[0, 4]$
- **B** $f(x) = x^3 - 3x^2 + 2x + 5$ ON $[0, 2]$
- **C** $f(x) = \sin 2x$ ON $[-1, 1]$
- **D** $f(x) = x\sqrt{x + 6}$ ON $[-6, 0]$

2 GIVEN $f(x) = 1 - x^{\frac{2}{3}}$, SHOW THAT $f(1) = f(-1)$ BUT THERE IS NO c IN $(-1, 1)$ SUCH THAT $f'(c) = 0$. WHY DOES THIS NOT CONTRADICT ROLLE'S THEOREM?

3 REPEAT QUESTION 2 FOR

$f(x) = (x - 1)^{-2}$, $f(0) = f(2)$ ON $[0, 2]$

4 VERIFY THAT THE FOLLOWING FUNCTIONS SATISFY THE CONDITIONS OF THE MEAN-VALUE THEOREM ON THE GIVEN INTERVAL. THEN FIND ALL VALUES OF c THAT SATISFY THE CONCLUSION OF THE MEAN-VALUE THEOREM.

- **A** $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$
- **B** $f(x) = x^3 + x - 1$, $[0, 2]$
- **C** $f(x) = \sqrt[3]{x}$, $[0, 1]$
- **D** $f(x) = \dfrac{x}{x + 2}$, $[1, 4]$

5 LET $f(x) = |x - 1|$.

SHOW THAT THERE IS NO VALUE OF c SUCH THAT $f(3) - f(0) = f'(c)(3 - 0)$.

WHY DOES THIS NOT CONTRADICT THE MEAN-VALUE THEOREM?

6 SHOW THAT THE EQUATION $x^5 + 10x + 3 = 0$ HAS EXACTLY ONE REAL ROOT.

Increasing and decreasing functions

UNDER THIS SUBTOPIC YOU CONSIDER INTERVALS IN WHICH THE GRAPH OF A FUNCTION RISES, FALLS OR A CONSTANT, AND ATTACH A MEANING TO IT. TO DO THIS, CONSIDER THE FOLLOWING

ACTIVITY 4.5

1 CONSIDER THE GRAPH OF THE FOLLOWING FUNCTION.

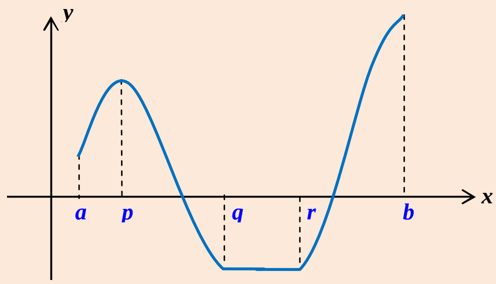


Figure 4.6

A DISCUSS WHETHER THE GRAPH OF f IS RISING OR FALLING AS YOU MOVE FROM LEFT TO RIGHT STARTING AT a.

B IDENTIFY THE INTERVALS IN WHICH THE GRAPH IS RISING AND FALLING.

C IDENTIFY THE INTERVALS IN WHICH THE GRAPH IS NEITHER RAISING NOR FALLING.

2 AGAIN CONSIDERING THE GRAPH OF f IN QUESTION 1 ABOVE,

A IF $x_1 < x_2$ ON $[a, p]$, IS IT TRUE THAT $f(x_1) < f(x_2)$?

B IF $x_1, x_2 \in [p, q]$ AND $x_1 < x_2$, WHICH OF THE FOLLOWING IS TRUE?

I $f(x_1) < f(x_2)$. **II** $f(x_1) > f(x_2)$.

C IF $x_1, x_2 \in [q, r]$; AND $x_1 < x_2$, WHICH ONE OF THE FOLLOWING IS TRUE?

I $f(x_1) < f(x_2)$ **II** $f(x_1) > f(x_2)$ **III** $f(x_1) = f(x_2)$

D TAKE $x_1, x_2 \in [r, b]$ WITH $x_1 < x_2$, COMPARE THE VALUES $f(x_1)$ AND $f(x_2)$.

NOW AS A SUMMARY OF THE ACTIVITY, YOU HAVE THE FOLLOWING DEFINITION.

Definition 4.4

LET f BE A FUNCTION ON AN INTERVAL I.

I IF FOR ANY x_1, x_2 IN I, $x_1 < x_2$ IMPLIES $f(x_1) \le f(x_2)$
f IS SAID TO BE **increasing** ON I.

II IF FOR ANY x_1, x_2 IN I, $x_1 < x_2$ IMPLIES $f(x_1) \ge f(x_2)$
f IS SAID TO BE **decreasing** ON I.

III IF FOR ANY x_1, x_2 IN I, $x_1 < x_2$ IMPLIES $f(x_1) < f(x_2)$
f IS SAID TO BE **strictly increasing** ON I.

IV IF FOR ANY x_1, x_2 IN I, $x_1 < x_2$ IMPLIES $f(x_1) > f(x_2)$
f IS SAID TO BE **strictly decreasing** ON I.

Example 8 BY LOOKING AT THE GRAPH OF ACTIVITY 4.5 ABOVE, IDENTIFY THE INTERVALS IN WHICH f IS INCREASING, DECREASING, STRICTLY INCREASING AND STRICTLY DECRE

Solution

I ON THE INTERVALS $[a, p]$ AND $[r, b]$ f IS STRICTLY INCREASING.

II ON THE INTERVAL $[p, q]$ f IS STRICTLY DECREASING.

III ON THE INTERVAL $[p, r]$ f IS DECREASING (BUT NOT STRICTLY)

IV ON THE INTERVAL $[q, b]$ f IS INCREASING. (BUT NOT STRICTLY)

How derivatives affect the shape of a graph

MANY APPLICATIONS OF CALCULUS DEPEND ON YOUR ABILITY TO DEDUCE FACTS ABOUT A FROM INFORMATION CONCERNING ITS DERIVATIVES. BECAUSE $f'(x)$ REPRESENTS THE SLOPE OF THE CURVE $y = f(x)$ AT THE POINT $(x, f(x))$, IT TELLS YOU THE DIRECTION IN WHICH THE CURVE PROCEEDS AT EACH POINT. SO IT IS REASONABLE TO EXPECT THAT $f'(x)$ WILL PROVIDE YOU WITH INFORMATION ABOUT $f(x)$.

IN THE PREVIOUS SECTION YOU HAVE SEEN THAT IF $f'(x) = 0$ FOR EACH x IN SOME INTERVAL I, THEN f IS A CONSTANT ON I. NOW WHAT DO YOU CONCLUDE IF $f'(x) > 0$ FOR EACH x IN I; OR IF $f'(x) < 0$ FOR EACH x IN I?

ACTIVITY 4.6

CONSIDER THE FOLLOWING GRAPH.

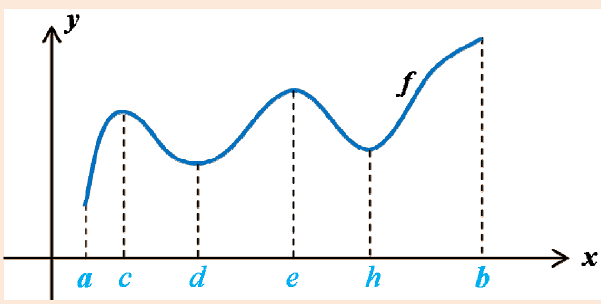


Figure 4.7

1 IF x IS ANY POINT IN (a, c), IS $f'(x) > 0$ OR $f'(x) < 0$? WHY? (HINT: RELATE $f'(x)$ TO THE SLOPE OF TANGENTS ON (a, c))

2 REPEAT IT FOR ANY x IN (c, d), (d, e), (e, h) AND (h, b). (ASSUME f IS DIFFERENTIABLE AT c, d, e, AND h).

AS A RESULT OF THE DISCUSSION IN ACTIVITY 4.6, YOU HAVE THE FOLLOWING TEST WHICH IS IMPORTANT IN IDENTIFYING THE INTERVALS IN WHICH A FUNCTION IS INCREASING OR DECR

Increasing and decreasing test

SUPPOSE THAT f IS CONTINUOUS ON AN INTERVAL I AND DIFFERENTIABLE IN THE INTERIOR OF I.

I IF $f'(x) \geq 0$ FOR ALL x IN THE INTERIOR OF I, THEN f IS INCREASING ON I.

II IF $f'(x) \leq 0$ FOR ALL x IN THE INTERIOR OF I, THEN f IS DECREASING ON I.

III IF $f'(x) > 0$ AND $f'(x) = 0$ ONLY FOR FINITE NUMBER OF POINTS ON I, THEN f IS STRICTLY INCREASING ON I.

IV IF $f'(x) < 0$ AND $f'(x) = 0$ ONLY FOR FINITE NUMBER OF POINTS ON I, THEN f IS STRICTLY DECREASING ON I.

Example 9 FIND WHERE THE FUNCTION $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ IS INCREASING AND WHERE IT IS DECREASING.

Solution $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$

YOU ARE GOING TO FIND INTERVALS IN WHICH $f'(x)$ IS POSITIVE OR NEGATIVE. USE SIGN CHARTS FOR THIS PURPOSE, AS FOLLOWS:

		–1		0		2	
$12x$	– – –		– – –	0	+ + + + + +		+ + +
$x - 2$	– – –		– – –		– – – – – –	0	+ + +
$x + 1$	– – –	0	+ + +		+ + + + + +		+ + +
$f'(x)$	– – –	0	+ + +	0	– – – – – –	0	+ + +

FROM THE SIGN CHART ONE CAN SEE THAT

I $f'(x) \geq 0$ ON $[-1, 0]$ AND $[2, \infty)$ AND $f'(x) = 0$ ONLY AT $x = -1, 0$ AND $x = 2$, THUS f IS STRICTLY INCREASING ON $[-1, 0]$ AND $[2, \infty)$.

II $f'(x) \leq 0$ ON $(-\infty, -1]$ AND $[0, 2]$ AND $f'(x) = 0$ ONLY AT $x = -1, 0$ AND 2, THUS f IS STRICTLY DECREASING ON $(-\infty, -1]$ AND $[0, 2]$.

Exercise 4.3

FIND INTERVALS IN WHICH f IS STRICTLY INCREASING OR STRICTLY DECREASING.

1 $f(x) = x^3 - 12x + 1$

2 $f(x) = x - 2\sin x$ ON $[0, 2\pi]$

3 $f(x) = x^3 - 3x^2 + 5$

4 $f(x) = 2x^3 - 3x^2 + 5$

5 $f(x) = x^4 - 6x^2$

6 $f(x) = 3x^5 - 5x^2 + 3$

7 $f(x) = x\sqrt{x^2 + 1}$

8 $f(x) = x\sqrt{x + 1}$

9 $f(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

10 $f(x) = x - 3x^{\frac{1}{3}}$

11	$f(x) = (x^2 - 1)^3$	**12**	$f(x) = \dfrac{x-1}{x^2+8}$
13	$f(x) = xe^x - 4$	**14**	$f(x) = 3 + \lvert x \rvert$
15	$f(x) = \dfrac{x^2}{x-4}$	**16**	$f(x) = \lvert x-3 \rvert - 5$
17	$f(x) = 2 - 3^{1-2x}$	**18**	$f(x) = \ln(3-2x)$
19	$f(x) = e^{x^2-1}$	**20**	$f(x) = \lvert \ln x \rvert$

Local extreme values of a function on its entire domain

RECALL THAT IF f HAS A LOCAL MAXIMUM OR MINIMUM AT c, THEN c MUST BE A CRITICAL NUMBER OF f; BUT NOT EVERY CRITICAL NUMBER GIVES RISE TO A MAXIMUM OR A MINIMUM. YOU THEREFORE NEED A TEST THAT WILL TELL YOU WHETHER f HAS A LOCAL MAXIMUM OR MINIMUM AT A CRITICAL NUMBER.

