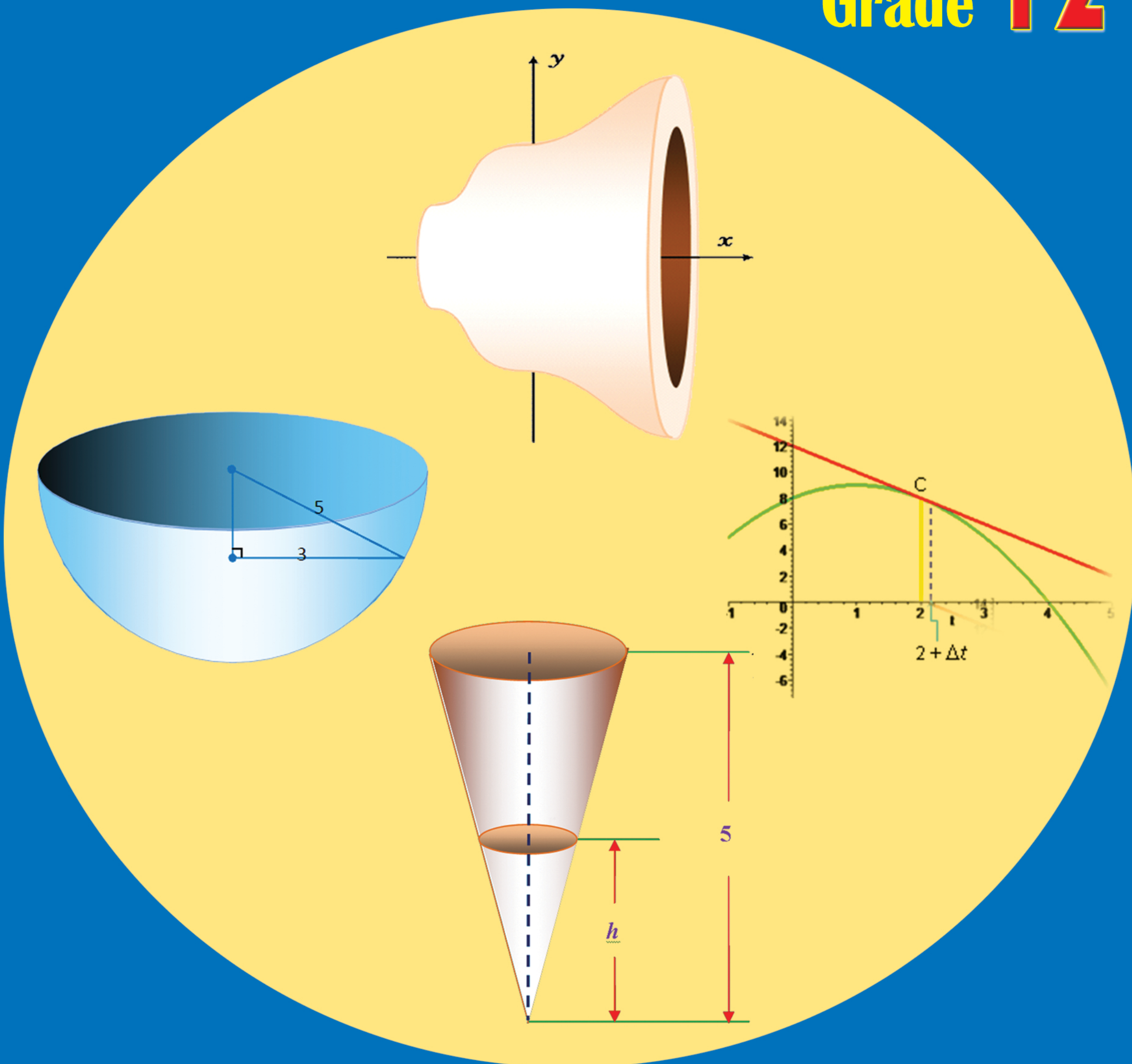
MATHEMATICS

TEACHER'S GUIDE

Grade 12



FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

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TEACHER'S GUIDE

GRADE 12

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INTRODUCTION

According to the Educational and Training Policy of the Federal Democratic Republic of Ethiopia, the second cycle of the secondary education and training will enable your students to choose subjects or areas of training which will prepare them adequately for higher education and for the world of work. The study of mathematics at this cycle, Grades 11 and 12, should contribute to your students' growth into good, balanced and educated individuals and members of society. At this cycle, your students should acquire the necessary mathematical knowledge and develop skills and competencies needed in their further studies, working life, hobbies, and all-round personal development. Moreover, the study of mathematics at this level shall significantly contribute to your students' lifelong learning and self development throughout their lives. These aims can be realized by closely linking mathematics learning with daily life, relating theorems with practice; paying attention to the practical application of mathematical concepts, methods and procedures by drawing examples from the fields of industry, agriculture and from sciences like physics, chemistry and engineering.

Mathematical study in grade 12 should be understood as the unity of imparting knowledge, developing abilities and skills and forming convictions, attitudes and habits. Therefore, the didactic-methodical conception has to contribute to all these sides of the educational process and to consider the specifics of students' age, the function of the secondary school level in the present and prospective developmental state of the country, the pre-requisites of the respective secondary school and the guiding principles of the subject mathematics.

Besides learning to think effectively and efficiently, your students come to understand how mathematics deals with their daily and routine lives and with lives of the people at large. Your students are also expected to realize the changing power of mathematics and its national and international significance.

To materialize the major goals stated above, encourage your students to apply high-level reasoning, and values to their daily life and to their understanding of the social, economic, and cultural realities of the surrounding context. This will in turn help the students to actively and effectively participate in the wider scope of the development activities of their nation. Your students are highly expected to gain solid knowledge of the fundamental theorems, rules and procedures of mathematics. It is also expected that your students should develop reliable skills for using this knowledge to solve problems independently and in groups. To this end, the specific objectives of mathematics learning at this cycle are to enable the students to:

- ✓ gain solid knowledge on mathematical concepts, theorems, rules and methods.
- ✓ appreciate the changing power, dynamism, structure and elegance of mathematics.
- ✓ apply mathematics in their daily life.
- ✓ understand the essential contributions of mathematics to the fields of engineering, science, agriculture and economics at large.
- ✓ work with this knowledge more independently in the field of problem solving.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The traditional teaching-learning paradigm has been replaced by active, participatory and student-centered model. A student-centered classroom atmosphere and approach stimulates student's inquiry. Your role as a teacher in such student-oriented approach would be a mentor who guides the students construct their own knowledge and skills. A primary goal when you teach fundamental basics is for the students to discover the concept by themselves, particularly as you recognize threads and patterns in the data and theorems that they encounter under the teacher's guidance and supervision.

You are also encouraged to motivate your students to develop personal qualities that will help them in real life. For example, encourage students' self confidence and their confidence in their knowledge, skills and general abilities. Motivate them to express their ideas and observations with courage and confidence. As the students develop personal confidence and feel comfortable on the subject, they would be motivated to address their material to groups and to express themselves and their ideas with strong conviction. Support students and give them chance to stand before the class and present their opinion, observation and work. Similarly, help the students by creating favorable conditions for them to come together in groups and exchange views and ideas about what they have worked out, investigated and about the material they have read. In this process, the students are given opportunities to openly discuss the knowledge they have acquired and to talk about issues raised in the course of the discussion. Always remember that teamwork is one of the acceptable ways of approach in a student-centered classroom setting.

This teacher's guide helps you only as a guide. It is very helpful for budgeting and breaking down your teaching time as you plan to approach specific topics. The guide also contains procedures to manage class activities, group discussions and reflections. Answers to the review questions are indicated at the end of each topic.

Every section of your teacher's guide includes student-assessment guidelines. Use the guidelines to evaluate your students' work. Based on your class's reality, you will give special attention to students who are working either above or below the standard level of achievement. Do an active follow up for each student's performance against the learning competencies presented in the guide. Be sure to consider both the standard competencies and the minimum competencies. ***Minimum requirement level*** is not the ***standard level of achievement***. To achieve the standard level, your students must successfully fulfill all of their grade-level's competencies.

When you identify students who are working either below the standard level or the minimum level, arrange extra support for them. For example, you can give them supplementary presentations and reviews of the materials in the class. Giving extra time to study and activities is recommended for those who are performing below the minimum level. You can also encourage high-level students by giving them advanced activities and extra exercises.

Some helpful references are listed at the end of this teacher's guide. For example, if you get an access internet, it could be a rich resource for you. Search for new web sites is well worth your time as you browse on the subject matter you need. Use one of the many search engines that exist-for example, Yahoo and Google are widely accepted.

Do not forget that, although this guide provides you with many ideas and guidelines, you are encouraged to be innovative and creative in the ways you put your students into practice. Use your own full capacity, knowledge and insights in the same way as you encourage your students to use theirs.

Contents

ACTIVE LEARNING AND CONTINUOUS ASSESSMENT REQUIRED!

Dear mathematics teacher! For generations the technique of teaching mathematics at any level was dominated by what is commonly called the direct instruction. That is, students are given the exact tools and formulas they need to solve a certain mathematical problem, sometimes without a clear explanation as to why, and they are told to do certain steps in a certain order and in turn are expected to do them as such at all times. This leaves little room for solving varying types of problems. It can also lead to misconceptions and students may not gain the full understanding of the concepts that are being taught.

You just sit back for a while and try to think the most common activities that you, as a mathematics teacher, are doing in the class.

> *Either you explain (lecture) the new topic to them, and expect your students to remember and use the contents of this new topic or you demonstrate with examples how a particular kind of problem is solved and students routinely imitate these steps and procedures to find answers to a great number of similar mathematical problems.*

But this method of teaching revealed little or nothing of the meaning behind the mathematical process the students were imitating.

We may think that teaching is telling students something, and learning occurs if students remember it. But research reveals that teaching is not "pouring" information into students' brain and expecting them to process it and apply it correctly later.

Most educationalists agree that learning is an active meaning-making process and students will learn best by trying to make sense of something on their own with the teacher as a guide to help them along the way. This is the central idea of the concept Active Learning.

> Active learning, *as the name suggests, is a process whereby learners are actively engaged (involved) in the learning process, rather than "passively" absorbing lectures. Students are rather encouraged to think, solve problems, do activities carefully selected by the teacher, answer questions, formulate questions of their own, discuss, explain, debate, or brainstorm, explore and discover, work cooperatively in groups to solve problems and workout projects.*

The design of the course materials (student textbooks and teachers guides) for mathematics envisages active learning to be dominantly used. With this strategy, we feel that you should be in a position to help students understand the concepts through relevant, meaningful and concrete activities. The activities should be carried out by students to explore the world of mathematics, to learn, to discover and to develop interest in the subject. Though it is your role to exploit the opportunity of using active learning at an optimal level, for the sake of helping you get an insight, we recommend that you do the following as frequently as possible during your teaching:

- Engage your students in more relevant and meaningful activities than just listening.
- Include learning materials having examples that relate to students life, so that they can make sense of the information.
- Let students be involved in dialog, debate, writing, and problem solving, as well as higher-order thinking, e.g., analysis, synthesis, evaluation.
- Encourage students’ critical thinking and inquiry by asking them thoughtful, open-ended questions, and encourage them to ask questions to each other.
- Have the habit of asking learners to apply the information in a practical situation. This facilitates personal interpretation and relevance.
- Guide them to arrive at an understanding of a new mathematical concept, formula, theorem, rule or any generalization, by themselves. You may realize this by giving them an activity in which students sequentially uncover layers of mathematical information one step at a time and discover new mathematics.
- Select assignments and projects that should allow learners to choose meaningful activities to help them apply and personalize the information. These need to help students undertake initiatives, discover mathematical results and even design new experiments to verify results.
- Let them frequently work in peers or groups. Working with other learners gives learners real-life experience of working in a group, and allows them to use their metacognitive skills. Learners will also be able to use the strengths of other learners, and to learn from others. When assigning learners for group work membership, it is advisable if it is based on the expertise level and learning style of individual group members, so that individual team members can benefit from one another's strengths.

In general, if mathematics is to develop creative and imaginative mathematical minds, you must overhaul your traditional methods of presentation to the more active and participatory strategies and provide learning opportunities that allow your students to be actively involved in the learning process. While students are engaged with activities, group discussions, projects, presentations and many others they need to be continuously assessed.

Continuous Assessment

You know that continuous assessment is an integral part of the teaching learning process. Continuous assessment is the periodic and systematic method of assessing and evaluating a person’s attributes and performance. Information collected from continuous behavioral change of students will help teachers to better understand their strengths and weaknesses in addition to providing a comprehensive picture of each student over a period of time. Continuous assessment will afford student to readily see his/her development pattern through the data. It will also help to strengthen the parent teacher relationship and collaboration. It is an ongoing process more than giving a test or exam frequently and recording the marks.

Continuous assessment enables you to assess a wide range of learning competencies and behaviors using a variety of instruments some of which are:

- ✓ Tests/ quizzes (written, oral or practical)
- ✓ Class room discussions, exercises, assignments or group works.
- ✓ Projects
- ✓ Observations
- ✓ Interview
- ✓ group discussions
- ✓ questionnaires

Different competencies may require different assessment techniques and instruments. For example, oral questions and interviews may serve to assess listening and speaking abilities. They also help to assess whether or not students are paying attention, and whether they can correctly express ideas. You can use oral questions and interviews to ask students to restate a definition, note or theorem, etc. Questionnaires, observations and discussions can help to assess the interest, participation and attitudes of a student. Written tests/exams can also help to assess student's ability to read, to do and correctly write answers for questions.

When to Assess

Continuous assessment and instruction are integrated in three different time frames namely, Pre-instruction, During-instruction and Post-instruction. To highlight each briefly

1. Pre-instruction assessment

This is to assess what students luck to start a lesson. Hence you should start a lesson by using opportunities to fill any observed gap. If students do well in the pre-instruction assessment, then you can begin instructing the lesson. Otherwise, you may need to revise important concepts.

The following are some suggestions to perform or make use of pre-instruction assessment.

i. assess whether or not students have the prerequisite knowledge and skill to be successful, through different approaches.

ii. make your teaching strategies motivating.

iii. plan how you form groups and how to give marks.

iv. create interest on students to learn the lesson.

2. Assessment during instruction:

This is an assessment during the course of instruction rather than before it is started or after it is completed. The following are some of the strategies you may use to assess during instruction.

i. observe and monitor students' learning.

ii. check that students are understanding the lesson. You may use varying approaches such as oral questions, asking students to do their work on the board, stimulate discussion, etc.

iii. identify which students need extra help and which students should be left alone.

iv. ask a balanced type of exercise problems according to the students ability, help weaker students and give additional exercise for fast students.

v. monitor how class works and group discussions are conducted

3. Post Instruction Assessment:

This is an assessment after instruction is completed. It is conducted usually for the purpose of documenting the marks and checking whether competencies are achieved. Based on the results students scored, you can decide whether or not there is anything the class didn't understand because of which you may revise some of the lessons or there is something you need to adjust on the approach of teaching. This also help you analyze whether or not the results really reflect what students know and what they can do, and decide how to treat the next lesson.

