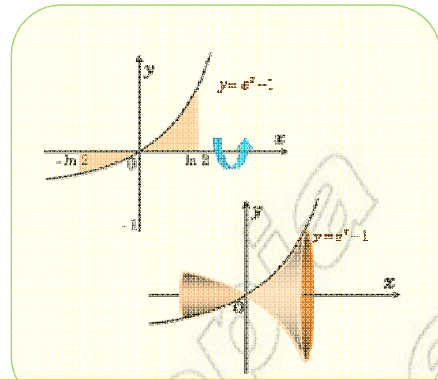


# Unit

# 5



## INTRODUCTION TO INTEGRAL CALCULUS

### Unit Outcomes:

*After completing this unit, you should be able to:*

- *understand the concept of definite integral.*
- *integrate polynomial functions, simple trigonometric functions, exponential and logarithmic functions.*
- *use the various techniques of integration to evaluate a given integral.*
- *use the fundamental theorem of calculus for computing definite integrals.*
- *apply the knowledge of integral calculus to solve real life mathematical problems.*

### Main Contents

#### 5.1 INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

#### 5.2 TECHNIQUES OF INTEGRATION

#### 5.3 DEFINITE INTEGRALS, AREA AND FUNDAMENTAL THEOREM OF CALCULUS

#### 5.4 APPLICATIONS OF INTEGRAL CALCULUS

*Key terms*

*Summary*

*Review Exercises*

## INTRODUCTION

YOU HAVE JUST SEEN **differential calculus**, WHICH IS ONE OF THE TWO BRANCHES OF CALCULUS. IN THIS UNIT YOU SHALL SEE THE OTHER BRANCH OF CALCULUS, CALLED INTEGRATION IS THE REVERSE PROCESS OF DIFFERENTIATION. IT IS THE PROCESS OF FINDING A FUNCTION ITSELF WHEN ITS DERIVATIVE IS KNOWN.

FOR EXAMPLE, IF THE SLOPE OF A TANGENT AT AN ARBITRARY POINT OF A CURVE IS KNOWN, IT IS POSSIBLE TO DETERMINE THE EQUATION OF THE CURVE USING THE METHOD OF INTEGRAL CALCULUS. IT IS POSSIBLE TO FIND DISTANCE OF A MOVING OBJECT IN TERMS OF TIME, IF ITS VELOCITY AND ACCELERATION IS KNOWN.

DIFFERENTIAL CALCULUS DEALS WITH RATE OF CHANGE OF FUNCTIONS, WHEREAS INTEGRAL CALCULUS DEALS WITH TOTAL SIZE OR VALUE SUCH AS AREAS ENCLOSED BY CURVES, VOLUMES OF SOLIDS, LENGTHS OF A CURVES, TOTAL MASS, TOTAL FORCE, ETC.

DIFFERENTIAL CALCULUS AND INTEGRAL CALCULUS ARE CONNECTED BY A THEOREM CALLED **fundamental theorem of calculus**.

IN INTEGRAL CALCULUS THERE ARE TWO KINDS OF INTEGRATIONS WHICH ARE CALLED THE **indefinite integral** OR THE ANTI DERIVATIVE AND THE **definite integral**.

THE INDEFINITE INTEGRAL OR THE ANTI DERIVATIVE INVOLVES FINDING THE FUNCTION WHEN ITS DERIVATIVE IS KNOWN.

THE DEFINITE INTEGRAL, DENOTED BY  $\int_a^b f(x) dx$  IS INFORMALLY DEFINED TO BE THE SIGNED AREA OF THE REGION IN THE PLANE BOUNDED BY THE CURVE  $y = f(x)$ , THE X-AXIS AND THE VERTICAL LINES  $x = a$  AND  $x = b$ .

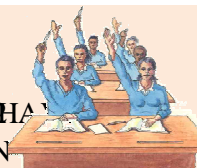
ONE OF THE MAIN GOALS OF THIS UNIT IS TO EXAMINE THE THEORY OF INTEGRAL CALCULUS AND TO INTRODUCE YOU TO ITS NUMEROUS APPLICATIONS IN SCIENCE AND ENGINEERING.

### 5.1 INTEGRATION AS REVERSE PROCESS OF DIFFERENTIATION

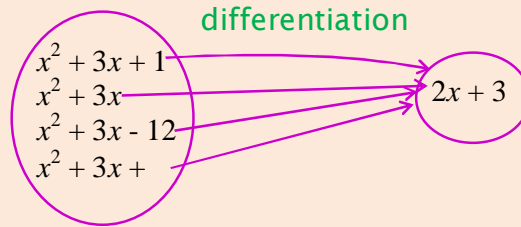
#### 5.1.1 The Concept of Indefinite Integral

#### ACTIVITY 5.1

- 1 FIND AT LEAST THREE DIFFERENT FUNCTIONS WHICH HAVE THE SAME DERIVATIVE. DESCRIBE SIMILARITIES (AND DIFFERENCES) BETWEEN THEM.
- 2 WRITE THE SET OF ALL FUNCTIONS WITH DERIVATIVE  $2x$ .



**3** CHECK THAT ALL OF THE FUNCTIONS:  
 $f(x) = x^2 + 3x + 1$ ,  $g(x) = x^2 + 3x$ ,  $h(x) = x^2 + 3x - 12$  AND  $k(x) = x^2 + 3x +$   
 HAVE THE SAME DERIVATIVE 2

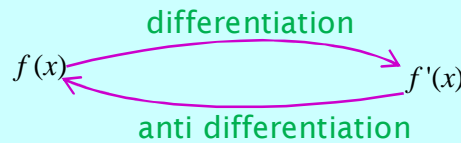


IN GENERAL  $\frac{d}{dx}(x^2 + 3x + c) = 2x + 3$  FOR ANY CONSTANT

DRAW THE GRAPHS OF  $f$  AND  $h$  TOGETHER, USING SAME PAIR OF AXES OF REFERENCE.

**Definition 5.1**

THE PROCESS OF FINDING FROM ITS DERIVATIVE IS SAID TO BE ANTI DIFFERENTIATION OR INTEGRATION. THIS IS SAID TO BE THE DERIVATIVE OF  $f'(x)$ .



Integration is the reverse operation of differentiation.

**Definition 5.2**

THE SET OF ALL ANTI DERIVATIVES (OF A FUNCTION) IS THE INDEFINITE INTEGRAL OF  $f(x)$ . THE INDEFINITE INTEGRAL IS DENOTED BY  $\int f(x)dx$  READ AS "THE INTEGRAL OF WITH RESPECT TO

- ✓ THE SYMBOL  $\int$  IS SAID TO BE THE INTEGRAL SIGN.
- ✓ THE FUNCTION IS SAID TO BE THE INTEGRAND OF THE INTEGRAL.
- ✓  $dx$  DENOTES THAT THE VARIABLE OF INTEGRATION IS
- ✓ IF A FUNCTION HAS AN INTEGRAL, THEN IT IS SAID TO BE INTEGRABLE.
- ✓ IF  $F'(x) = f(x)$ , THEN  $\int f(x)dx = F(x) + c$
- ✓  $\int f(x)dx$  IS READ AS, "THE INTEGRAL OF WITH RESPECT TO
- ✓  $c$  IS SAID TO BE THE CONSTANT OF INTEGRATION.

**Example 1**  $\int x dx = \frac{x^2}{2} + c$  BECAUSE  $\frac{d}{dx}\left(\frac{x^2}{2} + c\right) = \frac{2x}{2} + 0 = x$

**Note:**

**I**  $\int f'(x)dx = f(x) + c$

**II**  $\int \frac{d}{dx} f(x)dx = f(x) + c$

**III**  $\frac{d}{dx} \int f(x)dx = f(x)$

**Example 2**  $\int \frac{d}{dx}(4x+5)dx = \int 4dx = 4x + c$  BECAUSE  $\frac{d}{dx}(4x+c) = 4$

**Example 3** YOU KNOW THAT  $\frac{d}{dx}(x^6) = 6x^5 \Rightarrow \frac{1}{6} \frac{d}{dx}(x^6) = x^5$

$$\begin{aligned} \Rightarrow \int x^5 dx &= \int \frac{1}{6} \frac{d}{dx}(x^6) dx \\ &= \int \frac{d}{dx} \left( \frac{x^6}{6} \right) dx = \frac{x^6}{6} + c \end{aligned}$$

AGAIN  $\frac{d}{dx} \int x^5 dx = \frac{d}{dx} \left( \frac{x^6}{6} + c \right) = x^5$

**Integration of some simple functions**

**ACTIVITY 5.2**



**1** COPY AND FILL IN THE FOLLOWING TABLE

$f(x)$	4	$x$	$x^2$	$x^3$	$x^{10}$	$x^n$	SIN $x$	COS $x$	TAN $x$	COT $x$	$e^x$	$4^x$	LN $x$	LOG $x$
$f'(x)$														

**2** BY OBSERVING THE TABLE ABOVE, EVALUATE EACH OF THE FOLLOWING INTEGRALS.

**A**  $\int x^4 dx$

**B**  $\int \text{SIN}x dx$

**C**  $\int \text{COS}x dx$

**D**  $\int \text{SEC}^2x dx$

**E**  $\int \text{CSC}^2x dx$

**F**  $\int e^x dx$

**G**  $\int 4^x dx$

**H**  $\int \frac{1}{x} dx$

**I**  $\int \frac{1}{x \text{LN}10} dx$

IN THIS SECTION, YOU WILL SEE HOW TO FIND THE INTEGRALS OF CONSTANT, POWER, EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND SIMPLE TRIGONOMETRIC FUNCTIONS.

**The integration of a constant function**

$\int 0 dx = c$ , where  $c$  is a constant.

$\int cdx = cx + d$ , where  $c$  is a given constant and  $d$  is the constant of integration.

When  $c = 1$ ,  $\int dx = x + d$ .

**Integrating  $x^n$ , integration of a power function**

Differentiating  $x^{n+1}$  gives  $(n+1)x^n$ .

So  $\int (n+1)x^n dx = x^{n+1} + c$

Thus  $\int x^n dx = \frac{x^{n+1}}{n+1} + c; n \neq -1$ .

**Example 4** INTEGRATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO  $x$

**A** 4

**B**  $x^7$

**C**  $x^{-5}$

**D**  $x^{\frac{1}{2}}$

**E**  $x^{-\frac{3}{5}}$

**F**  $x^{-\frac{4}{3}}\sqrt{x}$

**Solution**

**A**  $\int 4 dx = 4x + c$

**B**  $\int x^7 dx = \frac{x^{7+1}}{7+1} + c = \frac{x^8}{8} + c$

**C**  $\int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + c = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$

**D**  $\int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

**E**  $\int \frac{1}{x^{-\frac{3}{5}}} dx = \frac{x^{-\frac{3}{5}+1}}{-\frac{3}{5}+1} + c = \frac{x^{\frac{2}{5}}}{\frac{2}{5}} + c = \frac{5x^{\frac{2}{5}}}{2} + c$

**F**  $\int x^{-\frac{4}{3}}\sqrt{x} dx = \int x^{-\frac{4}{3}+\frac{1}{2}} dx = \frac{x^{-\frac{5}{6}}}{-\frac{5}{6}} = -\frac{6}{5}\sqrt[6]{x}$

LET  $k$  BE A CONSTANT AND THEN  $\int kx^n dx = \frac{k}{n+1}x^{n+1} + c$ .

**Integrating  $(ax + b)^n$  with respect to  $x$** 

**Example 5** LET  $y = (3x + 5)^{10}$ , THEN USING THE SUBSTITUTION

$u = 3x + 5$ , WE HAVE  $u^{10}$ .

THEN,  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 10u^9 \times 3 = 3 \times 10 (3x + 5)^9$

$$\int 3 \times 10 (3x + 5)^9 dx = (3x+5)^{10} + c$$

IN GENERAL, BY APPLYING THE SAME TECHNIQUES WE HAVE

$$\frac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n \text{ SO THAT}$$

$$\int a(n+1)(ax+b)^n dx = (ax+b)^{n+1} + c.$$

$$\text{THUS } \int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \text{ WHERE } n \neq -1 \text{ AND } a \neq 0.$$

**Note:**

$$\int k(ax+b)^n dx = \frac{k}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1 \text{ AND } a \neq 0$$

**Example 6** INTEGRATE EACH OF THE FOLLOWING FUNCTIONS WITH RESPECT TO

**A**  $5x^6$       **B**  $\frac{1}{2x^4}$       **C**  $(2x-1)^{11}$       **D**  $(4x+3)^8$

**E**  $5(2-3x)^{\frac{1}{2}}$       **F**  $4\sqrt[3]{1-x^5}$       **G**  $(3x+5)^3\sqrt{3x+5}$

**Solution**

**A** USING  $\int kx^n dx = \frac{k}{n+1}x^{n+1} + c$ , YOU GET  $\int 5x^6 dx = \frac{5}{7}x^7 + c$

**B**  $\int \frac{1}{2x^4} dx = \int \frac{1}{2}x^{-4} dx = \frac{1}{2} \left( \frac{x^{-3}}{-3} \right) + c = -\frac{1}{6x^3} + c$

**C**  $\int (2x-1)^{11} dx = \frac{1}{2(11+1)}(2x-1)^{11+1} = \frac{(2x-1)^{12}}{24} + c$

**D**  $\int (4x+3)^8 dx = \frac{1}{4 \times 9}(4x+3)^9 + c = \frac{(4x+3)^9}{36} + c$

**E**  $\int 5(2-3x)^{\frac{1}{2}} dx$ . HERE  $k = 5$ ,  $a = -3$ ,  $n = \frac{1}{2}$

$$\text{HENCE } \int 5(2-3x)^{\frac{1}{2}} dx = \frac{5}{(-3)\left(\frac{1}{2}+1\right)}(2-3x)^{\frac{1}{2}+1} + c$$

$$= -\frac{10}{9}(2-3x)\sqrt{2-3x} + c$$

**F**  $\int 4\sqrt[3]{(1-x)^5} dx = \frac{4}{-1\left(\frac{5}{3}+1\right)}(1-x)^{\frac{5}{3}+1} + c = -\frac{3}{2}(1-x)^2\sqrt[3]{(1-x)^2} + c$

**G**  $\int (3x+5)^3\sqrt{3x+5} dx = \int (3x+5)^{\frac{7}{2}} dx = \frac{(3x+5)^{\frac{9}{2}}}{3 \times \frac{9}{2}} + c = \frac{2(3x+5)^4\sqrt{3x+5}}{27} + c$

**Exercise 5.1**

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO

- |           |                     |           |                     |           |                            |           |                        |
|-----------|---------------------|-----------|---------------------|-----------|----------------------------|-----------|------------------------|
| <b>1</b>  | $x^3$               | <b>2</b>  | $2x^4$              | <b>3</b>  | $x^{-3}$                   | <b>4</b>  | $x^{\frac{2}{5}}$      |
| <b>5</b>  | $\frac{4}{x^{1.5}}$ | <b>6</b>  | $6x^2\sqrt{x}$      | <b>7</b>  | $\frac{1}{8\sqrt[3]{x}}$   | <b>8</b>  | $(3x-1)^6$             |
| <b>9</b>  | $\sqrt[3]{1-2x}$    | <b>10</b> | $8\sqrt[4]{4-3x^3}$ | <b>11</b> | $\frac{3}{\sqrt[4]{4-5x}}$ | <b>12</b> | $(2x-3)^{\frac{1}{2}}$ |
| <b>13</b> | $(4x- )^{\sqrt{2}}$ |           |                     |           |                            |           |                        |

**Integration of exponential functions**

YOU SHOULD REMEMBER THAT

 HENCE,  $\int e^x dx = e^x + c$ . ALSO,  $\frac{d}{dx}(ke^x) = ke^x$ , HENCE  $\int ke^x dx = ke^x + c$ 

 SIMILARLY  $\frac{d}{dx}e^{kx} = ke^{kx} \Rightarrow \frac{1}{k} \frac{d}{dx}e^{kx} = e^{kx}$ . HENCE  $\int e^{kx} dx = \frac{e^{kx}}{k} + c$ 

 FOR  $a > 0$ ,  $\frac{d}{dx}a^x = a^x \ln a \Rightarrow \frac{1}{\ln a} \frac{d}{dx}a^x = a^x$ 

 HENCE  $\int a^x \ln a dx = a^x + c$ 

 THUS  $\int a^x dx = \frac{a^x}{\ln a} + c$ ;  $a > 0$  AND  $a \neq 1$ 
**Note:**

$$\int ka^x dx = \frac{k}{\ln a} a^x + c \quad \text{AND} \quad \int a^{kx} dx = \frac{a^{kx}}{k \ln a} + c$$

**Example 7** INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO

- |          |             |          |            |          |             |          |              |
|----------|-------------|----------|------------|----------|-------------|----------|--------------|
| <b>A</b> | $3e^x$      | <b>B</b> | $e^{2x}$   | <b>C</b> | $2^x$       | <b>D</b> | $e^{-x}$     |
| <b>E</b> | $5e^{1-2x}$ | <b>F</b> | $3^{4+2x}$ | <b>G</b> | $3e^{4+3x}$ | <b>H</b> | $\sqrt{e^x}$ |

**Solution**

**A**  $\int 3e^x dx = 3e^x + c$

**B**  $\int e^{2x} dx = \frac{e^{2x}}{2} + c$

**C**  $\int 2^x dx = \frac{2^x}{\ln 2} + c$

**D**  $\int e^{-x} dx = \int e^{(-1)x} dx = \frac{e^{-x}}{-1} + c = -e^{-x} + c$

**E**  $\int 5e^{1-2x} dx = \int 5e \times e^{-2x} dx = 5e \left( \frac{e^{-2x}}{-2} \right) + c = \frac{-5e^{1-2x}}{2} + c$

**F**  $\int 3^{4+2x} dx = \int 3^4 \times 3^{2x} dx = \int 81 \times 9^x dx = \frac{81 \times 9^x}{\text{LN } 9} + c$

**G**  $\int 3e^{4+3x} dx = \int 3e^4 \times e^{3x} dx = 3e^4 \times \frac{e^{3x}}{3} + c = e^{4+3x} + c$

**H**  $\int \sqrt{e^x} dx = \int e^{\frac{1}{2}x} dx = \frac{e^{\frac{1}{2}x}}{\frac{1}{2}} + c = 2e^{\frac{1}{2}x} + c = 2\sqrt{e^x} + c$

**Exercise 5.2**

FIND THE INTEGRAL OF EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO

- |           |              |           |                            |           |                           |          |                         |           |                 |
|-----------|--------------|-----------|----------------------------|-----------|---------------------------|----------|-------------------------|-----------|-----------------|
| <b>1</b>  | $e^{3x}$     | <b>2</b>  | $e^{-5x}$                  | <b>3</b>  | $5^{x+1}$                 | <b>4</b> | $2^{4-x}$               | <b>5</b>  | $e^{2-3x}$      |
| <b>6</b>  | $4e^{-1-2x}$ | <b>7</b>  | $\frac{5}{e^{+x}}$         | <b>8</b>  | $\sqrt{3^{x+5}}$          | <b>9</b> | $\frac{4e^4}{e^{4x+1}}$ | <b>10</b> | $\sqrt{e^{2x}}$ |
| <b>11</b> | $4^{3x-5}$   | <b>12</b> | $\frac{2^{1-3x}}{3^{x+1}}$ | <b>13</b> | $2^{x+3} \times 3^{4-2x}$ |          |                         |           |                 |

**Integration of  $\frac{1}{x}$**

IN  $\int x^n dx$ , YOU PUT A RESTRICTION. THUS, INTEGRATING  $\frac{1}{x} = \int x^{-1} dx$  CANNOT BE DONE USING THE RULE OF  $\frac{x^{n+1}}{n+1} + c$ . YOU RECALL THAT FOR  $\frac{d}{dx} \text{LN } x = \frac{1}{x}$

$\Rightarrow \int \frac{1}{x} dx = \text{LN } x + c$

WHAT HAPPENS IF  $x < 0$ ? LET  $x < 0$ , THEN  $-x > 0$  SO THAT  $\text{LN } (-x)$  IS DEFINED.

MOREOVER,  $\frac{d}{dx} \text{LN } (-x) = \frac{1}{-x} \frac{d}{dx} (-x) = \frac{-1}{-x} = \frac{1}{x}$  BY THE CHAIN RULE.