SUPPOSE f IS CONTINUOUS ON AN INTERVAL $[a, b]$ AND $a < c < b$ SUCH THAT f IS STRICTLY INCREASING ON $[a, c]$ AND f IS STRICTLY DECREASING ON $[c, b]$ AS IN FIGURE 4.8:

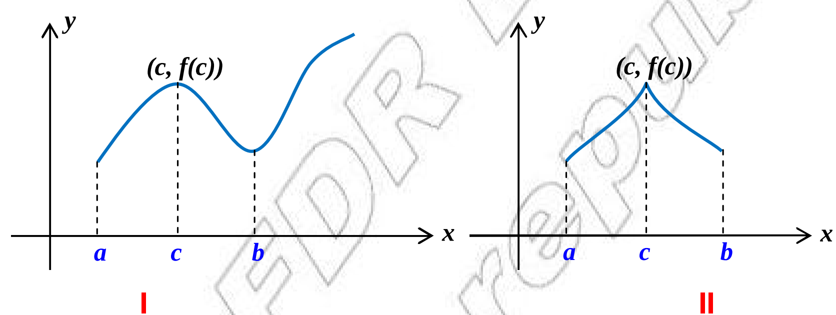


Figure 4.8

FROM FIGURE 4.8 IT IS CLEAR THAT

$$f(c) \geq f(x) \ \forall x \in [a, b].$$

THUS $f(c)$ IS A LOCAL MAXIMUM VALUE OF f ON (a, b)

Observe that:

$f'(x) > 0$ FOR EVERY $x \in (a, c)$; AND $f'(x) < 0$ FOR EVERY $x \in (c, b)$ IN BOTH OF THE GRAPHS OF FIGURE 4.8. IF f' SATISFIES THE ABOVE CONDITIONS, YOU SAY THAT f' CHANGES SIGN FROM POSITIVE TO NEGATIVE.

AGAIN SUPPOSE f IS CONTINUOUS ON AN INTERVAL $[a, b]$ AND $a < c < b$ SUCH THAT f IS STRICTLY DECREASING ON $[a, c]$ AND f IS STRICTLY INCREASING ON $[c, b]$ AS IN THE FIGURE 4.9 BELOW.

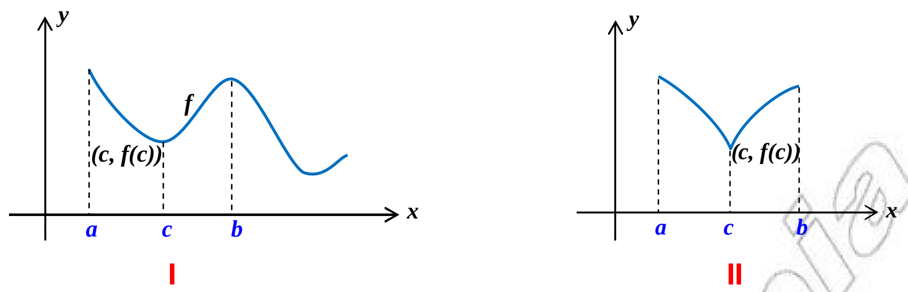


Figure 4.9

IT IS CLEAR THAT $f(c) \le f(x)$ FOR EVERY x IN (a, b) AND HENCE f HAS A LOCAL MINIMUM VALUE AT c.

Observe that:

$f'(x) < 0$ FOR EVERY x IN (a, c); AND $f'(x) > 0$ FOR EVERY x IN (c, b) IN BOTH OF THE GRAPHS IN FIGURE 4.9. IF f' SATISFIES THE ABOVE CONDITIONS, YOU SAY THAT f' CHANGES SIGN AT c FROM NEGATIVE TO POSITIVE.

THEREFORE, YOU CAN HAVE THE FOLLOWING TEST FOR LOCAL EXTREME VALUES OF A FUNCTION.

First derivative test for local extreme values of a function

SUPPOSE THAT c IS A CRITICAL NUMBER OF A CONTINUOUS FUNCTION, THEN

A IF f' CHANGES SIGN FROM POSITIVE TO NEGATIVE AT c, THEN f HAS A LOCAL MAXIMUM AT c.

B IF f' CHANGES SIGN FROM NEGATIVE TO POSITIVE AT c, THEN f HAS A LOCAL MINIMUM AT c.

C IF f' DOES NOT CHANGE SIGN AT c (THAT IS, f' IS POSITIVE ON BOTH SIDES OF c OR NEGATIVE ON BOTH SIDES), THEN f HAS NEITHER LOCAL MAXIMUM NOR MINIMUM AT c.

Example 10 FIND THE LOCAL MAXIMUM AND MINIMUM VALUES OF THE FUNCTION:

1 $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ **2** $g(x) = x + 2\sin x$ FOR $0 \le x \le 2\pi$

Solution

1 $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$

FROM THE SIGN CHART IN EXAMPLE 9 ONE CAN SEE THAT

$f'(x) \le 0$ ON $(-\infty, -1]$ AND $[0, 2]$ AND $f'(x) = 0$ ONLY AT $x = -1$, 0 AND 2.

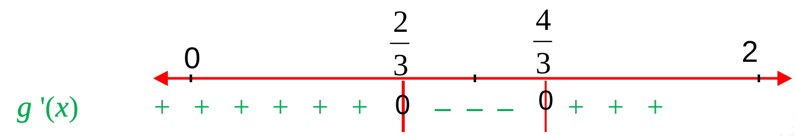
THUS, f IS STRICTLY DECREASING ON $(-\infty, -1]$ AND $[0, 2]$.

A f' CHANGES SIGN FROM NEGATIVE TO POSITIVE AT -1 AND 2

HENCE BOTH $f(-1) = 0$ AND $f(2) = -27$ ARE LOCAL MINIMUM VALUE.

B f' CHANGES SIGN FROM POSITIVE TO NEGATIVE AT 0 AND HENCE $f(0) = 5$ IS THE LOCAL MAXIMUM VALUE.

2 $g'(x) = 1 + 2\cos x$, $g'(x) = 0 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2}{3}\pi$ OR $x = \frac{4}{3}\pi$ IN $[0, 2\pi]$



A g' CHANGES SIGN FROM POSITIVE TO NEGATIVE AT $\frac{2}{3}$ AND HENCE

$$g\left(\frac{2}{3}\right) = \frac{2}{3} + 2\sin\left(\frac{2}{3}\right) = \frac{2}{3} + \sqrt{3}$$ IS A LOCAL MAXIMUM VALUE.

B g' CHANGES SIGN FROM NEGATIVE TO POSITIVE AT $\frac{4}{3}$ AND HENCE

$$g\left(\frac{4}{3}\right) = \frac{4}{3} + 2\sin\frac{4}{3} = \frac{4}{3} - \sqrt{3}$$ IS A LOCAL MINIMUM VALUE.

Exercise 4.4

FIND THE LOCAL MAXIMUM AND MINIMUM VALUES OF EACH OF THE FOLLOWING FUNCTIONS

1 $f(x) = x^2 - 12x + 1$ **2** $f(x) = x^3 - 3x^2 + 5$ **3** $f(x) = x - 2\sin x$ ON $[0, 2]$

4 $f(x) = \dfrac{x}{(1+x)^2}$ **5** $f(x) = x^3 - \dfrac{3}{2}x^2$ **6** $f(x) = x^3 - 12x$

7 $f(x) = (x^2 - 4)^{\frac{2}{3}}$ **8** $f(x) = x^3 - 3x^2 + 3x$ **9** $f(x) = -x^3 + 2$

10 $f(x) = 2x - 3x^{\frac{2}{3}}$

Concavity and inflection points

THIS SUBTOPIC FOCUSES ON THE IMPORTANCE OF THE SECOND DERIVATIVE IN IDENTIFY SHAPE OF THE CURVE.

IN THE PREVIOUS SECTION YOU HAVE USED THE FIRST DERIVATIVE TEST FOR INTER MONOTONICITY AND DETERMINING LOCAL MAXIMUM VALUES AND LOCAL MINIMUM VALU WILL SEE NOW THAT THE SECOND DERIVATIVE TEST IS ALSO IMPORTANT IN THE STUD BEHAVIOUR OF THE GRAPH OF f A FUNCTION

NOW CONSIDER THE FOLLOWING TWO GRAPHS OF INCREASING FUNCTIONS f AND g ON $[a, b]$.

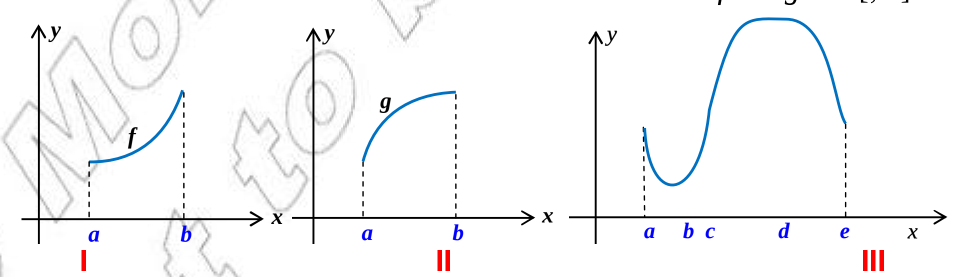
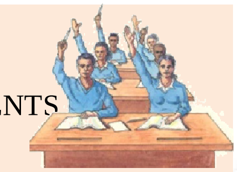


Figure 4.10

THE GRAPHS I AND II OF FIGURE 4.10 LOOK DIFFERENT BECAUSE THEY BEND IN DIFFERENT DIRECTIONS. YOU ARE GOING TO SEE HOW TO DISTINGUISH BETWEEN THESE TWO TY BEHAVIOUR. FOR THIS PURPOSE, FIRST TRY TO DO THE FOLLOWING ACTIVITY.

ACTIVITY 4.7



1. COPY THE CURVES IN FIGURE 4.10 AND TRY TO DRAW TANGENTS CURVES AT SEVERAL POINTS BETWEEN a AND b.
2. DOES THE CURVE OF f LIE ABOVE THE TANGENTS OR BELOW?
3. DOES THE CURVE OF g LIE ABOVE THE TANGENTS OR BELOW?

Definition 4.5

IF THE GRAPH OF A FUNCTION LIES ABOVE ALL OF ITS TANGENTS ON AN INTERVAL I, THEN concave upward ON I.

IF THE GRAPH OF A FUNCTION LIES BELOW ALL OF ITS TANGENTS ON AN INTERVAL I, THEN concave downward ON I.

Example 11 CONSIDER THE FOLLOWING GRAPH.

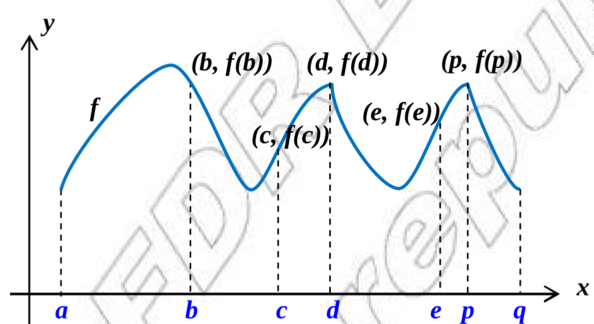


Figure 4.11

ON THE INTERVALS (a, b), (c, d), (e, p) THE GRAPH IS CONCAVE DOWNWARD.

ON THE INTERVALS (b, c), (d, e) AND (p, q) THE GRAPH IS CONCAVE UPWARD.