Forming and managing groups

You can form groups through various approaches: mixed ability, similar ability, gender or other social factors such as socioeconomic factors. When you form groups, however, care need to be taken in that you should monitor their effort. For example, if students are grouped by mixed ability the following problems may happen.

1. Mixed ability grouping may hold back high-ability students. Here, you should give enrichment activities for high ability students.
2. High ability students and low ability students might form a teacher-student relationship and exclude the medium ability students from group discussion. In this case you should group medium ability students together.

When you assign group work, the work might be divided among the group members, who work individually. Then the members get together to integrate, summarize and present their finding as a group project. Your role is to facilitate investigation and maintain cooperative effort.

Highlights about assessing students

You may use different instruments to assess different competencies. For example, consider each of the following competencies and the corresponding assessment instruments.

Competency 1. Define derivative of a function.

Instrument: Written question.

Question: State the definition of derivative of a function at point *c*.

Competency 2 - Students will calculate derivative of a function at a point.

Instrument: class work/homework/ quiz /test

Question: Find $f'(2)$where;

a. $f(x) = x^2 - 2x + 1$ b. $f(x) = e^{2x^2}$

Competency 3 – Apply derivatives in their daily life problems.

Instrument: Assignment/project.

Question: Some motor oil cans are cylindrical with cardboard sides and metal top and bottom. The standard volume for an oil can is one litre. If the metal costs twice as much for square cm as the cardboard, what dimensions should the can have to minimize the cost?

How often to assess

Here are some suggestions which may help you how often to assess.

- Class activities / class works: Every day (when convenient).
- Homework/Group work: as required.
- Quizzes: at the end of every one (or two) sub topics.
- Tests: at the end of every unit.
- Exams: once or twice in every semester.

How to Mark

The following are some suggestions which may help you get well prepared before you start marking:

- use computers to reduce the burden for record keeping.
- although low marks may diminish the students motivation to learn, don't give inflated marks for inflated marks can also cause reluctance.

The following are some suggestions on how to mark a semester's achievement.

1. One final semester exam 30%.
2. Tests 25%
3. Quizzes 10%
4. Homework 10%
5. Class activities, class work, presentation demonstration skills 15%
6. Project work, in groups or individually 10%.

Moreover

In a group work allow students to evaluate themselves as follows using format of the following type.

	A	B	C	D	
The ability to communicate					
The ability to express written works					
Motivation					
Responsibility					
Leadership quality					
Concern for others					
Participation					
Over all					

You can shift the leadership position or regroup the students according to the result of the self-evaluation. You can also consider your observation.

Reporting students' progress and marks to parents

Parents should be informed about their children's progress and performance in the class room. This can be done through different methods.

1. The report card: two to four times per year.
2. Written progress report: Per week/two weeks/per month/two months.
3. Parent – teacher conferences (as scheduled by the school).

The report should be about the student performance say, on tests, quizzes, projects, oral reports, etc that need to be reported. You can also include motivation or cooperation behavior. When presenting to parents your report can help them appraise fast learner, pay additional concern and care for low achieving student, and keep track of their child's education. In addition, this provides an opportunity for giving parents helpful information about how they can be partners with you in helping the student learn more effectively.

The following are some suggested strategies that may help you to communicate with parents concerning marks, assessment and student learning.

1. Review the student's performance before you meet with parents.
2. Discuss with parents the students good and poor performances.
3. Do not give false hopes. If a student has low ability, it should be clearly informed to his/her parents.
4. Give more opportunities for parents to contribute to the conversation.
5. Do not talk about other students. Don't compare the student with another student.
6. Focus on solutions

NB. ***All you need to do is thus plan what type of assessment and how many of each you are going to use beforehand (preferably during the beginning of the year/semester).***

UNIT 1 SEQUENCES AND SERIES

INTRODUCTION

In this unit, the first major outcome is to enable students understand the notion of sequences and series. The second one is to enable students to solve practical and real life problems.

In sequences and series, students will learn how to make predictions, decisions and generalizations from given patterns. Some of the sub-topics of the unit begin with opening problems and activities related to the content of the topic.

In teaching sequences and series, select the methods which give more time for active students' participation and less time for the lecture format.

Unit Outcomes

After completing this unit, students will be able to:

- *revise the notions of sets and functions.*
- *grasp the concept of sequence and series.*
- *compute any terms of sequences from given rule.*
- *find out possible rules (formula) from given terms.*
- *identify the types of sequences and series.*
- *compute partial and infinite sums of sequences.*
- *apply the knowledge of sequence and series to solve practical and real life problems.*

Suggested Teaching Aids in Unit 1

You know that students learn in a variety of different ways. Some are visually oriented and more inclined to acquire information from photographs or videos. Others do best when they hear instructions rather than read them. Teachers use teaching aids to provide these different ways of learning. Therefore, it is recommended that you may use models of planes, charts, calculators, logarithmic tables and computers for this unit. You can use also other types of materials as long as they help the learners to get the skills required.

Teaching Notes `

Under each sub-topic, a hint is given how to continue each sub-topic but your creativity is very crucial. The purpose of the teaching notes is to provide the teacher information to use activities, opening problems and group-works to motivate and guide students rather than lecturing. Now this unit begins with an opening problem which may motivate students to follow the unit attentively. Therefore, before passing on to any sub-topic of this unit make students discuss the opening problem.

Answers to Opening Problem

a. It can be reached just by counting the numbers in the following pattern

   0, 5, 10, 15, . . ., 50

   Thus, there are 11 rows

b. 100m, 90m, 80m, . . . , 0m

c. 21, 19, 17, . . . , 1

d. $21 + 19 + 17 + \ldots + 1 = 121$

Assessment

Use the opening problem to assess the background and the initiation the students have to continue the unit. It is better to use performance assessment methods such as engaging students in debate on real life problem like the opening problem given here.

The intention of the assessment is to provide a tool for the identification of students who are experiencing major difficulty. Such an assessment is effective for planning a remedial program. Assessment should provide the teacher with a valuable profile of each student’s strengths and weaknesses, and so enable the teacher to direct her/his teaching at the identified weaknesses.

1.1 SEQUENCES

Periods allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *revise the notion of sets and functions.*
- *explain the concepts sequence, term of a sequence, rule (formula of a sequence).*
- *compute any term of a sequence using rule (formula).*
- *draw graphs of finite sequences.*
- *determine the sequence, use recurrence relations such as $u_{n+1} = 2u_n+1$ given u_1.*
- *generate the Fibonacci sequence and investigate its uses (application) in real life.*

Vocabulary: Set, Relation, Function, Sequence, Terms of a sequence, Finite sequence, Infinite sequence, Recurrence relation

Introduction

Under this sub-unit we treat two subtopics: 1.1.1 Revision on sets and functions, and 1.1.2 Number sequences. These subtopics require the basic mathematical skills found in learning sets, relations and functions, graphing, evaluating a formula, stating domain and range, etc,.

1.1.1 Revision on Sets and Functions

This sub-topic begins by revising the concepts sets, relations and functions as Activity 1.1. This activity can be given to students as a small group work in order to create class room discussion. One advantage of classroom discussion is to investigate and to minimize the learning gaps among students. This teaching note allows for flexibility on your part as a teacher to plan for appropriate teaching methods that are suitable to individual learners in the class. You should always choose learner centered methods that encourage active participation of all learners. You should, wherever possible, encourage learners to work in groups. You should also ensure equal participation by all learners during group work.

Answer to Activity 1.1

1. a. A set is a collection of defined objects.

b. A set is finite, if it is empty or if it is equivalent to the set {1, 2, 3, ..., n }.

c. A set is infinite, if it is not finite.

d. A = B $\Leftrightarrow$ A $\subseteq$ B and B $\subseteq$ A.

A $\leftrightarrow$ B $\Leftrightarrow$ there is a one-to-one correspondence between A and B.

e. A set is countable, if it is either finite or equivalent to the set of natural numbers.

2. a. First, show the one-to-one correspondence.

$$\begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & . & . & . & n & . & . & . \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & . & . & . & \updownarrow & . & . & . \\ 0 & 1 & 2 & 3 & 4 & 5 & . & . & . & n-1 & . & . & . \end{array}$$

Then, define $f: \mathbb{N} \to \mathbb{W}$ by $f(n) = n - 1$

b. Rewrite 1, 2, 3, . . . as

$$\begin{array}{ccccccccccccccccccccccc} . & . & . & 2k+1 & . & . & . & 5 & 3 & 1 & 2 & 4 & 6 & 8 & . & . & . & 2k & . & . & . \\ . & . & . & \updownarrow & . & . & . & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & . & . & . & \updownarrow & . & . & . \\ . & . & . & -k & . & . & . & -2 & -1 & 0 & 1 & 2 & 3 & 4 & . & . & . & k & . & . & . \end{array}$$

Define $f: \mathbb{N} \to \mathbb{Z}$ by $f(k) = \begin{cases} \dfrac{k}{2}, \ if\ k\ is\ even. \\ -\dfrac{k-1}{2}, \ if\ k\ is\ odd. \end{cases}$

c. Define $f: \mathbb{N} \to$ E where E is the set of even integers

$$f(n) = \begin{cases} n, \ if\ n\ is\ even. \\ 1-n, \ if\ n\, is\ odd. \end{cases}$$

3. The function defined in 2(a), (b) and (c) are in a one-to-one correspondence. Therefore, each of them are equivalent to the set of natural numbers.

4. a. Domain = $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

b. Range = $\{2, 7, 23, 47, 119, 167, 287, 359, 527, 839\}$

c. Explain that the range doesn't contain all primes and that it contains some composites.

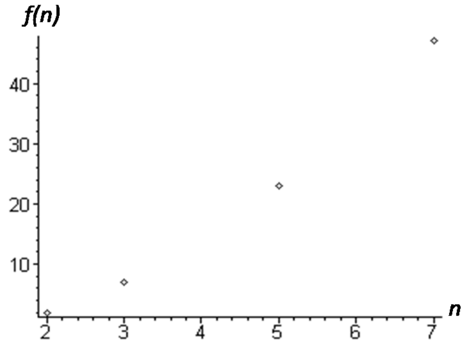


Figure 1.1

5. a. Notice that the n^{th} row consists of the numbers

$$\binom{n-1}{0},\ \binom{n-1}{1},\ \binom{n-1}{2},\ \ldots,\ \binom{n-1}{n-1}$$

Where $n = 1, 2, 3, \ldots$

$$\Rightarrow f(n) = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

$f(10) = 512$

$f(15) = 16384$

b. $f = \{ (1, 1), (2, 2), (3, 4), (4, 8), (5, 16), (6, 32)\}$

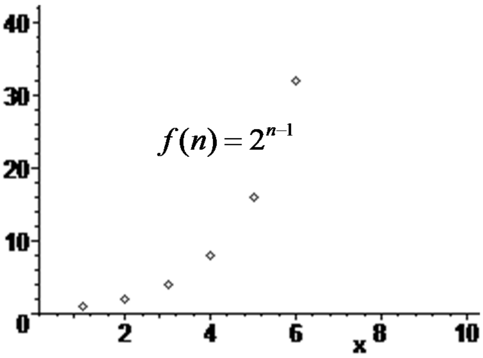


Figure 1.2

6. $f(n) = \dfrac{n!}{2^n}$

a. $f(1) = \dfrac{1}{2}$ b. $f(5) = \dfrac{15}{4}$ c. $f(10) = \dfrac{14175}{4}$

7. $q(1) = \pm 1$

$q(2) = \pm 1, \pm 2$

$q(3) = \pm 1, \pm 3$

$q(4) = \pm 1, \pm 2, \pm 4$

$q(5) = \pm 1, \pm 5$

$q(6) = \pm 1, \pm 2, \pm 3, \pm 6$

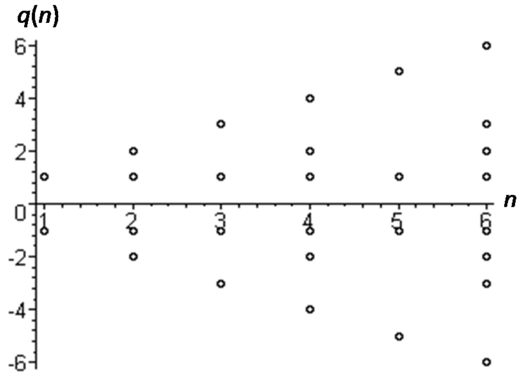


Figure 1.3

Assessment

Activity 1.1 can be used to assess the background of students. You should give homework, class-works and assess students by checking their exercise books.

1.1.2 Number Sequences

Activity 1.2 is an application problem designed to introduce the intuitive definition of a sequence, terms of a sequence, general term of a sequence and the relationship between the terms of a sequence.

Answer to Activity 1.2

1. The amount to be paid after n days delay forms the pattern (in Birr)

   203, 206, 209, . . . , $200 + 3n$

   a. 209 b. 230 c. $200 + 3n$

2. We think of a sequence as list of numbers that comes one after another by a given rule.

3. $a_{10} = 29$, $a_{15} = 44$, $a_{25} = 74$

After having discussed Activity 1.2, you are expected to encourage students define what a sequence is. You can guide them to define a sequence is a function whose domain is the subset of the set of natural numbers as given in the students' textbook followed by examples. You can make students identify finite sequences and infinite sequences and how formulas are important to list the terms of a sequence. Here, sometimes recursion formula is very important to determine the terms of a sequence. It is sometimes important to tell students the history of some famous mathematician and therefore, here we select one famous mathematician related to this topic named Leonardo Fibonacci attached to the Fibonacci sequence known by his name.

Assessment

Activity 1.2 can be used to assess the background of students so as make the students debate on the activities. You should give homework, class-works and assess students by checking their exercise books. For this purpose, you can use Exercise 1.1.

Answers to Exercise 1.1

1. a. $a_1 = 0.8,\ a_2 = 0.96\ ,\ a_3 = 0.992\ ,\ a_4 = 0.9984,\ a_5 = 0.99968$
   b. $a_1 = 1,\ a_2 = \frac{3}{5},\ a_3 = \frac{1}{2},\ a_4 = \frac{5}{11},\ a_5 = \frac{3}{7}$
   c. $a_1 = -3,\ a_2 = \frac{3}{2},\ a_3 = -1,\ a_4 = \frac{3}{4},\ a_5 = -\frac{3}{5}$
   d. $a_1 = 0,\ a_2 = -1,\ a_3 = 0\ ,\ a_4 = 1,\ a_5 = 0$
   e. $a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{2}{3},\ a_4 = \frac{3}{5},\ a_5 = \frac{5}{8}$
   f. $a_1 = 0\ ,\ a_2 = -1,\ a_3 = 0,\ a_4 = 5\ ,\ a_5 = 18$
   g. $a_1 = 0,\ a_2 = 2,\ a_3 = 0,\ a_4 = 2,\ a_5 = 0$
   h. $a_1 = 1,\ a_2 = 2,\ a_3 = \frac{9}{2},\ a_4 = \frac{32}{3},\ a_5 = \frac{625}{24}$
   i. $p_1 = 2,\ p_2 = 3,\ q_3 = 5,\ p_4 = 7,\ p_5 = 11$
   j. $q_1 = 1,\ q_2 = 3,\ q_3 = 6,\ q_4 = 10,\ q_5 = 15$
   k. $a_1 = -1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4,\ a_5 = 5$
   l. $a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{4}{5},\ a_4 = \frac{25}{41},\ a_5 = \frac{1681}{2306}$
2. Give this problem as group work.
   a. $a_n = 3n$, where n is a positive integer.
   b. $a_n = 5n - 3$, where n is a positive integer
   c. $a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$
   d. $a_n = \frac{(-1)^n n}{(n+1)^2}$ where n is a positive integer.
   e. $a_1 = 2,\ a_2 = 3$ and $a_{n+2} = a_n + a_{n+1}$ for $n \geq 1$.
   f. $(-1)^n + 1 = a_n$
   g. $2^{n-1} - 1 = a_n$
   h. $\frac{n^2}{n!} = a_n$
   i. $\sum_{i=1}^{n} \frac{2}{10^i} = a_n$
   j. $\frac{2^{n-1}}{n+1} = a_n$

1.2 ARITHMETIC SEQUENCE AND GEOMETRIC SEQUENCE

Periods allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define arithmetic progressions and geometric progressions.*
- *determine the terms of arithmetic and geometric sequences.*

Vocabulary: Arithmetic sequence, Common difference, Geometric sequence, Common ratio

Introduction

Under this sub-topic, you consider special type of sequences named arithmetic sequence and geometric sequence. These are simple sequences in which each of the terms can be determined given some of the terms.