$\Rightarrow$  FOR  $x < 0$   $\int \frac{1}{x} dx = \text{LN } (-x) + c$

THUS,  $\int \frac{1}{x} dx = \begin{cases} \text{LN } x + c, & \text{IF } x > 0 \\ \text{LN } (-x) + c, & \text{IF } x < 0 \end{cases} \Rightarrow \int \frac{1}{x} dx = \text{LN } |x| + c$



**Note:**

IF  $k$  IS A CONSTANT, THEN  $\int \frac{k}{x} dx = k \ln|x| + c$

**Example 8** EVALUATE

**A**  $\int \frac{3}{x} dx$

**B**  $\int \frac{1}{2x} dx$

**Solution**

**A** USING  $\int \frac{k}{x} dx = k \ln|x| + c$  YOU OBTAIN  $\int \frac{3}{x} dx = 3 \ln|x| + c$

**B**  $\int \frac{1}{2x} dx$ , HERE  $= \frac{1}{2}$

HENCE,  $\int \frac{1}{2x} dx = \frac{1}{2} \ln|x| + c = \ln\sqrt{|x|} + c$

NOW CONSIDER THE DERIVATIVE OF  $\ln(ax+b)$  WITH RESPECT TO  $x$  WHERE  $a \neq 0$ .

$$\begin{aligned} \frac{d}{dx} \ln(ax+b) &= \frac{1}{ax+b} \times \frac{d}{dx} (ax+b) && \text{(by the chain rule)} \\ &= \frac{a}{ax+b} \Rightarrow \frac{1}{a} \frac{d}{dx} \ln(ax+b) = \frac{1}{ax+b} \\ &\Rightarrow \int \frac{1}{a} \frac{d}{dx} (\ln(ax+b)) dx = \int \frac{1}{ax+b} dx \\ &\Rightarrow \int \frac{d}{dx} \left( \frac{1}{a} \ln(ax+b) \right) dx = \int \frac{1}{ax+b} dx \\ &\Rightarrow \int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c \end{aligned}$$

**Example 9** EVALUATE EACH OF THE FOLLOWING INTEGRALS.

**A**  $\int \frac{1}{4x+1} dx$

**B**  $\int \frac{5}{2-3x} dx$

**Solution**

**A** USING  $\int \frac{1}{ax+b} dx = \frac{\ln|ax+b|}{a} + c$ , YOU HAVE

$$\int \frac{1}{4x+1} dx = \frac{\ln|4x+1|}{4} + c$$

**B**  $\int \frac{5}{2-3x} dx = \frac{5 \ln|2-3x|}{-3} + c = -\frac{5}{3} \ln|2-3x| + c$

NOTE THAT  $\int \frac{1}{x} dx = \ln|x| + c = \ln|x| + \ln e^c = \ln|A x|$ ;  $e^c = A$

**Example 10** EVALUATE THE INTEGRAL  $\int \frac{1}{3x-1} dx$ ,

**Solution** 
$$\int \frac{1}{3x-1} dx = \frac{\text{LN} |3x-1|}{3} + c = \frac{1}{3} \text{LN} |3x-1| + c$$

$$= \text{LN} \sqrt[3]{3x-1} + c = e^c$$

**Exercise 5.3**

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO

- |                            |  |                             |
|----------------------------|--|-----------------------------|
| <b>1</b> $\frac{1}{3x}$    | <b>2</b> $\frac{5}{2x}$                          | <b>3</b> $\frac{2}{x+1}$    |
| <b>4</b> $\frac{3}{2x-1}$  | <b>5</b> $\frac{\sqrt{2}}{1-3x}$                 | <b>6</b> $\frac{4}{x-1}$    |
| <b>7</b> $\frac{-3}{5-2x}$ | <b>8</b> $\frac{1}{\left(\frac{1}{2}x+1\right)}$ | <b>9</b> $\frac{1}{x(x+1)}$ |

**5.1.2** Properties of Indefinite Integrals

**ACTIVITY 5.3**



**1** EVALUATE EACH OF THE FOLLOWING INTEGRALS.

- |  |  |
|--|--|
| <b>A</b> $\int (x^2 + \sqrt{x} - e^x) dx$  | <b>B</b> $\int x^2 dx + \int \sqrt{x} dx - \int e^x dx$  |
| <b>C</b> $\int (2x-1)^2 dx$  | <b>D</b> $4 \int x^2 dx - 4 \int x dx + \int dx$   |
| <b>E</b> $\int \left( 3x^{\frac{1}{2}} + x^{\frac{3}{2}} + \frac{1}{\sqrt{x}} - e^{-x} \right) dx$ | <b>F</b> $3 \int x^{\frac{1}{2}} dx + \int x^{\frac{3}{2}} dx + \int \frac{1}{\sqrt{x}} dx - \int e^{-x} dx$ |

**2** YOU REMEMBER THAT DIFFERENTIATION IS A DISTRIBUTIVE PROCESS OVER ADDITION. INTEGRATION DISTRIBUTIVE IN THE SAME WAY? JUSTIFY YOUR ANSWER BY CONSIDERING INTEGRALS IN PROBLEM 1 ABOVE.

**3** YOU KNOW THAT SEVERAL FUNCTIONS MAY HAVE THE SAME DERIVATIVE, FOR INSTANCE,  $x^2 + 2, x^2 + 3, \dots$  HAVE DERIVATIVE  $2x$ . WHAT IS THE GEOMETRICAL INTERPRETATION OF

THIS? USING ACTIVITY 5.3 AND WHAT YOU HAVE DONE SO FAR, YOU HAVE THE FOLLOWING PROPERTIES OF THE INDEFINITE INTEGRAL.

**Properties of the Indefinite Integral**

- 1  $\int f'(x)dx = f(x) + c$  OR  $\int \frac{d}{dx} f(x)dx = f(x) + c.$
- 2  $\frac{d}{dx} \int f(x)dx = f(x).$
- 3  $\int kf(x)dx = k \int f(x)dx.$
- 4  $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- 5  $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx.$

**Theorem 5.1**

IF TWO FUNCTIONS  $F(x)$  AND  $G(x)$  ARE ANTI DERIVATIVES OF THE FUNCTION  $f(x)$  ON THE INTERVAL  $[a, b]$ , THEN  $F(x) = G(x) + c$  FOR ALL  $x \in [a, b]$ , WHERE  $c$  IS AN ARBITRARY CONSTANT.

**Proof:**  $(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$   
 $\Rightarrow F(x) - G(x) = c \Rightarrow F(x) = G(x) + c$

WE WILL EXPLAIN BRIEFLY WHAT WE MEAN BY ARBITRARY CONSTANT

$$\int f(x)dx = G(x) + c$$

IF YOU DRAW ONE OF THE INTEGRAL CURVES TAKING  $c = 0$ , ALL THE OTHER INTEGRAL CURVES  $y = F(x) + c$  ARE OBTAINED BY SHIFTING THE CURVE IN ONE DIRECTION. THUS YOU OBTAIN A FAMILY OF (PARALLEL) CURVES.

THE FACT THAT THEY ARE PARALLEL CURVES MEANS THAT THEY HAVE EQUAL SLOPE AT (LOOK AT FIGURE 5.1 AND FIGURE 5.2)

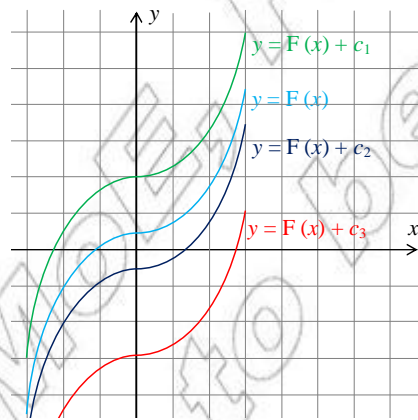


Figure 5.1

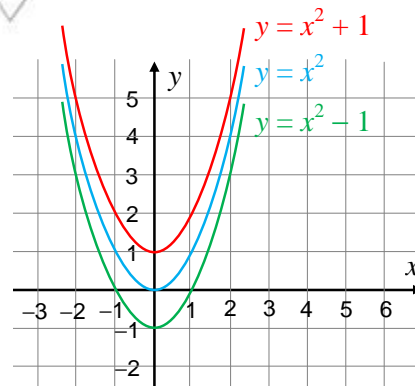


Figure 5.2

THE SLOPE OF EACH CURVE AT WHERE  $F'(a) = F_1'(a) = F_2'(a) = \dots$

**Example 11** Let  $f(x) = 2x$ . Then  $\int f(x) dx = x^2 + c$

THE SLOPE OF  $x^2$  AT  $x = 1$  IS  $\frac{dy}{dx} \Big|_{x=1} = 2(1) = 2$

SIMILARLY, THE SLOPE OF  $x^2$  AT  $x = 1$  IS 2, AND THE SLOPE OF  $x^2$  AT  $x = -1$  IS 2.  
[See **FIGURE 5.4**]

**Exercise 5.4**

EVALUATE EACH OF THE FOLLOWING INTEGRALS.

**1**  $\int \frac{d}{dx}(x^3) dx$

**2**  $\frac{d}{dx} \int x^3 dx$

**3**  $\int \left( x^6 + x^{\frac{1}{3}} - x^{-4} + x^{\frac{3}{2}} \right) dx$

**4**  $\int (\sqrt{x} - 3x^3 + x^{-2} + 2) dx$

**5**  $\int \frac{x^3 + x^2 + x + 1}{x^4} dx$

**6**  $\int \frac{(x+1)^2}{\sqrt{x}} dx$

**7**  $\int \frac{(z^4 + z^3 - 2z^2 + z + 1)}{z^2} dz$

**8**  $\int (x-1)(x^2 + x + 1) dx$

**9**  $\int \frac{(T^3 - 3T + 4)}{T} dT$

**10**  $\int \left( \frac{x+1}{x^2} \right) dx$

**11**  $\int \left( e^x - e^{-x} + \frac{1}{x} \right) dx$

**12**  $\int \frac{(e^x - 1)(e^x - 2)}{\sqrt{e^x}} dx$

**13**  $\int \left( 2x^3 + e^{2x} - \frac{1}{2x} \right) dx$

**14**  $\int e^x (1 - e^x)^2 dx$

**15**  $\int \left( 3^{1-2x} + \frac{1}{\sqrt{2^x}} + \frac{1}{e^{2x}} \right) dx$

**Integration of simple trigonometric functions**

YOU KNOW THAT  $\int \frac{d}{dx} f(x) dx = f(x) + c$

FROM **ACTIVITY 5.2** YOU OBSERVED THAT  $\frac{d}{dx} (\sin x) = \cos x$

$$\Rightarrow \int \frac{d}{dx} (\sin x) dx = \int \cos x dx$$

$$\Rightarrow \int \cos x dx = \sin x + c$$

THEREFORE, USING THE DERIVATIVES OF SIMPLE TRIGONOMETRIC FUNCTIONS YOU OBTAIN,

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \csc^2 x \, dx = -\cot x + c$$

SIMILARLY,  $\frac{d}{dx}(\sec x) = \sec x \tan x$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

THUS,  $\int \sec x \tan x \, dx = \sec x + c$

$$\int \csc x \cot x \, dx = -\csc x + c$$

USING THE PROPERTIES OF INDEFINITE INTEGRALS, YOU HAVE THE FOLLOWING INTEGRALS OF TRIGONOMETRIC FUNCTIONS.

$$\int k \sin x \, dx = -k \cos x + c$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + c; \text{ WHERE } a \neq 0$$

**Example 12**  $\int \cos(5x) \, dx = \frac{1}{5} \sin(5x) + c$  BECAUSE

$$\frac{d}{dx} \left( \frac{1}{5} \sin(5x) + c \right) = \frac{1}{5} \frac{d}{dx} \sin(5x) = \frac{1}{5} \times \cos(5x) \times 5 = \cos(5x).$$

**Example 13**  $\int \sec^2(3x+7) \, dx = \frac{1}{3} \tan(3x+7) + c$  BECAUSE

$$\frac{d}{dx} \left( \frac{1}{3} \tan(3x+7) + c \right) = \frac{1}{3} \frac{d}{dx} \tan(3x+7) = \frac{1}{3} \sec^2(3x+7) \times 3 = \sec^2(3x+7).$$

### Exercise 5.5

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO

**1**  $3 \sin(x)$

**2**  $\cos(2x)$

**3**  $\sin(4x-1)$

**4**  $3 \cos\left(x + \frac{\pi}{3}\right)$

**5**  $\sin(3x) + \cos(4x)$

**6**  $\sec^2(2x+1)$

**7**  $\csc(2x) \cot(2x)$

**8**  $\sec\left(x - \frac{\pi}{4}\right) \tan\left(x - \frac{\pi}{4}\right)$

## 5.2 TECHNIQUES OF INTEGRATION

IN DIFFERENTIAL CALCULUS YOU HAVE SEEN DIFFERENT RULES SUCH AS: THE ADDITION, PRODUCT, QUOTIENT AND CHAIN RULES. ALSO, IN THE REVERSE PROCESS, INTEGRATION, DIFFERENT METHODS. THE MOST COMMONLY USED METHODS ARE: SUBSTITUTION, PARTIAL AND INTEGRATION BY PARTS.

### 5.2.1 Integration by Substitution

**Integration by substitution** IS A COUNTER PART TO THE of differentiation. IT IS A METHOD OF FINDING INTEGRALS BY CHANGING VARIABLES. THE INTEGRAL EXPRESSED VARIABLE MAY BE SIMPLER TO EVALUATE OR CHANGED FROM THE UNFAMILIAR INTEGRAL TO A BETTER UNDERSTOOD FORM. THIS METHOD IS BASED ON A CHANGE OF VARIABLE EQUATION AND THE CHAIN RULE. THE CHANGE OF THE VARIABLE IS HELPFUL TO MAKE UNFAMILIAR INTEGRAL INTO AN INTEGRAL FORM YOU CAN RECOGNIZE.

CONSIDER  $\int 2x(x^2 + 1)^5 dx$

$$\text{LET } u = x^2 + 1, \text{ THEN } \frac{du}{dx} = \frac{d}{dx}(x^2 + 1) = 2x \Rightarrow du = 2x dx$$

$$\Rightarrow \int 2x(x^2 + 1)^5 dx = \int u^5 du = \frac{u^6}{6} + c$$

BUT  $u = x^2 + 1$ .

$$\text{THUS, } \int 2x(x^2 + 1)^5 dx = \frac{(x^2 + 1)^6}{6} + c$$

IN THIS INTEGRATION, YOU CHANGE THE VARIABLE FROM  $x$  TO  $u$ . YOU REMEMBER THIS AS A FUNCTION, THEN FOR A FUNCTION

$$\frac{d}{dx} f(u) = \frac{du}{dx} f'(u) \Rightarrow \int \frac{d}{dx} f(u) du = f(u) + c$$

$$\Rightarrow \int \frac{du}{dx} f'(u) du = f(u) + c \Rightarrow \int f'(u) \frac{du}{dx} dx = \int f'(u) du$$

**Example 1** FIND  $\int x\sqrt{x^2 + 5} dx$

**Solution** LET  $u = x^2 + 5$ , THEN  $\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x \Rightarrow \frac{1}{2} du = x dx$

$$\text{HENCE, } \int x\sqrt{x^2 + 5} dx = \int \sqrt{x^2 + 5} x dx = \frac{1}{2} \int \sqrt{u} du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3} u \sqrt{u} + c$$

$$\Rightarrow \int x\sqrt{x^2 + 5} dx = \frac{1}{3}(x^2 + 5)\sqrt{x^2 + 5} + c$$

**Example 2** FOR EACH OF THE FOLLOWING EXPRESSIONS, IDENTIFY THE VARIABLE OF SUBSTITUTION AND INTEGRATE WITH RESPECT TO

**A**  $x^2(5x^3 - 2)^9$       **B**  $\cos x e^{\sin x}$       **C**  $x e^{x^2}$

**D**  $\frac{x}{x^2 + 7}$       **E**  $\cos^3 x \sin x$       **F**  $\sqrt{x} \sqrt{1 + x\sqrt{x}}$

**Solution** REWRITE THE INTEGRALS USING A VARIABLE OF SUBSTITUTION.

**A**  $\int x^2(5x^3 - 2)^9 dx.$

HERE, THE FACTOR OF THE INTEGRAND IS THE DERIVATIVE OF  $(5x^3 - 2)$

$$\text{THUS } u = 5x^3 - 2 \Rightarrow \frac{du}{dx} = \frac{d}{dx}(5x^3 - 2) = 15x^2$$

$$\Rightarrow \frac{1}{15} du = x^2 dx \Rightarrow \int x^2(5x^3 - 2)^9 dx = \frac{1}{15} \int u^9 du = \frac{1}{15} \left( \frac{u^{10}}{10} \right) + c$$

$$\Rightarrow \int x^2(5x^3 - 2)^9 dx = \frac{1}{150} (5x^3 - 2)^{10} + c$$

**B**  $\int \cos x e^{\sin x} dx$

YOU KNOW THAT  $\frac{d}{dx}(\sin x) = \cos x$

HENCE  $u = \sin x \Rightarrow du = \cos x dx$

$$\Rightarrow \int \cos x e^{\sin x} dx = \int e^u du = e^u + c \Rightarrow \int \cos x e^{\sin x} dx = e^{\sin x} + c$$

**C**  $\int x e^{x^2} dx$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{1}{2} du = x dx$$

$$\Rightarrow \int x e^{x^2} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + c \Rightarrow \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$$

ALSO, OBSERVE THAT  $\frac{d}{dx}(e^{x^2}) = 2x e^{x^2}$

HENCE  $u = e^{x^2} \Rightarrow \frac{du}{dx} = 2x e^{x^2}$

$$\Rightarrow \frac{1}{2} du = x e^{x^2} dx \Rightarrow \int x e^{x^2} dx = \frac{1}{2} \int du = \frac{1}{2} u + c$$

$$\Rightarrow \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + c$$

**D**  $\int \frac{x}{x^2+7} dx; u = x^2+7$

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{1}{2} du = x dx \Rightarrow \int \frac{x}{x^2+7} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c$$

$$\Rightarrow \int \frac{x}{x^2+7} dx = \frac{1}{2} \ln|x^2+7| + c = \frac{1}{2} \ln\sqrt{x^2+7} + c$$

**E**  $\int \cos^3 x \sin x dx$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow -du = \sin x dx$$

$$\Rightarrow \int \cos^3 x \sin x dx = -\int u^3 du = \frac{-u^4}{4} + c \Rightarrow \int \cos^3 x \sin x dx = \frac{-\cos^4 x}{4} + c$$

**F**  $\int \sqrt{x} \sqrt{1+x\sqrt{x}} dx$

$$u = 1 + x\sqrt{x} \Rightarrow \frac{du}{dx} = \frac{d}{dx} \left( 1 + x^{\frac{3}{2}} \right) = \frac{3}{2} x^{\frac{1}{2}} \Rightarrow \frac{2}{3} du = \sqrt{x} dx$$

$$\int \sqrt{x} \sqrt{1+x\sqrt{x}} dx = \frac{2}{3} \int \sqrt{u} du = \frac{2}{3} \left( \frac{2}{3} \right) u^{\frac{3}{2}} + c = \frac{4}{9} u^{\frac{3}{2}} + c = \frac{4}{9} (1+x\sqrt{x})^{\frac{3}{2}} + c$$

**Example 3** FIND  $\int (3x-2)\sqrt{x+6} dx$ .

**Solution** HERE,  $3x-2$  IS NOT A CONSTANT TIMES THE DERIVATIVE OF  $x+6$ . BUT YOU CAN STILL USE SUBSTITUTION AS FOLLOWS.

$$u = x + 6 \Rightarrow x = u - 6 \Rightarrow 3x - 2 = 3(u - 6) - 2 = 3u - 20; u = x + 6 \Rightarrow du = dx$$

$$\text{THUS } \int (3x-2)\sqrt{x+6} dx = \int (3u-20)\sqrt{u} du = \int 3u\sqrt{u} - 20\sqrt{u} du$$

$$= 3 \int u^{\frac{3}{2}} du - 20 \int u^{\frac{1}{2}} du = 3 \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + c_1 - 20 \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c_2 = \frac{6}{5} u^2 \sqrt{u} - \frac{40}{3} u \sqrt{u} + c$$

$$\Rightarrow \int (3x-2)\sqrt{x+6} dx = \frac{6}{5} (x+6)^2 \sqrt{x+6} - \frac{40}{3} (x+6) \sqrt{x+6} + c.$$

**Example 4** EVALUATE  $\int \frac{3}{2x+1} dx$ .

**Solution**  $u = 2x + 1 \Rightarrow \frac{1}{2} du = dx$

$$\Rightarrow \int \frac{3}{2x+1} dx = \frac{3}{2} \int \frac{1}{u} du = \frac{3}{2} \ln|u| + c \Rightarrow \int \frac{3}{2x+1} dx = \frac{3}{2} \ln|2x+1| + c$$



**Example 5** EVALUATE THE FOLLOWING INTEGRALS

$$\mathbf{A} \quad \int f(x)f'(x)dx \qquad \mathbf{B} \quad \int \frac{f'(x)}{f(x)}dx, f(x) \neq 0$$

**Solution**  $u = f(x) \Rightarrow \frac{du}{dx} = \frac{d}{dx}f(x) = f'(x) \Rightarrow du = f'(x)dx$

$$\mathbf{A} \quad \int f(x)f'(x)dx = \int u du = \frac{u^2}{2} + c \Rightarrow \int f(x)f'(x)dx = \frac{(f(x))^2}{2} + c$$

$$\mathbf{B} \quad \int \frac{f'(x)}{f(x)}dx = \int \frac{1}{u} du = \ln|u| + c \Rightarrow \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

**Example 6** USING  $\int \frac{f'(x)}{f(x)}dx$ , SHOW THAT  $\int \tan x dx = -\ln|\cos x| + c$

**Solution**  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{(\cos x)'}{\cos x} dx = -\ln|\cos x| + c$

**Example 7** USING A SUITABLE IDENTIFY, FIND  $\int \sin^3 x dx$

**Solution** BY WRITING  $\cos^2 x = 1 - \sin^2 x$ , YOU HAVE,

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\Rightarrow \int \sin^3 x dx = \int \frac{1 - \cos(2x)}{2} dx = \int \frac{1}{2} dx - \frac{1}{2} \int \cos(2x) dx$$

$$\text{BUT } \int \cos(2x) dx = \frac{1}{2} \sin(2x) + c \quad \text{Explain!}$$

$$\Rightarrow \int \sin^3 x dx = \frac{1}{2}x - \frac{1}{4} \sin(2x) + c$$

**Example 8** FIND  $\int 2^{4x-1} dx$  USING THE METHOD OF SUBSTITUTION.

**Solution**  $u = 4x - 1 \Rightarrow \frac{du}{dx} = 4 \Rightarrow \frac{1}{4} du = dx$

$$\Rightarrow \int 2^{4x-1} dx = \frac{1}{4} \int 2^u du = \frac{1}{4} \left( \frac{2^u}{\ln 2} \right) = \frac{2^u}{\ln 16} + c \Rightarrow \int 2^{4x-1} dx = \frac{2^{4x-1}}{\ln 16} + c.$$