✍Note:

POINTS $(b, f(b))$, $(c, f(c))$, $(d, f(d))$, (E, $f(e)$) AND $(p, f(p))$ ARE POINTS ON THE GRAPH AT WHICH CONCAVITY CHANGES EITHER FROM CONCAVE UP TO CONCAVE DOWN OR FROM DOWN TO CONCAVE UP. SUCH TYPES OF POINTS ARE CALLED inflection points.

Definition 4.6

A POINT ON A CURVE IS CALLED AN inflection point, IF THE CURVE CHANGES EITHER FROM CONCAVE UP TO CONCAVE DOWN OR FROM CONCAVE DOWN TO CONCAVE UP.

NOW SEE HOW THE SECOND DERIVATIVE HELPS TO DETERMINE THE INTERVALS OF CONCA
INFLECTION POINTS.

Concavity test

LET f BE A FUNCTION WHICH IS TWICE DIFFERENTIABLE ON AN INTERVAL I, THEN

- **A** IF $f''(x) > 0$ FOR ALL x IN I, THE GRAPH OF f IS CONCAVE UPWARD ON I.
- **B** IF $f''(x) < 0$ FOR ALL x IN I, THE GRAPH OF f IS CONCAVE DOWNWARD ON I.

ANOTHER APPLICATION OF THE SECOND DERIVATIVE IS THE FOLLOWING TEST FOR MAXI
MINIMUM VALUES. IT IS A CONSEQUENCE OF THE CONCAVITY TEST.

The second derivative test

SUPPOSE f IS TWICE DIFFERENTIABLE AND f'' IS CONTINUOUS AT c.

- **A** IF $f'(c) = 0$ AND $f''(c) > 0$, THEN f HAS A LOCAL MINIMUM AT c.
- **B** IF $f'(c) = 0$ AND $f''(c) < 0$, THEN f HAS A LOCAL MAXIMUM AT c.

$f''(c) > 0$ NEAR c AND SO f IS CONCAVE UPWARD NEAR c. THIS MEANS THAT THE GRAPH OF f LIES ABOVE ITS HORIZONTAL TANGENT AT c AND SO f HAS A LOCAL MINIMUM AT c.

$f''(c) < 0$, NEAR c AND SO f IS CONCAVE DOWNWARD NEAR c. THIS MEANS THAT THE GRAPH OF f LIES BELOW ITS HORIZONTAL TANGENT AT c AND SO f HAS A LOCAL MAXIMUM AT c.

Example 12 DISCUS THE BEHAVIOUR OF THE CURVE $f(x) = x^4 - 4x^3$ WITH RESPECT TO CONCAVITY, POINTS OF INFLECTION, LOCAL MAXIMUM AND MINIMUM.

Solution $f'(x) = 4x^3 - 12x^2 \Rightarrow f'(x) = 4x^2(x-3)$

THUS $f'(x) = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow x = 0$ OR $x = 3$

NOW $f''(x) = 12x^2 - 24x$, $f''(0) = 0$ AND $f''(3) = 36 > 0$

SINCE $f'(3) = 0$ AND $f''(3) = 36 > 0$, $f(3) = -27$ IS A LOCAL MINIMUM VALUE BY THE SECOND DERIVATIVE TEST.

SINCE $f''(0) = 0$, THE SECOND DERIVATIVE TEST GIVES NO INFORMATION ABOUT THE CRI
NUMBER 0. BUT SINCE $f'(x) < 0$ FOR $x < 0$ AND ALSO FOR $0 < x < 3$, THE FIRST DERIVATIVE TEST TELLS US THAT f DOES NOT HAVE A LOCAL EXTREME VALUE AT 0.

TO DETERMINE INTERVALS OF CONCAVITY AND INFLECTION POINTS WE USE THE FO
SIGN CHART:

		0		2	
$12x$	– – – – – –	0	+ + + + + +		+ + +
$x - 2$	– – – – – –		– – – – – –	0	+ + +
$f''(x)$	+ + + + + +	0	– – – – – –	0	+ + +

THE POINTS WITH COORDINATES (0, 0) AND (2, –16) ARE INFLECTION POINTS.

THE GRAPH OF f IS CONCAVE UPWARD ON $(-\infty, 0)$ AND $(2, \infty)$ AND CONCAVE DOWNWARD ON (0, 2).

> **✍Note:**
>
> THE SECOND DERIVATIVE TEST IS INCONCLUSIVE WHEN $f''(c) = 0$. IN OTHER WORDS, AT SUCH A POINT THERE MIGHT BE A MAXIMUM, THERE MIGHT BE A MINIMUM, OR THERE MIGHT BE NEITHER. THIS TEST ALSO FAILS WHEN $f''(c)$ DOES NOT EXIST. IN SUCH CASES, THE FIRST DERIVATIVE TEST MUST BE USED. IN FACT, EVEN WHEN BOTH TESTS APPLY, THE FIRST DER TEST IS OFTEN THE EASIER ONE TO USE.

Example 13 DISCUSS THE BEHAVIOUR OF THE CURVE $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$ WITH RESPECT TO

A MONOTONICITY **B** RELATIVE EXTREME VALUES

C INFLECTION POINTS AND CONCAVITY.

Solution $$f'(x) = \left(\frac{2}{3}x^{\frac{-1}{3}}\right)(6-x)^{\frac{1}{3}} - x^{\frac{2}{3}}\frac{1}{3}(6-x)^{\frac{-2}{3}} = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

$f'(x) = 0$ WHEN $x = 4$ AND $f'(x)$ DOES NOT EXIST WHEN $x = 0$ OR $x = 6$.

HENCE, 0, 4 AND 6 ARE CRITICAL NUMBERS.

TO IDENTIFY THE EXTREME VALUE AND INTERVALS OF MONOTONICITY YOU USE THE SIG

		0		4		6	
$4-x$	+ + +		+ + + + + +	0	– – –		– – –
$x^{\frac{1}{3}}$	– – –	0	+ + + + + +		+ + +		+ + +
$(6-x)^{\frac{2}{3}}$	+ + +		+ + + + + +		+ + +	0	+ + +
$f'(x)$	– – –	$\nexists$	+ + + + + +	0	– – –	$\nexists$	– – –

FROM THE CHART $f'(x) > 0$ ON (0, 4)

HENCE f IS STRICTLY INCREASING ON [0, 4].

$f'(x) < 0$ ON $(-\infty, 0)$, (4, 6) AND $(6, \infty)$

HENCE f IS STRICTLY DECREASING ON $(-\infty, 0]$ AND $[4, \infty)$

f' CHANGES SIGN FROM NEGATIVE TO POSITIVE AT 0 AND HENCE

$f(0) = 0$ IS A LOCAL MINIMUM VALUE.

f' CHANGES SIGN FROM POSITIVE TO NEGATIVE AT $x = 4$ AND HENCE $f(4) = 2\sqrt[3]{4}$ IS A LOCAL MAXIMUM VALUE.

AT $x = 6$, f' DOES NOT CHANGE SIGN AND HENCE $f(6)$ IS NEITHER A LOCAL MAXIMUM VALUE NOR A LOCAL MINIMUM VALUE.

NOW TO CHECK CONCAVITY AND INFLECTION POINTS WE MAKE USE OF THE SECOND DERI

$$f''(x) = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

f'' DOES NOT EXIST AT $x = 0$ AND $x = 6$

TO DETERMINE CONCAVITY AND INFLECTION POINTS CONSIDER THE FOLLOWING CHAR

		0		6	
-8	− − −		− − − − − − − − −		− − −
$x^{\frac{4}{3}}$	+ + +	0	+ + + + + + + + +		+ + +
$(6-x)^{\frac{5}{3}}$	+ + +		+ + + + + + + + +	0	− − −
$f''(x)$	− − −	∄	− − − − − − − − −	∄	+ + +

$f''(x) > 0$ ON $(6, \infty)$.

HENCE THE GRAPH OF f IS CONCAVE UPWARD ON $(6, \infty)$.

$f''(x) < 0$ ON $(-\infty, 0)$ AND $(0, 6)$.

HENCE BY THE SECOND DERIVATIVE TEST, THE GRAPH OF f IS CONCAVE DOWNWARD ON $(-\infty, 0)$ AND $(0, 6)$.

f'' CHANGES SIGN AT $x = 6$, AND HENCE $(6, f(6)) = (6, 0)$ IS AN INFLECTION POINT.

Curve sketching

NOW, YOU ARE READY TO DEVELOP A PROCEDURE FOR CURVE SKETCHING. TO SKETCH THE GRAPH OF A GIVEN FUNCTION, WE NEED TO KNOW WHERE THE GRAPH CROSSES THE x-AXIS, THE y-AXIS, THE TURNING POINTS, AND INTERVALS IN WHICH THE GRAPH RISES AND FALLS.

Example 14 SKETCH THE GRAPH OF $f(x) = x^4 - 4x^3$.

Solution

A $f(x) = x^4 - 4x^3$ IS A POLYNOMIAL FUNCTION AND HENCE IT IS DEFINED FOR ALL REAL NUMBERS.

B y-INTERCEPT: IT IS THE VALUE OF y AT $x = 0$

THUS y-INTERCEPT $= f(0) = 0^4 - 4(0^3) = 0$

HENCE THE GRAPH CROSSES THE y-AXIS AT $(0, 0)$

C x-INTERCEPT: IT IS THE ZERO OF THE FUNCTION $f(x)$

WHICH MEANS $x^4 - 4x^3 = 0$

$$\Rightarrow x^3(x-4) = 0 \quad \Rightarrow x^3 = 0 \text{ OR } x - 4 = 0 \quad \Rightarrow x = 0 \text{ OR } x = 4$$

THEREFORE, $x = 0$ AND $x = 4$ ARE THE x-INTERCEPTS.

THAT MEANS THE GRAPH OF f CROSSES THE x-AXIS AT POINTS (0, 0) AND (4, 0).

D INTERVALS OF MONOTONICITY AND RELATIVE EXTREME VALUES

TO IDENTIFY INTERVALS IN WHICH f IS MONOTONIC, YOU NEED TO FIND THE DERIVATIVE OF f AND FIND CRITICAL NUMBERS.

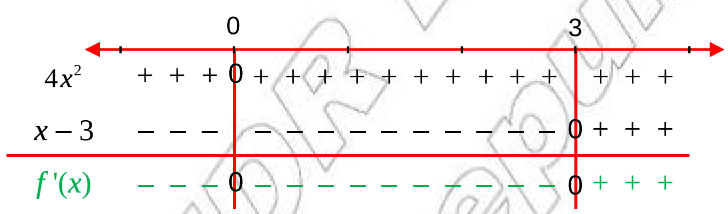
$$f(x) = x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow 4x^2 = 0 \text{ OR } x - 3 = 0$$

$$\Rightarrow x = 0 \text{ OR } x = 3$$

HENCE $x = 0$ AND $x = 3$ ARE CRITICAL NUMBERS OF f.

TO IDENTIFY INTERVALS OF MONOTONICITY AND EXTREME VALUES YOU USE THE FOLLOWING SIGN CHART.



IT CAN BE SEEN FROM THE CHART THAT:

I $f'(x) \le 0$ FOR ALL x IN THE INTERVAL $(-\infty, 3)$; THUS f IS STRICTLY DECREASING ON $(-\infty, 3]$.

II $f'(x) > 0$ FOR ALL x IN THE INTERVAL $(3, \infty)$; THUS f IS STRICTLY INCREASING ON $[3, \infty)$.

III THE SIGN OF f' CHANGES ONLY AT A CRITICAL NUMBER $x = 3$, WHERE IT CHANGES SIGN FROM NEGATIVE TO POSITIVE AND HENCE BY THE FIRST DERIVATIVE TEST, $f(3) = -27$ IS THE RELATIVE MINIMUM VALUE OF f.

E INTERVALS OF CONCAVITY AND INFLECTION POINTS.