1.2.1 Arithmetic Sequence

Teaching Notes

The opening problem under this sub-topic and Activity 1.3 help students find the next term of a sequence given the previous term. Thus, make students participate in the opening problem and Activity 1.3.

Answers to the Opening Problem

1. a. The n^{th} card number in R_1 is $3n - 2$

   Also, $3n - 2 \le 100 \Rightarrow n \le 34$ i.e $1, 4, 7, 10, \ldots, 3n - 2, \ldots, 100$

   Thus, there are 34 students in R_1.

   Similarly,

   b. The n^{th} card number in R_2 is $3n - 1$ and $3n - 1 \le 100 \Rightarrow n \le 33\frac{2}{3}$

   $\Rightarrow$ There are 33 students in R_2.

   c. The n^{th} card number in R_3 is $6n - 3$ and $6n - 3 \le 100 \Rightarrow n \le 17\frac{1}{6}$

   $\Rightarrow$ There are 17 students in R_3.

   d. The n^{th} card number in R_4 is $6n$ and $6n \le 100 \Rightarrow n \le 16\frac{2}{3}$

   $\Rightarrow$ There are 16 students in R_4.

2. No
3. No

Answers to Activity 1.3

1. a. 2 b. 5 c. 10 d. −10

2. Yes, it is by adding the constant term which is the difference between consecutive terms.

Assessment

The opening problem and Activity 1.3 can be used to assess the background of students. So make the students' debate on the opening problem and Activity 1.3 to assess them. You should give homework, class-works and assess students by checking their exercise books.

After having discussed the opening problem and Activity 1.3, give the definition of arithmetic sequence or arithmetic progression as given in the students' textbook. Then develop some properties of Arithmetic sequence from the definition like Theorem 1.1 and illustrate by examples like those examples and Activity 1.4 given in the students' textbook.

Answers to Activity 1.4

1. a. $b - c = c - a \Rightarrow 2c = a + b \Rightarrow c = \dfrac{a+b}{2}$

   b. $\dfrac{10+15}{2} = 12.5$

2. Let the 5 – arithmetic means between 4 and 13 be m_1, m_2, m_3, m_4 and m_5. Then, the total number of terms in the sequence is 7.

   Hence, $a_1 = 4$; $a_7 = 13$

   $\Rightarrow 4 + 6d = 13$

   $\Rightarrow d = 1.5$

   $\Rightarrow$ The arithmetic means are 5.5, 7, 8.5, 10, 11.5

Assessment

Activity 1.4 can be used to assess the background of students. So make the students' debate on Activity 1.4 in order to assess students. You should give students homework, class-works and assess students by checking their exercise books. For this purpose, you may use Exercise 1.2.

Answers to Exercise 1.2

1. Except (d), all are arithmetic

   Here you can include oral questions such as:

What is the common difference?

Which one is a constant sequence?

Which sequence is increasing / decreasing?

2. a. Show that the common difference is –4.

   b. $a_n = 97 + (n-1) \times (-4) = 101 - 4n$

   c. Motivate students to answer this question orally.

   Next to this, prove the statement mathematically,

   $$a_n = 60 \Rightarrow 101 - 4n = 60 \Rightarrow 4n = 41$$

   $$\Rightarrow n = \frac{41}{4}$$

   But $\frac{41}{4}$ is not in the domain of a sequence.

   Hence, 60 is not a term in the sequence.

3. $a_n = 7n - 3$

   a. $a_{n+1} - a_n = 7(n + 1) - 3 - (7n - 3) = 7$ constant. The difference between any two terms is 7.

   b. $a_{75} = 7(75) - 3 = 522$

   c. $a_n \geq 528 \Rightarrow 7n - 3 \geq 528 \Rightarrow n \geq 75\frac{7}{6}$

   The smallest of such natural numbers is 76.

   Hence, $a_{76} = 529$

4. Given $A_3 = 12$ and $A_9 = 14$

   $A_n = A_1 + (n - 1)d$, where d is the common difference.

   Thus, $A_3 = 12 = A_1 + 2d$ and $A_9 = 14 = A_1 + 8d$

   Solving $\begin{cases} A_1 + 2d = 12 \\ A_1 + 8d = 14 \end{cases}$

   We obtain $d = \frac{1}{3}$ and $A_1 = \frac{34}{3}$

   Therefore $A_{30} = A_1 + (30 - 1)d$

   $$= \frac{34}{3} + 29 \times \frac{1}{3} = 21$$

5. $A_4 = 8$ and $A_8 = 10$, using the above technique show that $d = \frac{1}{2}$ and $A_1 = 6.5$

6. $A_4 = A_1 + 3d$, $A_4 = 5$ and $d = 6$ (given)

$5 = A_1 + 18$

$\Rightarrow A_1 = -13$ and $A_9 = 35$

7. $A_1 = 202$ and $A_{30} = \frac{461}{3}$

8. $A_p = q$ and $A_q = p$, then

$A_1 + (p - 1)\,d = q$

$A_1 + (q - 1)\,d = p$

$\Rightarrow (p - 1 - q + 1)\,d = q - p$

$\Rightarrow d = -1$

$\Rightarrow A_1 = q - (p - q)\,d$

$\Rightarrow A_1 = q - (p - 1)(-1)$

$= p + q - 1$

$\Rightarrow A_n = p + q - 1 + (n - 1)(-1)$

$= p + q - n$

$\Rightarrow A_{p+q} = p + q - (p + q) = 0$

9. A student may not observe that the first whole number divisible by 7 is 0. Determine the largest whole number less than 1000 that is divisible by 7.

$A_1 = 0$, $d = 7$, $A_n = 7n - 7$

$7n < 1007$

$\Rightarrow n < 143\frac{6}{7}$

$\Rightarrow n = 143$

10. When n – arithmetic means are inserted between a and b, then the resulting terms of the sequence are $a, m_1, m_2, m_3, \ldots, m_n, b$. . . . a total of $(n + 2)$ terms

$\Rightarrow b = a + (n + 1)\,d \Rightarrow d = \dfrac{b-a}{n+1}$.

11. $A_1 = 18{,}000.00$, $d = 1{,}500.00$. The beginning of the 11th year is the same as the end of the 10th year.

Thus, $A_{10} = A_1 + 9d$

$= 18{,}000.00 + 9 \times 1{,}500.00 = 31{,}500.00$

Hence, his annual salary at beginning of the 11th year is Birr 31,500.

1.2.2 Geometric Sequence

The purpose of the opening problem and Activity 1.5 is to introduce the existence of another type of sequence whose rule is governed by a common ratio instead of a common difference. Ask orally whether students could write the next term or not. Before defining what a geometric sequence is, please try to make the students do the opening problem and activity 1.5. You are required to guide and help students to do both the opening problem and the activity.

Answers to the Opening Problem

In 2001, the population is 75,000,000. The rate at which the population increases is 2%. Hence, in 2002, the population will, approximately, be

$$75000\,000 + 75000\,000 \times \frac{2}{100} = 75000000\left(1+\frac{2}{100}\right)$$

Similarly, the population at 2000 + k will be approximately

$$75000000\left(1+\frac{2}{100}\right)^{k-1}$$

a. In 2020, the population is expected to be, approximately

$$75{,}000{,}000\left(1+\frac{2}{100}\right)^{19} = 109{,}260{,}838.$$

b. $2\,(75{,}000{,}000) = 75{,}000{,}000\left(1+\frac{2}{100}\right)^{k-1}$

$\Rightarrow \quad 2 = \left(1+\frac{2}{100}\right)^{k-1}$

$\Rightarrow \quad 2 = (1.02)^{k-1}$

$\Rightarrow \quad k-1 = \dfrac{\log 2}{\log 1.02}$

$\Rightarrow \quad k = 1 + \dfrac{\log 2}{\log 1.02}$

$\Rightarrow \quad k = 36.00278880$

Thus, it takes about 36 years to double the population.

Answers to Activity 1.5

1. a. 2 b. $\frac{1}{10}$ c. $\frac{-1}{3}$ d. $\frac{1}{2}$
2. Yes, we can find the second term by multiplying the first term by the given ratio; the third by multiplying the second by the ratio, the fourth by multiplying the third by the ratio, and so on.

After having discussed Activity 1.5, give the definition of geometric sequence as it is given in the students' textbook and ask students to develop some properties of geometric sequence like Theorem 1.2 and Activity 1.6. Use examples on page 14 students' textbook for illustrative purpose.

Answer to Activity 1.6

1. a. $\frac{c}{a}=\frac{b}{c} \Rightarrow c^2 = ab \Rightarrow c = \sqrt{ab}$, since $c > 0$.

   b. The geometric mean between 4 and 8 is $\sqrt{4 \times 8} = 4\sqrt{2}$

2. When geometric means m_1, m_2 and m_3 are inserted between 0.4 and 5, then the number of the resulting terms are 0.4, m_1, m_2 , m_3 , 5 is 5.

   $\Rightarrow G_1 = 0.4$ and $G_5 = 5$
   $\Rightarrow 0.4r^4 = 5$
   $\Rightarrow r^4 = \frac{25}{2} \Rightarrow r = \sqrt[4]{\frac{25}{2}}$

   $\Rightarrow$ The geometric means are $m_1 = 0.4\sqrt[4]{\frac{25}{2}}$, $m_2 = \sqrt{2}$, $m_3 = \frac{\sqrt{10}}{\sqrt[4]{2}}$

Assessment

Activity 1.5 and 1.6 can be used to assess the background of students. So make the students present their work on the blackboard and observe them debating on it. You should give homework, class-works and assess students by checking their exercise books. For this purpose, you may use Exercise 1.3 and 1.4. Here you can use the puzzle problems for more able students who are more inclined to mathematics.

Answers to Exercise 1.3

1. a. $(-2)^{n-1}$ b. 5 c. It is not geometric

   d. $9\left(-\frac{1}{3}\right)^{n-1}$ e. It is not geometric f. $(2x)^n$

   g. $\left(5+\sqrt{5}\right)\left(\frac{1}{\sqrt{5}}\right)^{n-1}$

2. 135, 405, 1215 and $G_{10} = r^9$. $G_1 = 3^9 (5) = 5 \times 3^9 = 98, 415$.
3. 5^{-6}
4. 2401
5. x, $4x + 3$, $7x + 6$ is geometric sequence

   $\Rightarrow \frac{4x+3}{x} = \frac{7x+6}{4x+3} \Rightarrow x^2+2x+1 \Rightarrow x=-1$

   The resulting terms are: –1, –1, –1

PUZZLE

The purpose of the puzzle is to increase the students' power of imagination and appreciation of the lesson. This problem is best if each student tries it individually and if students discuss it in groups.

g_1 = Birr 0.01 and $r = 2$. Hence $g_n = 0.01 \times 2^{n-1}$

$\Rightarrow g_{30} = 0.01 \times 2^{29} = 5368709.12.$

The money the society invests on the 30th day is Birr 5368709.12.

The total amount of money they invest is Birr 10737418.23.

Answers to Exercise 1.4

1.
   a. Arithmetic with $A_1 = 4$ and $d = 3$
   b. Neither
   c. Neither
   d. Neither
   e. Geometric with $G_1 = 2$ and $r = -2$
   f. Geometric with $G_1 = \frac{4}{3}$ and $r = 6$
   g. Arithmetic with $A_1 = 3$ and $d = -2$
   h. Neither
   i. Neither
   j. Geometric with $G_1 = \frac{4}{343}$ and $r = \frac{4}{7}$

2.
   a. $A_n = 5n - 2$, $d = 5$
   b. $A_n = \frac{61n - 406}{5}$, $d = \frac{61}{5}$
   c. $A_n = \frac{1}{2}n + 6$, $d = \frac{1}{2}$

3.
   a. $G_4 = 80$
   b. $G_6 = -972$
   c. $G_3 = r^2 G_1$ and $G_6 = r^5 G_1$

$$\Rightarrow \frac{G_6}{G_3} = \frac{216}{1} \Rightarrow \frac{r^5 G_1}{r^2 G_1} = 216$$

$$\Rightarrow r^3 = 216 \Rightarrow r = 6$$

$$\Rightarrow G_1 = \frac{1}{36}$$

   d. $r = -\frac{1}{\sqrt{3}}$, $G_1 = -1$

$$G_8 = \left(\frac{1}{\sqrt{3}}\right)^7, G_n = -1\left(-\frac{1}{\sqrt{3}}\right)^{n-1}$$

4. $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 \Rightarrow a - 2\sqrt{ab} + b \geq 0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

5. This problem is going to be solved using calculators, computers or logarithm tables. Considering the sequence as a geometric sequence, we have the n^{th} term to be

$G_n = 4 \times 3^{n-1}$

$G_n \geq 20000$

$\Rightarrow\ 4 \times 3^{n-1} \geq 20000$

$\Rightarrow\ 3^{n-1} \geq 5000$

$\Rightarrow\ (n-1) \geq \log_3 5000$

$\Rightarrow\ n \geq 8.75268$

$\Rightarrow$ The smallest such n is 9.

Note that, $G_8 = 8748$

$G_9 = 26244$

6. $G_n = 10\left(\frac{1}{2}\right)^{n-1}$

$G_n \geq 0.0001$

$\Rightarrow\ 10\left(\frac{1}{2}\right)^{n-1} < 0.0001$

$\Rightarrow\ \left(\frac{1}{2}\right)^{n-1} < 0.00001$

$\Rightarrow\ n-1 > \log_{0.5} 0.00001$

$\Rightarrow\ n-1 > \frac{\log 0.00001}{\log 0.5}$

$\Rightarrow\ n-1 > 16.6096$

$\Rightarrow\ n > 17.6096$

$\Rightarrow$ The smallest of such n is 18.