*Can you do this without using substitution?*

LOOK AT THE FOLLOWING.

$$\int 2^{4x-1} dx = \int \frac{16^x}{2} dx = \frac{1}{2} \int 16^x dx = \frac{1}{2} \left( \frac{16^x}{\ln 16} \right) + c = \frac{2^{4x-1}}{\ln 16} + c$$

**Example 9** FIND  $\int x^2 \cos(x^3 + 1) dx$

**Solution**  $u = x^3 + 1 \Rightarrow \frac{1}{3} du = x^2 dx \Rightarrow \int x^2 \cos(x^3 + 1) dx = \frac{1}{3} \int \cos u du = \frac{1}{3} \sin u + c$

$$\int x^2 \cos(x^3 + 1) dx = \frac{1}{3} \sin(x^3 + 1) + c$$

**Example 10** EVALUATE  $\int \frac{x}{\sqrt{x^2 + a^2}} dx$

**Solution** LET  $u = x^2 + a^2 \Rightarrow \frac{du}{dx} = \frac{d}{dx}(x^2 + a^2) = 2x \Rightarrow \frac{1}{2} du = x dx$

$$\text{THUS, } \int \frac{x}{\sqrt{x^2 + a^2}} dx = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \sqrt{u} + c \Rightarrow \int \frac{x}{\sqrt{x^2 + a^2}} dx = \sqrt{x^2 + a^2} + c.$$

**Example 11** EVALUATE  $\int \frac{1}{x \ln x} dx$

**Solution** IN THE PRODUCT  $\left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right)$ , THE FACTOR  $\frac{1}{x}$  IS THE DERIVATIVE OF  $\ln x$

THEREFORE,  $\ln x$  IS  $u$  SO THAT  $\frac{du}{u} = \frac{1}{u} du = \frac{1}{\ln x} \left(-\frac{1}{x^2}\right) dx = -\frac{1}{x^2 \ln x} dx$

### Exercise 5.6

**1** INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS WITH RESPECT TO  $x$

**A**  $2x(x^2 + 1)^3$

**B**  $x\sqrt{x^2 + 4}$

**C**  $x^2\sqrt{x^3 + 1}$

**D**  $(2x + 1)\sqrt{x^2 + x + 9}$

**E**  $\sin x \cos x$

**F**  $(2x + 3)e^{(x^2 + 3x + 4)}$

**G**  $\sin x e^{\cos x}$

**H**  $(x + 2)\sqrt{x - 3}$

**2** FIND EACH OF THE FOLLOWING INTEGRALS USING THE SUBSTITUTION METHOD

**A**  $\int \sqrt{3x - 2} dx; u = 3x - 2$

**B**  $\int x\sqrt{1 - 5x^2} dx; u = 1 - 5x^2$

**C**  $\int \sin(x^2) dx; u = x^2$

**D**  $\int (1 - 4x) dx; u = 1 + x$

**E**  $\int x(x^2 - 3)^5 dx; u = x^2 - 3$

**F**  $\int x^2(2 + 3x^3) dx; u = 3x^3 + 2$

**G**  $\int e^x \sqrt{1 + e^x} dx; u = 1 + e^x$

**H**  $\int \sin x \cos x dx; u = \cos x$

**I**  $\int \sqrt{4x - 3} dx; u = x - 3$

**J**  $\int \frac{1}{(1 - x)^{\frac{1}{3}}} dx; u = 1 - x$

**K**  $\int 3^{\frac{1}{x}} x^{-2} dx; u = \frac{1}{x}$

**M**  $\int \cos(3x) dx; u = 3x$

**O**  $\int \frac{4x-5}{2x^2-5x+4} dx;$   
 $u = 2x^2 - 5x + 4$

**Q**  $\int (3+2x)^{12} dx; u = 3+2x$

**S**  $\int \sin(x^2) dx; u = x^2$

**U**  $\int \frac{x}{\sqrt{x^2+5}} dx; u = x^2+5$

**L**  $\int 3^{0.6x} dx; u = 0.6x + \pi$

**N**  $\int x \sin(x^2) dx; u = x^2$

**P**  $\int \frac{x+1}{\sqrt{x+3}} dx; u = x+3$  OR  $u = x+1$

**R**  $\int \tan^2 x dx; u = \tan x$

**T**  $\int 5^{x\sqrt{x}} \sqrt{x} dx; u = x\sqrt{x}$

**V**  $\int (2x-3) \sqrt{x+3} dx; u = x+3$

**3** EVALUATE EACH OF THE FOLLOWING INTEGRALS.

**A**  $\int x^3(x^4+5) dx$

**B**  $\int \left(1 - \frac{1}{x^2}\right) \left(x + \frac{1}{x}\right) dx$

**C**  $\int (2^x) x dx$

**D**  $\int \cos x dx$

**E**  $\int \sin x \sqrt{1 - \cos x} dx$

**F**  $\int e^x \sqrt{4+e^x} dx$

**G**  $\int (ax+b)^n dx$

**H**  $\int \cos(x+\beta) dx$

**I**  $\int 3^x (1-3^{x+1})^9 dx$

**J**  $\int \frac{4}{4x-2} dx$

**K**  $\int \frac{1}{ax+b} dx$

**L**  $\int \frac{1}{(ax+b)^n} dx$

**M**  $\int \frac{x}{\sqrt{x^2+1}} dx$

**N**  $\int x^2(x^3-8) dx$

**O**  $\int \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

**P**  $\int \frac{2^{\sqrt{y}}}{\sqrt{y}} dy$

**Q**  $\int x\sqrt{3+5x} dx$

**R**  $\int \frac{\sin t}{\sqrt{3+\cos t}} dt$

**S**  $\int \frac{\sin(2t)}{1-\cos(2t)} dt$

**T**  $\int xe^{x^2+7} dx$

**U**  $\int \frac{4x-1}{1-2x+4x^2} dx$

**V**  $\int (3x+1)(3x^2+2x+5)^6 dx$

**W**  $\int (x-1) \sqrt{(x^2-2x+3)^2} dx$

**X**  $\int \cos^2 x dx$

## 5.2.2 Integration by partial fractions

DECOMPOSITION OF A RATIONAL EXPRESSION INTO PARTIAL FRACTIONS WAS DISCUSSED IN THIS SECTION, TO FIND THE INTEGRALS OF SOME RATIONAL EXPRESSIONS, YOU USE FRACTIONS ALONG WITH THE METHOD OF SUBSTITUTION.

### ACTIVITY 5.4



- 1 DECOMPOSE EACH OF THE FOLLOWING RATIONAL EXPRESSIONS INTO PARTIAL FRACTIONS

**A**  $\frac{1}{x(x+1)}$

**B**  $\frac{x}{x^3 - 3x + 2}$

**C**  $\frac{2x-3}{(x-1)^2}$

**D**  $\frac{x^3}{x^2 - 4x + 3}$

**E**  $\frac{x+2}{x^2(x-3)}$

**F**  $\frac{x^2 + 2x + 3}{(x+1)(x^2 - 4)}$

**G**  $\frac{x-1}{(x+1)^2(x+2)}$

- 2 CONSIDER THE INTEGRAL OF THE RATIONAL EXPRESSION  $\frac{x+3}{x+1}$  AS

$1 + \frac{2}{x+1}$  BY USING LONG DIVISION.

$$\Rightarrow \int \frac{x+3}{x+1} dx = \int \left( 1 + \frac{2}{x+1} \right) dx = x + 2 \int \frac{1}{x+1} dx = x + 2 \ln|x+1| + c$$

USING THIS TECHNIQUE OF INTEGRATION, FIND EACH OF THE FOLLOWING INTEGRALS.

**A**  $\int \frac{x+2}{x+3} dx$

**B**  $\int \frac{x+2}{4x-3} dx$

**C**  $\int \frac{x}{4x+5} dx$

**D**  $\int \frac{4x-5}{5x-4} dx$

**E**  $\int \frac{1}{(2x-1)^4} dx$

**F**  $\int \left( \frac{x+1}{x-3} \right)^3 dx$

- 3 YOU KNOW THAT  $\int \left( \frac{1}{x+2} + \frac{3}{x-1} \right) dx = \int \frac{1}{x+2} dx + \int \frac{3}{x-1} dx = \ln|x+2| + 3 \ln|x-1| + c$

CAN YOU EVALUATE THIS INTEGRAL BY SUMMING UP THE EXPRESSIONS?

$$\text{I.E., } \int \left( \frac{1}{x+2} + \frac{3}{x-1} \right) dx = \int \frac{x-1+3(x+2)}{(x+2)(x-1)} dx = \int \frac{4x+5}{(x+2)(x-1)} dx$$

FROM ACTIVITY 5.4 YOU HAVE SEEN THAT DECOMPOSITION INTO PARTIAL FRACTIONS TOGETHER WITH SUBSTITUTION ENABLES YOU TO EVALUATE THE INTEGRALS OF SOME RATIONAL EXPRESSIONS.

**Example 12** FIND  $\int \frac{x+5}{x^2+4x+3} dx$

**Solution** USING PARTIAL FRACTIONS, YOU OBTAIN,

$$\begin{aligned} \frac{x+5}{x^2+4x+3} &= \frac{A}{x+1} + \frac{B}{x+3} \Rightarrow \int \frac{x+5}{x^2+4x+3} dx = \int \left( \frac{A}{x+1} + \frac{B}{x+3} \right) dx \\ &= A \ln|x+1| + B \ln|x+3| + c = 2 \ln|x+1| - \ln|x+3| + c \end{aligned}$$

**Example 13** FIND  $\int \frac{x^3+2x^2-x-7}{x^2+x-2} dx$ .

**Solution** THE RATIONAL EXPRESSION IS AN IMPROPER FRACTION. FACTORIZING THE DENOMINATOR WE USE LONG DIVISION, TO OBTAIN

$$\begin{aligned} \int \frac{x^3+2x^2-x-7}{x^2+x-2} dx &= \int \left( x+1 - \frac{5}{x^2+x-2} \right) dx \\ &= \frac{x^2}{2} + x - 5 \int \left( \frac{A}{x+2} + \frac{B}{x-1} \right) dx = \frac{x^2}{2} + x - 5 \left( A \ln|x+2| + B \ln|x-1| \right) + c \\ &= \frac{x^2}{2} + x + \frac{5}{3} \ln|x+2| - \frac{5}{3} \ln|x-1| + c = \frac{x^2}{2} + x + \frac{5}{3} \left[ \ln|x+2| - \ln|x-1| \right] + c \\ &= \frac{x^2}{2} + x + \frac{5}{3} \ln \left| \frac{x+2}{x-1} \right| + c \end{aligned}$$

**Example 14** EVALUATE  $\int \frac{dx}{x^2-9}$

**Solution** USING PARTIAL FRACTIONS YOU HAVE

$$\int \frac{dx}{x^2-9} = \int \frac{A}{x-3} dx + \int \frac{B}{x+3} dx = A \ln|x-3| + B \ln|x+3| + c$$

FROM PARTIAL FRACTIONS WE CALCULATE THE INDIVIDUAL VALUES A =

$$\Rightarrow \int \frac{dx}{x^2-9} = \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + c = \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| + c$$

### Exercise 5.7

USE THE METHOD OF SUBSTITUTION ALONG WITH PARTIAL FRACTIONS TO EVALUATE THE FOLLOWING INTEGRALS.

**1**  $\int \frac{x}{x+5} dx$

**2**  $\int \frac{4x+1}{x^2-3x+2} dx$

**3**  $\int \frac{x^2-x-2}{x^2+x-2} dx$

<b>4</b> $\int \frac{x^2 + 4}{x^2 - 1} dx$	<b>5</b> $\int \frac{3x + 5}{x + 2} dx$	<b>6</b> $\int \frac{x}{x^2 - 2x - 8} dx$
<b>7</b> $\int \frac{x}{(x^2 - 3x - 8)^2} dx$	<b>8</b> $\int \frac{x^3}{(x + 1)^2(x + 2)} dx$	<b>9</b> $\int \frac{1}{(x + 2)^2} dx$
<b>10</b> $\int \frac{x^2 + 2x - 3}{x^2(x^2 - 5x + 6)} dx$		

### 5.2.3 Integration by parts

THE PRODUCT RULE FOR DIFFERENTIATION IS

$$\frac{d}{dx}(f(x) \cdot g(x)) = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x)$$

THIS FORM CANNOT BE EXPRESSED AS  $\frac{du}{dx}$

HENCE, IT CANNOT BE INTEGRATED BY THE METHOD OF SUBSTITUTION.

INTEGRATION BY PARTS IS A METHOD WHICH IS A COUNTER PART OF THE PRODUCT RULE OF DIFFERENTIATION.

INTEGRATING BOTH SIDES OF THE ABOVE EXPRESSIONS GIVES,

$$\begin{aligned} \int \frac{d}{dx}(f(x) \cdot g(x)) dx &= \int g(x) \frac{d}{dx} f(x) dx + \int f(x) \frac{d}{dx} g(x) dx \\ \Rightarrow f(x) \cdot g(x) &= \int g(x) \frac{d}{dx} f(x) dx + \int f(x) \frac{d}{dx} g(x) dx \\ \Rightarrow \int f(x) \frac{d}{dx} g(x) dx &= f(x) \cdot g(x) - \int g(x) \frac{d}{dx} f(x) dx. \end{aligned}$$

### ACTIVITY 5.5



- DIFFERENTIATE EACH OF THE FOLLOWING EXPRESSIONS. SELECT TO
 

<b>A</b> $x \ln x - x + 4$	<b>B</b> $x e^x - e^x - 7$
<b>C</b> $x \cos x - \cos x + 5$	<b>D</b> $e^x (\sin x + \cos x)$
<b>E</b> $x^2 \ln x - x^2$	
- USING THE RESULT FROM 1 ABOVE, EVALUATE EACH OF THE FOLLOWING INTEGRALS.
 

<b>A</b> $\int \ln x dx$	<b>B</b> $\int x e^x dx$	<b>C</b> $\int x \sin x dx$
<b>D</b> $\int e^x \sin x dx$	<b>E</b> $\int x \ln x dx$	
- SUPPOSE YOU WANT TO INTEGRATE  $\int x^2 \sin x dx$ , WHICH METHOD ARE YOU GOING TO APPLY?

**Note:**

LET  $u$  AND  $v$  BE FUNCTIONS. IF  $u = u(x)$  AND  $v = v(x)$ .

THEN,  $\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx} \Rightarrow u \frac{dv}{dx} = \frac{d}{dx}(uv) - v \frac{du}{dx}$

$\Rightarrow \int u \frac{dv}{dx} dx = \int \frac{d}{dx}(uv) dx - \int v \frac{du}{dx} dx \Rightarrow \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

IN SHORT  $\int u dv = uv - \int v du$

IN THIS METHOD, YOU SHOULD BE ABLE TO CHOOSE “PARTS”

**Examples 15** EVALUATE  $\int x e^x dx$

**Solution** HERE  $u = x$  AND  $v = e^x$ .

NOW, DECIDE WHICH PART SHOULD BE  $u$  AND WHICH PART SHOULD BE  $v$ .

SUPPOSE  $u = x$  AND  $v = e^x$ , THEN

$\frac{du}{dx} = 1$ , AND  $\int dv = \int e^x dx \Rightarrow v = e^x$

$\Rightarrow \int x e^x dx = uv - \int v \frac{du}{dx} dx = x e^x - \int e^x dx = x e^x - e^x + c$

IF  $u = e^x$  AND  $v = x$ .

THEN,  $\frac{du}{dx} = e^x$  AND  $v = x^2 = \int x e^x dx = uv - \int v \frac{du}{dx} dx = e^x \cdot x^2 - \int x^2 e^x dx$

THIS IS MORE COMPLEX THAN THE ORIGINAL INTEGRAL. HENCE, IT IS SOMETIMES HELPFUL TO CONSIDER  $e^x$  TO BE THE POLYNOMIAL FACTOR.

IN THE EXPRESSIONS THE POLYNOMIAL FACTOR.

**Example 16** EVALUATE  $\int x \ln x dx$

**Solution** IN  $\ln x$ , WHAT IS THE POLYNOMIAL FACTOR?

LET  $u = \ln x$  AND  $v = x$ . THEN  $du = \frac{1}{x}$  AND  $dv = dx$ .

THUS,  $\int x \ln x dx = x \ln x - \int x \left(\frac{1}{x}\right) dx = x \ln x - \int dx = x \ln x - x + c$

**Example 17** EVALUATE  $\int \log_2 x dx$

**Solution** NOTE THAT  $\log_2 x = \frac{\ln x}{\ln 2}$

HENCE  $\int \log_2 x dx = \int \frac{\ln x}{\ln 2} dx = \frac{1}{\ln 2} \int \ln x dx = \frac{1}{\ln 2} (x \ln x - x) + c$

**Note:**

If  $a > 0$  AND  $a \neq 1$ ,

$$\int \log_a x \, dx = \int \frac{\ln x}{\ln a} \, dx = \frac{1}{\ln a} \int \ln x \, dx$$

$$= \frac{1}{\ln a} (x \ln x - x) + c$$

**Example 18** EVALUATE  $\int \log(3x+1) \, dx$

**Solution** LET  $u = 3x + 1$ , THEN

$$\frac{du}{dx} = 3 \Rightarrow \frac{1}{3} du = dx$$

$$\Rightarrow \int \log(3x+1) \, dx = \int \frac{\ln(3x+1)}{\ln 10} \, dx = \frac{1}{3 \ln 10} \int \ln u \, du$$

$$= \frac{1}{3 \ln 10} (u \ln u - u) + c$$

$$= \frac{1}{3 \ln 10} ((3x+1) \ln(3x+1) - (3x+1)) + c$$

$$= \frac{1}{3 \ln 10} ((3x+1) \ln(3x+1) - 3x - 1) + c$$

**Example 19** EVALUATE  $\int x \sin x \, dx$

**Solution**  $u = x \Rightarrow du = dx$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x \Rightarrow \int x \sin x \, dx = -x \cos x - \int -\cos x \, dx$$

$$= -x \cos x + \sin x + c$$

**Example 20** EVALUATE  $\int x \ln x \, dx$

**Solution**  $u = \ln x \Rightarrow du = \frac{1}{x}$  AND  $v = x \Rightarrow dv = \frac{x^2}{2}$

$$\Rightarrow \int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$$



CAN YOU ASSUME  $u = \ln x$  AND  $dv = dx$ ?

IF YOU SET  $u = \ln x$  AND  $dv = dx$

$$\Rightarrow v = x \ln x - x$$

$$\begin{aligned} \text{THEN } \int x \ln x \, dx &= x(x \ln x - x) - \int (x \ln x - x) \, dx \\ &= x^2 \ln x - x^2 - \int x \ln x \, dx + \int x \, dx \end{aligned}$$

$$\Rightarrow 2 \int x \ln x \, dx = x^2 \ln x - x^2 + \frac{x^2}{2} + c$$

$$\Rightarrow \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c$$

ALTHOUGH THIS GIVES YOU THE CORRECT ANSWER, IT IS SAFER TO SET

**Example 21** EVALUATE  $\int x^r \ln x \, dx$  WHERE  $r$  IS A REAL NUMBER DIFFERENT FROM  $-1$ .

**Solution** WHAT HAPPENS IF  $r = -1$ ? ARE YOU GOING TO USE BY PARTS?

$$\text{IF } r = -1, \text{ THEN } \int x^r \ln x \, dx = \int \frac{\ln x}{x} \, dx$$

BY THE METHOD OF SUBSTITUTION YOU HAVE,

$$u = \ln x \Rightarrow du = \frac{1}{x} \, dx,$$

$$\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + c \Rightarrow \int \frac{\ln x}{x} \, dx = \frac{(\ln x)^2}{2} + c$$

$$\text{IF } r \neq -1, \text{ THEN } u = \ln x \Rightarrow du = \frac{1}{x} \, dx$$

$$dv = x^r \, dx \Rightarrow v = \frac{x^{r+1}}{r+1}.$$

$$\text{THEN } \int x^r \ln x \, dx = uv - \int v \, du$$

$$= (\ln x) \frac{x^{r+1}}{r+1} - \int \frac{x^{r+1}}{r+1} \left( \frac{1}{x} \right) dx$$

$$= \frac{x^{r+1}}{r+1} \ln x - \frac{x^{r+1}}{(r+1)^2} + c$$

**Example 22** EVALUATE  $\int_0^3 x \ln x \, dx$

**Solution** 
$$\int x^2 \ln x \, dx = \frac{1}{LN 3} \int x^2 \ln x \, dx$$

$$= \frac{1}{LN 3} \left[ \frac{x^3}{3} \ln x - \frac{x^3}{9} \right] + c \quad \text{Why?}$$

**Example 23** FIND  $\int e^x \sin x \, dx$

**Solution** CHOOSE  $u = e^x$  AND  $v = \sin x$

THEN  $du = e^x \, dx$  AND  $dv = \cos x$ .

$$\Rightarrow \int e^x \sin x \, dx = -e^x \cos x - \int -\cos x \, dx = -e^x \cos x + \int \cos x \, dx$$

$\int e^x \cos x \, dx$  HAS THE SAME FORM AS  $\int e^x \sin x \, dx$

HENCE YOU APPLY INTEGRATION BY PARTS FOR A SECOND TIME.

$$u = e^x \Rightarrow du = e^x \, dx \text{ AND } v = \cos x \Rightarrow dv = -\sin x$$

$$\Rightarrow \int \cos x e^x \, dx = e^x \sin x - \int \sin x \, dx$$

$$\text{BUT } \int e^x \sin x \, dx = -e^x \cos x + \int \cos x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, dx$$

BY COLLECTING LIKE TERMS, YOU OBTAIN

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + c$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

In the integral  $\int f(x)g(x) \, dx$ , IF  $f(x)$  IS A TRANSCENDENTAL FUNCTION (EXPONENTIAL, TRIGONOMETRIC OR LOGARITHMIC FUNCTION) AND  $g(x)$  IS A POLYNOMIAL FUNCTION, USE THE SUBSTITUTION  $u = f(x)$  AND  $dv = g(x) \, dx$  FOR INTEGRATION BY PARTS.