TO IDENTIFY INTERVALS OF CONCAVITY AND INFLECTION POINTS, YOU MAKE USE OF THE SECOND DERIVATIVE;

$$f'(x) = 4x^3 - 12x^2, \quad f''(x) = 12x^2 - 24x, \quad f''(x) = 0 \Rightarrow 12x(x-2) = 0$$

$$\Rightarrow 12x = 0 \text{ OR } x - 2 = 0 \quad \Rightarrow x = 0 \text{ OR } x = 2$$

TO IDENTIFY INTERVALS OF CONCAVITY AND INFLECTION POINTS YOU USE THE FOLLOWING

	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, \infty)$
$12x$	− − − − − − −	0	+ + + + + +		+ + +
$x-2$	− − − − − − −		− − − − − − −	0	+ + +
$f''(x)$	+ + + + + +	0	− − − − − − −	0	+ + +

AS CAN BE SEEN FROM THE SIGN CHART

I $f''(x) > 0$ FOR ALL x IN THE INTERVALS $(-\infty, 0)$ AND $(2, \infty)$; THUS BY THE SECOND DERIVATIVE TEST, THE GRAPH OF F IS CONCAVE UPWARD ON $(-\infty, 0)$ AND $(2, \infty)$.

II $f''(x) < 0$ ON $(0, 2)$ AND HENCE BY THE SECOND DERIVATIVE TEST THE GRAPH OF f IS CONCAVE DOWNWARD ON $(0, 2)$.

III THE POINTS AT WHICH CONCAVITY CHANGES ARE CALLED INFLECTION POINTS.

THEREFORE, $(0, f(0)) = (0, 0)$ AND $(2, f(2)) = (2, -16)$ ARE THE INFLECTION POINTS OF THE GRAPH OF f.

NOW USING THE ABOVE INFORMATION, YOU CAN SKETCH THE GRAPH OF f AS FOLLOWS.

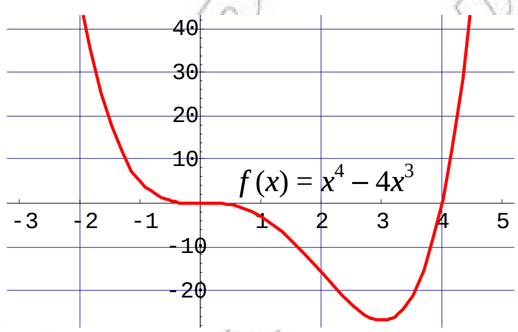


Figure 4.12

Example 15 SKETCH THE GRAPH OF $f(x) = \dfrac{x}{x^2+1}$.

Solution

A $f(x) = \dfrac{x}{x^2+1}$ IS DEFINED FOR ALL REAL NUMBERS.

B y-INTERCEPT: IT IS THE VALUE OF f AT $x = 0$

THUS y-INTERCEPT $= f(0) = 0$

HENCE THE GRAPH CROSSES THE y-AXIS AT $(0, 0)$

C x-INTERCEPT: IT IS THE ZERO OF THE FUNCTION $f(x)$

WHICH MEANS $\dfrac{x}{x^2+1} = 0 \Rightarrow x = 0.$

THEREFORE, $x = 0$ IS THE x – INTERCEPT.

THAT MEANS THE GRAPH OF f CROSSES THE x-AXIS ONLY AT (0, 0).

D INTERVALS OF MONOTONICITY AND RELATIVE EXTREME VALUES

TO IDENTIFY INTERVALS IN WHICH f IS MONOTONIC YOU NEED TO FIND THE DERIVATIVE OF f AND FIND CRITICAL NUMBERS.

$$f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{\left(x^2+1\right)^2}$$

$$f'(x) = 0 \Rightarrow \frac{1-x^2}{\left(x^2+1\right)^2} = 0 \Rightarrow \frac{(1-x)(1+x)}{\left(x^2+1\right)^2} = 0$$

$$\Rightarrow x = 1 \text{ OR } x = -1 .$$

HENCE $x = 1$ AND $x = -1$ ARE CRITICAL NUMBERS OF f.

TO IDENTIFY INTERVALS OF MONOTONICITY AND EXTREME VALUES YOU USE THE FO SIGN CHART.

		-1		1	
$1-x$	+ + + +		+ + + +	0	– – – –
$x+1$	– – – –	0	+ + + +		+ + + +
$\left(x^2+1\right)^2$	+ + + +		+ + + +		+ + + +
$f'(x)$	– – – –	0	+ + + +	0	– – – –

IT CAN BE SEEN FROM THE CHART THAT:

I $f'(x) < 0$ FOR ALL x IN THE INTERVALS $(-\infty,-1)$ AND $(1,\infty)$; THUS f IS STRICTLY DECREASING ON $(-\infty,-1]$ AND $[1,\infty)$.

II $f'(x) > 0$ FOR ALL x IN THE INTERVAL $(-1,1)$; THUS f IS STRICTLY INCREASING ON $[-1,1]$

III THE SIGN OF f' CHANGES AT $x = -1$ AND $x = 1$. IT CHANGES SIGN FROM NEGATIVE TO POSITIVE AT $x = -1$ AND HENCE BY THE FIRST DERIVATIVE TEST, $f(-1) = \dfrac{-1}{2}$ IS THE RELATIVE MINIMUM VALUE OF f. IT ALSO CHANGES SIGN FROM POSITIVE TO NEGATIVE AT $x = 1$ AND HENCE BY THE FIRST DERIVATIVE TEST, $f(1) = \dfrac{1}{2}$ IS A RELATIVE MAXIMUM VALUE OF f.

E INTERVALS OF CONCAVITY AND INFLECTION POINTS.

TO IDENTIFY INTERVALS OF CONCAVITY AND INFLECTION POINTS, YOU MAKE USE OF THE SECOND DERIVATIVE;

$$f'(x) = \frac{1-x^2}{\left(x^2+1\right)^2}, \qquad f''(x) = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}, \qquad f''(x) = 0 \Rightarrow \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3} = 0$$

$$\Rightarrow x = 0 \text{ OR } x = \sqrt{3} \text{ OR } x = -\sqrt{3}.$$

TO IDENTIFY INTERVALS OF CONCAVITY AND INFLECTION POINTS YOU USE THE FOLLOWING

	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$2x$	$-\ -\ -$		$-\ -\ -$	0	$+\ +\ +$		$+\ +\ +$
$x-\sqrt{3}$	$-\ -\ -$		$-\ -\ -$		$-\ -\ -$	0	$+\ +\ +$
$x+\sqrt{3}$	$-\ -\ -$	0	$+\ +\ +$		$+\ +\ +$		$+\ +\ +$
$\left(x^2+1\right)^3$	$+\ +\ +$		$+\ +\ +$		$+\ +\ +$		$+\ +\ +$
$f''(x)$	$-\ -\ -$	0	$+\ +\ +$	0	$-\ -\ -$	0	$+\ +\ +$

AS IT CAN BE SEEN FROM THE SIGN CHART

I $f''(x) < 0$ FOR ALL x IN THE INTERVALS $(-\infty,-\sqrt{3})$ AND $(0,\sqrt{3})$; THUS BY THE SECOND DERIVATIVE TEST THE GRAPH OF f IS CONCAVE DOWNWARD ON $(-\infty,-\sqrt{3})$ AND $(0,\sqrt{3})$.

II $f''(x) > 0$ ON $(-\sqrt{3},0)$ AND $(\sqrt{3},\infty)$ AND HENCE BY THE SECOND DERIVATIVE TEST, THE GRAPH OF f IS CONCAVE UPWARD ON $(-\sqrt{3},0)$ AND $(\sqrt{3},\infty)$.

III INFLECTION POINTS OF THE GRAPH OF f ARE $(0, 0)$, $\left(-\sqrt{3},\frac{-\sqrt{3}}{4}\right)$ AND $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$.

NOW USING THE ABOVE INFORMATION, YOU CAN SKETCH THE GRAPH OF f AS FOLLOWS:

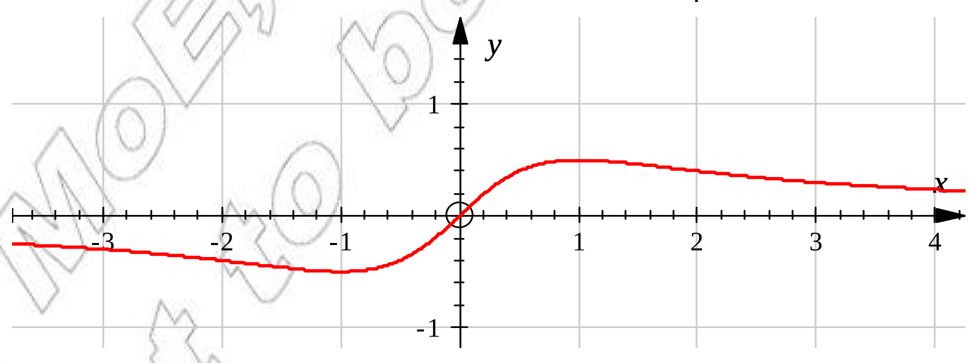


Figure 4.13

Exercise 4.5

SKETCH THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS BY INDICATING THE FOLLOWIN

A DOMAIN OF THE FUNCTION **B** INTERCEPTS (y-INTERCEPT AND x-INTERCEPT)

C ASYMPTOTES (IF ANY) **D** INTERVALS OF MONOTONICITY

E LOCAL EXTREME VALUES **F** INTERVALS OF CONCAVITY

G INFLECTION POINTS

1 $f(x) = x^3 - 12x$ **2** $f(x) = e^x$ **3** $f(x) = \ln x$

4 $f(x) = \dfrac{4}{1 + x^2}$ **5** $f(x) = \dfrac{1}{4}x^4 - 2x^2$ **6** $f(x) = \dfrac{2x - 6}{x^2 - 9}$

7 $f(x) = x^3 - \dfrac{3}{2}x^2 + 6x$ **8** $f(x) = \dfrac{1}{2^x - 1}$

4.2 MINIMIZATION AND MAXIMIZATION PROBLEMS

THE METHODS YOU HAVE LEARNED IN THIS UNIT FOR FINDING EXTREME VALUES HAVE I APPLICATIONS IN MANY AREAS OF LIFE. A BUSINESSPERSON WANTS TO MINIMIZE COST MAXIMIZE PROFITS. A TRAVELLER WANTS TO MINIMIZE TRANSPORTATION TIME. YOU PRINCIPLES IN OPTICS WHICH STATES THAT LIGHT FOLLOWS THE PATH THAT TAKES THE LI THIS SECTION YOU WILL SOLVE PROBLEMS SUCH AS MAXIMIZING AREAS, VOLUMES AND PRO MINIMIZING DISTANCES, TIME, AND COSTS. LET US SEE THE FOLLOWING EXAMPLES:

Example 1 FIND TWO NONNEGATIVE REAL NUMBERS WHOSE SUM IS 18 AND WHOSE PRODU IS MAXIMUM.

Solution THERE ARE MANY PAIRS OF NUMBERS WHOSE SUM IS 18, FOR INSTANCE,

(1, 17), (2 , 16), (3, 15), (4, 14), (5, 13), (6, 12), (7, 11), (8, 10), (9, 9),

(5.2, 12.8), (6.5, 11.5), . . . , ETC.

ALL THESE PAIRS HAVE DIFFERENT PRODUCTS, AND YOU CANNOT LIST ALL SUCH PAIR ALL THE PRODUCTS. AS A RESULT YOU FAIL TO GET THE MAXIMUM PRODUCT IN DOI INSTEAD OF LISTING SUCH PAIRS AND PRODUCTS YOU TAKE TWO VARIABLES SAY x AND y SUCH THAT $x \geq 0$, $y \geq 0$, AND $x + y = 18$ WITH THE PRODUCT xy MAXIMUM.