$\therefore G_{18} = 0.000076293$

7. When 4-arithmetic means are inserted between 2 and 20, then the common difference $d = \frac{20-2}{5} = 3.6$

Therefore, the arithmetic means are 5.6, 9.2, 12.8, 16.4

When 5 - geometric means are inserted between 2 and 20, then $2r^6 = 20$

$\Rightarrow r = \sqrt[6]{10}$

$\Rightarrow$The geometric means are $2\sqrt[6]{10}$, $2\sqrt[3]{10}$, $2\sqrt{10}$, $2\sqrt[3]{100}$, $2\sqrt[6]{10^5}$

8. Ask some students to explain that $xy = 16$ and $x + y = 10$

$\Rightarrow$Either $x = 8$ and $y = 2$, or $x = 2$ and $y = 8$.

9. $\ln(g_{n+1}) - \ln(g_n) = \ln\left(\frac{g_{n+1}}{g_n}\right) = \ln r.$

The difference between each consecutive term is a constant which is $\ln r$.

Hence, $\{\ln\ g_n\}$ is arithmetic.

1.3 THE SIGMA NOTATION AND PARTIAL SUMS

Periods allotted: 6 periods

Competencies

At the end of this sub-unit, students will be able to:

- *use the sigma notation for sums.*
- *find the n^{th} partial sum of a sequence.*
- *use the symbol for the sum of sequences.*
- *compute partial sums of arithmetic and geometric progressions.*
- *apply partial sum formula to solve problems of science and technology.*

Vocabulary: Sigma notation, Partial sum

Sums of Arithmetic Progressions and Sums of Geometric Progressions

Introduction

In the previous sections, we are interested in the individual terms of a sequence. In this section, you are going to describe the process of taking sums of terms of a sequence. Thus this sub-topic is devoted to finding partial sums of a sequence.

Teaching Notes

Under this sub-topic you guide and help students to get the sum of the first *n* term of arithmetic and geometric progressions. First, you start with the sum of the first n terms of arithmetic progression. You can start this sub-topic by the opening problem given at the beginning of this topic. This opening problem give highlights about partial sum. And therefore, you are required to guide and help the students do this opening problem individually as it is concerned with individual real life-problem and make them discuss their work in groups or as a whole class.

Answers to the Opening Problem

The sequence of parents, grandparents, great grandparents, and so forth is found to be $2, 2^2, 2^3, 2^4, \ldots$

The number of

i. parents = 2
ii. 1^{st} grand parents (grandparents) = 2^2
iii. 2^{nd} grand parents (great grandparent) = 2^3
.
.
.
xi. 10^{th} grand parents = 2^{11}

Therefore, you are consisting of $2 + 2^2 + 2^3 + \ldots + 2^{11} = 4096$ persons.

After having discussed the opening problem, you can discuss how to form sums of a sequence and use the sigma notation to represent the sum. You can illustrate it by using examples given in the students' textbook.

This sub-unit aims to introduce the sigma notation which stands for the sum of terms of a sequence. If need arise, proof properties of $\sum$ which is given in the students' textbook. You can precede the proof as follows:

To proof (1), first ask the students the distributive property of multiplication over addition.

$$c\,(a + b) = ca + cb$$

Thus, $\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ca_3 + \ldots\ldots + ca_n$

$$= c\,[a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_n]$$

$$= c\sum_{k=1}^{n} a_k$$

2. $\sum_{k=1}^{n}(a_k + b_k) = a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + \ldots + a_n + b_n$

$$= a_1 + a_2 + \ldots + a_n + b_1 + b_2 + \ldots\ldots + b_n$$

(commutative property of addition)

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

3. $\sum_{k=1}^{n}(a_k - b_k) = a_1 - b_1 + a_2 - b_2 + a_3 - b_3 + \ldots + a_n - b_n$

$$= a_1 + a_2 + \ldots\ldots + a_n - (b_1 + b_2 + \ldots\ldots + b_n)$$

$$= \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Assessment

The opening problem may be used to assess students' background and readiness on how to find sums of terms of a sequence. So, make the students present their work either for group or for the whole class. You should give homework, class-works and assess students by checking their exercise books. For this purpose, you may use Exercise 1.5.

Answers to Exercise 1.5

1. This problem is designed for group work. Here, if the need arises, the group can begin by finding S_1, S_2, S_3 and so on to find the indicated sum; and try to get the patterns for some of the sums.

a. $S_6 = 3 + 7 + 11 + 15 + 19 + 23 = 78$

b. $S_5 = -8 + -3 + 2 + 7 + 12 = 10$

c. $-2 + 0 + 2 + 4 + 6 + 8 = 18 = S_6$

d. $S_6 = 1 + 2 + 4 + 8 + 16 + 32 = 63$

$S_{10} = S_6 + 64 + 128 + 256 + 512$

$= 63 + 64 + 128 + 256 + 512 = 1023$

$S_{20} = S_{10} + a_{11} + a_{12} + \ldots + a_{20} = 1{,}048{,}575$

e. $S_6 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{63}{32} = \frac{2 \times 32 - 1}{32}$

$$S_{10} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{28} + \frac{1}{256} + \frac{1}{521} = \frac{63}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512}$$

$$= \frac{1008 + 8 + 4 + 2 + 1}{512}$$

$$= \frac{1023}{512} = \frac{2 \times 512 - 1}{512}$$

From the above patterns, we get $S_{20} = \frac{2 \times 2^{19} - 1}{2^{19}} = \frac{2^{20} - 1}{2^{19}} = 2 - \frac{1}{2^{19}}$

$$S_{100} = \frac{2 \times 2^{19} - 1}{2^{99}} = 2 - \frac{1}{2^{99}}$$

In general $S_n = \frac{2 \times 2^{n-1} - 1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

f. $a_n = 3n + 1$, then $a_1 = 4$, $a_2 = 7$, $a_3 = 10$, $a_4 = 13$, $a_5 = 16$, $a_6 = 19$ and so on, thus

$S_6 = 4 + 7 + 10 + 13 + 16 + 19 = 69$

$S_{10} = S_6 + 22 + 25 + 28 + 31 = 69 + 47 + 59 = 175$, $S_{20} = 650$

g. $a_n = 2n - 1$

$S_6 = 1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

$S_{10} = 1 + 3 + 5 + 7 + \ldots + 11 + 13 + 15 + 17 + 19 = 100 = 10^2$

$S_{20} = 1 + 3 + 5 + \ldots + 39 = 400 = 20^2$

$S_n = n^2$

h. $a_n = \log\left(\frac{n}{n+1}\right)$

$$S_6 = \log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \log\frac{4}{5} + \log\frac{5}{6} + \log\frac{6}{7}$$

$$= \log\left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\right) = \log\frac{1}{7} = -\log 7$$

$$S_{10} = \log\frac{1}{2} + \log\frac{2}{3} + \ldots + \log\frac{10}{11}$$

$$= \log\left(\frac{1}{2}\times\frac{2}{3}\times\ldots\ldots\times\frac{9}{10}\times\frac{10}{11}\right) = -\log 11$$

Similarly, $S_{20} = -\log 21$

$S_{100} = -\log 101$

$S_n = -\log(n+1)$

i. $a_n = \frac{n}{n+1} - \frac{n+1}{n+2} \Rightarrow a_n = \frac{1}{n+2} - \frac{1}{n+1}$. (using partial fraction method)

Look at the Pattern.

$$S_n = \left(\frac{1}{3}-\frac{1}{2}\right) + \left(\frac{1}{4}-\frac{1}{3}\right) + \left(\frac{1}{5}-\frac{1}{4}\right) + \left(\frac{1}{6}-\frac{1}{5}\right) + \left(\frac{1}{7}-\frac{1}{6}\right) + \ldots\left(\frac{1}{n+2}-\frac{1}{n+1}\right)$$

$$\Rightarrow S_n = -\frac{1}{2} + \frac{1}{n+2}$$

$$\Rightarrow S_6 = -\frac{1}{2} + \frac{1}{8} = \frac{-3}{8}, \text{ for } n = 6$$

$$S_{10} = -\frac{1}{2} + \frac{1}{12} = \frac{-5}{12}, \text{ for } n = 10$$

$$S_{20} = -\frac{1}{2} + \frac{1}{22} = \frac{-10}{22}, \text{ for } n = 20$$

$$S_{100} = -\frac{1}{2} + \frac{1}{102} = -\frac{50}{102}, \text{ for } n = 100$$

2. a. $\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5 = 15$

b. $4(3)+4(4)+4(5)+4(6)=4(3+4+5+6)=72=\sum_{k=1}^{4}4(k+2)$

c. $\sum_{k=1}^{6}5(k-1)=5(0+1+2+3+4+5)=75$

d. $\sum_{k=1}^{8}3k=3(1+2+3+4+5+6+7+8)=108$

e. $\sum_{k=2}^{6}k^2=4+9+16+25+36=90$

f. $\sum_{k=3}^{5}k^3=27+64+125=216$

g. $\sum_{k=1}^{5}4=4+4+4+4+4=20$

h. $\sum_{k=3}^{10}7=7+7+7+7+7+7+7+7=56$

i. $\sum_{m=1}^{10}\left[\frac{2}{m}-\frac{2}{m+1}\right]=2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{10}-\frac{1}{11}\right)\right]=\frac{20}{11}$

j. $\sum_{n=1}^{8}\log_3\left(\frac{n+1}{n}\right)=\log_3 2+\log_3\frac{3}{2}+\log_3\frac{4}{3}+\ldots+\log_3\frac{9}{8}$

$$=\log_3\left(2\times\frac{3}{2}\times\frac{4}{3}\times\ldots\times\frac{9}{8}\right)=\log_3 9=2$$

k. $\sum_{k=1}^{6}\log_8 2^k=\sum_{k=1}^{6}k\log_8 2=\sum_{k=1}^{6}\frac{k}{3}=\frac{1}{3}\left[1+2+3+4+5+6\right]=7$

3. a. $\sum_{k=1}^{5}4k$ b. $\sum_{k=1}^{8}(3k-1)$

c. $\sum_{k=1}^{7}2k^2$ d. $\sum_{k=1}^{10}2k+5$

4. a. $\sum_{n=1}^{5}(4n-2)$ b. $\sum_{n=1}^{4}5^n$ c. $\sum_{n=1}^{26}(2n-1)$

d. $\sum_{n=1}^{17}(3n+1)$ e. $\sum_{n=1}^{99}\frac{1}{n(n+1)}$ f. $\sum_{n=1}^{10}\frac{2n}{4n+1}$

After having discussed Exercise 1.5, you can consider the partial sum of Arithmetic and geometric sequence. As it is given in the students' text, begin first with the arithmetic one, specially finding the sum of the first n natural numbers, since natural numbers are arithmetic sequence with first term 1 and common difference 1.

Please tell the students the historical note given in the student textbook page 21. Derive a general formula for the sum of the first n term of positive integers, and then, step by step show them how to derive the general formula of the sum of the first n terms of Arithmetic progression as it is given in the students textbook. Illustrate it by using those examples given in the students' textbook.

After having discussed the sum of the first n terms of arithmetic progression, in a similar fashion, guide and help the students to find the sum of the first n – terms of geometric progression. For this purpose, you can follow the procedures given in students' textbook and illustrate with examples.

Assessment

You can assess students by asking oral questions while deriving formulas of partial sums of a sequence. You should also give them homework, class-works and assess students by checking their exercise books. For this purpose, you may use Exercise 1.6.

Answers to Exercise 1.6

1. Given $A_1 = 4, d = 5 \quad S_8 = ?$

   Using the general formula $S_n = \frac{n}{2}\left[2A_1 + (n-1)d\right]$

   $$S_8 = \frac{8}{2}\ (2 \times 4 + 7 \times 5) = 4\ (8 + 35) = 4 \times 43 = 172$$

2. $$S_{10} = \frac{10}{2}\ (2 \times 8 + 9\,(-1)) = 5\ [16 - 9] = 35$$

3. Given $A_4 = 2$ and $A_7 = 17$, $S_7 = ?$

   $$\begin{cases} A_7 = A_1 + 6d \\ A_4 = A_1 + 3d \end{cases} \Rightarrow \begin{cases} A_7 - A_4 = 3d \\ 17 - 2 = 3d \Rightarrow d = 5 \end{cases}$$

   Thus $A_1 = A_4 - 3d \Rightarrow A_1 = 2 - 3 \times 5 = -13$

   Hence using formula $S_n = n\left(\frac{A_1 + A_n}{2}\right)$

   $$S_7 = 7\left(\frac{-13+17}{2}\right) = 14$$

4. Given $G_1 = 4$ and $r = 5$

$$S_n = \frac{G_1(1-r^n)}{1-r}$$

$$S_8 = \frac{4(1-5^8)}{1-5} = 5^8 - 1 = 390624$$

$$S_{12} = \frac{4\ (1\ -\ 5^{12})}{-4} = 5^{12} - 1 = 244{,}140{,}624$$

$$S_{20} = \frac{4\ (1\ -\ 5^{20})}{-4} = 5^{20} - 1 = 95{,}367{,}431{,}640{,}624$$

$$S_{100} = \frac{4\ (1\ -\ 5^{100})}{-4} = 5^{100} - 1 = 7.889 \times 10^{69}$$

In general $S_n = 5^n - 1$

Hence, as n gets "larger and larger" $5^n - 1$ gets also "larger and larger"

5. Given $G_1 = 4$ and $r = \frac{2}{3}$

Using the formula $S_n = \frac{G_1(1-r^n)}{1-r}$

$$S_8 = 4\ \frac{\left(1-\left(\frac{2}{3}\right)^8\right)}{1-\frac{2}{3}} = 3 \times 4\left(1-\left(\frac{2}{3}\right)^8\right) = 12\left(1-\left(\frac{2}{3}\right)^8\right) = 11.53$$

$$S_{12} = \frac{4\left(1-\left(\frac{2}{3}\right)^{12}\right)}{1-\frac{2}{3}} = 12\left(1-\left(\frac{2}{3}\right)^{12}\right) = 11.9$$

$$S_{20} = 12\left(1-\left(\frac{2}{3}\right)^{20}\right) = 11.996,\ S_{100} = 12\left(1-\left(\frac{2}{3}\right)^{100}\right) = 12$$

In general $S_n = 12\left(1-\left(\frac{2}{3}\right)^n\right)$, as n gets larger and larger $\left(\frac{2}{3}\right)^n$ gets smaller and smaller. So S_n becomes closer to 12.

6. $S_{10} = 165$, $A_1 = 3$, $A_{10} = ?$

$S_n = n\left(\frac{A_1 + A_n}{2}\right)$

$S_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$

Thus $165 = \frac{10}{2}(3 + A_{10})$

$\Rightarrow \frac{165}{5} = 3 + A_{10} \Rightarrow A_{10} = 33 - 3 = 30$

7. $S_{20} = \frac{20}{2}(A_1 + A_{20})$

$\Rightarrow 910 = 10(A_1 + A_{20})$

$\Rightarrow 91 = A_1 + 95 \Rightarrow A_1 = -4$

8. $S_{16} = 368, A_1 = 1$, $A_8 = ?$

First we get A_{16}, $S_{16} = \frac{16}{2}(A_1 + A_{16})$

$\Rightarrow 368 = 8(1 + A_{16})$

$\Rightarrow 46 = 1 + A_{16}$

$\Rightarrow A_{16} = 45$

Now $A_n = A_1 + (n-1)d$

$A_{16} = A_1 + 15d$

$45 = 1 + 15d$

$15d = 44 \Rightarrow d = \frac{44}{15}$

Thus, $A_8 = A_1 + 7d = 1 + 7\left(\frac{44}{15}\right) = \frac{15+308}{15} = \frac{323}{15}$

9. $S_n = 969, A_1 = 9$ and $d = 6$, $n = ?$

$\Rightarrow A_n = 9 + 6(n-1) = 6n + 3$

$S_n = \frac{n}{2}(A_1 + A_n) \Rightarrow 969 = \frac{n}{2}(9 + 6n + 3)$

$\Rightarrow 3n^2 + 6n - 969 = 0$

$\Rightarrow n = -19$ or $n = 17$

$\Rightarrow n = 17$ since $n > 0$

10. The smallest three digit whole number divisible by 13 is 104.

Thus, $a_1 = 104$ and $d = 13$.