**Exercise 5.8**

INTEGRATE EACH OF THE FOLLOWING EXPRESSIONS USING THE METHOD OF INTEGRATION BY PARTS.

- |                           |                              |                              |
|---------------------------|------------------------------|------------------------------|
| <b>1</b> $x e^{1-x}$      | <b>2</b> $x \cos x$          | <b>3</b> $x e^{3x+1}$        |
| <b>4</b> $x^2 e^x$        | <b>5</b> $4x \sin x$         | <b>6</b> $e^x \cos x$ (2)    |
| <b>7</b> $e^{3x} \sin x$  | <b>8</b> $e^{-x} \sin x$ (2) | <b>9</b> $\ln x$ (4)         |
| <b>10</b> $x^3 \ln x$     | <b>11</b> $e^x (x + 2)$      | <b>12</b> $x^2 \sin x$       |
| <b>13</b> $x^2 \ln x$ (2) | <b>14</b> $x \ln(x); n > 0$  | <b>15</b> $x \sin(x); n > 0$ |

## 5.3 DEFINITE INTEGRALS, AREA AND THE FUNDAMENTAL THEOREM OF CALCULUS



### OPENING PROBLEM

THE AREA UNDER THE CURVE OF  $y = 5 - 2x^2 + 4x^3 - x^5$  FROM  $x = -\frac{1}{2}$  TO  $x = 1\frac{1}{2}$  IS DIVIDED INTO  $n$  STRIPS. EACH STRIP IS APPROXIMATED BY A RECTANGLE AS SHOWN WITH THE FOLLOWING FIGURE.

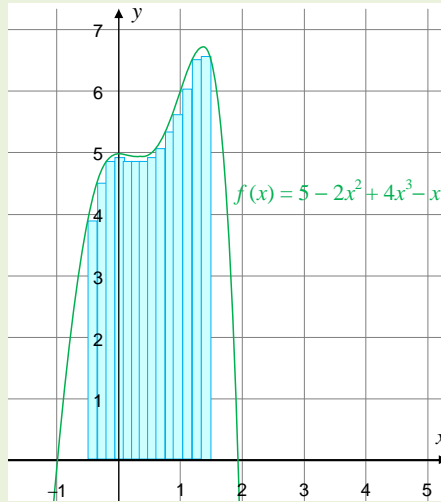


Figure 5.3

WHAT IS THE LIMIT OF THE SUM OF THE AREAS OF ALL APPROXIMATE RECTANGLES AS  $n \rightarrow \infty$ ?

### 5.3.1 The Area of a Region under a Curve

FROM GEOMETRY, YOU KNOW HOW TO DETERMINE THE AREAS OF CERTAIN PLANE FIGURES: TRIANGLES, RECTANGLES, PARALLELOGRAMS, TRAPEZIUMS, DIFFERENT REGULAR POLYGONS, COMBINATIONS OF PARTS OF CIRCLES AND POLYGONS.

IN THIS TOPIC, YOU SHALL DETERMINE THE AREA OF A REGION UNDER THE CURVE OF A NON-CURVED FUNCTION  $f(x)$  THAT IS CONTINUOUS ON A CLOSED INTERVAL. WE WILL DIVIDE THE REGION INTO  $n$  STRIPES APPROXIMATED BY RECTANGLES OF UNIFORM WIDTH

WHERE  $\Delta x = \frac{b-a}{n}$  FORMED BY VERTICAL LINES THROUGH  $x_0, x_1, x_2, \dots, x_n = b$ ; WHERE

$$a = x_0 < x_1 < x_2 < \dots < x_n = b, x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots = (x_n - x_{n-1}) = \Delta x$$

LOOK AT FIGURE 5.4

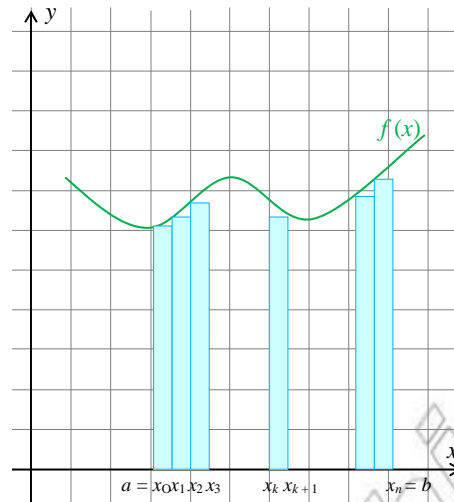


Figure 5.4

AS THE VALUE OF  $n$  GETS LARGER AND LARGER THE RECTANGLES GET THINNER AND THINNER. THE RECTANGLES RISE UP TO FILL IN THE REGION.

THUS, THE AREA OF THE REGION WILL BE THE LIMITING VALUE OF THE SUM OF THE AREA OF THE RECTANGLES. THIS IS ONE OF THE DIFFERENT TECHNIQUES OF FINDING THE AREA OF A REGION UNDER A CURVE.

## ACTIVITY 5.6



- 1 LET  $x_0, x_1, x_2, \dots, x_n \in [a, b]$  WITH  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ . THE FINITE SET  $P = \{x_0, x_1, x_2, \dots, x_n\}$  IS SAID TO BE A PARTITION OF  $[a, b]$ .  
FOR INSTANCE  $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$  IS A PARTITION OF  $[0, 1]$ .  
FIND AT LEAST THREE DIFFERENT PARTITIONS OF  $[0, 1]$
- 2 THE  $n$  - SUB INTERVALS IN WHICH THE PARTITION  $P$  DIVIDES  $[a, b]$  ARE  $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ .  
THE LENGTH OF  $k^{\text{th}}$  SUB INTERVAL  $[x_{k-1}, x_k]$  IS  $x_k - x_{k-1}$ .  
A DIVIDE  $[0, 1]$  INTO 5 - SUB INTERVALS OF EQUAL LENGTHS  
B DIVIDE  $[3, 5]$  INTO 10 - SUB INTERVALS OF EQUAL LENGTHS.  
C DIVIDE  $[0, 1]$  INTO 7 SUB INTERVALS OF EQUAL LENGTHS.
- 3 CONSIDER THE AREA UNDER THE CURVE  $y = \sin x$  FROM  $x = 0$  TO  $x = 1$ . DIVIDE THE INTERVAL  $[0, 1]$  INTO  $n$  - SUB INTERVALS EACH OF LENGTH  $\frac{1}{n}$ . FIGURE 5.5 BELOW SHOWS A SKETCH OF THE INSCRIBED RECTANGLE.

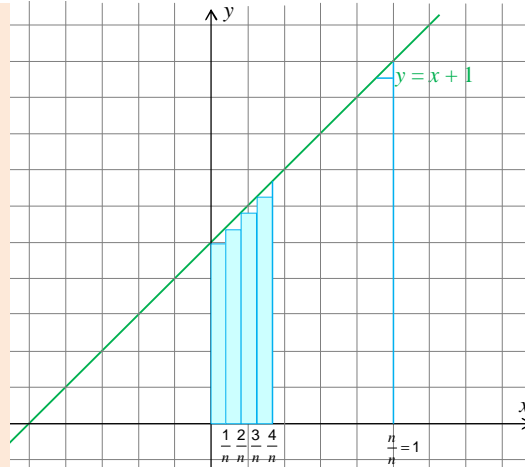


Figure 5.5

- I FIND THE SUM OF THE AREAS OF THE RECTANGLES WHEN  
**A**  $n = 3$     **B**  $n = 5$     **C**  $n = 10$
- II FIND THE LIMITING VALUE OF THE SUM OF THE AREAS OF THE RECTANGLES, AS  $n$  REPEAT PROBLEM 3, IF THE RECTANGLES ARE CIRCUMSCRIBED INSTEAD OF BEING INSCRIBED. LOOK AT FIGURE 5.6

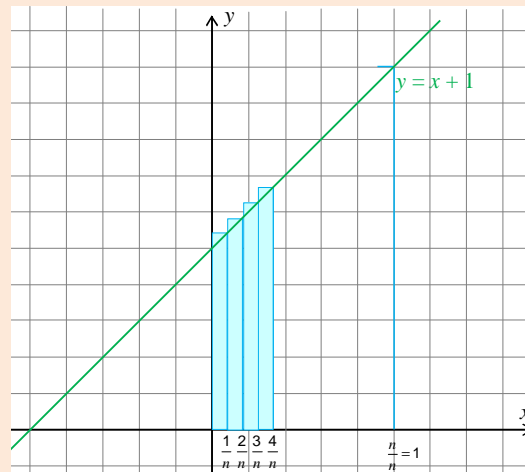


Figure 5.6

USING THE CONCEPT DEVELOPED IN THE ACTIVITY, CONSIDER A NON-NEGATIVE AND CONTINUOUS FUNCTION  $y = f(x)$  ON THE INTERVAL  $[a, b]$  OF THE  $x$ -AXIS BETWEEN THE LINES  $a$  AND  $b$  IS CALCULATED AS FOLLOWS.

DIVIDE THE INTERVAL INTO SUB INTERVALS

$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$  EACH OF LENGTH  $\frac{b-a}{n}$

LET  $n$  RECTANGLES EACH OF WIDTH  $\frac{b-a}{n}$  BE INSCRIBED IN THE REGION AS SHOWN IN

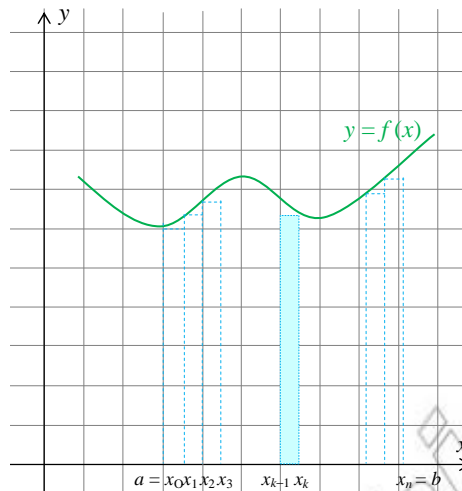


Figure 5.7

LET  $z_k \in [x_{k-1}, x_k]$  SUCH THAT  $f(z_k)$  IS THE HEIGHT OF THE RECTANGLE.

LET  $\Delta A_k$  BE THE AREA OF THE RECTANGLE.

$$\text{THEN } \Delta A_k = \left( \frac{b-a}{n} \right) f(z_k)$$

LET  $A_n$  BE THE SUM OF THE RECTANGLES.

$$\text{THEN } A_n = \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n \frac{b-a}{n} f(z_k) = \left( \frac{b-a}{n} \right) \sum_{k=1}^n f(z_k)$$

THE AREA OF THE REGION IS THE LIMITING VALUE OF  $A_n$ .

$$\text{I.E. } A = \lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n f(z_k)$$

### Definition 5.3

- 1 THE SUM  $\sum_{k=1}^n f(z_k) \Delta x$  IS SAID TO BE THE INTEGRAL SUM OF THE FUNCTION OVER THE INTERVAL  $[a, b]$ .
- 2 IF  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k) \Delta x$  EXISTS AND IS EQUAL TO I, THEN I IS SAID TO BE THE DEFINITE INTEGRAL OVER THE INTERVAL AND IS DENOTED BY  $\int_a^b f(x) dx$ .  $a$  AND  $b$  ARE SAID TO BE THE Lower and upper limits OF INTEGRATION, RESPECTIVELY.

**Example 1** FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF THE FUNCTION  $y = x^2$  AND THE  $x$ -AXIS BETWEEN THE LINES  $x = 1$  AND  $x = 2$ .

**SOLUTION**

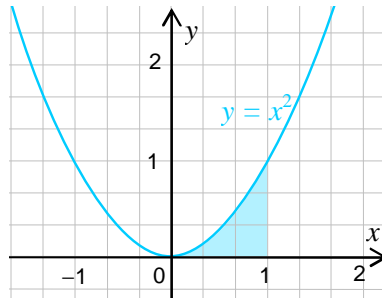


Figure 5.8

USING THE DEFINITION, CALCULATE THE AREA OF THE REGION AS FOLLOWS.

$$A = \int_a^b f(x) dx \Rightarrow \int_a^b x^2 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k) \Delta x$$

WHERE  $\Delta x = \frac{1-0}{n} = \frac{1}{n}$  AND  $z_k = \frac{k-1}{n} \Rightarrow f(z_k) = \left(\frac{k-1}{n}\right)^2$

$$\begin{aligned} \Rightarrow \sum_{k=1}^n f(z_k) \Delta x &= \sum_{k=1}^n \left(\frac{k-1}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3} \sum_{k=1}^n (k-1)^2 \\ &= \frac{1}{n^3} [0+1+2^2+3^2+\dots+(n-1)^2] \\ &= \frac{1}{n^3} \frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{1}{6n^3} [2n^3 - 3n^2 + n] \\ &= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \\ \Rightarrow A = \int_a^b x^2 dx &= \lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3} \end{aligned}$$

**Theorem 5.2 Estimate of the definite integral**

IF THE FUNCTION IS CONTINUOUS ON THE INTERVAL  $[a, b]$ , THEN  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(z_i) \Delta x$  EXISTS.

THAT IS, THE DEFINITE INTEGRAL  $\int_a^b f(x) dx$  EXISTS.

**Example 2** SHOW THAT  $\int_0^{\frac{\pi}{2}} \sin x dx$  EXISTS.

**Solution**  $f(x) = \sin x$  IS CONTINUOUS ON  $\left[0, \frac{\pi}{2}\right]$ .

THUS, BY THE ABOVE THEOREM, THE DEFINITE INTEGRAL EXISTS.

**Example 3** SHOW THAT  $\int_{-1}^2 \frac{1}{x} dx$  DOESN'T EXIST.

**Solution**  $f(x) = \frac{1}{x}$  IS DISCONTINUOUS AT

$\Rightarrow f$  IS NOT CONTINUOUS ON  $[-1, 2] \Rightarrow \int_{-1}^2 \frac{1}{x} dx$  DOESN'T EXIST.

**Exercise 5.9**

- 1 ESTIMATE THE AREA OF THE REGION BOUNDED BY THE GRAPH OF  $y = \sqrt{x}$  BETWEEN THE LINES  $x = 0$  AND  $x = 2$ , BY DIVIDING THE INTERVAL  $[0, 2]$  INTO
  - A 5 - SUB INTERVALS
  - B 10 - SUB INTERVALS
  - C  $n$  - SUB INTERVAL, OF EQUAL LENGTHS.
- 2 USING THE DEFINITION OF AREA UNDER A CURVE, DETERMINE THE AREA OF THE REGION ENCLOSED BY THE CURVE  $f(x)$  AND THE X-AXIS BETWEEN THE LINES  $x = a$  AND  $x = b$ , WHEN
  - A  $f(x) = 3x - 1; a = 1, b = 3$
  - B  $f(x) = x^3; a = 1, b = 2$
  - C  $f(x) = x^2 - 4x; a = 0, b = 4$
  - D  $f(x) = x^2 - 2x + 1; a = 0, b = 2$ ,
- 3 IN EACH OF THE FOLLOWING DETERMINE WHETHER OR NOT THE FUNCTION EXISTS ON THE GIVEN INTERVAL.
  - A  $f(x) = \tan x; \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
  - B  $f(x) = \cos x; \left[-\pi, \frac{3}{2}\right]$
  - C  $f(x) = |x|; [-3, 1]$
  - D  $f(x) = \frac{x}{x^2 - 1}; [-2, 2]$
  - E  $f(x) = \frac{2x}{x^2 - 9}; [-4, 4]$
  - F  $f(x) = \frac{x+1}{x^2 - 4}; [-1, 1]$
- 4 USING THE DEFINITION, EVALUATE EACH OF THE FOLLOWING INTEGRALS.
  - A  $\int_1^5 4 dx$
  - B  $\int_0^3 x dx$
  - C  $\int_{-1}^1 (x^2 + 1) dx$
  - D  $\int_{-1}^1 \frac{1}{x^2} dx$
  - E  $\int_{-1}^1 x^3 dx$
- 5 EVALUATE  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx$ . WHAT IS THIS DEFINITE INTEGRAL REPRESENTING?
- 6 USING THE FACT  $\sum_{x=1}^n x^2 = \sum_{s=1}^n s^2$  SHOW THAT  $\int_a^b f(x) dx = \int_a^b f(s) ds$
- 7 IN FIGURE 5.9, THE AREA OF THE REGION IS EQUAL TO
 
$$A = A_1 + A_2$$
 USE THIS FACT TO EXPLAIN THAT  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ .



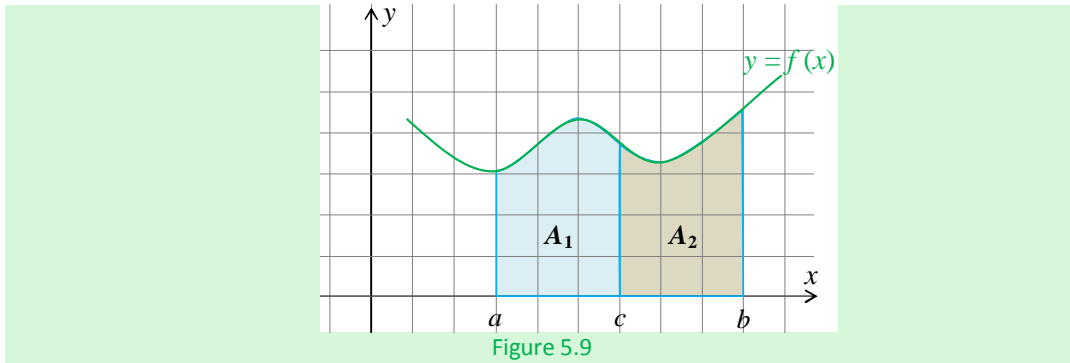


Figure 5.9

### 5.3.2 Fundamental Theorem of Calculus

FUNDAMENTAL THEOREM OF CALCULUS IS THE STATEMENT WHICH ASSERTS THAT DIFFERENTIATION AND INTEGRATION ARE INVERSE OPERATIONS OF EACH OTHER. TO UNDERSTAND THIS, LET  $f$  BE A CONTINUOUS FUNCTION ON  $[a, b]$ . IF YOU FIRST INTEGRATE  $f$  AND THEN DIFFERENTIATE THE RESULT YOU CAN RETRIEVE BACK THE ORIGINAL FUNCTION. THE NEXT THEOREM ALLOWS YOU TO EVALUATE THE DEFINITE INTEGRAL BY USING THE ANTI DERIVATIVE OF THE FUNCTION TO BE INTEGRATED.

#### Theorem 5.3 Fundamental theorem of calculus

IF  $f$  IS CONTINUOUS ON THE CLOSED INTERVAL  $[a, b]$  AND  $F$  IS AN ANTI DERIVATIVE (OR INDEFINITE INTEGRAL) OF  $f$  OF

THAT IS,  $F'(x) = f(x)$  FOR ALL  $x \in [a, b]$ , THEN  $\int_a^b f(x) dx = F(b) - F(a)$

**Example 4** EVALUATE  $\int_1^4 x dx$

**Solution** THIS VALUE IS CALCULATED USING THE DEFINITION OF DEFINITE INTEGRALS. HERE YOU USE FUNDAMENTAL THEOREM OF CALCULUS

THE INDEFINITE INTEGRAL,

$$F(x) = \int x dx = \frac{x^2}{2} + c \Rightarrow \int_1^4 x dx = F(4) - F(1) = \left(\frac{4^2}{2} + c\right) - \left(\frac{1^2}{2} + c\right) = \frac{15}{2}$$

OBSERVE THAT EVALUATING THE DEFINITE INTEGRAL USING THE INTEGRAL SUM IS LENGTHY AND COMPLICATED AS COMPARED TO USING THE FUNDAMENTAL THEOREM OF CALCULUS

#### Note:

IN EVALUATING  $F(b) - F(a)$ , THE CONSTANT OF INTEGRATION CANCELS OUT.