SINCE $x + y = 18$, THEN $y = 18 - x$. $(0 \leq x \leq 18, 0 \leq y \leq 18)$

THUS YOU WANT TO MAXIMIZE $x(18 - x) = 18x - x^2$.

CONSIDER $f(x) = 18x - x^2$, WHICH IS CONTINUOUS ON [0, 18] AND DIFFERENTIABLE ON (0, 18).

$f'(x) = 18 - 2x$ $\qquad f'(x) = 0 \Rightarrow x = 9$

THE MAXIMUM OCCURS EITHER AT END POINTS OR AT CRITICAL NUMBERS. THUS EVAL THE VALUES OF THE FUNCTION AT CRITICAL NUMBERS AND END POINTS, YOU GET,

$$f(0) = 0\ ,\ f(18) = 0\ \text{ AND }\ f(9) = 81$$

COMPARING THESE VALUES, $x = 9$ GIVES THE MAXIMUM PRODUCT. HENCE $x = 9$ AND $y = 9$ ARE THE TWO REAL NUMBERS WHOSE SUM IS 18 AND WHOSE PRODUCT IS MAXIMUM.

Example 2 A FARMER HAS 240 M OF FENCING MATERIAL AND WANTS TO FENCE A RECTANGULAR FIELD THAT BORDERS A STRAIGHT RIVER. (NO FENCE IS NEEDED ALONG THE RIV WHAT ARE THE DIMENSIONS OF THE FIELD THAT HAS THE LARGEST AREA?

Solution YOU NEED TO FENCE ALONG THE THREE SIDES OF A RECTANGULAR FIELD.

FOR EXAMPLE, YOU MAY HAVE 240 = 100 + 100 + 40 = 80 + 80 + 80 = 90 + 90 + 60 AS POSSIBILITIES FOR THE THREE SIDES.

YOU CAN LIST A LOT OF POSSIBILITIES; BUT THE PROBLEM IS WHICH POSSIBILITY GIV MAXIMUM AREA.

THUS INSTEAD OF LISTING THE POSSIBILITIES, YOU CONSIDER THE GENERAL CASE: YOU MAXIMIZE THE AREA A OF THE RECTANGULAR REGION. LET x AND y BE THE WIDTH AND DEPTH OF THE RECTANGLE.

THEN EXPRESS A IN TERMS OF x AND y AS:

$$A = xy$$

WE WANT TO EXPRESS A AS A FUNCTION OF JUST ONE VARIABLE, SO ELIMINATE x BY EXPRESSING IT IN TERMS OF y. TO DO THIS, YOU USE THE GIVEN INFORMATION THAT THE TOTA LENGTH OF THE FENCING IS 240 M.

$$2y + x = 240$$

$$\Rightarrow \quad x = 240 - 2y$$

$$A(y) = (240 - 2y)\,y = 240y - 2y^2; \quad 0 \le y \le 120$$

$A(y) = 240y - 2y^2$ IS CONTINUOUS ON $[0, 120]$ AND DIFFERENTIABLE ON $(0, 120)$

$$A'(y) = 240 - 4y$$

$$A'(y) = 0 \quad \Rightarrow 240 - 4y = 0 \ \Rightarrow\ y = 60$$

HENCE $y = 60$ IS A CRITICAL NUMBER.

TO GET THE MAXIMUM AREA, YOU CALCULATE THE VALUE OF A AT $y = 60$ (THE CRITICAL NUMBER),

$y = 0$ AND $y = 120$ (THE TWO END POINTS): $A(0) = 0 = A(120)$ AND $A(60) = 7200$

THEREFORE $A(60) = 7200$ IS THE LARGEST VALUE.

HENCE $y = 60$ M AND $x = 120$ M ARE THE DIMENSIONS OF THE FIELD THAT GIVE THE MAXIMUM AREA.

Example 3 A CYLINDRICAL CAN IS TO BE MADE TO HOLD 10 LITRES OF OIL. FIND THE DIMENSIONS THAT WILL MINIMIZE THE COST OF THE METAL TO MANUFACTURE THE CAN.

Solution

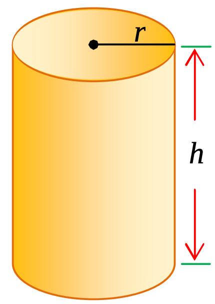


Figure 4.14

IN ORDER TO MINIMIZE THE COST OF THE METAL, YOU HAVE TO MINIMIZE THE TOTAL SURFACE AREA OF THE CYLINDER. YOU SEE THAT THE SIDES ARE MADE FROM A RECTANGULAR S DIMENSIONS 2 r (CIRCUMFERENCE OF THE BASE CIRCLE) AND h. SO THE TOTAL SURFACE AREA IS GIVEN BY

$$A = 2\ r^2 + 2\ rh$$

TO ELIMINATE H YOU USE THE FACT THAT THE VOLUME IS GIVEN AS:

V = 10 LITRES = 10,000 CM^3

$$\Rightarrow\ r^2h = 10000 \qquad \Rightarrow h = \frac{10000}{r^2}$$

$$\Rightarrow A(r) = 2\ r^2 + 2\ r\left(\frac{10000}{r^2}\right) = 2\ r^2 + \frac{20,000}{r} \Rightarrow A'(r) = 4\ r - \frac{20,000}{r^2}$$

$$A'(r) = 0 \Rightarrow 4\ r - \frac{20,000}{r^2} = 0 \Rightarrow r = 10\left(\sqrt[3]{\frac{5}{\ }}\right)$$

APPLYING THE SECOND DERIVATIVE TEST $A''(r) = 4 + \frac{40,000}{r^3} > 0$ FOR ANY $r > 0$, AND

HENCE $r = 10\left(\sqrt[3]{\frac{5}{\ }}\right)$ GIVES THE MINIMUM VALUE.

THUS THE VALUE OF h CORRESPONDING TO $r = 10\left(\sqrt[3]{\frac{5}{\ }}\right)$ IS $20\left(\sqrt[3]{\frac{5}{\ }}\right)$

THUS, TO MINIMIZE THE COST OF THE CAN, THE RADIUS SHOULD BE $10\left(\sqrt[3]{\frac{5}{\ }}\right)$ CM AND THE HEIGHT SHOULD BE $2r = 20\left(\sqrt[3]{\frac{5}{\ }}\right)$ CM.

Example 4 A HOME GARDENER ESTIMATES THAT IF 16 APPLE TREES ARE PLANTED, THE AVERAGE YIELD WILL BE 80 APPLES PER TREE. BUT BECAUSE OF THE SIZE OF THE GARDEN, FOR EACH ADDITIONAL TREE PLANTED THE YIELD WILL DECREASE BY 4 APPLES PER TREE. HOW MANY TREES SHOULD BE PLANTED TO MAXIMIZE THE TOTAL YIELD OF APPLES? WHAT IS THE MAXIMUM YIELD?

Solution TO SOLVE THIS PROBLEM CONSIDER THE FOLLOWING:

A IF ONLY 16 APPLE TREES ARE PLANTED, THEN WHAT IS THE TOTAL AVERAGE YIELD?

B IF 17 APPLE TREES (ONE ADDITIONAL TREE) ARE PLANTED, THEN WHAT IS THE TOTAL AVERAGE YIELD?

C IF 18 APPLE TREES (TWO ADDITIONAL TREES) ARE PLANTED, THEN WHAT IS THE TOTAL AVERAGE YIELD?

D IN GENERAL, IF 16 + x APPLE TREES (x ADDITIONAL TREES) ARE PLANTED, THEN WHAT IS THE TOTAL AVERAGE YIELD?

NOW TO COME TO THE SOLUTION YOU CONSIDER THE GENERAL CASE (D) AND ASSUME x ADDITIONAL APPLE TREES, ARE PLANTED. THUS THE TOTAL YIELD WILL BE (16 + x)(80 − 4x), SINCE FOR EACH ADDITIONAL APPLE TREE PLANTED, THE YIELD WILL DECREASE BY 4 A TREE. THUS, YOU ARE GOING TO MAXIMIZE THE FUNCTION:

$$f(x) = (16 + x)(80 - 4x) = 1280 + 16x - 4x^2 \text{ ON } [0, \infty).$$

$$f'(x) = 16 - 8x \qquad f'(x) = 0 \Rightarrow 16 - 8x = 0 \quad \Rightarrow x = 2$$

THUS $x = 2$ IS THE ONLY CRITICAL NUMBER.

$f''(x) = -8 < 0$ AND HENCE BY THE SECOND DERIVATIVE TEST, THE FUNCTION HAS MAXIMUM VALUE AT CRITICAL NUMBER 2.

THEREFORE, 18 TREES SHOULD BE PLANTED TO GET THE MAXIMUM YIELD:

$f(2) = 18 \times 72 = 1296$ IS THE MAXIMUM YIELD.

Example 5 A MANUFACTURER WANTS TO DESIGN AN OPEN BOX THAT HAS A SQUARE BASE AND A SURFACE AREA OF 48 SQ UNITS AS SHOWN IN THE FIGURE BELOW. WHAT DIMENSIONS WILL PRODUCE A BOX WITH A MAXIMUM VOLUME?

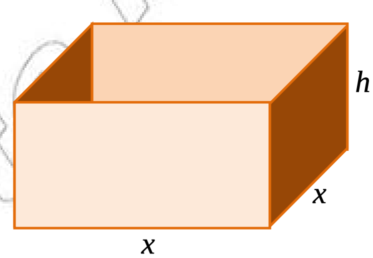


Figure 4.14

Solution BECAUSE THE BASE OF THE BOX IS SQUARE, THE VOLUME V OF THE BOX IS GIVEN BY:

$$V = x^2h$$

THE SURFACE AREA S OF THE OPEN BOX IS GIVEN BY:

S = (AREA OF BASE) + (AREA OF FOUR FACES)

$S = x^2 + 4xh$

BECAUSE V IS TO BE OPTIMIZED, IT HELPS TO EXPRESS V AS A FUNCTION OF JUST ONE VARIABLE.

I.E., $h = \dfrac{S - x^2}{4x} = \dfrac{48 - x^2}{4x}$ (SINCE S = 48 SQ.UNITS)

THUS, $V(x) = x^2\left(\dfrac{48 - x^2}{4x}\right) = 12x - \dfrac{1}{4}x^3$

$V'(x) = 12 - \dfrac{3}{4}x^2$

$V'(x) = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

SINCE x IS THE DIMENSION OF THE BOX, IT IS NON-NEGATIVE AND HENCE $x = 4$ IS THE ONLY CRITICAL NUMBER.

$V''(x) = \dfrac{-3}{2}x < 0 \;\; \forall x > 0$ SO, $V(4)$ IS A MAXIMUM BY THE SECOND DERIVATIVE TEST.

THEREFORE, $x = 4$ AND $h = 2$ GIVES THE MAXIMUM VOLUME, AND WHICH IS

$V = (4^2)(2) = 32$ CUBIC UNITS

Example 6 FIND THE POINTS ON THE GRAPH OF $f(x) = 1 - x^2$ THAT ARE CLOSEST TO O(0, 0)

Solution LOOK AT THE GRAPH OF $f(x) = 1 - x^2$

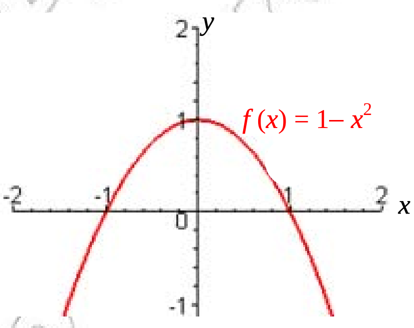


Figure 4.15

ANY POINT ON THE GRAPH IS OF THE FORM $(x, 1 - x^2)$

HENCE $d = \sqrt{(x-0)^2 + (1 - x^2 - 0)^2} = \sqrt{x^2 + (1 - x^2)^2}$

d IS A MINIMUM WHENEVER THE NUMBER UNDER THE RADICAL IS A MINIMUM.