Hence $a_n = 104 + (n-1) \times 13 = 91 + 13n$.

To determine the largest three digit number divisible by 13, solve

$91 + 13n < 1000$

$\Rightarrow \quad 13n < 909$

$\Rightarrow \quad n < 69.92307692$

$\Rightarrow \quad S_{69} = 37674 = \frac{69}{2}(104 + 988) = \frac{n}{2}(a_1 + a_n)$

11. Let $m_1, m_2, m_3, \ldots, m_n$ be n – arithmetic means that are inserted between a and b.

Then $m_1 = a + d$ and $m_n = b - d$.

$$\sum_{i=1}^{n} m_i = \frac{n}{2}(m_1 + m_n)$$

$$= \frac{n}{2}(a + d + b - d) = \frac{n}{2}(a + b)$$

12. $\begin{cases} a_4 = 84 \\ a_{10} = 60 \end{cases} \Rightarrow \begin{cases} a_1 + 3d = 84 \\ a_1 + 9d = 60 \end{cases}$

$\Rightarrow d = -4$ and $a_1 = 96$

$\Rightarrow a_n = 100 - 4n$

The sum is the maximum if we add all non - negative terms.

Let the smallest positive term be a_n, then $a_n \geq 0$

$\Rightarrow \quad 100 - 4n \geq 0$

$\Rightarrow \quad n \leq 25$

$\Rightarrow \quad$ The maximum sum is $S_{25} = \frac{25}{2}(96 + 0) = 1200$

13. If a, A_1, A_2, b is an arithmetic sequence, then,

$A_1 - a = b - A_2$

$\Rightarrow A_1 + A_2 = a + b$

If a, G_1, G_2, b is a geometric sequence, then,

$$\frac{G_1}{a} = \frac{b}{G_2}$$

$\Rightarrow \quad G_1G_2 = ab$

$$\Rightarrow \frac{A_1 + A_2}{G_1G_2} = \frac{a + b}{ab}$$

14. a. 1190 b. $-\dfrac{101}{420}$ c. $\dfrac{2448}{15625}$

d. $\dfrac{155}{21}$ e. $-\dfrac{893}{2520}$ f. 2870

15. Her loss −100, −60, −20,… forms an arithmetic sequence with

$A_1 = -100$ and $d = 40$.

Her loss (profit) at the end of 2 years and 7 months (31 months) is

$$S_{31} = \frac{31}{2}\left(2\times(-100)+30\times40\right) = \frac{31}{2}\left(-200+1200\right) = 31\times500 = \text{Birr } 15{,}500$$

∴ Her capital at the end of 2 years and 7 months

= Birr 3,000 + Birr 15,500 = Birr 18,500

16. Let P_n be the population of the city after n years.

Then $P_1 = 400000 + 400000 \times 0.03 = 400000\ (1.03)$

$P_2 = 400000\ (1.03) + 400000 \times (1.03) \times 0.03$

$= 400000\ (1.03)^2$

Hence, it can be shown that $P_n = 400000\ (1.03)^n$

a. $P_4 = 400000\ (1.03)^4 \approx 450204$

b. $P_{10} = 400000\ (1.03)^{10} \approx 537567$

17. Let A_n be the amount of money in organization A at the end of the n^{th} year. The investment in A forms an arithmetic sequence with first term

$A_1 = 10{,}300$ and $d = 300$

$\Rightarrow A_n = 10{,}300 + (n-1) \times 300 = 10{,}000 + 300n$.

Let G_n be the amount of money in organization B at the end of n – years. The investment in B forms a geometric sequence with $G_1 = 16{,}800$ and common ratio $r = 1.05$.

$G_n = 16000\ (1.05)^n$

a. $A_{10} = 10{,}000 + (300 \times 10) = 10{,}000 + 3{,}000 = 13{,}000$.

$G_{10} = 16{,}000\ (1.05)^{10} = 26062.31$

b. $A_n = 10{,}000 + 300n$, $G_n = 16{,}000\ (1.05)^n$

c. The amount in A cannot exceed the amount in B any time.

18. Let b_n be the amount of money the n^{th} buyer paid. Then $b_1 = 20000$, $b_2 = 12000$, $b_3 = 7200$, ---

    It forms a geometric sequence $\{G_n\}$ with $G_1 = 20000$ and $r = 0.6$.

    Let t_n be the amount of tax paid by the n^{th} buyer.

    Then $t_1 = 20000\ (20\%) = 4000$, $t_2 = 12000\ (20\%) = 2400, \ldots,$

    $t_n = 0.2\ (0.6)^{n-1} \times 20000.$

    $\Rightarrow S_n = \dfrac{0.2(20000)}{1-0.6} = 10000.$ $\Rightarrow$ The total amount of money that will be collected will be Birr 10000.

1.4 INFINITE SERIES

Periods allotted: 4 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define a series.*
- *decide whether a given geometric series is divergent or convergent.*
- *show how infinite series can be divergent or convergent.*
- *show how recurring decimals converge.*

Vocabulary: Infinite series, Geometric series, Convergent series, Divergent series

Introduction

This sub-unit aims at introducing an infinite sum of terms of a sequence and to introduce informally the idea of limit of sequence just by using phrases such as, if n becomes "larger and larger" or as n tends to infinity. This idea is then used to describe convergence and divergence of the series.

Teaching Notes

In this section you need not introduce the concept of a limit by using phrases such as, if n becomes "larger and larger" or as n tends to infinity. At this moment, we may not use the symbol $\lim\limits_{n\to\infty} S_n$. At this level, the students are expected to identify the partial sum S_n of the terms of the sequence a_n.

If a_1 , a_2 , $a_3, \ldots$ terms of a sequence, then the partial sum is given by

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

Assessment

Activity 1.7 and opening problem at the beginning of the sub-topic can be used to assess the background of students, so make the students debate on the activities and opening problems. You may give the activity and opening problem as a group work so that representatives may present their work for the whole class. But you have to ensure the participation of all students through questioning the group members while the representatives present. You should give homework, class-works and assess students by checking their exercise books. For this purpose, you can use Exercise 1.7

Answer to Opening Problem

a. $\left(16+2\left(\frac{0.81\times 16}{1-0.81}\right)\right)=152.42\text{m}$ b. $1+2\left(\frac{0.9}{1-0.9}\right)=1+18=19\,\text{sec}.$

Answer to Activity 1.7

1, 2 and 3. The answers to a, c and d are given in the student textbook.

b. $S_n = n^2$ as $n\to\infty$, $S_n\to\infty$

e. $S_n=\dfrac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$ as $n\to\infty$, $S_n\to\frac{1}{2}$

f. $S_n=\dfrac{2\left(1-2^n\right)}{1-2}=-2\left(1-2^n\right)$ as $n\to\infty$, $S_n\to\infty$

Answer to Exercise 1.7

1. a. $2+1+\frac{1}{2}+\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}2\left(\frac{1}{2}\right)^{n-1}=\dfrac{2}{1-\frac{1}{2}}=4$

b. $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\ldots=\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}=\dfrac{1}{1-\frac{2}{3}}=3$

c. $\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\ldots=\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^{n-1}=\dfrac{\frac{1}{5}}{1-\frac{1}{2}}=\frac{2}{5}$

d. $\frac{1}{5} - \frac{1}{10} + \frac{1}{20} - \frac{1}{40} + \ldots = \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{5}}{1+\frac{1}{2}} = \frac{2}{15}$

e. $\frac{1}{5} + \frac{4}{15} + \frac{16}{45} + \ldots = \infty$ because $r = \frac{4}{3} > 1$.

f. $\frac{70}{9}$ g. $\frac{64}{3}$ h. ∞

i. $\frac{15}{2}$ j. $\frac{2}{5}$ k. $\frac{243}{512}$

2. a. $\frac{0.4}{1-0.1} = \frac{4}{9}$

b. $0.3 + \frac{0.07}{1-0.1} = \frac{3}{10} + \frac{7}{90} = \frac{17}{45}$

c. $3.23 + \frac{0.0054}{1-0.01} = \frac{323}{100} + \frac{3}{550} = \frac{3559}{1100}$

d. $13.452 + \frac{0.000981}{1-0.001} = \frac{13452}{1000} + \frac{109}{111000} = \frac{1493281}{111000}$

3. $5^{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots} = 5^2 = 25$

4. $5^r + 5^{2r} + 5^{3r} + \ldots = \frac{1}{4}$

$\Rightarrow \frac{5^r}{1-5^r} = \frac{1}{4}$

$\Rightarrow 4(5^r) = 1 - 5^r$

$\Rightarrow 4(5^r) + 5^r = 1$

$\Rightarrow 5^r\,(4 + 1) = 1$

$\Rightarrow 5^r = \frac{1}{5} = 5^{-1}$

$\Rightarrow r = -1$

5. $3^r.3^{r^2}.3^{r^3}\ldots = 3$

$\Rightarrow 3^{r+r^2+r^3+\ldots} = 3$

$\Rightarrow r + r^2 + r^3 + \ldots. = 1$

$\Rightarrow \frac{r}{1-r} = 1 \Rightarrow r = \frac{1}{2}$

6. The total distance that could be covered by the ball is

$$2\frac{h}{1-r} - h = \frac{2h - h + rh}{1-r}$$

$$= h\left(\frac{1+r}{1-r}\right)$$

1.5 APPLICATIONS OF ARITHMETIC PROGRESSIONS AND GEOMETRIC PROGRESSIONS

Periods allotted: 2 periods

Competency

At the end of this sub-unit, students will be able to:

- *discuss the applications of arithmetic and geometric progressions and series in science and technology and daily life.*

Vocabulary: Binomial series

Introduction

This sub-unit is devoted to the application of arithmetic and geometric progressions or geometric series (binomial series) that are associated with real life situations.

Teaching Notes

In this section, you need to illustrate real life problems with examples. You can use examples given in the students' textbook.

Assessment

You should give homework, class-works and assess students by checking their exercise books. For this purpose, you can use Exercise 1.8 and make the students do in group specially questions 8 to 15 so that the more able students may help the less able students.

Answers to Exercise 1.8 (Application Problems)

1. A_1 = Birr 25, 250, d = Birr 250, $n = 3 \times 2 = 6$.

   $\Rightarrow A_7$ = Birr (25, 250 + 6(250)) = Birr 26750

2. $A_1 = 1000$, $d = 200$, $A_8 = 1000 + 7 \times 200 = 2400$

For problems 3-8, motivate the students to use calculators or computers.

3. The value at the end of the n^{th} year (in Birr) is

$$g_n = 28{,}000\left(1-\frac{1}{10}\right)^n = 28{,}000\,(0.9)^n$$

$$\Rightarrow g_4 = 28{,}000\,(0.9)^4 = 18{,}370.80$$

4. At the end of the n^{th} year, the value of the boat (in Birr) is

$$g_n = 34{,}000\,(1 - 0.12)^n$$

$$\Rightarrow g_5 = 34{,}000\,(1 - 0.12)^5 = 17942.88517$$

5. The population of the town at the end of the n^{th} year is

$g_n = 100{,}000\ (1.025)^n$

$\Rightarrow g_{10} = 128{,}008.45$

6. The money at the end of the n^{th} year (in Birr) is $g_n = 3500\ (1.06)^n$

a. $g_1 = 3{,}710.00$ b. $g_2 = 3{,}932.6$

c. $g_3 = 4{,}168.556$ d. $g_4 = 4{,}418.6\ 6936$

e. $g_n = 3{,}500\ (1.06)^n$ f. Yes, $r = 1.06$

7. a. $A_n = 31{,}100 + (n–1) \times 1200$ b. $B_n = 35{,}100 + (n–1) \times 900$, $B_{11} = 44{,}100$

$A_{11} = 43{,}100$

c. $S_{11} = 408{,}100$ d. $S_{11} = 435600$

e. $\sum_{i=1}^{11} B_i - \sum_{i=1}^{11} A_i = 27{,}500$

8. $A_n = 17 + 3(n–1)$

$S_{38} = (17 + 128) \times \frac{38}{2} = 2{,}755$

9. $A_n = 10{,}000 – 500\ (n–1)$

$A_{18} = 1500$

$S_{18} = \frac{18}{2}\left(2 \times 10{,}000 + (17)(-500)\right) = 103{,}500$

10. $A_n = A_1 – (n – 1) \times 250$

$A_{10} = A_1 – 9 \times 250$

$S_{10} = \frac{10}{2}\left(A_1 + A_{10}\right)$

$\Rightarrow 13{,}250 = 5\ (A_1 + A_1 – 9 \times 250)$

$\Rightarrow 13{,}250 = 5(2A_1 – 9 \times 250)$

$\Rightarrow A_1 = 2{,}450$

11. $g_1 = 20$, $r = \frac{1}{2}$; $g_n = 20\left(\frac{1}{2}\right)^{n-1}$; $g_5 = 20\left(\frac{1}{2}\right)^4 = \frac{20}{16} = 1.25\text{m}$

12. $g_1 = 60$; $r = \frac{1}{3}$

$$S_5 = 60 + 2\left(20 + \frac{20}{3} + \frac{20}{9} + \frac{20}{27}\right) = 119.2592593\text{m}$$

$$S_{10} = 60 + \left(40 + \frac{40}{3} + \frac{40}{9} + ...+ G_9\right)$$

$$\text{Let } H_9 = \left(40 + \frac{40}{3} + \frac{40}{9} + ...+ G_9\right) = 40\frac{\left(\left(\frac{1}{3}\right)^9 - 1\right)}{\frac{1}{3} - 1} = 59.997$$

$\therefore S_{10} = 60 + H_9 = 60 + 59.997 = 119.997$ m.

13. $$S_n = \frac{120\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} - 60 = 180\left(1 - \frac{1}{3^n}\right) - 60$$

$S_\infty = 180 - 60 = 120$

14. Use the binomial series expansion given in students' textbook and you take the same procedure as examples in page34.

a. $$(x-4)^{-7} = \frac{-1}{4^7}\left[1 + -7\left(\frac{-x}{4}\right) + \frac{(-7)(-8)}{2!}\left(\frac{-x}{4}\right)^2 + ...\right]$$

$$= \frac{-1}{4^7}\left[1 + \frac{7}{4}x + \frac{7}{4}x^2 + \frac{21}{16}x^3 + ...\right]$$

The series converges for $|x| < 4$.

b. $$(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{3\times1}{2!2^2}x^2 + \frac{3\times1\times-1}{3!2^3}x^3 + \frac{3\times1\times-1\times-3}{4!2^4}x^4 + ...$$

The series converges for $|x| < 1$.

c. $$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{\frac{-1}{2}}$$

$$= \frac{1}{2}\left(1 + \frac{1}{8}x + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + ...\right)$$

The series converges for $|x| < 4$.