THEREFORE, YOU CAN WRITE  $\int_a^b f(x) dx = F(b) - F(a)$

**Example 5** EVALUATE  $\int_1^3 (x^3 + x + 1) dx$

**Solution** 
$$\int_1^3 (x^3 + x + 1) dx = \left. \frac{x^4}{4} + \frac{x^2}{2} + x \right|_1^3 = \left( \frac{3^4}{4} + \frac{3^2}{2} + 3 \right) - \left( \frac{1}{4} + \frac{1}{2} + 1 \right)$$

$$= \frac{81}{4} + \frac{9}{2} + 3 - \frac{1}{4} - \frac{1}{2} - 1 = \frac{80}{4} + \frac{8}{2} + 2 = 26$$

**Example 6** EVALUATE  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x dx$

**Solution** 
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x dx = -\cos x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\left[ \cos \frac{\pi}{3} - \cos \frac{\pi}{6} \right] = -\left[ \frac{1}{2} - \frac{\sqrt{3}}{2} \right] = \frac{\sqrt{3}-1}{2}$$

**Example 7** FIND THE AREA OF THE REGION BOUNDED BY THE ARC OF THE SINE FUNCTION BETWEEN  $x = 0$  AND  $x = \frac{\pi}{2}$ .

**Solution** THE AREA OF THIS REGION IDENTIFIED TO BE THE VALUE OF THE DEFINITE INTEGRAL  $\int_0^{\frac{\pi}{2}} \sin x dx$ .

$$\Rightarrow A = \int_0^{\frac{\pi}{2}} \sin x dx = -\cos x \Big|_0^{\frac{\pi}{2}} = -\left[ \cos \frac{\pi}{2} - \cos 0 \right] = -[0 - 1] = 1$$

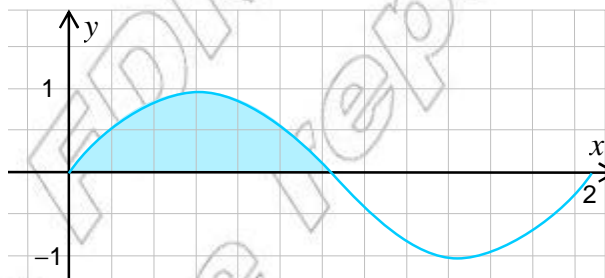


Figure 5.10

**Example 8** EVALUATE  $\int_{-1}^0 e^x dx$

**Solution** 
$$\int_{-1}^0 e^x dx = e^x \Big|_{-1}^0 = e^0 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$$

**Example 9** EVALUATE  $\int_1^e \ln x dx$

**Solution** 
$$\int_1^e \ln x dx = x \ln x - x \Big|_1^e = e \ln e - e - (1 \times \ln 1 + 1) = e - e - (0 + 1) = -1$$

## Properties of the definite integral

### ACTIVITY 5.7



LET  $f(x) = x^2$  AND  $g(x) = 1 - \frac{1}{x}$ .

1 EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS.

**A**  $\int_1^3 (f(x) + g(x)) dx$       **B**  $\int_{-2}^3 f(x) dx$

**C**  $\int_3^1 f(x) dx + \int_1^3 f(x) dx$       **D**  $\int_3^3 f(x) dx$

**E**  $4 \int_{-2}^3 f(x) dx$       **F**  $\int_1^4 g(x) dx + \int_4^{10} g(x) dx - \int_1^{10} g(x) dx$

2 LET  $f$  AND  $g$  BE CONTINUOUS FUNCTIONS ON THE CLOSED INTERVAL  $[a, b]$ .

**A** EVALUATE  $\int_a^a f(x) dx$       **B** EXPRESS  $\int_b^a f(x) dx$  IN TERMS OF  $\int_a^b f(x) dx$

**C** IN THE INDEFINITE INTEGRAL YOU LEARNED THAT

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx \text{ AND } \int kf(x) dx = k \int f(x) dx.$$

DOES THIS PROPERTY HOLD TRUE FOR DEFINITE INTEGRALS? JUSTIFY YOUR ANSWER BY PRODUCING EXAMPLES.

**D** YOU ALSO LEARNED THAT  $\sum_{i=1}^n a_i = \sum_{i=1}^k a_i + \sum_{i=k+1}^n a_i$  FOR  $k < n$ . DOES THE EQUALITY

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \text{ FOR } a \leq c < b \text{ HOLD TRUE?}$$

**E** IN DIFFERENTIAL CALCULUS YOU SAW THAT  $\frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$ .

GIVE AN EXAMPLE TO SHOW THAT  $\int_a^b \left( \frac{d}{dx} f(x) \right) \left( \frac{d}{dx} g(x) \right) dx \neq \int_a^b f(x) dx \cdot \int_a^b g(x) dx$ .

SHOW THAT  $\int_a^b \frac{f(x)}{g(x)} dx \neq \frac{\int_a^b f(x) dx}{\int_a^b g(x) dx}$  BY PRODUCING EXAMPLES.

### Properties of the definite Integral

IF  $f$  AND  $g$  ARE CONTINUOUS ON  $[a, b] \subseteq \mathbb{R}$  AND  $c \in [a, b]$  THEN

**1**  $\int_a^a f(x) dx = 0$

**2**  $\int_b^a f(x) dx = -\int_a^b f(x) dx$

**3**  $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

**4**  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

**5**  $\int_a^b kf(x) dx = k \int_a^b f(x) dx$

**Example 10** EVALUATE EACH OF THE FOLLOWING INTEGRALS FROM THE AB

**A**  $\int_3^3 (x^3 + 1) dx$       **B**  $\int_{-\frac{4}{4}} \sin x dx$       **C**  $\int_1^2 \left(x - \frac{1}{x^2}\right)^2 dx$

**D**  $\int_1^{\sqrt{2}} \frac{x}{x^2 + 1} dx + \int_{\sqrt{2}}^5 \frac{x}{x^2 + 1} dx$       **E**  $\int_{-1}^1 e^{+3x} dx$

**Solution**

**A** BY PROPERTY 1,  $\int_3^3 (x^3 + 1) dx = 0$

**B** BY PROPERTY 2,

$$\begin{aligned} \int_{-\frac{4}{4}} \sin x dx &= -\int^{\frac{4}{4}} \sin x dx = \cos x \Big|_{-\frac{4}{4}}^{\frac{4}{4}} = \cos \left(\frac{\pi}{4}\right) - \cos \left(-\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} - (-1) = \frac{\sqrt{2}}{2} + 1 \end{aligned}$$

**C** BY PROPERTY 3 AND PROPERTY 5,

$$\begin{aligned} \int_1^2 \left(x - \frac{1}{x^2}\right)^2 dx &= \int_1^2 \left(x^2 - \frac{2}{x} + \frac{1}{x^4}\right) dx \\ &= \int_1^2 x^2 dx - 2 \int_1^2 \frac{1}{x} dx + \int_1^2 \frac{1}{x^4} dx = \frac{x^3}{3} \Big|_1^2 - 2 \ln|x| \Big|_1^2 - \frac{1}{3x^3} \Big|_1^2 \\ &= \left(\frac{8}{3} - \frac{1}{3}\right) - 2[\ln 2 - \ln 1] - \left[\frac{1}{3(2^3)} - \frac{1}{3}\right] \\ &= \frac{7}{3} - 2 \ln 2 - \frac{7}{24} = \frac{21}{8} - 2 \ln 2 \end{aligned}$$

**D** BY PROPERTY 4,

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{x}{x^2 + 1} dx + \int_{\sqrt{2}}^5 \frac{x}{x^2 + 1} dx &= \int_1^5 \frac{x}{x^2 + 1} dx \\ &= \frac{1}{2} \ln|x^2 + 1| \Big|_1^5 \\ &= \frac{1}{2} (\ln 26 - \ln 2) = \frac{1}{2} \ln 13 \end{aligned}$$

**E**  $\int_{-1}^1 e^{+3x} dx = \int_{-1}^1 e^{3x} dx = \frac{e^{3x}}{3} \Big|_{-1}^1$

$$\begin{aligned} \int_{-1}^1 e^{+3x} dx &= \int_{-1}^1 e^{3x} dx = e \int_{-1}^1 e^{3x} dx = e \left. \frac{e^{3x}}{3} \right|_{-1}^1 = e \left( \frac{e^3}{3} - \frac{e^{-3}}{3} \right) = \frac{e^{-3}}{3} (e^6 - 1) \\ &= e \left( \frac{e^3}{3} - \frac{e^{-3}}{3} \right) = \frac{e^{-3}}{3} (e^6 - 1). \end{aligned}$$

## Change of variable

IN EVALUATING THE INDEFINITE INTEGRAL  $\int f(x) dx$ , THE METHODS YOU HAVE BEEN USING ARE: SUBSTITUTION, PARTIAL FRACTIONS AND INTEGRATION BY PARTS.

IN THE SUBSTITUTION METHOD, THE COMPOSITION OF THE FUNCTION IS THE DERIVATIVE OF (

$$\Rightarrow \int_a^b f(g(x)) \cdot g'(x) dx = F(g(b)) - F(g(a))$$

TO EVALUATE THE DEFINITE INTEGRAL BY THE METHOD OF SUBSTITUTION, YOU TRANSFORM THE INTEGRAND AS WELL AS THE LIMITS OF INTEGRATION.

FOR THIS PROCESS YOU HAVE THE FOLLOWING THEOREM.

### Theorem 5.4 Change of variables

IF THE FUNCTION IS CONTINUOUS ON A CLOSED INTERVAL  $[a, b]$ , AND THE SUBSTITUTION FUNCTION  $u = g(x)$  IS DIFFERENTIABLE WITH  $g(a) = c$  AND  $g(b) = d$ , THEN

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

## ACTIVITY 5.8



1 IF  $-1 \leq x \leq 2$ , FIND THE INTERVALS OF VALUES FOR

**A**  $u = 3x - 1$

**B**  $u = \sqrt{x+1}$

**C**  $u = e^{x^2+1}$

**D**  $u = 1 - x\sqrt{x^2-1}$

**E**  $u = x^2 - x - 2$

2 IF  $\int_a^b \frac{2x}{\sqrt{x^2+4}} dx = \int_c^d \frac{du}{\sqrt{u}}$ , FIND THE VALUES OF  $a, b, c, d$ .

3 EVALUATE  $\int_1^2 (2x+1)\sqrt{x^2+x-2} dx$  USING THE METHOD OF SUBSTITUTION.

**Example 11** EVALUATE THE INTEGRAL  $\int_1^2 xe^{x^2} dx$

**Solution** USING INTEGRATION BY SUBSTITUTION,

$$u = x^2, \frac{du}{dx} = 2x \Rightarrow \frac{1}{2} du = x dx$$

AS  $x$  VARIES FROM 1 TO 2,  $u(x)$  VARIES FROM  $u(1) = 1$  TO  $u(2) = 2^2 = 4$ .

$$\int_1^2 xe^{x^2} dx = \frac{1}{2} \int_1^4 e^u du = \frac{1}{2} e^u \Big|_1^4 = \frac{1}{2} (e^4 - e)$$

**Example 12** EVALUATE THE INDEFINITE INTEGRAL  $\int \sqrt{2x^2 + 5} dx$ .

**Solution** HERE  $u = g(x) = 2x^2 + 5$ ,  $g(-3) = 2(-3)^2 + 5 = 23$ ,

$$g(1) = 2(1)^2 + 5 = 7$$

$$\frac{du}{dx} = \frac{d}{dx}(2x^2 + 5) = 4x \Rightarrow \frac{1}{4} du = x dx$$

$$\int_{-3}^1 x\sqrt{2x^2 + 5} dx = \frac{1}{4} \int_{23}^7 \sqrt{u} du = \frac{1}{4} \left( \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right)_{23}^7 = \frac{1}{6} (7\sqrt{7} - 23\sqrt{23}).$$

**Example 13** EVALUATE  $\int_0^{\frac{\pi}{3}} \cos^3 x \sin x dx$

**Solution** THE DERIVATIVE OF  $\cos x$  IS  $-\sin x$  WHICH IS A FACTOR OF THE INTEGRAND.

HENCE  $u = g(x) = \cos x$ .

$$\Rightarrow -du = \sin x dx$$

$$\int_0^{\frac{\pi}{3}} \cos^3 x \sin x dx = -\int_{g(\frac{\pi}{3})}^{g(0)} u^3 du = -\int_1^{\frac{1}{2}} u^3 du = -\frac{u^4}{4} \Big|_1^{\frac{1}{2}} = -\left(\frac{1}{64} - \frac{1}{4}\right) = \frac{15}{64}$$

### Exercise 5.10

IN EXERCISES 1–15 EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS USING FUNDAMENTAL THEOREM OF CALCULUS. IN THE EXERCISES,

**1**  $\int_{-1}^4 3 dx$

**2**  $\int_a^b dx$

**3**  $\int_{-1}^5 -x dx$

**4**  $\int_1^2 x^5 dx$

**5**  $\int_a^b x^n dx$

**6**  $\int_2^1 (x^3 + 5x^2 - 1) dx$

**7**  $\int_1^2 \sqrt{2x-1} dx$

**8**  $\int_0^1 2^{3-x} dx$

**9**  $\int_0^{\frac{\pi}{2}} \sin^2 x dx$

**10**  $\int_2^3 \frac{1}{x} dx$

**11**  $\int_0^1 \frac{9}{4x+1} dx$

**12**  $\int_0^{\sqrt{e}} \sin(x^2 + 3) dx$

**13**  $\int_{\frac{1}{2}}^2 x \sin x dx$

**4**  $\int_e^{e^3} x \ln x dx$

**15**  $\int_1^3 \frac{1}{2x^2 + x - 1} dx$

IN EXERCISES 16 – 25 EVALUATE EACH OF THE FOLLOWING DEFINITE INTEGRALS USING SINGLE CHANGE OF VARIABLES.

**16**  $\int_{-1}^1 \frac{2x+3}{(x^3+3x+4)^6} dx$

**17**  $\int_{-1}^{\frac{1}{2}} (4x+3)^{10} dx$

**18**  $\int_{\sqrt{2}}^3 x\sqrt{x^2+7} dx$

- 19**  $\int_1^2 x^2(x^3 - 3)^5 dx$       **20**  $\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$       **21**  $\int_0^2 (x-2)\sqrt{x+1} dx$
- 22**  $\int_{-1}^0 \frac{x+4}{3x+1} dx$       **23**  $\int_{-4}^3 \frac{x+1}{x^2-x-6} dx$       **24**  $\int_{-1}^1 \frac{3t^2-1}{e^{t^3-t}} dt$
- 25**  $\int_{-1}^1 \frac{e^x}{1+e^x} dx$
- 26** YOU KNOW THAT  $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$ . IS  $f(x)$  INTEGRABLE ON  $[2, 4]$ ? IF SO, FIND  $\int_{-2}^2 \frac{1}{x^2} dx$ .
- 27** LET  $f$  BE AN EVEN FUNCTION WHICH IS CONTINUOUS FOR ANY REAL NUMBER THEN, SHOW THAT  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ . USE  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x dx$  TO VERIFY YOUR WORK
- 28** IF  $f$  IS AN ODD FUNCTION THAT IS CONTINUOUS FOR ANY REAL NUMBER SHOW THAT  $\int_{-a}^a f(x) dx = 0$
- VERIFY YOUR CONCLUSION BY COMPUTING THE FOLLOWING INTEGRALS
- A**  $\int_{-3}^3 (x^3 + x) dx$       **B**  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx$       **C**  $\int_{-1}^1 \frac{x}{x^2+1} dx$

## 5.4 APPLICATIONS OF INTEGRAL CALCULUS

IN THIS SECTION, YOU SHALL SEE SOME OF THE MATHEMATICAL AND PHYSICAL APPLICATIONS OF INTEGRAL CALCULUS. IN THE MATHEMATICAL APPLICATION YOU CALCULATE THE AREA BOUNDED BY CURVES OF CONTINUOUS FUNCTIONS DEFINED ON A CLOSED INTERVAL  $[a, b]$  AND THE VOLUME OF A SOLID OF REVOLUTION.

IN THE PHYSICAL APPLICATIONS, YOU CALCULATE THE WORK DONE BY A VARIABLE FORCE, STRAIGHT LINE, ACCELERATION, VELOCITY AND DISPLACEMENT.

### 5.4.1 The Area Between Two Curves

YOU CALCULATED THE AREA OF SOME REGIONS UNDER THE GRAPHS OF A NON-NEGATIVE FUNCTION ON  $[a, b]$ , WHEN THE DEFINITE INTEGRAL WAS DEFINED. HOWEVER THE FOCUS WAS TO EVALUATE THE INTEGRAL RATHER THAN TO CALCULATE AREA. HERE, YOU USE THIS CONCEPT IN ORDER TO DETERMINE THE AREA OF A REGION WHOSE UPPER AND LOWER BOUNDARIES ARE CONTINUOUS FUNCTIONS ON A GIVEN CLOSED INTERVAL  $[a, b]$

## ACTIVITY 5.9



- 1 USING THE DEFINITION OF THE DEFINITE INTEGRAL, CALCULATE THE AREA OF THE REGION BOUNDED BY THE GRAPH OF

  - A  $y = x$  AND THE  $x$ -AXIS BETWEEN  $x = 0$  AND  $x = 1$ .
  - B  $y = x^2 + 1$  AND THE  $x$ -AXIS BETWEEN  $x = 0$  AND  $x = 1$ .
- 2 USING THE RESULTS FROM 1, AND YOUR KNOWLEDGE OF THE AREA OF A SHADED PART, FIND THE AREA OF THE REGION BOUNDED BY THE GRAPHS OF  $y = x^2 + 1$  AND  $y = x$  BETWEEN  $x = 0$  AND  $x = 1$ .

WE EXTEND THE PROBLEM TO FIND THE AREA OF AN ARBITRARY REGION ENCLOSED BY THE GRAPHS OF CONTINUOUS FUNCTIONS.

**Example 1** FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF THE FUNCTION

$$f(x) = x^2 - 3x + 2 \text{ AND THE } x\text{-AXIS BETWEEN } x = 0 \text{ AND } x = 3.$$

**Solution** LOOK AT THE GRAPH BETWEEN  $x = 0$  AND  $x = 3$ .

LET  $A_1$ ,  $A_2$  AND  $A_3$  BE THE AREAS OF THE PARTS OF THE REGION BETWEEN  $x = 0$  AND  $x = 1$ ,  $x = 1$  AND  $x = 2$ , AND  $x = 2$  AND  $x = 3$ , RESPECTIVELY.

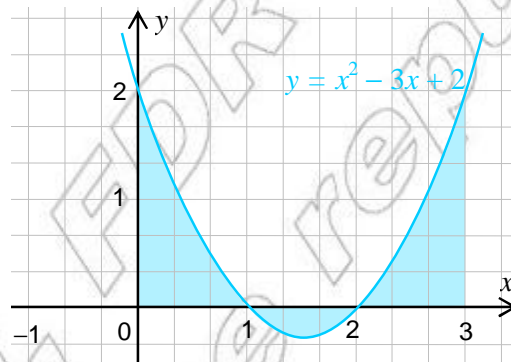


Figure 5.11

THE PART OF THE REGION BETWEEN  $x = 1$  AND  $x = 2$  IS BELOW THE  $x$ -AXIS.

$$\Rightarrow A_2 = -\int_1^2 (x^2 - 3x + 2) dx = -\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_1^2 = 4 - \frac{23}{6} = \frac{1}{6}$$

$$\text{WHEREAS } A_1 = \int_0^1 (x^2 - 3x + 2) dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_0^1 = \frac{5}{6} \text{ AND}$$

$$A_3 = \int_2^3 (x^2 - 3x + 2) dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_2^3 = \frac{5}{6}$$



THEREFORE, THE AREA A OF THE REGION IS

$$A = A_1 + A_2 + A_3 = \frac{11}{6}$$

WHAT WOULD HAVE HAPPENED, IF YOU HAD SIMPLY TRIED TO CALCULATE

$$A = \int_0^3 (x^2 - 3x + 2) dx?$$

**Example 2** FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF THE  $\sin x$ -AXIS BETWEEN  $x = -\frac{\pi}{2}$  AND  $x = 2$ .

**Solution**

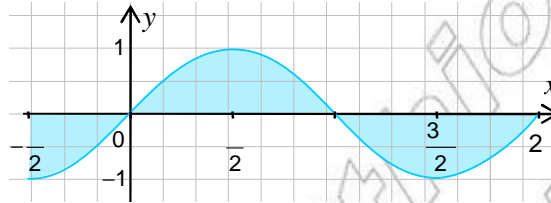


Figure 5.12

FROM THE GRAPH YOU HAVE THE AREA

$$\begin{aligned} A &= -\int_{-\frac{\pi}{2}}^0 \sin x \, dx + \int_0^{\frac{\pi}{2}} \sin x \, dx - \int_{\frac{\pi}{2}}^2 \sin x \, dx \\ &= -\cos x \Big|_{-\frac{\pi}{2}}^0 - \cos x \Big|_0^{\frac{\pi}{2}} + \cos x \Big|_{\frac{\pi}{2}}^2 \\ &= \cos 0 + (\cos 0 - \cos \frac{\pi}{2}) + (\cos 2 - \cos \frac{\pi}{2}) \\ &= 1 + 1 - (-1) + 1 - (-1) = 5 \end{aligned}$$

**Example 3** FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF  $y = x^5$  AND THE  $x$ -AXIS BETWEEN  $x = -1$  AND  $x = 1$ .