THUS, YOU MINIMIZE $g(x) = x^2 + (1 - x^2)^2$

$$g(x) = x^2 + 1 - 2x^2 + x^4 = 1 - x^2 + x^4$$

$$g'(x) = -2x + 4x^3$$

$$g'(x) = 0 \Rightarrow 2x\left(2x^2 - 1\right) = 0 \Rightarrow x = 0 \text{ OR } x = \pm\frac{\sqrt{2}}{2}$$

THEREFORE, $0, -\frac{\sqrt{2}}{2}$ AND $\frac{\sqrt{2}}{2}$ ARE CRITICAL NUMBERS.

TO CHECK WHETHER THESE NUMBERS GIVE A MINIMUM DISTANCE, YOU USE THE SECOND DERIVATIVE TEST.

$$g''(x) = -2 + 12x^2 > 0 \text{ FOR } x = \frac{\sqrt{2}}{2} \text{ AND } x = -\frac{\sqrt{2}}{2}$$

$$g\left(\frac{\sqrt{2}}{2}\right) = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = g\left(-\frac{\sqrt{2}}{2}\right)$$

THUS, $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ AND $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ ARE THE CLOSEST POINTS TO (0, 0).

BUT THE CRITICAL NUMBER $x = 0$ DOES NOT MINIMIZE THE DISTANCE. WHY?

Example 7 SUPPOSE THE TOTAL COST $C(x)$ (IN THOUSANDS OF BIRR) FOR MANUFACTURING x DESKTOP COMPUTERS PER MONTH IS GIVEN BY THE FUNCTION

$$C(x) = 575 + 25x - \frac{1}{4}x^2,\ 0 \le x \le 50$$

A FIND THE MARGINAL COST AT A PRODUCTION LEVEL OF x COMPUTERS PER MONTH.

B USE THE MARGINAL COST FUNCTION TO APPROXIMATE THE COST OF PRODUCING THE 31ST COMPUTER.

C USE THE TOTAL COST FUNCTION TO FIND THE EXACT COST OF PRODUCING THE 31ST COMPUTER.

Solution

A SINCE MARGINAL COST IS THE DERIVATIVE OF THE COST FUNCTION $C(x)$, YOU HAVE

$$C'(x) = 25 - \frac{1}{2}x$$

B THE MARGINAL COST AT A PRODUCTION LEVEL OF 30 COMPUTERS IS

$$C'(30) = 25 - \frac{1}{2}\times 30 = 10$$

OR BIRR 10,000 PER COMPUTER.

THAT MEANS AT A PRODUCTION LEVEL OF 30 COMPUTERS PER MONTH, THE TOTAL COST IS INCREASING AT THE RATE OF BIRR 10,000 PER COMPUTER.

HENCE THE COST OF PRODUCING THE 31ST COMPUTER IS APPROXIMATELY BIRR 10,000.

c THE EXACT COST OF PRODUCING THE 31ST COMPUTER IS

$$\begin{pmatrix}\text{TOTAL COST OF}\\ \text{PRODUCING 31}\\ \text{COMPUTERS}\end{pmatrix} - \begin{pmatrix}\text{TOTAL COST OF}\\ \text{PRODUCING 30}\\ \text{COMPUTERS}\end{pmatrix} = C(31) - C(30)$$

$$= 1{,}109.75 - 1{,}100 = 9.75 \text{ OR BIRR } 97$$

As a summary from what you have seen in solving problems by the application of differential calculus, the greatest challenge is often to convert the real-life word problem into a mathematical maximization or minimization problem, by setting up the function that is to be maximized or minimized. The following guideline adapted to particular situation may help.

1 Understand the problem

THE FIRST STEP IS TO READ THE PROBLEM CAREFULLY UNTIL IT IS CLEARLY UNDERS YOURSELF:

What is the unknown? What are the given quantities? What are the given conditions?

2 Draw a diagram (if necessary)

IN MOST PROBLEMS, IT IS USEFUL TO DRAW A DIAGRAM AND IDENTIFY THE GIVE REQUIRED QUANTITIES ON THE DIAGRAM.

3 Introduce notation

ASSIGN A SYMBOL TO THE QUANTITY THAT IS TO BE MAXIMIZED OR MINIMIZED AND S SYMBOLS FOR THE UNKNOWNS.

4 Express the quantity which is going to be optimized in terms of the unknowns.

5 If the quantity which is going to be optimized is expressed as a function of more than one unknown in step 4,

USE THE GIVEN INFORMATION TO FIND RELATIONSHIPS (IN THE FORM OF EQUATIONS) THESE UNKNOWNS. THEN USE THESE EQUATIONS TO ELIMINATE ALL BUT ONE OF THE UN THE EXPRESSION. THUS THE QUANTITY WHICH IS GOING TO BE OPTIMIZED WILL BE EXPRES A FUNCTION OF ONE UNKNOWN; WRITE THE DOMAIN OF THIS FUNCTION AND USE THE ME SOLVING MAXIMIZATION AND MINIMIZATION PROBLEMS TO GET THE QUANTITY OPTIMIZE

Exercise 4.6

1. THE PRODUCT OF TWO POSITIVE NUMBERS IS 288. FIND THE NUMBERS WHICH MINIMIZE THE SUM OF THE SECOND NUMBER AND TWICE THE FIRST NUMBER.
2. FIND THE POINTS ON THE GRAPH OF THE FUNCTION THAT ARE CLOSEST TO THE GIVEN POINT.

   a $f(x) = x^2 - 4$; $(0, 2)$ **b** $f(x) = x^2 + 1$; $(0, 4)$

   c $f(x) = x^2$; $(2, 1)$

3 What positive number minimizes the sum of x and its reciprocal?

4 Find the length and width of a rectangle with perimeter 100 m that maximize the area.

5 A farmer has a 200 m fencing material to enclose two adjacent sides of a rectangular field. What dimensions should be used so that the enclose area will be a maximum?

6 A dairy farmer plans to enclose a rectangular pasture adjacent to a river. To provide enough grass for the herd, the pasture must have an area of 180,000 m^2. No fencing is required along the river. What dimensions will use the smallest amount of fencing?

7 Find the length and width of a rectangle with area 64 m^2 that gives minimum perimeter.

8 The combined perimeter of a circle and a square is 16. Find the dimensions of the circle and square that produce a minimum total area.

9 A ten meter wire is to be used to form a square and a circle.

a Express the sum of the areas of the square and the circle as a function $A(x)$ of the side of the square x.

b Identify the domain of $A(x)$.

c How much wire should be used for the square and how much for the circle in order to enclose the smallest total area?

10 A company has determined that its total revenue (in Birr) for a product can be modeled by $R(x) = -x^3 + 450x^2 + 52{,}500x$ where x is the number of units produced (and sold). What production level will yield a maximum revenue?

11 Find the number of units that must be produced to minimize the cost function $C(x) = 0.008x^2 + 2x + 304$. What is the minimum cost?

12 A mass connected to a spring moves along the x-axis so that its x-coordinate at time t is given by

$$x(t) = \sin 2t + \sqrt{3}\cos 2t.$$

What is the maximum distance of the mass from the origin?

13 The body temperature (in degree centigrade) of a patient t hours after taking a fever reducing drug is given by

$$C(t) = 37 + \frac{4}{\sqrt{t+1}}$$

Find $C(3)$ and $C'(3)$. Give a brief verbal interpretation of these results.

4.3 RATE OF CHANGE

IN THE PREVIOUS SECTIONS YOU HAVE SEEN DERIVATIVES AS RATES OF CHANGE I.E. $f'(x)$ IS THE RATE OF CHANGE OF THE FUNCTION f WITH RESPECT TO x AT THE POINT $(x, f(x))$. IN THIS SECTION, YOU WILL SEE THAT THERE ARE MANY REAL-LIFE APPLICATIONS OF RATES OF CHANGE. A FEW AR ACCELERATION, POPULATION GROWTH RATES, UNEMPLOYMENT RATES, PRODUCTION RATE FLOW RATES. ALTHOUGH RATES OF CHANGE OFTEN INVOLVE CHANGE WITH RESPECT TO TIM INVESTIGATE THE RATE OF CHANGE OF ONE VARIABLE WITH RESPECT TO ANY OTHER RELAT

WHEN DETERMINING THE RATE OF CHANGE OF ONE VARIABLE WITH RESPECT TO ANOTHER, BE CAREFUL TO DISTINGUISH BETWEEN AVERAGE AND INSTANTANEOUS RATES OF CHA DISTINCTION BETWEEN THESE TWO RATES OF CHANGE IS COMPARABLE TO THE DISTINCTIO THE SLOPE OF THE SECANT LINE THROUGH TWO POINTS ON A GRAPH AND THE SLOPE OF TH LINE AT ONE POINT ON THE GRAPH.

THE SLOPE OF THE TANGENT LINE IS THE DERIVATIVE OF A FUNCTION AT THE GIVEN POIN REGARDED AS THE INSTANTANEOUS RATE OF CHANGE:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{INSTANTANEOUS RATE OF CHANGE}$$

BUT THE SLOPE OF A SECANT LINE IS DETERMINED BY TWO POINTS GIVEN ON THE LINE; REGARDED AS THE AVERAGE RATE OF CHANGE:

$$\frac{f(b) - f(a)}{b - a} = \text{AVERAGE RATE OF CHANGE}$$

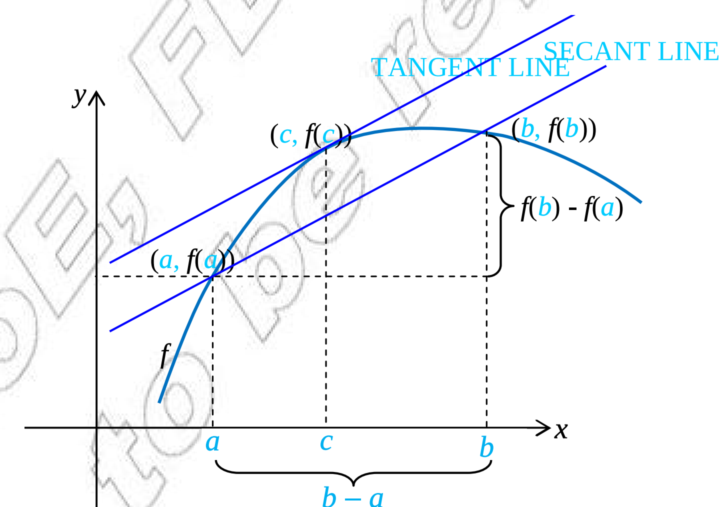


Figure 4.16

Example 1 THE CONCENTRATION C (IN MILLIGRAMS PER MILLILITRE) OF A DRUG IN A PATI BLOOD STREAM IS MONITORED AT 10-MINUTE INTERVALS FOR 2 HRS, WHERE MEASURED IN MINUTES, AS SHOWN IN THE TABLE. FIND THE AVERAGE RATI CHANGE OVER EACH INTERVAL.