15. Again you use the binomial series expansion given on page 33 in students' textbook and you follow the same procedure as in examples on page35 approximations of square roots, cube roots and so on. You may tell students to use calculators. Hint is given on how to do it is given below.

a. $\sqrt{5} = \sqrt{4+1} = 2\left(1+\frac{1}{4}\right)^{\frac{1}{2}}$ and use binomial series expansion to get approximately equal to 2.24 to the nearest hundredth, 2.236 to the nearest thousandth, and 2.2361 to the nearest ten thousandth.

b. $\sqrt[3]{9} = \sqrt[3]{8+1} = 2\left(1+\frac{1}{8}\right)^{\frac{1}{3}}$ and use binomial series expansion and approximate to the nearest hundredth, thousandth, ten thousandth .

c. $\sqrt[4]{17} = \sqrt[4]{16+1} = 2\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$ and use binomial series expansion and approximate to the nearest hundredth, thousandth, ten thousandth.

Answer to Review Exercises on Unit 1

1. a. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$ b. 0, 1, 4, 9, 16

c. –1, 2, –6, 24, –120 d. $\frac{1}{2}, \frac{5}{7}, \frac{4}{5}, \frac{11}{13}, \frac{7}{8}$

e. $\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{3}, \frac{\sqrt{3}}{8}, \frac{1}{10}$ f. $-\frac{1}{2}, \frac{1}{7}, \frac{1}{2}, \frac{13}{19}, \frac{11}{14}$

2. a. $-2, -\frac{1}{2}, -2, -\frac{1}{2}, -2$ b. $1, 3, 3, 1, \frac{1}{3}$

c. 1, 1, 1, 1, 1 d. 0, 1, 1, 2, 3

3. a. $3n + 1$ b. $4n - 5$

c. $\frac{1}{2}n - 2$ d. $\frac{2n\sqrt{2}}{3}$

4. a. $G_n = (-1)\left(-\frac{1}{4}\right)^{n-1}$ b. $G_3 = \frac{2}{9}$ and $G_5 = \frac{2}{243}$

$$\Rightarrow \frac{G_3}{G_5} = \frac{r^2G_1}{r^4G_1} = \frac{\left(\frac{2}{9}\right)}{\left(\frac{2}{243}\right)} = 27$$

$$\Rightarrow r^2 = \frac{1}{27} \Rightarrow r = \pm\left(\frac{1}{3\sqrt{3}}\right)$$

$$\Rightarrow G_1 = 6$$

$$\Rightarrow G_n = 6\left(\frac{1}{3\sqrt{3}}\right)^{n-1} \text{ or } G_n = 6\left(-\frac{1}{3\sqrt{3}}\right)^{n-1}$$

c. $rG_1 = \frac{\sqrt{2}}{2}$ and $r^3 G_1 = \sqrt{2}$

Taking the ratio, gives $\frac{rG_1}{r^3G_1} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{2}}$

$\Rightarrow \frac{1}{r^2} = \frac{1}{2} \Rightarrow r = \pm\sqrt{2}$

But $rG_1 = \frac{\sqrt{2}}{2} \Rightarrow G_1 = \pm\frac{1}{2}$

$\Rightarrow G_n = \frac{1}{2}\sqrt{2}^{n-1}$

d. $G_1 = 0.15$ and $G_3 = 0.0015$

$\Rightarrow 0.15r^2 = 0.0015 \Rightarrow r^2 = 0.01 \Rightarrow r = \pm 0.1$

$\Rightarrow G_n = 0.15(0.1)^{n-1}$ or $G_n = 0.15(-0.1)^{n-1}$

5. a. 155 b. –342 c. $\frac{28747}{3465}$

d. $\sum_{k=0}^{10}\left(3+(-1)^k\right) = \sum_{k=0}^{10}3 + \sum_{k=0}^{10}(-1)^k = 33+1 = 34$ e. 2732

f. $\sum_{k=1}^{5}\frac{2^{k+5}}{3^{k-1}} = \sum_{k=1}^{5}96\left(\frac{2}{3}\right)^k = \frac{13504}{81}$

g. $\sum_{k=1}^{\infty}\left(\frac{3^k + 2^k}{6^k}\right)^2 = \sum_{k=1}^{\infty}\frac{9^k + 2(6^k) + 4^k}{36^k}$

$$= \sum_{k=1}^{\infty}\left(\frac{1}{4^k}\right) + \sum_{k=1}^{\infty}\frac{2}{6^k} + \sum_{k=1}^{\infty}\frac{1}{9^k}$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} + 2\left(\frac{\frac{1}{6}}{1-\frac{1}{6}}\right) + \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{1}{3} + \frac{2}{5} + \frac{1}{8} = \frac{103}{120}$$

h. $\frac{243}{16}$ i. $\frac{10457207475}{524288}$ j. $\frac{1}{972}$

k. $\frac{44633821}{1048576}$ l. -11715

6. One may start solving this problem by the following oral question.

   What is the smallest whole number which leaves remainder 2 when it is divided by 5?

   Then $A_1 = 2$, $d = 5$

   $\Rightarrow A_n = 5n - 3$

   $A_n < 100 \Rightarrow 5n - 3 < 100$

   $\Rightarrow 5n < 103 \Rightarrow n < 20\frac{3}{5}$

   $\Rightarrow n = 20$

   $\Rightarrow S_{20} = \frac{20}{2}(2 \times 2 + 19 \times 5) = 990$

7. a. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{2}{1+\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{1+\sqrt{2}} = 4 - 2\sqrt{2}$

   b. $\sum_{n=1}^{\infty} 9\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{9}{1-\frac{1}{\sqrt{3}}} = \frac{9\sqrt{3}}{\sqrt{3}-1} = \frac{9}{2}\left(3+\sqrt{3}\right)$

   c. Divergent series

   d. $\begin{cases} \infty, \text{ if } |x| \geq 1 \\ \frac{1}{1-x}, \text{ if } |x| < 1 \end{cases}$

   e. ∞

8. 495

9. $\sum_{i=1}^{20} A_i = 950$ and $\sum_{i=1}^{20} A_{i+20} = 0$

$$\Rightarrow \sum_{i=1}^{40} A_i = 950 + 0 = 950$$

Solving the system $\begin{cases} 10\left(2A_1 + 19d\right) = 950 \\ 20\left(2A_1 + 39d\right) = 950 \end{cases}$

gives $A_1 = \dfrac{1121}{16}$

$d = -\dfrac{19}{8}$

$\Rightarrow A_n = \dfrac{1159}{16} - \dfrac{19}{8}n.$

10. $8 + m_1 + m_2 + \ldots + m_n + 44 = 338 \Rightarrow s_{n+2} = \dfrac{n+2}{2}\left(A_1 + A_{n+2}\right)$

$\dfrac{n+2}{2}\left(8 + 44\right) = 338$

$\Rightarrow n = 11$ and $d = 3$

11. Let the cost of the car at the end of n – years be C_n. Then; $\{C_n\}$ is an arithmetic sequence with $C_1 = 125{,}000 - 4{,}000 = 121{,}000$ and $d = -4{,}000$

Then $C_n = 121{,}000 + (n - 1) \times (-4{,}000)$

$= 125{,}000 - 4{,}000n$

$G_n = \dfrac{75 \times 125{,}000}{100} = 75 \times 1{,}250 = 93{,}750$

$G_n \leq 93{,}750$

$\Rightarrow 12{,}500 - 4{,}000n \leq 75 \times 1{,}250 \Rightarrow n \geq 7.8125$

$\Rightarrow$ It takes 8 years to make a loss of 25% of its value.

12. $A_n = 1{,}000{,}000 + (n - 1) \times 3{,}000{,}000$

$= 700{,}000 + 300{,}000n$

$S_{20} = \dfrac{20}{2}(2{,}000{,}000 + 19 \times 300{,}000) = 77{,}000{,}000$

13. $A_1 = 100{,}000$ and $d = 8{,}000$

$A_n = 100{,}000 + (n - 1) \times 8{,}000$

$S_{30} = \dfrac{30}{2}(200{,}000 + 29 \times 8{,}000)$

$=$ Birr 6,480,000

14. Birr 91,000

15. Let P_n be the amount of money that the n^{th} person earns. Let t_n be the amount of tax the n^{th} person pays.

Then $P_1 = 10{,}000$, $P_2 = (0.6)\,(10{,}000) = 6{,}000$, $P_3 = 0.6(6{,}000) = 3600$

$$\Rightarrow P_n = (0.6)^{n-1} \times 10{,}000$$

$$\Rightarrow t_n = (0.2)\,(0.6)^{n-1} \times 10{,}000$$

$$\Rightarrow \text{The total amount of tax} = \text{Birr } \frac{t_1}{1-r} = \text{Birr } \frac{2{,}000}{1-0.6} = \text{Birr } 5{,}000$$

16. a. $$(9+x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{x}{9}\right)^{\frac{1}{2}} = 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

$$= 3\left[1+\frac{1}{2}\left(\frac{x}{9}\right)+\frac{1}{2}\,\frac{\left(\frac{1}{2}-1\right)\left(\frac{x}{9}\right)^2}{2!}+\frac{1}{2}\,\frac{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{9}\right)^3+\ldots\right]$$

The series conveges provided that $|x| < 9$

b. $$(1+5x)^{-\frac{5}{2}} = 1+\left(\frac{-5}{2}\right)(5x)+\frac{\frac{(-5)}{2}\left(\frac{-5}{2}-1\right)(5x)^2}{2!}+\frac{\left(\frac{-5}{2}\right)\left(\frac{-5}{2}-1\right)\left(\frac{-5}{2}-2\right)(5x)^3}{3!}+\ldots$$

The series converges provided that $|x| = \frac{1}{5}$.

c. Replace n *by* $\frac{3}{2}$ and x by $\frac{-x}{2}$ in the binomial series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots.$$

to get $(2-x)^{\frac{3}{2}} = \sqrt{8}\left(1-\frac{x}{2}\right)^{\frac{3}{2}} = \sqrt{8}\left[1-\frac{3}{4}x+\frac{3}{32}x^2+\frac{1}{128}x^3+\ldots\right]$ and it converges for $|x| < 2$.

UNIT 2 INTRODUCTION TO LIMITS AND CONTINUITY

INTRODUCTION

In this unit, there are three major outcomes. The first major outcome is to enable students to understand the notion of limits and continuity.

The second one is to enable students to solve practical and real world life problems.

Therefore, they need to know how to use limits to study rates of changes to approximate, and determine areas and many other things.

The third one constitutes the concepts, limits and continuity is fundamental for the coming units, namely, **differential calculus and integral calculus**.

This unit will present each topic clearly by producing illustrative examples and graphical descriptions.

The unit is presented in four sub-units. In each sub-unit there are subtopics.

The subtopics begin with opening problems and activity exercises which are believed to open class room discussion and introduce unfamiliar terms.

There are also calculator works to explore limits of functions and approximate real roots of continuous functions.

The exercises that are in the textbook are to be used for continuous assessment depending on the level of the knowledge of the students.

Unit Outcomes

After completing this unit, students will be able to:

- *understand the concept "limit" intuitively.*
- *find out the limit of a number sequence.*
- *determine the limit of a given function.*
- *determine continuity of a function over a given interval.*
- *apply the concept of limits to solve real life mathematical problems.*
- *develop a suitable ground for dealing with differential and integral calculus.*

Suggested Teaching Aids in Unit 2

2.1 Charts on graphs of

- ✓ Convergent /divergent sequences
- ✓ Monotonic sequences
- ✓ Bounded sequences

2.2 Charts on graphs of functions displaying

- ✓ Limit of a function at a point and functional value
- ✓ One side limits
- ✓ Limit at infinity
- ✓ Infinite limits and asymptotes

2.3 Charts on graphs of

- ✓ A function continuous at a point
- ✓ Discontinuous at a point when
  - $f(a)$ is undefined
  - $\lim_{x \to a} f(x)$ doesn't exist.
  - $\lim_{x \to a} f(x) \neq f(a)$
- ✓ The intermediate value theorem
- ✓ The extreme value theorem

2.1 LIMITS OF SEQUENCES OF NUMBERS

Periods Allotted: 12 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *define upper and lower bound of number sequences.*
- *find out the least upper (greatest lower) bound of sequences.*
- *define limit of a number sequence.*
- *consolidate their knowledge on the concept of sequences stressing on the concept of null sequence.*
- *apply theorems on the convergence of bounded sequences.*
- *prove theorems about the limit of the sum of two convergent sequences.*
- *apply theorems on the limit of the difference, product, quotient of two convergent sequences.*

Vocabulary: Sequences of numbers, Upper bounds, Lower bounds, Least upper bound (lub), Greatest lower bound (glb), Limits, Monotonic sequences, Convergence

Introduction

This sub-unit deals with manipulating the limits of sequence of numbers. There are different techniques and properties as discussed in the textbook. Students must know that a monotone and bounded sequence converges but this fact doesn't tell the value of the limit. The advantage of this fact is that, once the existence of the limit is known, then students should attempt to evaluate it. This sub-unit is again divided into three sub-topics.

Teaching Notes

- ✓ Give different exercise problems including Activity 2.1 on finding minimum and maximum elements of given sets.
- ✓ Define terms such as: least upper bound, greatest lower bound of number sequences, increasing and decreasing sequences.
- ✓ Give illustrative examples on how to find the least upper bound and the greatest lower bound.
- ✓ Discuss the concept of "limit of a sequence" by using simple and appropriate examples such as

$$\lim_{n\to\infty} \frac{1}{n},\ \lim_{n\to\infty}\left(8-\frac{1}{n}\right).$$

- ✓ Introduce the concept of "null sequence" with the help of examples such as.

$$\left\{\frac{1}{n}\right\},\left\{\frac{100}{n^2+1}\right\},\left\{\frac{1}{2^n}\right\}.$$

- ✓ Discuss the convergence of monotonic sequences and theorems on convergence of bounded and monotonic sequences.

Discuss the properties of convergence of monotonic sequences using appropriate examples.

Before you start teaching limits of sequences of numbers, discuss the opening problem with the class or group. This will help you to assess the students prior knowledge in convergence of sequences. It also helps to indicate the type of activities that will interest and motivate students.

Answers to the Opening Problem

a. As n tends to infinity, almost every point on the circle is becoming a vertex of the polygon so that, the lengths of each side of the polygon approaches to 0.

b. The polygon is approaching the circle.

c. The polygon will never reach the circle.

Activity 2.1 is designed to introduce upper bounds and lower bounds of number sequence based on the students prior knowledge of the minimum and maximum elements of a set.

Answer to Activity 2.1

This activity can be given as group work/or in pairs.

Students, within a group may first do the activity individually and then discuss with a partner.

1. The problems are designed to introduce the concepts boundedness, least upper bound (lub) and greatest lower bound (glb).