**Solution**

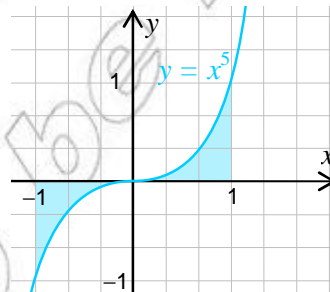


Figure 5.13

FROM THE SYMMETRY OF THE REGION, YOU HAVE THE AREA

$$A = 2 \int_0^1 x^5 \, dx = 2 \left( \frac{x^6}{6} \right) \Big|_0^1 = \frac{1}{3}$$

**Example 4** FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF  $y = x^3 - 2x^2 + x$  AND THE X-AXIS BETWEEN  $x = -1$  AND  $x = 2$ .

**Solution**

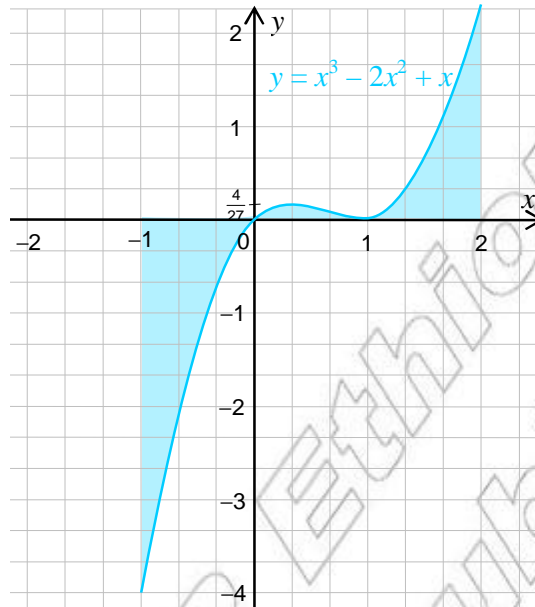


Figure 5.14

$$f(x) = x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2$$

LET THE AREA OF THE PART OF THE REGION UNDER THE

$$\text{THEN, } A_1 = -\int_{-1}^0 (x^3 - 2x^2 + x) dx = -\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{-1}^0$$

$$= -\left[0 - \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2}\right)\right] = \frac{17}{12} \text{ SQUARE UNITS}$$

LET THE AREA OF THE REGION ABOVE BE  $A_2$  THEN

$$A_2 = \int_0^2 (x^3 - 2x^2 + x) dx = \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_0^2 = \left(\frac{16}{4} - \frac{16}{3} + 2\right) = \frac{2}{3}$$

$\Rightarrow$  THE AREA OF THE REGION IS

$$A = A_1 + A_2 = \frac{17}{12} + \frac{2}{3} = \frac{25}{12}$$

**Example 5** Let  $f(x) = \begin{cases} 2^x, & \text{if } x \leq 1; \\ 1 + \frac{1}{x}, & \text{if } x > 1. \end{cases}$

FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF  $f(x)$  BETWEEN  $x = -1$  AND  $x = 2$ .

**Solution** YOU FIRST SHOW THAT THE FUNCTION IS CONTINUOUS ON  $[-1, 2]$ . LOOK AT THE GRAPH OF  $f(x)$  IN FIGURE 5.15. THE FUNCTION IS CONTINUOUS ON  $[-1, 2]$ .

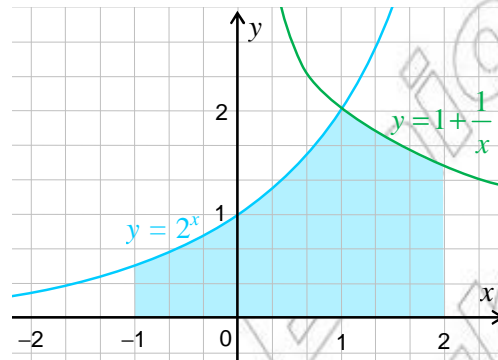


Figure 5.15

THE UPPER PART OF THE REGION IS BOUNDED BY THE GRAPHS OF TWO FUNCTIONS,  $y = 2^x$  AND  $y = 1 + \frac{1}{x}$  INTERSECTING AT  $(1, 2)$ .

LET  $A_1$  BE THE AREA OF THE REGION BETWEEN THE LINES  $x = -1$  AND  $x = 1$  AND  $A_2$  BE THE AREA OF THE REGION BETWEEN THE LINES  $x = 1$  AND  $x = 2$ .

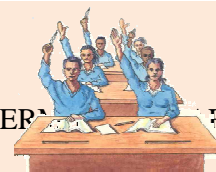
$$A_1 = \int_{-1}^1 2^x dx = \frac{2^x}{\ln 2} \Big|_{-1}^1 = \frac{1}{\ln 2} \left( 2 - \frac{1}{2} \right) = \frac{3}{2 \ln 2} = \frac{3}{2 \ln 2}$$

$$A_2 = \int_1^2 \left( 1 + \frac{1}{x} \right) dx = x + \ln|x| \Big|_1^2 = 2 + \ln 2 - (1 + \ln 1) = 1 + \ln 2$$

$$\Rightarrow \text{THE AREA OF THE REGION } A = A_1 + A_2 = \frac{3}{2 \ln 2} + 1 + \ln 2$$

## ACTIVITY 5.10

- USING YOUR KNOWLEDGE OF SHADED AREA, DETERMINE THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF  $f(x) = 2^x$  AND  $g(x) = 1$  AND THE LINES  $x = -1$  AND  $x = 3$ .



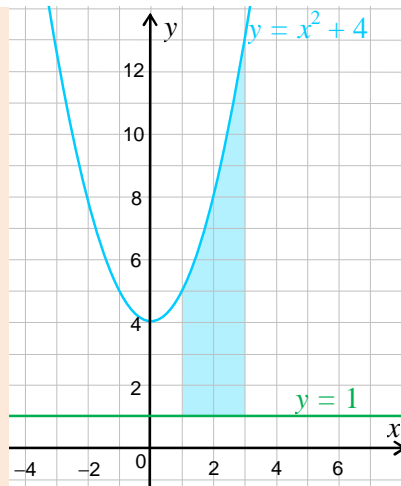


Figure 5.16

2 CONSIDER THE FOLLOWING REGION

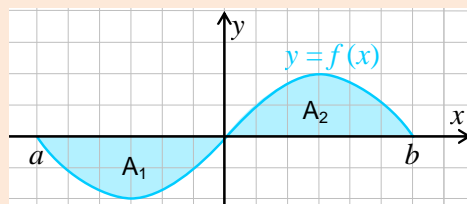


Figure 5.17

EXPRESS THE AREA OF THE REGION IN TERMS OF THE INTEGRAL OF THE BOUNDARIES  $y = f(x)$  AND  $y = 0$ .

**Theorem 5.5**

SUPPOSE  $f$  AND  $g$  ARE CONTINUOUS FUNCTIONS ON  $[a, b]$  WITH  $f(x) \geq g(x)$  ON  $[a, b]$ . THE AREA BOUNDED BY THE CURVES  $y = f(x)$  AND  $y = g(x)$  BETWEEN THE LINES  $x = a$  AND  $x = b$  IS

$$A = \int_a^b (f(x) - g(x)) dx.$$

**Example 6** FIND THE AREA OF THE REGION ENCLOSED BY THE CURVES  $f(x) = x - 3$ .

**Solution** THE FIRST STEP IS TO DRAW THE GRAPHS OF BOTH FUNCTIONS USING THE SAME SCALE. YOU SOLVE THE EQUATION  $f(x) = g(x)$  TO GET THE INTERSECTION POINTS OF THE GRAPHS.

$$x^2 - x - 6 = x - 3$$

$$\Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = 3 \text{ OR } x = -1$$

$$f(x) \geq g(x) \text{ ON } [-1, 3]$$

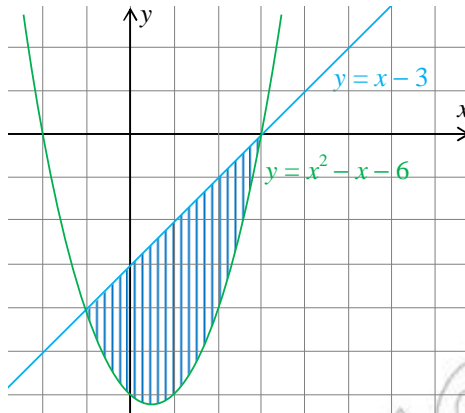


Figure 5.18

**Note:**

THE HEIGHT OF EACH INFINITESIMAL RECTANGLE WITHIN THE SHADED REGION IS EQUAL TO (

⇒ THE AREA OF THE REGION IS

$$\begin{aligned}
 A &= \int_{-1}^3 ((x-3) - (x^2 - x - 6)) dx = \int_{-1}^3 (-x^2 + 2x + 3) dx = -\frac{x^3}{3} + x^2 + 3x \Big|_{-1}^3 \\
 &= \frac{-27}{3} + 9 + 9 - \left( -\left(\frac{-1}{3}\right) + 1 + 3(-1) \right) = 9 - \left( \frac{1}{3} + 1 - 3 \right) = \frac{32}{3}
 \end{aligned}$$

**Example 7** FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF  $g(x) = 6 - x$  BETWEEN THE LINES  $x = 3$  AND  $x = -1$ .

**Solution** THE FIRST STEP IS TO DRAW THE GRAPHS OF BOTH FUNCTIONS USING THE SAME COORDINATE AXES.

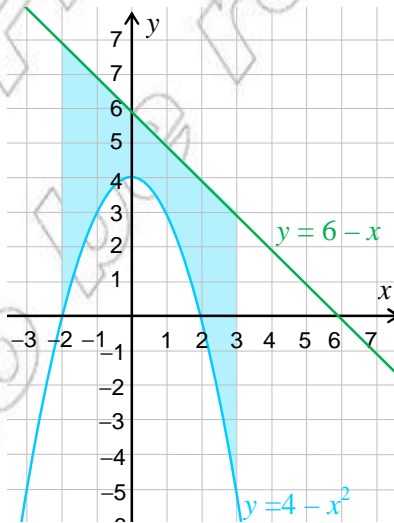


Figure 5.19

$$g(x) \geq f(x) \text{ ON } [2, 3]$$

$$\begin{aligned} \Rightarrow \text{THE AREA} &= \int_{-2}^3 ((6-x) - (4-x^2)) dx = \int_{-2}^3 (x^2 - x + 2) dx \\ &= \left. \frac{x^3}{3} - \frac{x^2}{2} + 2x \right|_{-2}^3 = \frac{27}{3} - \frac{9}{2} + 6 - \left[ \frac{-8}{3} - \frac{4}{2} - 4 \right] = \frac{115}{6} \end{aligned}$$

**Example 8** FIND THE AREA OF THE REGION IN THE FIRST QUADRANT WHICH IS ENCLOSED BY THE Y-AXIS AND THE CURVES  $f(x) = \cos x$  AND  $g(x) = \sin x$ .

**Solution** LOOK AT THE GRAPHS OF BOTH FUNCTIONS.

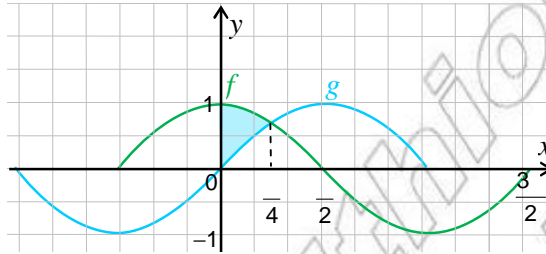


Figure 5.20

THE CURVES MEET AT  $\frac{\pi}{4}$  AND  $\cos x \geq \sin x$  ON  $\left[0, \frac{\pi}{4}\right]$ .

THEREFORE THE REQUIRED AREA IS

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx = \left. \sin x + \cos x \right|_0^{\frac{\pi}{4}} \\ &= \sin \frac{\pi}{4} + \cos \frac{\pi}{4} - (\sin 0 + \cos 0) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1 \end{aligned}$$

**Example 9** FIND THE AREA ENCLOSED BY THE GRAPHS OF  $f(x) = x^2 - 1$

**Solution** THE FIRST STEP IS TO DRAW BOTH GRAPHS.

SOLVE THE EQUATION

$x^3 - x = x^2 - 1$  TO FIND OUT THE INTERSECTION POINTS OF THE GRAPHS.

$$x^3 - x = x^2 - 1$$

$$\Rightarrow x^3 - x^2 - x + 1 = 0 \Rightarrow x^2(x-1) - (x-1) = 0 \Rightarrow (x^2-1)(x-1) = 0 \Rightarrow x = \pm 1$$

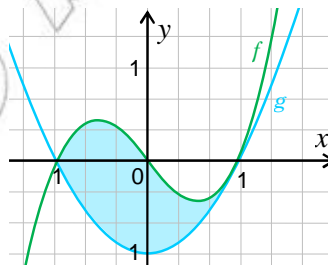


Figure 5.21

THE REQUIRED AREA  $\int_{-1}^1 (x^3 - x - (x^2 - 1)) dx = \int_{-1}^1 (x^3 - x^2 - x + 1) dx$

$$= \left. \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x \right|_{-1}^1 = \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 - \left( \frac{1}{4} + \frac{1}{3} - \frac{1}{2} - 1 \right) = \frac{4}{3}$$

**Example 10** FIND THE AREA OF THE REGION ENCLOSED BY THE CURVES  $f(x) = 2x^2 - x^3$  AND  $g(x) = 4 - x^2$ .

**Solution** THE FIRST STEP IS TO DETERMINE THE INTERSECTION POINTS OF THE GRAPHS AND THEN TO DRAW BOTH GRAPHS.

THUS,  $2x^2 - x^3 = 4 - x^2 \Rightarrow x^3 - 3x^2 + 4 = 0$

USING THE RATIONAL ROOT TEST THE ZEROS ARE  $x = -1$  AND  $x = 2$ .

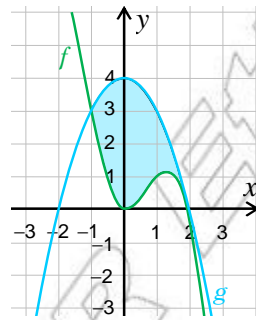


Figure 5.22

THE REQUIRED AREA  $\int_{-1}^2 (4 - x^2) - (2x^2 - x^3) dx = \int_{-1}^2 (x^3 - 3x^2 + 4) dx$

$$= \left. \frac{x^4}{4} - x^3 + 4x \right|_{-1}^2 = \left( \frac{2^4}{4} - 2^3 + 4(2) \right) - \left( \frac{(-1)^4}{4} + 1 - 4 \right)$$

$$= (4 - 8 + 8) - \left( \frac{1}{4} + 1 - 4 \right) = \left( 4 + 3 - \frac{1}{4} \right) = \frac{27}{4}$$

**Example 11** FIND THE AREA ENCLOSED BY THE GRAPH OF  $f(x) = |x|$  AND THE X-AXIS BETWEEN THE VERTICAL LINES  $x = -4$  AND  $x = 3$ .

**Solution** DO YOU THINK THAT  $\int_{-4}^3 f(x) dx$  EXISTS? *Explain!*

$f(x) = |x|$  IS CONTINUOUS ON  $[-4, 3]$ .

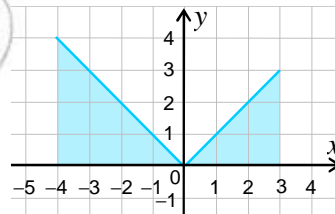


Figure 5.23

YOU KNOW THAT  $f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

THUS, THE AREA  $\int_{-4}^3 |x| dx = \int_{-4}^0 |x| dx + \int_0^3 |x| dx$

$$= \int_{-4}^0 (-x) dx + \int_0^3 x dx = -\frac{x^2}{2} \Big|_{-4}^0 + \frac{x^2}{2} \Big|_0^3 = -\left(0 - \frac{(-4)^2}{2}\right) + \left(\frac{3^2}{2} - 0\right) = \frac{25}{2}$$

**Example 12** DETERMINE THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF  $x = 9 - 2y^2$  AND  $x = -y^2$ .

**Solution** HERE THE CURVES ARE OPENING IN THE NEGATIVE DIRECTION OF THE X-AXIS. THE REGION IS SYMMETRICAL WITH RESPECT TO THE Y-AXIS.

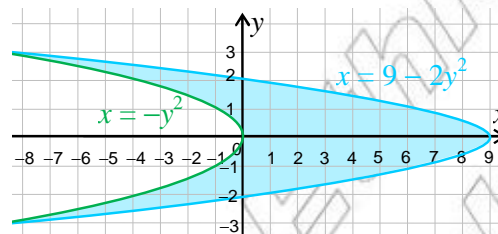


Figure 5.24

YOU SOLVE  $y^2 = 9 - 2y^2$  IN ORDER TO DETERMINE THE INTERSECTION POINTS OF THE GRAPHS. THUS,  $y^2 = 9 - 2y^2 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

THE REQUIRED AREA IS FOUND BY INTEGRATING WITH RESPECT TO Y.

$$A = 2 \int_0^3 ((9 - 2y^2) + y^2) dy = 2 \left( 9y - \frac{y^3}{3} \right) \Big|_0^3 = 2(27 - 9) = 36$$

**Example 13** FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPH OF  $y = x^2 + 1$  AND THE LINE  $y = 5$ .

**Solution** FROM THE GRAPH, YOU SEE THAT THE LINE CROSSES THE CURVE AT  $x = \pm 2$ .

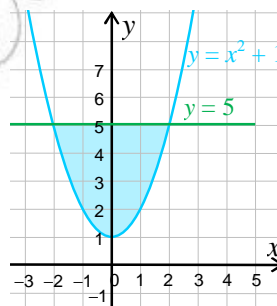


Figure 5.25



$x^2 + 1 \leq 5$  FOR ALL  $[-2, 2]$ . THEREFORE, THE REQUIRED AREA IS

$$A = \int_{-2}^2 (5 - (x^2 + 1)) dx = 4x - \frac{x^3}{3} \Big|_{-2}^2 = 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}.$$

### Exercise 5.11

1 FIND THE AREA OF THE REGION ENCLOSED BY THE FUNCTION AND THE X-AXIS FROM  $x = a$  TO  $x = b$  WHEN

**A**  $f(x) = x$ ;  $x = -3$  AND  $x = 2$ .      **B**  $f(x) = 12 - 3x^2$ ;  $x = -4$  AND  $x = 3$ .

**C**  $f(x) = x^3$ ;  $x = -1$  AND  $x = 1$ .      **D**  $f(x) = 2^x$ ;  $x = -1$  AND  $x = 4$ .

**E**  $f(x) = \ln x$ ;  $x = \frac{1}{e}$  AND  $x = e^2$ .      **F**  $f(x) = \sin x$ ;  $x = \frac{\pi}{4}$  AND  $x = \frac{7}{4}\pi$ .

**G**  $f(x) = \frac{1}{x}$ ;  $x = \frac{1}{10}$  AND  $x = 1$ .      **H**  $f(x) = x^2 + 4$ ;  $x = -1$  AND  $x = \frac{1}{2}$ .

2 FIND THE AREA OF THE REGION ENCLOSED BY THE GRAPHS OF

**A**  $f(x) = x^2$  AND  $g(x) = \sqrt{x}$ .      **B**  $f(x) = |x|$  AND  $g(x) = x^2$ .

**C**  $f(x) = 3x^2 - 4$  AND  $g(x) = 2x^2$ .      **D**  $f(x) = x^3 - 4x$  AND  $g(x) = -3x^2$ .

### 5.4.2 Volume of Revolution



#### OPENING PROBLEM

A HEMISPHERICAL BOWL OF RADIUS 5M CONTAINS SOME WATER. IF THE RADIUS OF THE SURFACE OF THE WATER IS 3M, WHAT IS THE VOLUME OF THE WATER?

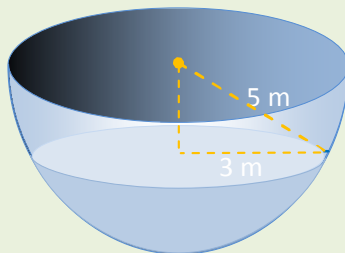


Figure 5.26

IN THIS SECTION, YOU WILL APPLY INTEGRATION TO DETERMINE THE VOLUME OF A SOLID BY CONSIDERING CROSS SECTIONS. IN THE STUDY OF PLANTS OR ANIMALS, VERY THIN CROSS SECTIONS ARE PREPARED BY SCIENTISTS. DURING EXAMINATION IN A TRANSMISSION ELECTRON MICROSCOPE (TEM), THE ELECTRON BEAM CAN PENETRATE IF THE SLICED SPECIMEN IS EXTREMELY THIN. BECAUSE ONLY THE ELECTRONS THAT PASS THROUGH THE SPECIMEN ARE RECORDED.