A [0, 10] **B** [0, 40] **C** [100, 120]

t	0	10	20	30	40	50	60	70	80	90	100	110	120
c	0	2	17	37	55	73	89	103	111	113	113	103	68

Solution

A FOR THE INTERVAL [0, 10], THE AVERAGE RATE OF CHANGE IS

$$\frac{\Delta c}{\Delta t}=\frac{2-0}{10-0}=\frac{2}{10}=0.2 \text{ MG PER ML/ MI}$$

B FOR THE INTERVAL [0, 40], THE AVERAGE RATE OF CHANGE IS

$$\frac{\Delta c}{\Delta t}=\frac{55-0}{40-0}=\frac{55}{40}=\frac{11}{8} \text{ MG PER ML/ MI}$$

C FOR THE INTERVAL [100, 120], THE AVERAGE RATE OF CHANGE IS

$$\frac{\Delta c}{\Delta t}=\frac{68-113}{120-100}=\frac{-45}{20}=\frac{-9}{4} \text{ MG PER ML/MIN}$$

Example 2 IF A FREE-FALLING OBJECT IS DROPPED FROM A HEIGHT OF 100 M AND RESISTANCE IS NEGLECTED, THE HEIGHT h (IN METRE) OF THE OBJECT AT TIME t (IN SECONDS) IS GIVEN BY $h(t) = -16t^2 + 100$.

I FIND THE AVERAGE VELOCITY OF THE OBJECT OVER

A [1, 2] **B** [1, 1.5] **C** [0, 2]

II FIND THE INSTANTANEOUS RATE OF CHANGE AT

A $t = 1$ SEC **B** $t = 2$ SEC **C** $t = 3$ SEC **D** $t = 1.5$ SEC

Solution

I **A** $h(1) = 84$, $h(2) = 36$

AVERAGE VELOCITY OVER [1, 2] IS GIVEN BY:

$$\frac{h(2)-h(1)}{2-1}=\frac{36-84}{1}=-48\,m/\text{SEC}$$

B $h(1) = 84$, $h(1.5) = 64$

AVERAGE VELOCITY OVER [1, 1.5] IS GIVEN BY

$$\frac{h(1.5)-h(1)}{1.5-1}=\frac{64-84}{0.5}=-40 \text{ M/SEC}$$

c $h(0) = 100$, $h(2) = 36$

AVERAGE VELOCITY OVER [0, 2] IS GIVEN BY

$$\frac{h(2)-h(0)}{2-0} = \frac{36-100}{2} = -32\, m/\text{SEC}$$

ii $h(t) = -16t^2 + 100 \Rightarrow h'(t) = -32t$.

THUS, THE INSTANTANEOUS RATES OF CHANGE ARE GIVEN AS FOLLOWS:

a $h'(1) = -32$ M/SEC **b** $h'(2) = -64$ M/SEC

c $h'(3) = -96$ M/SEC **d** $h'(1.5) = -48$ M/SEC

Exercise 4.7

1 THE HEIGHT h (IN METERS) OF A FREE-FALLING OBJECT AT TIME t (IN SECONDS) IS GIVEN BY $h(t) = -16t^2 + 180$. FIND

i THE AVERAGE VELOCITY OF THE OBJECT OVER THESE INTERVALS

a [0, 1] **b** [1, 2] **c** [2, 3] **d** [1, 5]

ii THE INSTANTANEOUS VELOCITY OF THE OBJECT AT

a $t = 0.5$ SEC **b** $t = 1$ SEC **c** $t = 1.5$ SEC **d** $t = 2$ SEC

2 THE POPULATION OF A DEVELOPING RURAL AREA HAS BEEN GROWING ACCORDING TO THE MODEL $P(t) = 22t^2 + 52t + 10{,}000$, WHERE t IS TIME IN YEARS, WITH $t = 0$ REPRESENTING THE YEAR 2000 E.C.

a EVALUATE P FOR $t = 0$, $t = 5$, $t = 8$ AND $t = 10$. EXPLAIN THESE VALUES.

b DETERMINE THE POPULATION GROWTH RATE, $\frac{dP}{dt}$.

c EVALUATE $\frac{dP}{dt}$ FOR THE SAME VALUES AS IN PART **a**. EXPLAIN YOUR RESULTS.

Related rates

IN THIS SECTION, YOU WILL STUDY PROBLEMS INVOLVING VARIABLES THAT ARE CHANGING WITH RESPECT TO TIME. IF TWO OR MORE SUCH VARIABLES ARE RELATED TO EACH OTHER, THEN THEIR RATES OF CHANGE WITH RESPECT TO TIME ARE ALSO RELATED.

FOR INSTANCE, SUPPOSE THAT x AND y ARE RELATED BY THE EQUATION $y = 2x$. IF BOTH VARIABLES ARE CHANGING WITH RESPECT TO TIME THEN THEIR RATES OF CHANGE WILL ALSO BE RELATED, BY

$$\frac{dy}{dt} = 2\frac{dx}{dt}$$

Examining two rates that are related

Example 3 A STONE IS DROPPED INTO A CALM POOL OF WATER, CAUSING RIPPLES IN THE F OF CONCENTRIC CIRCLES, AS SHOWN IN THE FIGURE BELOW. THE RADIUS r OF THE OUTER RIPPLE IS INCREASING AT A CONSTANT RATE OF 1 CM PER SECOND. WHE RADIUS IS 4 CM, AT WHAT RATE IS THE TOTAL AREA A OF THE DISTURBED V CHANGING?

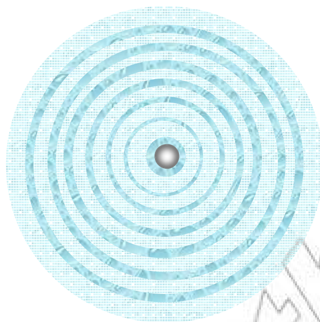


Figure 4.17

Solution THE RADIUS r, AND THE AREA A OF A CIRCLE ARE RELATED AS FOLLOWS:

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

WHEN $r = 4$ AND $\frac{dr}{dt} = 1$, WE HAVE $2\pi r\frac{dr}{dt} = 2\pi(4)(1) = 8\pi\ \text{CM}^2/\text{SEC}$.

THEREFORE, THE AREA IS CHANGING AT THE RATE OF $8\pi\ \text{CM}^2/\text{SEC}$.

Example 4 AIR IS BEING PUMPED INTO A SPHERICAL BALLOON AT THE RATE OF $4.5\text{CM}^3/\text{MIN}$. FIND THE RATE OF CHANGE OF THE RADIUS WHEN THE RADIUS IS 2CM.

Solution LET r BE THE RADIUS OF THE SPHERE, THEN THE VOLUME V OF THE SPHERE IS GIVEN BY $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(2)^2}\times 4.5 \ (\text{SINCE } \frac{dV}{dt} = 4.5)$$

$$= \frac{4.5}{16\pi}\text{CM}/\text{MIN} \approx 0.09\,\text{CM}/\text{MIN}$$

Example 5 A LADDER 5M LONG RESTS AGAINST A VERTICAL WALL. IF THE BOTTOM OF THE SLIDES AWAY FROM THE WALL AT A RATE OF $\frac{1}{4}$ M/SEC, HOW FAST IS THE TOP OF THE LADDER SLIDING DOWN THE WALL WHEN THE BOTTOM OF THE LADDER IS 3M FR THE WALL?

Solution

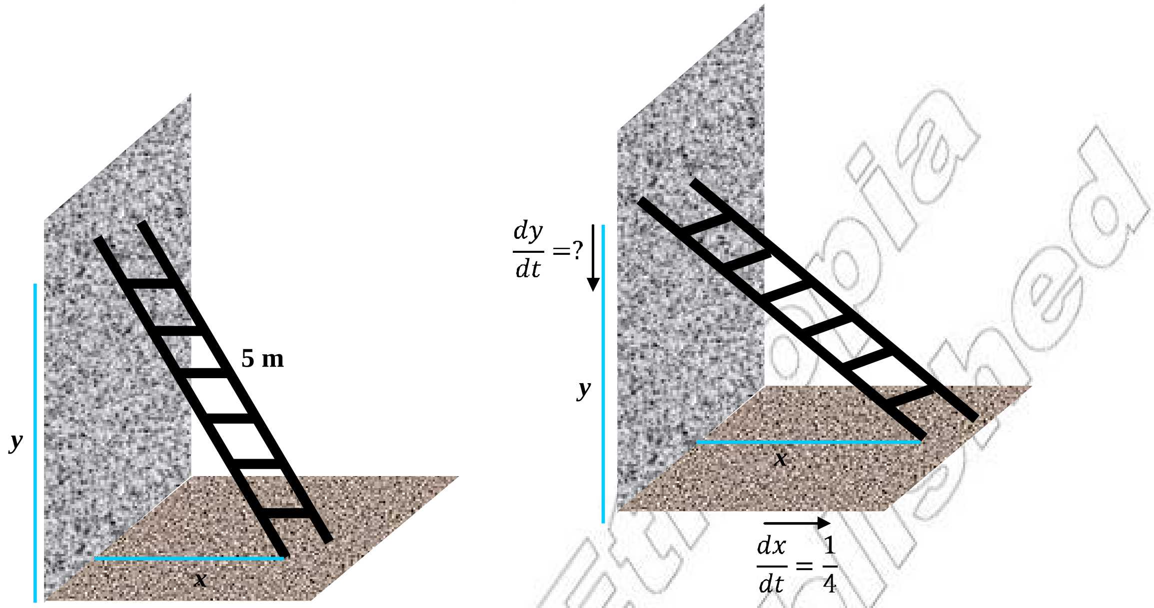


Figure 4.18

$$3^2 + y^2 = 25 \Rightarrow y^2 = 16$$

$$\Rightarrow y = 4$$

$$x^2 + y^2 = 25 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-3}{4}\left(\frac{1}{4}\right) = \frac{-3}{16}\text{ M/ SEC.}$$

THE FACT THAT $\frac{dy}{dt}$ IS NEGATIVE MEANS THAT THE DISTANCE FROM THE TOP OF THE LADDE

THE GROUND IS DECREASING AT A RATE OF $\frac{3}{16}$ M/SEC. IN OTHER WORDS, THE TOP OF THE LADDER

IS SLIDING DOWN THE WALL AT A RATE OF $\frac{3}{16}$ M/SEC.

Example 6 A WATER TANK IS IN THE SHAPE OF AN INVERTED CIRCULAR CONE WITH B RADIUS 3 M AND HEIGHT 5 M. IF WATER IS BEING PUMPED INTO THE TANK AT RATE OF 2 M^3/MIN, FIND THE RATE AT WHICH THE WATER LEVEL IS RISING WHEN THE WATER IS 3 M DEEP.

Solution

$$\frac{r}{h}=\frac{3}{5}\Rightarrow h=\frac{5}{3}r \;\Rightarrow r=\frac{3}{5}h$$

$$V=\frac{1}{3}\; r^2h\;=\;\frac{1}{3}\;\left(\frac{3}{5}h\right)^2 h\;=\;\frac{3}{25}\; h^3$$

$$\frac{dV}{dt}=\frac{9}{25}\; h^2\frac{dh}{dt}\;=\;\frac{1}{3}\; r^2\left(\frac{5}{3}r\right)$$

$$\Rightarrow \frac{dh}{dt}=\frac{25}{9\;h^2}\frac{dv}{dt}=\frac{25}{9\;(3)^2}\times 2\text{M/ MIN}=\frac{50}{81}\text{ M/ MIN}$$

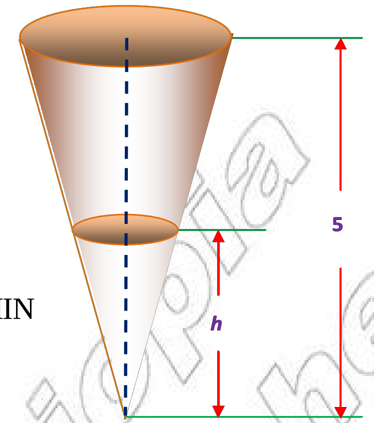


Figure 4.19

Example 7 GIVEN $x^2y + xy = 6$,

A FIND THE RATE OF CHANGE OF x WITH RESPECT TO y.