   The problems can be given as oral questions.

	a	b	c	d	e	f	g
Minimum element	1	–1	–3	No	–1	No	No
Maximum element	10	1	No	1	2	4	No

2. There are several values for *m* and *k*.

   Students are expected to estimate the least value of *m* and the largest value of *k*.

   Here, you can ask some students to write and explain the values of *m* and *k* on the black board.

a. $2^n + 1 \geq 3$

b. $0 < \frac{1}{3^n} \leq \frac{1}{3}$

c. $-2 \leq (-1)^n \left(1+\frac{1}{n}\right) \leq 1.5$

d. $1 < \frac{n+1}{n} \leq 2$

e. $7 < 7 + \frac{1}{n} \leq 8$

f. $0.9 \leq \frac{10^n - 1}{10^n} < 1$

2.1.1 Upper Bounds and Lower Bounds

After giving solutions to Activity 2.1, encourage the students to reach the definition of upper bounds and lower bounds.

Next, discuss the examples that have been given with the class or group. After defining lub and glb and exhaustively discussing the worked examples, give the students Exercise 2.1 as class work or homework. Here, you should ask some students to list a few upper bounds and a few lower bounds either orally or on the blackboard. Some of the problems like questions 8,9 and 10 can be given as group work. You can form the group on the basis of similar ability, mixed ability, gender or other social factors.

In doing these exercises, students must be able either to describe the first few terms of the sequence and decide lub, and glb from its trend or describe it graphically. In particular the rational expressions need to be described graphically. For those problems which are beyond mental calculation, students need to use calculators.

Additional exercise problems for high ability students

1. Plot the point $\left(n, \frac{n}{2^n}\right)$ in the coordinate system and demonstrate to the class that the sequence $\left(\frac{n}{2^n}\right)$ has lub $\frac{1}{2}$ and glb 0 which are $\frac{1}{2}$ and 0 respectively.

   Also, $\lim\limits_{n\to\infty}\left(\frac{n}{2^n}\right)=0$.

2. Draw the graphs of $f(x) = nx^2 + (n + 1)x + 1$ for different values of n,
   For n =1, 2, 3, … and demonstrate to the class that as n tends to infinity, the axis of the parabola converges to the line $x=-\frac{1}{2}$ and one of the zeros converges to 0

Solution:

1.

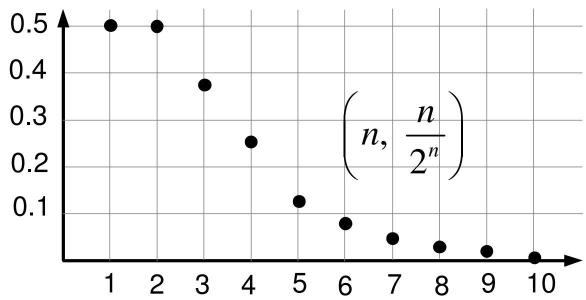


Figure 2.1

2. The zeros of $f(x) = nx^2 + (n + 1)x + 1$ are $-1, \frac{-1}{n}$.

   As n tends to infinity $\frac{-1}{n}$ tends to 0.

   The axis of the parabola is $x=\frac{-b}{2a} = \frac{-(n+1)}{2n}$

   As n tends to infinity, $\frac{-(n+1)}{2n}$ tends to $-\frac{1}{2}$

   $\Rightarrow$ the axis tends to $x=-\frac{1}{2}$

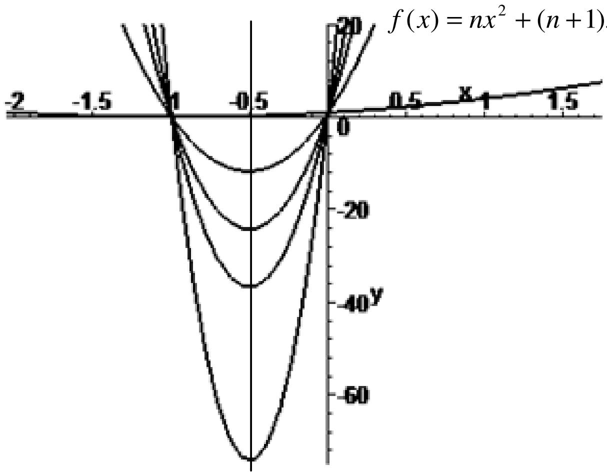


Figure 2.2

Assessment

In order to assess the understanding of your students, you can ask oral questions on definition of upper bound, lower bound of sequences. You can also use the examples to assess students on the determination of lub and glb of sequences of numbers. Depending on the situation, it is also possible to give some of the exercises from the student text as homework and class work to determine the limit of a sequence and check solutions.

For example, find $\lim\limits_{n\to\infty} \dfrac{6n-5}{2n+4}$

You can also ask students to determine whether or not a given sequence is null sequence, monotonic sequence, convergent or divergent sequence.

For example, which of the following sequences are

a. Monotonic b. Convergent c. Null

$$\left\{2-\frac{1}{n}\right\}, \left\{\frac{n}{n+1}\right\}, \left\{\left(\frac{-1}{5}\right)^n\right\}, \left\{1-\left(\frac{1}{n}\right)\right\}$$

$\{2^n\}$ etc.

You may also give various exercises problems regarding the application of the theorems on finding limits of differences, products, quotients as class work and homework and check solutions.

Answer to Exercise 2.1

The following table contains the sets of upper bounds and lower bounds but students are expected to list a few of the numbers.

	The set of upper bounds	**The set of lower bounds**	**lub**	**glb**
1	$\left[\frac{1}{5}, \infty\right)$	$\left(-\infty, -\frac{1}{4}\right]$	$\frac{1}{5}$	$-\frac{1}{4}$
2	$[1, \infty)$	$(-\infty, 0]$	1	0
3	$[3, \infty)$	$(-\infty, 1]$	3	1
4	$(-1)^n\left(1-\frac{1}{n}\right) = \begin{cases} 1-\frac{1}{n}, & \text{if } n \text{ is even} \\ \frac{1}{n}-1, & \text{if } n \text{ is odd} \end{cases}$ $[1, \infty)$	$(-\infty, -1]$	1	-1
5	$\left[-\frac{2}{7}, \infty\right)$	$\left(-\infty, \frac{-3}{2}\right]$	$-\frac{2}{7}$	$\frac{-3}{2}$
6	$\left[\sqrt{2}, \infty\right)$	$(-\infty, -2]$	$\sqrt{2}$	-2
7	$\left[\frac{5}{2}, \infty\right)$	$(-\infty, -4]$	$\frac{5}{2}$	-4

8. In this problem, students must use calculators or computers.

   The first few terms are (approximated)

   1, 1.414, 1.442, 1.414, 1.379, 1.348, 1.32

   lub = $\sqrt[3]{3} \approx 1.442$

   glb = 1

9. $$\frac{n!}{n^n} = \frac{n\,.\,(n-1)\,(n-2)\,(n-3)\,.\,.\,.\,.\,1}{n\,.\,n\,.\,n\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,n}$$

   $$\Rightarrow\quad 0 < 1 \times \frac{n-1}{n}\,.\,\frac{n-2}{n}\,.\,\frac{n-3}{n}\,.\,.\,.\,.\,.\,\frac{1}{n} \le 1$$

   The terms are: $\frac{1}{1}, \frac{2\,.\,1}{2 \times 2} = \frac{1}{2}, \frac{3.\,2.\,1}{3.\,3.\,3} = \frac{2}{9}, \frac{4.3.2.1}{4.4.4.4} = \frac{3}{32}$

   The set of lower bounds is $(-\infty, 0]$ and the set of upper bounds is $[1, \infty)$.

   $\Rightarrow$ lub = 1, glb = 0

10. The terms of the sequence are

    $2, 2, \frac{4}{3}, \frac{2}{3}$

    Notice that, if $a_n = \frac{2^n}{n!}$, then $a_n = \frac{2}{n}a_{n-1}$

From this we obtain,

$(-\infty,\ 0]$ and $[2,\ \infty)$ are the set of lower bounds and the set of upper bounds respectively.

$\Rightarrow$ lub = 2, glb = 0

Monotonic Sequences

Using the concept of increasing and decreasing functions, let students try to give the definitions of increasing and decreasing sequences.

You may first list some functions on the black board for example, write

$f_n = n,\ f_n = \dfrac{1}{n},\ f_n = \dfrac{(-1)^n}{n}$ and ask the students to identify the functions which are either increasing or decreasing.

Next discuss the worked examples with the class or group and give Exercise 2.2 as class work or home work.

You should choose a balanced type of exercise problems according to the students' ability.

Answers to Exercise 2.2

1. A graphical description is the most important tool for the sequences in a, b, e and f.

The sequences in a, b, c, and d are decreasing where as those in e and f are increasing. Problems 2, 3 and 4 may be given as class discussion. You may use these problems to motivate students to produce as many examples as possible.

2. Consider the following sequence to open the class room discussion.

   $\left\{\dfrac{(-1)^n}{n}\right\}$

3. $\{(-1)^n\}$

4. Never

5. Only the sequences, a. $\left\{n + \dfrac{1}{n}\right\}$ and h. $\left\{\ln\left(\dfrac{1}{n}\right)\right\}$ are not bounded

You can describe both of them graphically. Use the graphs of $y = x + \dfrac{1}{x}$ for $n + \dfrac{1}{n}$

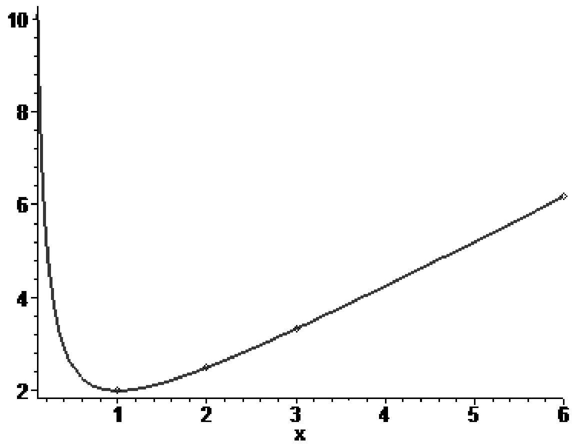


Figure 2.3

Use the graph of $y = \ln\left(\dfrac{1}{x}\right) = -\ln x$ for $\ln\left(\dfrac{1}{n}\right)$

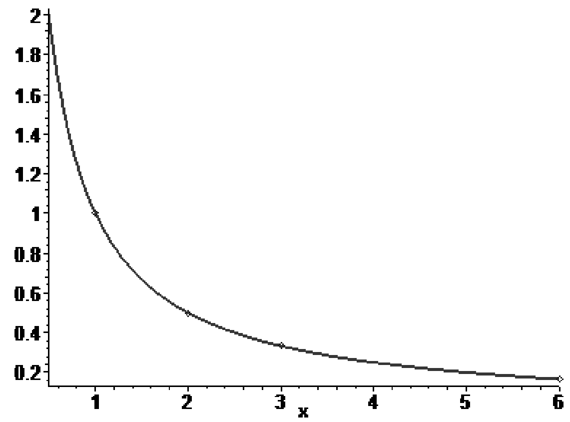


Figure 2.4

6. Use the problems to summarize the concept of convergence. The sequences converge to

a. 3 b. $\dfrac{2}{3}$ c. 0

d. Students should be able to recall that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Thus, $\dfrac{1 + 3 + 5 + \ldots + (2n-1)}{6n^2+1} = \dfrac{n^2}{6n^2+1}$

$\Rightarrow$ the sequence is converging to $\dfrac{1}{6}$.

e. $\dfrac{2^{n+1}}{5^{n-4}} = \dfrac{2^n \times 2}{5^n \times 5^{-4}} = 5^4 \times 2\left(\dfrac{2}{5}\right)^n$ converges to 0.

f. 0 g. 0 h. 1

2.1.2 Limits of Sequences

To start the lesson, discuss the opening problem with the class. Next, discuss Activity 2.2 orally. Here, the discussion should be directed to finding the limit of a sequence. In teaching this subtopic, draw graphs of convergent sequences or use a prepared wall chart.

After discussing the worked examples with the class or group, give Exercise 2.3 as class work or home work.

In doing this exercise, let the students present the solutions on the black board. Check the solutions and identify the students that need extra help.

Answers to the Opening Problem

1. $\frac{1}{n} = \frac{1}{101}, \frac{1}{102}, \frac{1}{103}, \ldots, \frac{1}{100+n}$
2. $N = 100,001$

Answers to Activity 2.2

The problems are expected to be oral questions.

Only $\{(-1)^n\}$ and $\{2^n\}$ do not converge. The rest of the sequences converge to

1. 0	2. 0	3. 4
4. 0	5. 0	6. 1

Answers to Exercise 2.3

This exercise requires the knowledge of convergence of sequences and the properties of some rational functions and simple trigonometric functions.

1. a. 0 b. 0 c. 0
   d. 1 e. 1 f. 1
   g. $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$ because as n – tends to infinity, still $-1 \le \cos(n) \le 1$
   h. $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$ because as n – tends to infinity $\frac{1}{n}$ tends to 0 so that $\cos\left(\frac{1}{n}\right)$ tends to cos (0) = 1.
   i. $\lim_{x\to\infty}\left(n+\frac{1}{n}\right) = \infty$ j. $\lim_{n\to\infty}\left(\frac{1+n}{2+n}\right) = 1$
   k. does not exist l. $-\frac{1}{2}$ m. $\lim_{n\to\infty}\left(n-\frac{10}{n}\right) = \infty$

n. $(-1)^n \dfrac{(n-1)}{n+1} = (-1)^n \left(1-\dfrac{2}{n+1}\right)$ as n tends to infinity, $1-\dfrac{2}{n+1}$ tends to 1 so that $(-1)^n \left(1-\dfrac{2}{n+1}\right)$ tends to $\pm$ 1.Thus, the limit doesn't exist.

o. $0.\overline{6}$ or $\dfrac{2}{3}$

2. Question 2 could be easier, if you remind the students to recall the properties of rational functions and geometric sequences.

a.	null	b.	not	c.	null	d.	null	e.	null
f.	not	g.	null	h.	not	i.	null		

2.1.3 Convergence Properties of Sequences

You can start the lesson by asking a slow learner student to answer question 1 in Activity 2.3.

Next ask any student or group of students to demonstrate the solution of question 2 on the black board either graphically or algebraically.

Following Activity 2.3, discuss the worked example and state Theorem 2.1.

After discussing the worked examples give Exercise 2.4 as class work or homework according to the student's ability. Here, you should assess the appropriate strategies they are using to answer each question.

Check the solutions and identify which students need extra help and which students should be left alone.

Answers to Activity 2.3

This is an activity for students to reach the following two conclusions.

i. A bounded monotonic sequence converges.
ii. An unbounded monotonic sequence diverges.

Also, use this activity to introduce the basic limit theorems.