SUPPOSE A REGION ROTATES ABOUT A STRAIGHT LINE AS SHOWN THEN A SOLID FIGURE, CALLED A **solid of revolution**, WILL BE FORMED (SEE 5.27)

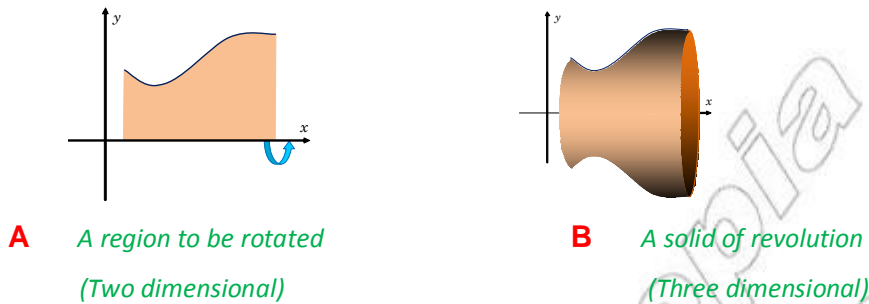


Figure 5.27

## ACTIVITY 5.11



IDENTIFY THE TYPE OF THE SOLID SO FORMED WHEN THE GIVEN REGION ROTATES ABOUT THE Y-AXIS

- 1 THE REGION BOUNDED BY THE LINE  $y = 5$  AND THE Y-AXIS BETWEEN  $x = -1$  AND  $x = 1$ ;
- 2 THE REGION BOUNDED BY THE LINE  $y = 1$  AND THE Y-AXIS BETWEEN  $x = -1$  AND  $x = 1$ ;
- 3 THE SEMICIRCLE  $y^2 = 1 - x^2$ ;  $0 \leq y \leq 1$  AND THE Y-AXIS BETWEEN  $x = -1$  AND  $x = 1$ ;
- 4 THE REGION BETWEEN THE LINE  $y = 3$  AND THE Y-AXIS BETWEEN  $x = -1$  AND  $x = 1$ ;
- 5 THE REGION BOUNDED BY  $y = \sqrt{4 - x^2}$  AND THE Y-AXIS BETWEEN  $x = -2$  AND  $x = 2$ .

FROM THE ABOVE ACTIVITY, YOU HAVE SEEN DIFFERENT SOLIDS FORMED BY ROTATING AN AREA ABOUT A STRAIGHT LINE. IN GENERAL, A SOLID OF REVOLUTION IS A THREE DIMENSIONAL OBJECT FORMED BY ROTATING AN AREA ABOUT A STRAIGHT LINE. THE NEXT TASK IS TO FIND THE VOLUME OF SUCH A SOLID OF REVOLUTION. THE VOLUME OF A SOLID OF REVOLUTION IS SAID TO BE **Volume**. THE LINE ABOUT WHICH THE AREA ROTATES IS AN **Axis of Symmetry**.

NOW, CONSIDER THE FOLLOWING SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION BOUNDED BY THE CURVE  $y = f(x)$  AND THE X-AXIS FROM  $x = a$  TO  $x = b$ .

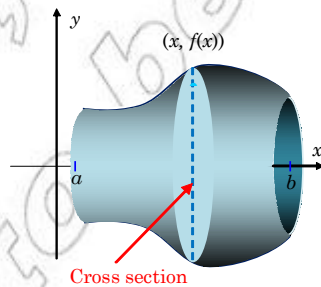


Figure 5.28

EVERY CROSS SECTION WHICH IS PERPENDICULAR TO THE X-AXIS IS A CIRCULAR REGION WITH RADIUS  $r = f(x)$ . THUS, THE AREA OF THE CROSS SECTION IS

## How to determine the volume of a solid of revolution

DIVIDE THE SOLID OF REVOLUTION INTO EQUALLY SPACED CROSS SECTIONS WHICH ARE PERPENDICULAR TO THE AXIS OF ROTATION. SEE

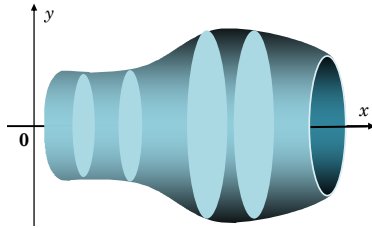


Figure 5.29



Figure 5.30

AS THE CUTS GET CLOSE ENOUGH, THEN THE SECTIONS APPROXIMATELY BE A CYLINDRICAL SOLID. SEE FIGURE 5.30

LET  $V_k$  BE THE VOLUME OF THE SECTIONS, THEN

$$V_k = r^2 h, \text{ WHERE } r = f(x_k) \text{ AND } h = \Delta x$$

$$\Rightarrow V_k = (f(x_k))^2 \Delta x$$

LET  $\Delta V$  BE THE SUM OF THE VOLUMES OF THE SECTIONS.

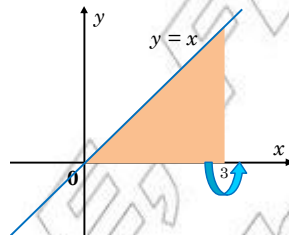
$$\text{THEN } \Delta V = \sum_{k=1}^n V_k.$$

THE VOLUME OF THE SOLID OF REVOLUTION IS

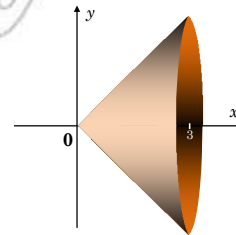
$$V = \lim_{\Delta x \rightarrow 0} \Delta V = \lim_{n \rightarrow \infty} \sum_{k=1}^n V_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n (f(x_k))^2 \Delta x = \int_a^b (f(x))^2 dx$$

**Example 14** FIND THE VOLUME GENERATED WHEN THE AREA BOUNDED BY THE  $x$ -AXIS FROM 0 TO  $x = 3$  IS ROTATED ABOUT THE  $y$ -AXIS.

**SOLUTION**



**A** Rotating the region about the  $y$ -axis gives the solid as shown in the figure on the right. Using the definite integral the volume  $V$  is determined as follows



**B** The solid of revolution is a right circular cone with radius and height each 3 units long.

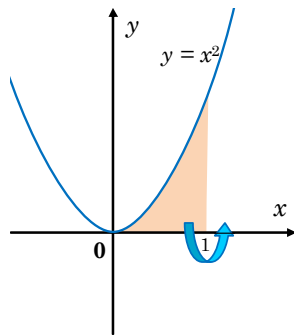
Figure 5.31

$$V = \int_0^3 x^2 dx = \left. \frac{x^3}{3} \right|_0^3 = 9$$

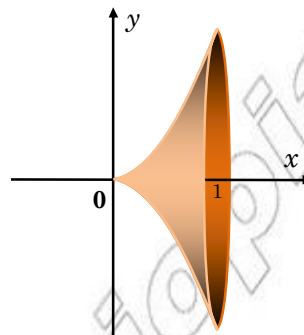
CHECK THAT YOU ARRIVE AT THE SAME RESULT FOR THE VOLUME OF THE CONE.

**Example 15** FIND THE VOLUME OF THE SOLID GENERATED BY THE REGION BOUNDED BY THE GRAPH OF  $y = x^2$  AND THE  $y$ -AXIS BETWEEN  $x = 0$  AND  $x = 1$  ABOUT THE  $y$ -AXIS.

**Solution**



**A** The region to be rotated



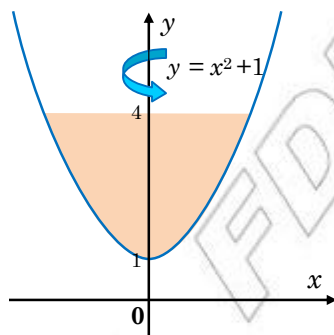
**B** The solid of revolution

Figure 5.32

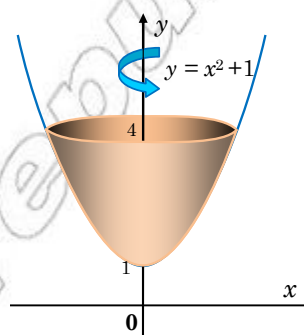
$$V = \int_0^1 (x^2)^2 dx = \frac{x^5}{5} \Big|_0^1 = \frac{1}{5}$$

**Example 16** THE AREA BOUNDED BY THE GRAPH OF  $y = x^2 + 1$  AND THE LINE  $y = 4$  ROTATES ABOUT THE  $y$ -AXIS, FIND THE VOLUME OF THE SOLID GENERATED.

**Solution**



**A** The region being rotated about the  $y$ -axis.



**B** The solid of revolution

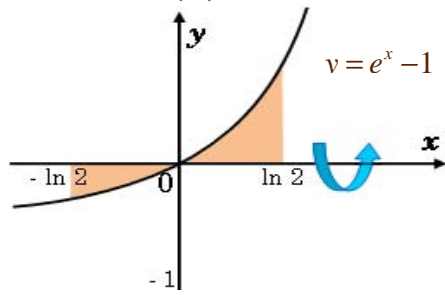
Figure 5.33

$y = x^2 + 1 \Rightarrow x = \pm \sqrt{y-1}$ . HERE, YOU HAVE HORIZONTAL CROSS SECTIONS.

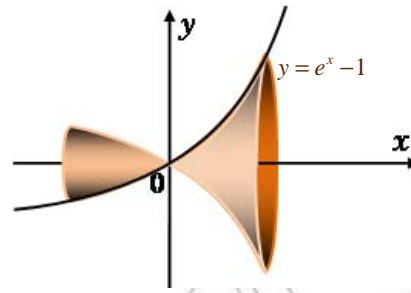
$$V = \int_1^4 (\sqrt{y-1})^2 dx = \int_1^4 (y-1) dy = \left( \frac{y^2}{2} - y \right) \Big|_1^4 = \left( \frac{16}{2} - 4 \right) - \left( \frac{1}{2} - 1 \right) = \frac{9}{2}$$

**Example 17** FIND THE VOLUME OF THE SOLID OF REVOLUTION ABOUT THE  $y$ -AXIS, WHICH IS THE REGION ENCLOSED BY  $y = x - 1$  AND THE  $x$ -AXIS FROM  $x = \ln\left(\frac{1}{2}\right)$  TO  $x = \ln(2)$  ROTATES.

**Solution**  $\text{LN}\left(\frac{1}{2}\right) = -\text{LN}(2)$



**A** The region to be rotated



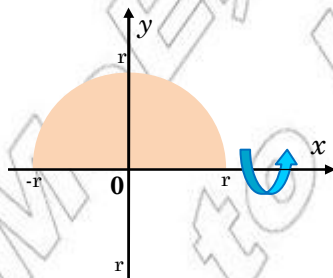
**B** The solid of revolution

Figure 5.34

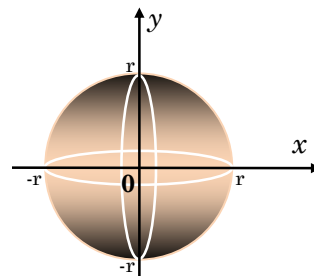
$$\begin{aligned}
 V &= \int_{-\text{LN}(2)}^{\text{LN}(2)} (e^x - 1)^2 dx \\
 &= \int_{-\text{LN}(2)}^{\text{LN}(2)} (e^{2x} - 2e^x + 1) dx = \left( \frac{e^{2x}}{2} - 2e^x + x \right) \Big|_{-\text{LN}(2)}^{\text{LN}(2)} \\
 &= \left( \frac{e^{2\text{LN}2}}{2} - 2e^{\text{LN}2} + \text{LN}2 - \left( \frac{e^{-2\text{LN}2}}{2} - 2e^{-\text{LN}2} - \text{LN}2 \right) \right) \\
 &= \left( 2 - 4 + \text{LN}2 \left( \frac{1}{8} - 1 + \text{LN}2 \right) \right) \\
 &= \left( -2 + \text{LN}2 \left( -\frac{7}{8} - \text{LN}2 \right) \right) = \left( -2 + 2\text{LN}2 \left( \frac{7}{8} - \text{LN}2 \right) \right)
 \end{aligned}$$

**Example 18** USING THE VOLUME OF A SOLID OF REVOLUTION, SHOW THAT THE VOLUME OF A SPHERE OF RADIUS  $r$  IS  $\frac{4}{3}\pi r^3$ .

**Solution** IN ACTIVITY 5.11 YOU SHOULD HAVE SEEN THAT A SPHERE OF RADIUS  $r$  IS GENERATED WHEN THE SEMICIRCULAR REGION  $y \leq \sqrt{r^2 - x^2}$  REVOLVES AROUND THE  $y$ -AXIS.



**A** The semicircular region to be rotated



**B** A sphere of radius  $r$

Figure 5.35

$$x^2 + y^2 = r^2 ; 0 \leq y \leq r \Rightarrow y = \sqrt{r^2 - x^2}$$

THE VOLUME

$$\begin{aligned} V &= \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx = \int_{-r}^r (r^2 - x^2) dx = \left( r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left( r^2 (r) - \frac{r^3}{3} - \left( r^2 (-r) - \frac{(-r)^3}{3} \right) \right) = \pi \left( r^3 - \frac{r^3}{3} - \left( -r^3 + \frac{r^3}{3} \right) \right) = \frac{4}{3} \pi r^3 \end{aligned}$$

**Example 19** FIND THE VOLUME OF WATER IN A SPHERICAL BOWL WITH RADIUS 5 M AND WITH MAXIMUM DEPTH IS 2 M.

**Solution** FROM FIGURE 5.36, YOU CAN DETERMINE THE RADIUS OF THE SURFACE OF THE WATER WHICH IS 4 M.

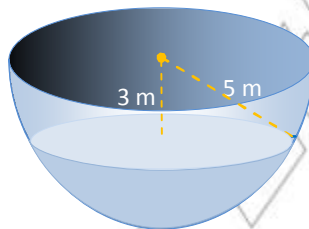


Figure 5.36

THE HEMISPHERE CAN BE GENERATED BY THE QUARTER OF THE CIRCULAR REGION.

$x^2 + y^2 = 25 ; 0 \leq x \leq 5$  AND  $-5 \leq y \leq 0$  REVOLVING ABOUT THE Y-AXIS

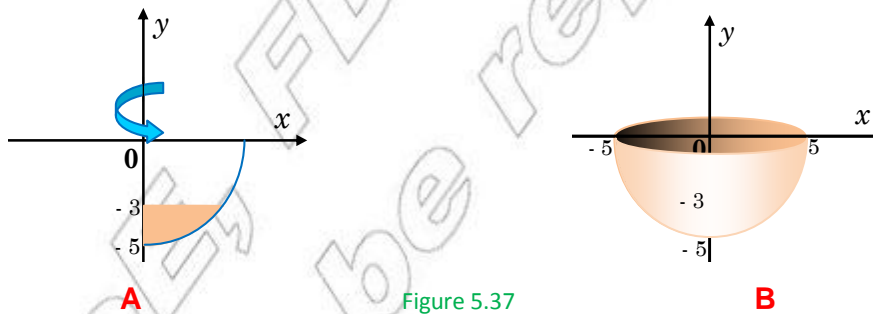


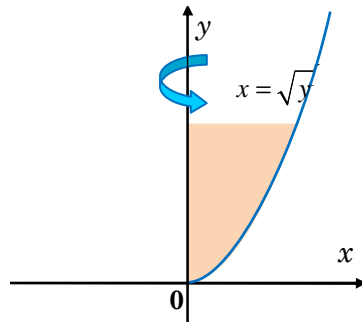
Figure 5.37

THE VOLUME OF THE WATER

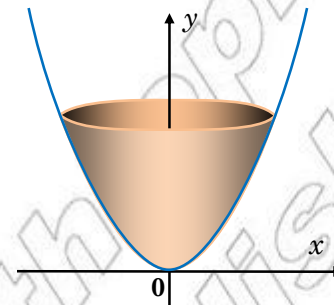
$$\begin{aligned} &\int_{-5}^{-3} (\sqrt{25 - y^2})^2 dy = \int_{-5}^{-3} (25 - y^2) dy \\ &= \pi \left( 25y - \frac{y^3}{3} \right) \Big|_{-5}^{-3} = \pi \left( 25(-3) - \left( \frac{-27}{3} \right) - \left( -125 + \frac{125}{3} \right) \right) \\ &= \frac{52}{3} \text{ CM.} \end{aligned}$$

**Example 20** FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION ENCLOSED BY  $\sqrt{y}$  AND THE  $y$ -AXIS FROM  $y = 0$  TO  $y = 4$ .

**SOLUTION** 
$$V = \int_0^4 (\sqrt{y})^2 dy = \int_0^4 y dy = \left( \frac{y^2}{2} \right) \Big|_0^4 = 8$$



**A** The region to be rotated about the  $y$ -axis



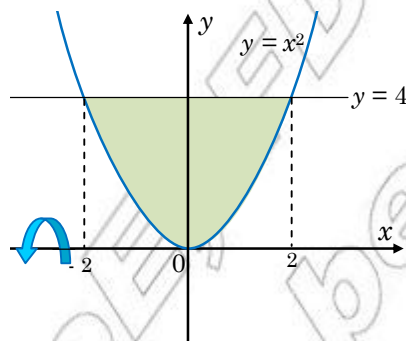
**B** The solid of revolution

Figure 5.38

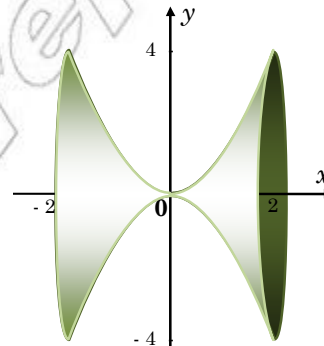
**Example 21** IF THE REGION BOUNDED BY  $y = x^2$  AND THE LINE  $y = 4$  ROTATES ABOUT THE  $x$ -AXIS, FIND THE VOLUME OF THE SOLID OF REVOLUTION.

**Solution** THE FIRST STEP IS TO DETERMINE THE INTERSECTION OF THE CURVE AND THE LINE AND THEN SKETCH BOTH GRAPHS TOGETHER.

$$x^2 = 4 \Rightarrow x = \pm 2$$



**A** The region to be rotated about the  $x$ -axis



**B** The solid of revolution

Figure 5.39

THE SOLID OF REVOLUTION IS A CYLINDER THAT IS GENERATED BY THE AREA BOUNDED BY  $y = x^2$  AND THE  $x$ -AXIS FROM  $x = -2$  TO  $x = 2$ .

LET  $V_1$  BE THE VOLUME OF VACANT SPACE.

$$\text{THEN } V_1 = \int_{-2}^2 (x^2)^2 dx = \left. \frac{x^5}{5} \right|_{-2}^2 = \left( \frac{32}{5} - \left( -\frac{32}{5} \right) \right) = \frac{64}{5}$$

LET  $V_2$  BE THE VOLUME OF THE CYLINDER, THEN

$$V_2 = \int_{-2}^2 4^2 dx = 16 \left. x \right|_{-2}^2 = 16 (2 - (-2)) = 64$$

THUS, THE VOLUME OF THE REQUIRED SOLID IS

$$V = V_2 - V_1 = 64 - \frac{64}{5} = \frac{256}{5}$$

OBSERVE THAT

$$\begin{aligned} V &= V_2 - V_1 = \int_{-2}^2 4^2 dx - \int_{-2}^2 (x^2)^2 dx = \int_{-2}^2 (4^2 - (x^2)^2) dx \\ &= \int_{-2}^2 (4^2 - x^4) dx = \int_{-2}^2 (16 - x^4) dx = \int_{-2}^2 16 dx - \int_{-2}^2 x^4 dx \\ &= \left. 16x \right|_{-2}^2 - \left. \frac{x^5}{5} \right|_{-2}^2 = (32 + 32) - \left( \frac{32}{5} + \frac{32}{5} \right) \\ &= 64 - \frac{64}{5} = \frac{256}{5} \end{aligned}$$

FROM THE ABOVE OBSERVATION, CAN YOU SEE HOW TO CALCULATE THE VOLUME OF A SOLID OF REVOLUTION GENERATED BY AN AREA ENCLOSED BY TWO CURVES?

CONSIDER THE REGION ENCLOSED BY THE CURVES  $f(x)$  AND  $g(x)$  BETWEEN  $a$  AND  $b$ .

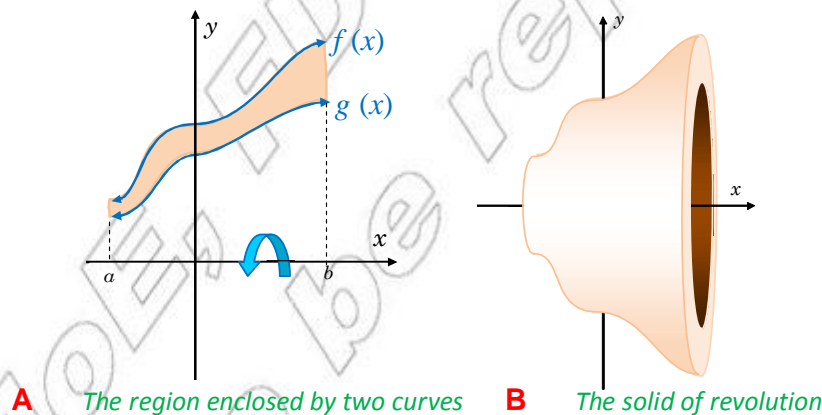


Figure 5.40

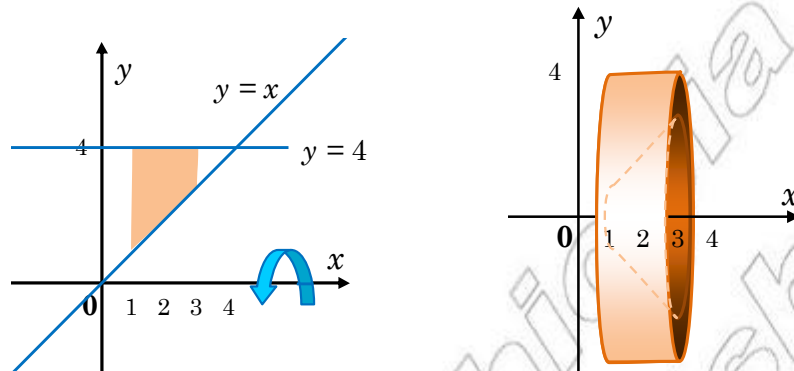
USING THE CONCEPT OF THE VOLUME OF A SOLID OF REVOLUTION, YOU HAVE THE VOLUME OF THE SOLID OF REVOLUTION TO BE

$$V = \int_a^b ((f(x))^2 - (g(x))^2) dx$$



**Example 22** FIND THE VOLUME OF SOLID OF REVOLUTION WHICH IS ABOUT THE  $y$ -AXIS BY REVOLVING THE AREA BETWEEN THE LINES  $y = 4$  AND  $y = x$  FROM  $x = 1$  TO  $x = 3$ .