B FIND THE RATE OF CHANGE OF y WITH RESPECT TO x.

Solution

A IN THIS CASE WE ASSUME x IS DIFFERENTIABLE WITH RESPECT TO y.

THUS $\frac{d}{dy}\left(x^2y+xy\right)=\frac{d}{dy}(6)$

$$\Rightarrow x^2 + x + 2xy\,\frac{dx}{dy}+y\frac{dx}{dy}=0$$

$$\Rightarrow (2xy + y)\frac{dx}{dy}=-x^2-x$$

$$\Rightarrow \frac{dx}{dy}=\frac{-x^2-x}{2xy+y}$$

B $\frac{d}{dx}\left(x^2y+xy\right)=\frac{d}{dx}(6)$

$$\Rightarrow y(2x) + x^2\,\frac{dy}{dx}+y+x\frac{dy}{dx}=0$$

$$\Rightarrow (x^2 + x)\,\frac{dy}{dx}=-2xy-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2xy-y}{x^2+x}=\frac{1}{\dfrac{-2xy-y}{x^2+y}}=\frac{1}{\dfrac{dx}{dy}}$$

Example 8 THE TOTAL SALES S (IN THOUSANDS OF COPIES OF MOVIES) FOR A HOME VIDEO MOVIE t MONTHS AFTER THE MOVIE IS INTRODUCED ARE GIVEN BY:

$$S(t)=\frac{125t^2}{t^2+100}$$

A FIND THE RATE OF CHANGE OF SALES, $S'(t)$, AT TIME t.

B FIND $S(10)$ AND $S'(10)$. GIVE A BRIEF VERBAL INTERPRETATION OF THESE VALUES.

C USE THE RESULTS FROM **B** ABOVE TO ESTIMATE THE TOTAL SALES AFTER 11 MONTHS.

Solution

A

$$S'(t)=\frac{(t^2+100)(125t^2)'-125t^2(t^2+100)'}{(t^2+100)^2}$$

$$=\frac{(250t)\left(t^2+100\right)-(2t)125t^2}{\left(t^2+100\right)^2}=\frac{250t^3+25000t-250t^3}{\left(t^2+100\right)^2}$$

$$=\frac{25,000t}{(t^2+100)^2}$$

B $S(10)=\dfrac{125(10)^2}{10^2+100}=62.5$, AND

$$S'(10)=\frac{25,000(10)}{\left(10^2+100\right)^2}=6.25$$

THE TOTAL SALES AFTER 10 MONTHS ARE 62,500 COPIES OF MOVIES, AND SALES ARE INCREASING AT THE RATE OF 6,250 COPIES PER MONTH.

C THE TOTAL SALES WILL INCREASE BY APPROXIMATELY 6,250 COPIES DURING THE NEXT MONTH. THUS, THE ESTIMATED TOTAL SALES AFTER 11 MONTHS ARE

62,500 + 6,250 = 68,750 COPIES OF THE MOVIE.

Exercise 4.8

1. THE RADIUS r OF A CIRCLE IS INCREASING AT A RATE OF 3 CM/MIN. FIND THE RATE OF CHANGE OF THE AREA WHEN

   A r = 8 CM **B** r = 12 CM

2. THE RADIUS r OF A SPHERE IS INCREASING AT A RATE OF 3 CM/MIN. FIND THE RATE OF CHANGE OF THE VOLUME WHEN

   A r = 2 CM **B** r = 3 CM

3. A 10 M LADDER IS LEANING AGAINST A HOUSE. THE BASE OF THE LADDER IS PULLED AWAY FROM THE HOUSE AT A RATE OF $\frac{1}{4}$ M/SEC. HOW FAST IS THE TOP OF THE LADDER MOVING DOWN THE WALL WHEN THE BASE IS

   A 6 M FROM THE HOUSE? **B** 8 M FROM THE HOUSE?

   C 9 M FROM THE HOUSE?

4 FIND $\frac{dy}{dx}$ AND $\frac{dx}{dy}$ ASSUMING THAT y IS DIFFERENTIABLE WITH RESPECT TO x AND x IS ALSO DIFFERENTIABLE WITH RESPECT TO y.

A $x^2 + y^2 = 25$ **B** $3xy + y^2x - x^2y = 10$

C $x + xy^2 - y = xy$ **D** $xy + x^2y^2 = x^3y^3$

E $x \sin y + y \cos x = xy$

5 A SPHERICAL BALLOON IS INFLATED WITH GAS AT THE RATE OF 20 CM^3/MIN. HOW FAST IS THE RADIUS OF THE BALLOON CHANGING AT THE INSTANT WHEN THE RADIUS IS

A 1 CM? **B** 2 CM? **C** 3 CM?

6 THE RADIUS r OF A RIGHT CIRCULAR CONE IS INCREASING AT A RATE OF 2 CM/MIN. THE HEIGHT h OF THE CONE IS RELATED TO THE RADIUS BY $h = 3r$. FIND THE RATE OF CHANGE OF THE VOLUME WHEN

A r = 3CM **B** r = 6 CM

Key Terms

absolute maximum	decreasing function	monotonicity
absolute minimum	extreme values	relative maximum
concave downward	first derivative test	relative minimum
concave upward	increasing function	Rolle's theorem
concavity	inflection point	second derivative test
critical number	mean-value theorem	

Summary

AFTER STUDYING THIS UNIT, YOU SHOULD KNOW THE DEFINITION OF THE FOLLOWING TECHN AND HAVE ACQUIRED THE SKILLS TO FIND THEM OR TEST THEM.

1 ***Critical number***

SUPPOSE f IS DEFINED AT c AND EITHER $f'(c) = 0$ OR $f'(c)$ DOES NOT EXIST. THEN THE NUMBER c IS CALLED A critical number OF f AND THE POINT WITH COORDINATES $(c, f(c))$ ON THE GRAPH OF f IS CALLED A critical point; THIS CRITICAL POINT IS EITHER A valley OR A peak OF THE GRAPH.

2 ***Absolute maximum and absolute minimum***

LET f BE A FUNCTION DEFINED ON A SET S THAT CONTAINS c. THEN

$f(c)$ IS AN absolute maximum OF f ON S IF $f(c) \geq f(x)$ FOR ALL x IN S.

$f(c)$ IS AN absolute minimum OF f ON S IF $f(c) \leq f(x)$ FOR ALL x IN S.

3 ***Relative maximum and relative minimum***

THE FUNCTION f IS SAID TO HAVE A relative maximum AT c, IF $f(c) \geq f(x)$ FOR ALL x IN AN OPEN INTERVAL CONTAINING c.

THE FUNCTION f IS SAID TO HAVE A relative minimum AT c, IF $f(c) \leq f(x)$ FOR ALL x IN AN OPEN INTERVAL CONTAINING c.

4 ***First derivative test***

LET f BE A FUNCTION WHICH IS CONTINUOUS AND DIFFERENTIABLE ON AN INTERVAL I. THEN

A ***First derivative test for local extreme values***

IF f' CHANGES SIGN FROM POSITIVE TO NEGATIVE AT c, THEN f HAS A local maximum VALUE AT c FOR SOME CRITICAL NUMBER c.

IF f' CHANGES SIGN FROM NEGATIVE TO POSITIVE AT c, THEN f HAS A local minimum VALUE AT c FOR SOME CRITICAL NUMBER c.

B ***First derivative test for intervals of monotonicity***

IF $f'(x) > 0$ ON I, THEN f IS strictly increasing ON I; IF $f'(x) < 0$ ON I, THEN f IS strictly decreasing ON I.

5 ***Second derivative test***

LET f BE A FUNCTION SUCH THAT $f'(c) = 0$ AND THE SECOND DERIVATIVE EXISTS ON AN OPEN INTERVAL CONTAINING c. THEN

A ***Second derivative test for local extreme values***

IF $f''(c) > 0$ THEN $f(c)$ IS A LOCAL MINIMUM VALUE ON f.

IF $f''(c) < 0$ THEN $f(c)$ IS A LOCAL MAXIMUM VALUE ON f.

IF $f''(c) = 0$, THEN THE TEST FAILS.

B ***Second derivative test for intervals of concavity***

IF $f''(x) > 0$ FOR ALL x IN I THEN THE GRAPH OF f IS concave upward ON I.

IF $f''(x) < 0$ FOR ALL x IN I THEN THE GRAPH OF f IS concave downward ON I.

6 ***Inflection point***

THE POINT AT WHICH CONCAVITY CHANGES, EITHER FROM CONCAVE UP TO CONCAVE DOWN OR FROM CONCAVE DOWN TO CONCAVE UP IS CALLED AN inflection point.

Review Exercises on Unit 4

1 FOR EACH OF THE FOLLOWING FUNCTIONS FIND: CRITICAL NUMBERS, LOCAL EXTREME V INTERVALS OF MONOTONICITY, INTERVALS OF CONCAVITY AND INFLECTION POINTS.

A $f(x) = x^4 - 8x^2 + 6$ **B** $f(x) = x^3 + 3x^2 - 9x + 5$

C $f(x) = \dfrac{2x}{x^2 + 1}$ **D** $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$

2 FIND THE ABSOLUTE MAXIMUM AND MINIMUM VALUES OF EACH OF THE FOLLOWING FUNCTIONS ON THE INDICATED INTERVALS.

A $f(x) = x^4 - 8x^2 + 6$; $[-3, 3]$ **B** $f(x) = x^3 + 3x^2 - 9x + 5$; $[-2, 2]$

C $f(x) = \dfrac{2x}{x^2 + 1}$; $[1, 2]$ **D** $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$; $[-3, 1]$

3 A BOX IS TO HAVE A SQUARE BASE, AN OPEN TOP, AND A VOLUME OF 32 M^3. FIND THE DIMENSIONS OF THE BOX THAT USE THE LEAST AMOUNT OF MATERIAL.

4 DETERMINE THE POINT(S) ON $f(x) = x^2 + 1$ THAT ARE CLOSEST TO THE POINT (0, 2).

5 FIND THE MAXIMUM AND MINIMUM OF THE FUNCTION $f(x) = \cos 2x - 2 \sin x$ FOR $0 \le x \le 2$.

6 A WINDOW WHOSE BOTTOM IS A RECTANGLE AND TOP IS A SEMICIRCLE BEING BUILT. IT IS 12M OF FRAMING MATERIALS, THEN WHAT MUST BE THE DIMENSION OF THE WINDOW

7 DETERMINE THE AREA OF THE LARGEST RECTANGLE THAT CAN BE INSCRIBED IN A RADIUS 9 M.

8 WATER IS BEING POURED INTO A CONICAL VASE AT A RATE OF 18 CM^3/SEC. THE DIAMETER OF THE CONE IS 30 CM AND ITS HEIGHT IS 25 CM. AT WHAT RATE IS THE WATER LEVEL R WHEN ITS DEPTH IS 20 CM?

9 TWO POLES, ONE 6 M TALL AND THE OTHER 15 M TALL, ARE 20 M APART. A WIRE IS ATTA TO THE TOP OF EACH POLE AND IS ALSO STAKED TO THE GROUND SOMEWHERE BETW TWO POLES. WHERE SHOULD THE WIRE BE STAKED SO THAT THE MINIMUM AMOUNT OF IS USED?

10 A CAR TRAVELLING NORTH AT 48 KM/HR IS APPROACHING AN INTERSECTION. A T TRAVELLING EAST AT 60 KM/HR IS MOVING AWAY FROM THE SAME INTERSECTION. HOW DISTANCE BETWEEN THE CAR AND THE TRUCK CHANGING WHEN THE CAR IS 9 M FRO INTERSECTION AND THE TRUCK IS 40 M FROM THE INTERSECTION?