**Solution**



**A** The region to be rotated

**B** The solid of revolution

Figure 5.41

USING THE FORMULA  $\int_a^b ((f(x))^2 - (g(x))^2) dx$ ; YOU HAVE

$$V = \int_1^3 (4^2 - x^2) dx = \left( 16x - \frac{x^3}{3} \right) \Big|_1^3 = (48 - 9) - \left( 16 - \frac{1}{3} \right) = \frac{70}{3}$$

**Example 23** IF THE REGION ENCLOSED BY THE GRAPHS OF  $y = x^2$  FROM  $x = 0$  TO  $x = 1$  ROTATES ABOUT THE  $y$ -AXIS, FIND THE VOLUME OF THE SOLID OF REVOLUTION.

**Solution**

$$V = \int_0^1 (x^2 - (x^2)^2) dx = \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2}{15}$$

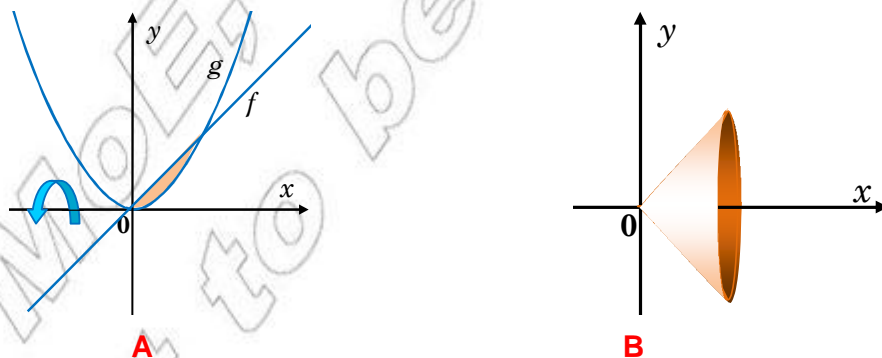


Figure 5.42

**Example 24** *Work done by a variable force*

THE WORK DONE BY A FORCE THROUGH A DISPLACEMENT OF IS

$$\int_{x_1}^{x_2} |F| dx$$

FIND THE WORK DONE WHEN A PARTICLE IS MOVED THROUGH A DISPLACEMENT OF 10 M

ALONG A SMOOTH HORIZONTAL SURFACE BY A FORCE  $F$ , OF MAGNITUDE

WHERE  $x$  IS THE DISPLACEMENT OF THE PARTICLE FROM ITS INITIAL POSITION, IN METRES

**Solution**

$$\text{WORK DONE} = \int_0^{10} |F| dx = \int_0^{10} \left(9 - \frac{1}{2}x\right) dx = 9x - \frac{x^2}{4} \Big|_0^{10} = 90 - \frac{100}{4} = 65.$$

**Motion of a particle in a straight line**

SUPPOSE A PARTICLE MOVES ALONG A STRAIGHT LINE AS ITS INITIAL POINT.

THE VELOCITY IS THE RATE AT WHICH THE DISPLACEMENT INCREASES WITH RESPECT TO TIME

$$\Rightarrow v = \frac{ds}{dt} \Rightarrow \int v dt = \int ds \quad \Rightarrow s = \int v dt$$

THE ACCELERATION IS THE RATE AT WHICH THE VELOCITY INCREASES WITH RESPECT TO TIME

$$\Rightarrow a = \frac{dv}{dt} \Rightarrow \int a dt = \int dv \Rightarrow v = \int a dt$$

**Example 25** SUPPOSE A PARTICLE MOVES ALONG A STRAIGHT LINE WITH AN ACCELERATION OF  $3.5t$ . WHEN  $t = 2$  SEC, IT HAS A DISPLACEMENT OF 10 M FROM O AND A VELOCITY OF 15 M/SEC. FIND THE VELOCITY AND THE DISPLACEMENT WHEN  $t = 4$  SEC.

**Solution** USING THE GIVEN INFORMATION YOU HAVE,

$$v = \int a dt = \int 3.5t dt = \frac{3.5}{2}t^2 + c. \text{ BUT } v(2) = 15 \Rightarrow 15 = \frac{3.5}{2}(2)^2 + c \Rightarrow c = 8.$$

$$v = \frac{7}{4}t^2 + 8.$$

$$\text{ALSO } s = \int v dt \Rightarrow s = \int \left(\frac{7}{4}t^2 + 8\right) dt = \frac{7}{12}t^3 + 8t + c$$

$$\text{BUT } (2) = 10 \Rightarrow 10 = \frac{7}{12}(2)^3 + 8(2) + c$$

$$\Rightarrow c = -\frac{32}{3} \Rightarrow s = \frac{7}{12}t^3 + 8t - \frac{32}{3}$$

THEREFORE, WHEN

**a** THE VELOCITY  $= \frac{7}{4}(5)^2 + 8 = 51.75 \text{ M/SEC}$

**b** THE DISPLACEMENT  $= \frac{7}{12}(5)^3 + 8(5) - \frac{32}{3} = 102.25 \text{ M}$

### Exercise 5.12

- 1** FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION ENCLOSED BY THE GIVEN FUNCTION AND THE VERTICAL LINES.

**A**  $y = 2x$ ;  $x = 0$  AND  $x = 1$       **B**  $y = x^2 + 1$ ;  $x = -1$  AND  $x = 2$

**C**  $y = e^x$ ;  $x = 1$  AND  $x = 2$       **D**  $y = \sin x$ ;  $x = \frac{\pi}{3}$  AND  $x = \frac{\pi}{2}$

**E**  $y = |x|$ ;  $x = -3$  AND  $x = 1$       **F**  $y = 2^x$ ;  $x = -2$  AND  $x = 3$

**G**  $y = x^3$ ;  $x = -1$  AND  $x = 2$
- 2** FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION ENCLOSED BY THE GRAPHS OF THE GIVEN FUNCTIONS.

**A**  $f(x) = 4x - x^2$  AND  $g(x) = 3$

**B**  $f(x) = x^3$  AND  $g(x) = x$

**C**  $f(x) = \sin x$  AND  $g(x) = \cos x$  FROM  $x = 0$  TO  $x = \frac{\pi}{2}$

**D**  $f(x) = x^2$ ,  $g(x) = |x|$  FROM  $x = -2$  TO  $x = 2$
- 3** USING THE VOLUME OF REVOLUTION, PROVE THAT THE VOLUME OF A RIGHT CIRCULAR CONE OF RADIUS  $r$  AND HEIGHT  $h$  IS  $\frac{1}{3}\pi r^2 h (R^2 + rR + r^2)$ .
- 4** A PARTICLE P STARTS FROM A POINT A WITH VELOCITY 0 MOVING ALONG A STRAIGHT LINE AB WITH AN ACCELERATION OF  $2 \text{ M/SEC}^2$ . FIND

**A** THE ACCELERATION      **B** THE VELOCITY AND

**C** THE DISPLACEMENT AFTER TEN SECONDS



## Key Terms

acceleration	differentiation	partial fraction
anti derivative	displacement	substitution
area	fundamental theorem	velocity
by parts	indefinite integral	volume of revolution
definite integral	integration	work done by force



## Summary

### 1 Anti derivative or Indefinite integral

LET  $F(x)$  BE A FUNCTION, THEN

- ✓  $F(x)$  IS SAID TO BE AN ANTIDERIVATIVE OF  $f(x)$ .
- ✓ THE SET OF ANTIDERIVATIVES IS SAID TO BE THE INDEFINITE INTEGRAL OF  $f(x)$ .
- ✓ THE INDEFINITE INTEGRAL IS DENOTED BY  $\int f(x) dx$
- ✓ IFF  $F(x)$  AND  $G(x)$  ARE ANTI DERIVATIVES, THEN THE DIFFERENCE BETWEEN  $F(x)$  AND  $G(x)$  IS A CONSTANT.

### 2 The Integral of Some Functions

#### The Integral of power functions

- I  $\int x^r dx = \frac{x^{r+1}}{r+1} + c; r \neq -1.$
- II IF  $r = -1$ , THEN  $\int \frac{1}{x} dx = \text{LN}|x| + c$ .
- III  $\int kx^r dx = k \int x^r dx$

#### The Integral of trigonometric functions

- I  $\int \text{COS} x dx = \text{SIN} x + c$
- II  $\int \text{SIN} x dx = -\text{COS} x + c$
- III  $\int \text{SEC} x dx = \text{TAN} x + c$
- IV  $\int \text{SEC} x \text{TAN} x dx = \text{SEC} x + c$
- V  $\int \text{TAN} x dx = -\text{LN}|\text{COS} x| + c$
- VI  $\int \text{CSC} x dx = -\text{COT} x + c$

#### The Integral of exponential functions

- I  $\int e^x dx = e^x + c$
- II  $\int a^x dx = \frac{a^x}{\text{LN} a} + c; a > 0 \text{ AND } a \neq 1$

#### The integral of logarithmic functions

- I  $\int \text{LN} x dx = x \text{LN} x - x + c$
- II  $\int \text{LOG}_a x dx = \frac{1}{\text{LN} a} (x \text{LN} x - x) + c$

### 3 The Integral of a sum or difference of functions

- I  $\int kf(x) dx = k \int f(x) dx$
- II  $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$

#### 4 Techniques of Integration

##### Integration by substitution

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du; \text{ WHERE } u = g(x).$$

$$\text{I} \quad \int f'(x) f(x) dx = \frac{(f(x))^2}{2} + c \quad \text{II} \quad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

##### Integration by parts

$$\int u \frac{d}{dx} v = uv - \int v \frac{du}{dx}$$

#### 5 Fundamental Theorem of Calculus

$$\text{IF } f(x) = F'(x), \text{ THEN } \int_a^b f(x) dx = F(b) - F(a)$$

#### 6 Properties of definite integrals

- I  $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
- II  $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- III IF  $f(x) \geq 0$  ON  $[a, b]$ , THEN  $\int_a^b f(x) dx \geq 0$
- IV  $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- V  $\int_a^a f(x) dx = 0$
- VI  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx; a \leq c < b.$
- VII IF  $u = g(x), \int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$

#### 7 Applications of the definite integral

- I THE AREA BOUNDED BY TWO CONTINUOUS CURVES  $y = f(x)$  AND  $y = g(x)$  ON  $[a, b]$  WITH  $f(x) \geq g(x) \forall x \in [a, b]$  IS

$$A = \int_a^b (f(x) - g(x)) dx.$$

- II THE VOLUME OF A SOLID OF REVOLUTION GENERATED BY REVOLVING THE REGION BOUNDED BY  $y = f(x)$  AND  $y = g(x)$  WITH  $f(x) \geq g(x) \forall x \in [a, b]$  ABOUT THE  $x$ -AXIS IS

$$V = \int_a^b ((f(x))^2 - (g(x))^2) dx.$$



## Review Exercises on Unit 5

IN EXERCISES 1 – 60 INTEGRATE THE EXPRESSION WITH RESPECT TO

<b>1</b>	$\frac{1}{2}x$	<b>2</b>	$2x + 5$	<b>3</b>	$x^2 - 3x + 2$	<b>4</b>	$3x^5$
<b>5</b>	$x^7$	<b>6</b>	$x^{\frac{3}{2}}$	<b>7</b>	$x^{\frac{1}{3}}$	<b>8</b>	$x^{\frac{1}{3}} + x\sqrt{x} - \frac{1}{2x} + x$
<b>9</b>	$2x^{-3}$	<b>10</b>	$x^{\frac{2}{5}}$	<b>11</b>	$\sin(x + 2)$	<b>12</b>	$4^x - \sin x$
<b>13</b>	$\tan(x - 4)$	<b>14</b>	$x(x^2 + 1)$	<b>15</b>	$\sqrt{2x + 7}$	<b>16</b>	$(3x + 5)^{13}$
<b>17</b>	$(x^2 - 1)^3$	<b>18</b>	$(x + 1)(x^2 + 2x + 5)^{10}$	<b>19</b>	$\frac{x}{x + 1}$	<b>20</b>	$\frac{1}{x^2 - 16}$
<b>21</b>	$x\sqrt{(x^2 + 4)^5}$	<b>22</b>	$\frac{x}{x^2 - 2x - 3}$	<b>23</b>	$2^{4x + 3}$	<b>24</b>	$x \log \sqrt{x^2 + 1}$
<b>25</b>	$\sin^{(n)}(x) \cos x$	<b>26</b>	$\frac{3^{x+1}}{5^{1-4x}}$	<b>27</b>	$\frac{\ln x}{x}$	<b>28</b>	$x \sin(x^2)$
<b>29</b>	$x x $	<b>30</b>	$\sqrt{6 + x}$	<b>31</b>	$\frac{\sqrt{x + x}}{x\sqrt[3]{x}}$	<b>32</b>	$(1 + 2^x)^2$
<b>33</b>	$\frac{2\sqrt{x}}{\sqrt{x}}$	<b>34</b>	$\frac{2x + 1}{4^{x^2 + x + 1}}$	<b>35</b>	$2^x 2x\sqrt{1 + 2^x}$	<b>36</b>	$\frac{(e^{x+3})(3^{x+5})}{2^{3x-2}}$
<b>37</b>	$\frac{\cos x}{3 + \sin x}$	<b>38</b>	$\frac{1}{x^2} \cos\left(\frac{1}{x}\right)$	<b>39</b>	$\frac{\sec^2 \sqrt{x}}{\sqrt{x}}$	<b>40</b>	$\frac{\sin x}{\cos^3 x}$
<b>41</b>	$xe^{x^2}$	<b>42</b>	$x^{-2}e^{\frac{1}{x}}$	<b>43</b>	$\frac{e^{\frac{1}{x}}}{x^2}$	<b>44</b>	$\frac{x + 1}{x^2 + 2x + 4}$
<b>45</b>	$\frac{4}{(x + 3)^2}$	<b>46</b>	$\frac{x^2}{x + 3}$	<b>47</b>	$(2x + 1)(x^2 + x + 3)^{10}$	<b>48</b>	$x\sqrt{9 + x^3}$
<b>49</b>	$\cos x e^{\sin x}$	<b>50</b>	$\frac{x}{\sqrt{x^2 + 5x}}$	<b>51</b>	$(1 + 2e^x)^2$	<b>52</b>	$\sin^3\left(\frac{x}{3}\right)$
<b>53</b>	$\frac{3x}{x^2 - 1}$	<b>54</b>	$\frac{3x^2}{x^2 - 9}$	<b>55</b>	$\frac{x}{(x - 2)(x + 1)^2}$	<b>56</b>	$\frac{3x + 2}{(x + 3)^2}$
<b>57</b>	$\frac{4}{x^2(x + 1)^2}$	<b>58</b>	$\frac{2x^2 + 1}{(x + 1)^2(x + 3)}$	<b>59</b>	$\frac{x^3 + 1}{x^2(x - 4)}$	<b>60</b>	$\frac{x}{(x^2 - 1)(x + 3)}$

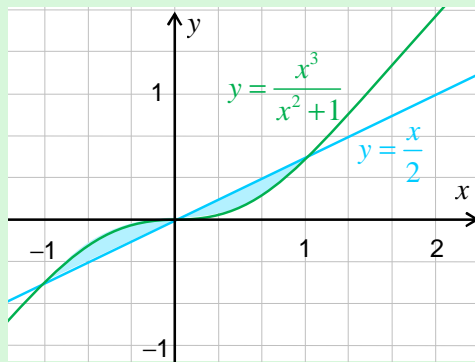
IN EXERCISES 61 – 85 EVALUATE THE DEFINITE INTEGRAL.

- |           |   |           |   |           |   |
|-----------|---|-----------|---|-----------|---|
| <b>61</b> | $\int_a^b dx$   | <b>62</b> | $\int_{e-1}^{e+1} 4dx$                                    | <b>63</b> | $\int_2^3 (x-5) dx$                                 |
| <b>64</b> | $\int_1^2 6x^3 dx$                                      | <b>65</b> | $\int_0^1 e^x dx$   | <b>66</b> | $\int_1^4 \sqrt{x} dx$                              |
| <b>67</b> | $\int_{\sqrt{2}}^3 3^x dx$                              | <b>68</b> | $\int_1^8 x^{\frac{1}{3}} dx$                             | <b>69</b> | $\int_1^3 \sqrt{x} \left(1 - \frac{1}{x}\right) dx$ |
| <b>70</b> | $\int_{-1}^1 e^{x+3} dx$                                | <b>71</b> | $\int_0^1 3^{2x+5} dx$                                    | <b>72</b> | $\int_{\frac{1}{2}}^1 2^{3x-2} dx$                  |
| <b>73</b> | $\int_0^1 \frac{1}{x+1} dx$                             | <b>74</b> | $\int_{-2}^2 (e^x + e^{-x}) dx$                           | <b>75</b> | $\int_{\frac{1}{n}}^1 e^{nx} dx$                    |
| <b>76</b> | $\int_2^3 \frac{x}{x+5} dx$                             | <b>77</b> | $\int_0^3 x\sqrt{x^2+1} dx$                               | <b>78</b> | $\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$         |
| <b>79</b> | $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$      | <b>80</b> | $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(x) \cos(x) dx$ | <b>81</b> | $\int_0^1 \frac{t}{4^{t^2-1}} dt$                   |
| <b>82</b> | $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{4 + \cos(x)} dx$ | <b>83</b> | $\int_{-2}^1 (x+1)\sqrt{x+2} dx$                          | <b>84</b> | $\int_1^0 x(8x^2 - 1)^6 dx$                         |
| <b>85</b> | $\int_1^2 \frac{2x-3}{(x^2-3x+1)} dx$                   |           |   |           |   |

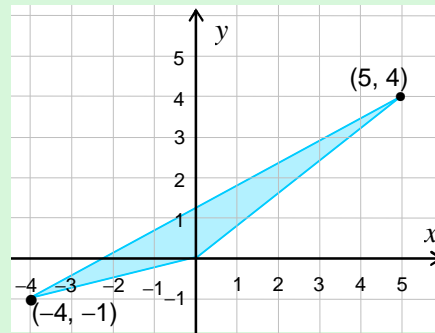
 IN EXERCISES 86 – 97 FIND THE AREA OF THE REGION BOUNDED BY THE GRAPH OF  $f(x)$  AND THE LINES  $x = a$  AND  $x = b$ .

- |           |  |           |   |
|-----------|--|-----------|---|
| <b>86</b> | $f(x) = 4; a = -1, b = 2$                                      | <b>87</b> | $f(x) = 3x; a = -3, b = -1$                     |
| <b>88</b> | $f(x) = 3x + 1; a = 0, b = 3$                                  | <b>89</b> | $f(x) = 2x^2 + 1; a = 0, b = 3$                 |
| <b>90</b> | $f(x) = 1 - 4x^2; a = -1, b = 1$                               | <b>91</b> | $f(x) = x^3; a = -\frac{1}{2}, b = 2$           |
| <b>92</b> | $f(x) = e^x; a = -1, b = 4$                                    | <b>93</b> | $f(x) = \frac{x}{x+1}; a = -\frac{1}{2}, b = 3$ |
| <b>94</b> | $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}; a = \frac{1}{4}, b = 4$ | <b>95</b> | $f(x) = \ln(x); a = \frac{1}{e}, b = e$         |
| <b>96</b> | $f(x) = x^3 - 2x^2 - 5x + 6; a = -2, b = 3$                    | <b>97</b> | $f(x) =  x^2 - 1 ; a = -3, b = 2$               |

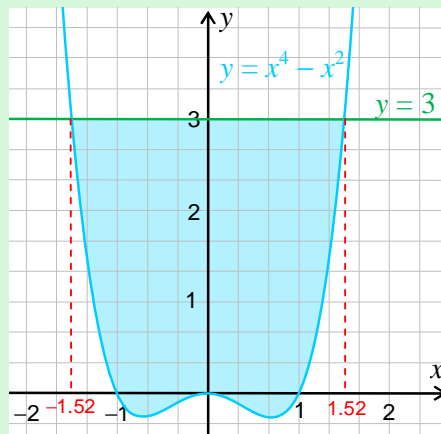
98 FIND EACH OF THE FOLLOWING SHADED AREAS.



A



B



C

Figure 5.43

99 FIND THE AREA OF THE REGION ENCLOSED BY

A  $f(x) = x, y = \frac{1}{x}$  AND  $y = 4$

B  $f(x) = 4 - x^2$  AND  $g(x) = 3x$ .

100 FIND THE VOLUME OF THE SOLID GENERATED WHEN THE REGION ENCLOSED BY THE GIVEN CURVES AND LINES IS ROTATED ABOUT THE

A  $y = 4x - x^2$

B  $y = x^3 + 1; x = -1, x = 2$

101 FIND THE VOLUME OF THE SOLID GENERATED WHEN THE REGION BOUNDED BY  $y = x^2 + 2$  ROTATES ABOUT THE

102 FIND THE VOLUME OF THE SOLID OF REVOLUTION GENERATED WHEN THE REGION ENCLOSED BY  $y = e^x$ , THE  $y$ -AXIS AND THE LINE  $y = 1$  ROTATES ABOUT THE