1. a. 5 (bounded) b. $-\infty$ (not bounded)
   c. 0 (bounded) d. 0 (bounded)
   e. $\nexists$ (doesn't exist) bounded f. ∞ (not bounded)
2. a. Bounded and decreasing and $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)=1$ which is the glb.
   b. Bounded and increasing and $\lim\limits_{n\to\infty}\left(3-\dfrac{2}{n}\right) = 3$ that is the lub.
   c. Unbounded but decreasing.

d. Bounded and decreasing with $\lim\limits_{n\to\infty} 2^{1-n} = \lim\limits_{n\to\infty} \frac{2}{2^n} = 0$ that is the glb.

e. Bounded and decreasing with $\lim\limits_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$ the glb.

f. Unbounded but decreasing.

Answers to Exercise 2.4

Most of these limits can be found at a glance. Some of them need mental calculation. Students should be able to use the properties of convergence of sequences. In this exercise, the solution of question 19, needs a real mathematical thinking. It will be best if it is given as group work.

1. $\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{3}{n+2}\right)=\lim\limits_{n\to\infty}\frac{1}{n}+\lim\limits_{n\to\infty}\frac{3}{n+2}=0+0=0$

2. $\lim\limits_{n\to\infty}\left(\frac{3^n+2^n}{6^n}\right)=\lim\limits_{n\to\infty}\frac{3^n}{6^n}+\lim\limits_{n\to\infty}\frac{2^n}{6^n}$

$$=\lim\limits_{n\to\infty}\left(\frac{1}{2^n}+\lim\limits_{n\to\infty}\frac{1}{3^n}\right)=0+0=0$$

3. $\lim\limits_{n\to\infty}\sqrt{3}^n = \lim\limits_{n\to\infty}3^{\frac{n}{2}}=\infty$

4. $\lim\limits_{n\to\infty}\frac{25}{n+10}=\lim\limits_{n\to\infty}\frac{\frac{25}{n}}{1+\frac{10}{n}}=\frac{\lim\limits_{n\to\infty}\frac{25}{n}}{1+\lim\limits_{n\to\infty}\frac{10}{n}}=\frac{0}{1+0}=0$

5. $\lim\limits_{n\to\infty}\frac{n^2+1}{30n+100}=\infty$

6. $\lim\limits_{n\to\infty}\left(\frac{1+n+n^2}{n}\right)=\lim\limits_{n\to\infty}\left(\frac{1}{n}+1+n\right)=0+1+\infty=\infty$

7. $\lim\limits_{n\to\infty}\left(\frac{-3}{5}\right)^n=0$ because $-1<\frac{-3}{5}<1$.

8. $\lim\limits_{n\to\infty}\left(20+\left(-\frac{1}{3}\right)^n\right)=20+\lim\limits_{n\to\infty}\left(-\frac{1}{3}\right)^n=20+0=20$

9. $\lim\limits_{n\to\infty}\left(\left(\frac{1}{3}\right)^n-n\right)=\lim\limits_{n\to\infty}\left(\frac{1}{3}\right)^n-\lim\limits_{n\to\infty}n=0-\infty=-\infty$

10. $\lim\limits_{n\to\infty}\frac{(3n+1)^2}{2n^2+3n+1}=\lim\limits_{n\to\infty}\frac{9n^2+6n+1}{2n^2+3n+1}$

$$= \lim_{n\to\infty} \frac{\left(\frac{9n^2+6n+1}{n^2}\right)}{\left(\frac{2n^2+3n+1}{n^2}\right)} = \lim_{n\to\infty} \left(\frac{9 + \frac{6}{n} + \frac{1}{n^2}}{2 + \frac{3}{n} + \frac{1}{n^2}}\right) = \frac{9}{2}$$

11. $$\lim_{n\to\infty} \frac{\sqrt{n^2+5}}{n+1} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^2+5}}{n}\right)}{\left(\frac{n+1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^2+5}{n^2}}}{1 + \frac{1}{n}} = \frac{\lim\limits_{n\to\infty} \sqrt{1 + \frac{5}{n^2}}}{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)} = 1$$

12. $$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right)\times\left(\frac{5n-2}{6n+1}\right) = \lim_{n\to\infty} \frac{2n+3}{2n+5} \times \lim_{n\to\infty} \frac{5n-2}{6n+1}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2n+3}{n}\right)}{\left(\frac{2n+5}{n}\right)} \lim_{n\to\infty} \frac{\left(\frac{5n-2}{n}\right)}{\left(\frac{6n+1}{n}\right)}$$

$$= \frac{\lim\limits_{n\to\infty} \left(2 + \frac{3}{n}\right)}{\lim\limits_{n\to\infty} \left(2 + \frac{5}{n}\right)} \times \frac{\lim\limits_{n\to\infty} \left(5 - \frac{2}{n}\right)}{\lim\limits_{n\to\infty} \left(6 + \frac{1}{n}\right)}$$

$$= \frac{2}{2} \times \frac{5}{6} = \frac{5}{6}$$

13. $$\lim_{n\to\infty}\left(\frac{1 + 2^2+3^2+\ldots+n^2}{n^3}\right) = \lim_{n\to\infty}\frac{n\ (n+1)\ (2n+1)}{6n^3} = \frac{1}{3}$$

14. $$\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = 0$$

15. $$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}+1}\right) = 0$$

16. Rationalizing the denominator gives,

$$\lim_{n\to\infty} \frac{n+3}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{(n+3)\left(\sqrt{n}-1\right)}{n-1} = \lim_{n\to\infty} \frac{n\sqrt{n}-n+3\sqrt{n}-3}{n-1}$$

$$= \lim_{n\to\infty} \frac{\left(\dfrac{n\sqrt{n} - n + 3\sqrt{n} - 3}{n}\right)}{\left(\dfrac{n-1}{n}\right)}$$

$$= \frac{\lim\limits_{n\to\infty}\left(\sqrt{n} - 1 + 3\dfrac{\sqrt{n}}{n} \dfrac{-3}{n}\right)}{\lim\limits_{n\to\infty}\left(1 - \dfrac{1}{n}\right)} = \frac{\infty - 1 + 0 - 0}{1 - 0} = \infty$$

17. $\lim\limits_{n\to\infty} \left(\dfrac{1}{2}\right)^{1-\frac{1}{2n}} = \lim\limits_{n\to\infty} 2^{\frac{1}{2n}-1} = \lim\limits_{n\to\infty} \dfrac{\sqrt{2}^{\frac{1}{n}}}{2} = \dfrac{1}{2}$

18. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}-3}{n+2} = \lim\limits_{n\to\infty} \dfrac{\left(\dfrac{\sqrt{n^2+1}-3}{n}\right)}{\left(\dfrac{n+2}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{\left(\dfrac{\sqrt{n^2+1}}{n} - \dfrac{3}{n}\right)}{1 + \dfrac{2}{n}} = 1$

19. This is to show that the converses of the properties of convergence of sequences do not hold true.

   a. Consider $a_n = n + \dfrac{1}{n}$ and $b_n = 2 - n$, then

   $a_n + b_n = n + \dfrac{1}{n} + 2 - n = 2 + \dfrac{1}{n}$ which converges to 2.

   But neither $n + \dfrac{1}{n}$, nor $2 - n$ converges.

   b. Consider similar sequences such as

   $a_n = (-1)^n$, $b_n = \dfrac{(-1)^n}{2}$

   $\Rightarrow a_nb_n = (-1)^n \cdot \dfrac{(-1)^n}{2} = \dfrac{(-1)^{2n}}{2} = \dfrac{\left((-1)^2\right)^n}{2} = \dfrac{1}{2}$

   $\Rightarrow \lim\limits_{n\to\infty}\left(a_n b_n\right) = \dfrac{1}{2}$. But neither $(-1)^n$ nor $\dfrac{(-1)^n}{2}$ converges

20. This is an example to show that both $\lim\limits_{n\to\infty} 2^n$ and $\lim\limits_{n\to\infty} n!$ do not exist but the quotient $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$, exists.

   This can be shown as $\dfrac{2^n}{n!} = \dfrac{2}{n} \times \dfrac{2}{n-1} \times \dfrac{2}{n-2} \times \ldots \times \dfrac{2}{2} \times \dfrac{2}{1}$

2.2 LIMITS OF FUNCTIONS

Periods Allotted: 6 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *define limit of a function.*
- *determine the limit of a given function at a point.*
- *find out the limit of the sum, difference, product and quotient of two functions.*

Vocabulary: Limit of a function at a point, Limit value, Functional value, Basic limit theorem, Limit at infinity, None-existence of limits, One side limits, Right hand limit, Left hand limits, Two side limits

Introduction

This topic includes the definition and calculation of various limits of functions by simplifications and by graphical displays.

The basic limit theorems have important connection with the convergence properties of sequences of numbers. In addition, one side limits, limits at infinity and infinite limits are presented carefully.

After the students gain the necessary skill with limits of functions, application problems in daily life are discussed.

Teaching Notes

Teaching this sub-unit may require graphical demonstration knowledge of the informal definition of limit of a function at a point. This knowledge may extend to one side limits, limits at infinity and infinite limits.

Here are some teaching strategies you might want to use:

✓Discuss the behavior of certain functions in an open interval about a point (not possibly containing the point) using graphs or calculators.

✓Illustrate how to use the definition of limit for determining the limit of a given function using different examples based on the intuitive definition of a limit.

For example, $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$, $\lim\limits_{x\to 1}\left(x^2+5x+1\right)$, $\lim\limits_{x\to 1}\dfrac{1}{x-1}$

✓Help students to use calculators to verify and estimate limits.

✓Discuss the basic limit theorems with the help of examples.

✓Discuss the different techniques of evaluating limits such as factorizing expressions, rationalizing expressions, using substitutions, etc.

To start the lesson, discuss Activity 2.4 orally. Then guide the students to give the intuitive definition of limit of a function at a point.

Use a prepared wall chart or draw a graph on the board when you discuss the definitions, activities and the worked examples.

Give Exercise 2.5 as class work or home work to assess the competencies in limits of functions at a point.

Before proceeding to the basic limit theorems, students should discuss question 16.

After stating the basic limit theorems, discuss the worked examples. After instruction give Exercise 2.6 as class work or home work.

Here, questions 1, 2 and 3 are not difficult. You can therefore assign slow learners to demonstrate the solutions on the blackboard.

Question 4 can be given as a group work.

Some of the worked examples have difficult nature. Problems of the following types should be discussed with fast learner students.

1. $\lim_{x\to 2} \dfrac{\sqrt{1+\sqrt{1+x}}-\sqrt{3}}{x-2}$. 2. $\lim_{x\to 4} \dfrac{\sqrt{4+\sqrt{1+x}}-\sqrt{3}}{x-4}$

For slow learner students, select the following types of problems.

1. $\lim_{x\to 2} \dfrac{x-4}{\sqrt{x}-2}$ 2. $\lim_{x\to \sqrt{7}} \dfrac{x^2-7}{x+\sqrt{7}}$

Limits at Infinity

In teaching limits at infinity, ask the students to do Activity 2.6 individually and demonstrate the solutions on the board.

Next, define limits at infinity and discuss the examples that have been worked out.

Non-Existence of Limits

Students have seen different conditions in which a limit fails to exist. The example $f(x)=\sin\left(\dfrac{\pi}{x}\right)$ discussed in the textbook is used to show that $\lim_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$ doesn't exists.

Ask students to draw the graph of f on selected intervals so that they can predict its nature near the origin.

One Side Limits

In teaching one side limits, encourage students to draw graphs of functions. Activity 2.7 is designed to introduce one side limit.

Discuss the definitions and the worked examples with the class or group.

Infinite limits and vertical asymptotes are also discussed in this subtopic.

Ask students to demonstrate the vertical asymptotes of graphs of different functions which they know other than rational functions.

For example, ask some students to sketch the graphs of the reciprocal functions, sec x, csc x, cot x, $\frac{1}{e^x - 1}$, on the blackboard.

Finally, give students Exercise 2.7 as class work or home work.

Assessment

You can use any one of the following in order to assess the understanding of the students.

- ✓ Asking oral questions to repeat the informal definition of limit of a function at a point.
- ✓ During instruction, use the examples to assist students how they are evaluating limits.
- ✓ Ask students to repeat the theorems in their own words.
- ✓ Give the exercises as homework or class work and check the solutions.

Answers to Activity 2.4

1. i. $\mathbb{R} \setminus \{-2, -1, 3\}$

   ii. $f(-2), f(-1)$ and $f(3)$ are undefined $f(2) = \frac{13}{3}$, $f(4) = 1$

   iii. a. 0 b. 1 c. 2 d. 2

   e. 3 f. $\frac{13}{3}$ g. 4 h. -2

   i. 1 j. $-\infty$

2. Both $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

Answers to Activity 2.5

1. $\lim\limits_{x\to a} f(x)$ is the number $f(x)$ approaching, as x-approaches a, but $f(a)$ is the value of $f(x)$ exactly at $x = a$.

2. Students are expected to produce several examples. You may start the discussion by giving $f(x) = \frac{|x|}{x}$ and show that $\lim_{x \to 0} \frac{|x|}{x}$ doesn't exist.

3. The limit $\lim_{x \to a} f(x)$ exists and it is $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

4. a. 2 b. 3.5 c. doesn't exist d. 3.5
   e. 2 f. 3 g. 4 h. 4

In this Exercise questions 5, 8, 9, 10, 11 and 12 can be given as group work. Also, the abilities in factorization, rationalization and graphing are essential. For some of the problems they need to use calculators.

Answers to Exercise 2.5

1. 27
2. 0
3. doesn't exist
4. 1
5. doesn't exist. Here ask students to sketch the graphs on a paper and show to the class during discussion.
6. $\lim_{x \to 1} \frac{x-1}{x^2+x-2} = \lim_{x \to 1} \frac{x-1}{(x+2)(x-1)} = \lim_{x \to 1} \frac{1}{x+2} = \frac{1}{3}$
7. $\lim_{x \to 2} \frac{x-2}{x^2-x-2} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+1)} = \lim_{x \to 2} \frac{1}{x+1} = \frac{1}{3}$
8. $\lim_{x \to -3} \frac{x^3+27}{x+3} = \lim_{x \to -3} \frac{(x+3)(x^2-3x+9)}{x+3}$

   $= \lim_{x \to -3} (x^2-3x+9) = 27$
9. $\lim_{x \to 1} \frac{x^4-1}{x^6-1} = \lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{(x^2-1)(x^4+x^2+1)} = \lim_{x \to 1} \frac{x^2+1}{x^4+x^2+1} = \frac{2}{3}$
10. Question 10 needs substitution as shown below.

    Factorizing $x-1 = \left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$ is also possible. But it is not advisable for this exercise.

    Let $y = \sqrt[3]{x}$, then $x = y^3$ and as $x \to 1$, $y \to 1$.

Therefore, $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x-1} = \lim\limits_{y\to 1}\dfrac{y-1}{y^3-1}$

$$= \lim_{y\to 1}\frac{y-1}{(y-1)\left(y^2+y+1\right)} = \lim_{y\to 1}\frac{1}{y^2+y+1} = \frac{1}{3}$$

11. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)} = \lim\limits_{x\to 4}\dfrac{1}{\sqrt{x}+2} = \dfrac{1}{4}$

12. Define $\dfrac{x-4|x|}{x} = \begin{cases} \dfrac{x-4x}{x} = -3, \text{ if } x > 0 \\ \dfrac{x-(-4x)}{x} = 5, \text{ if } x < 0 \end{cases}$

Then $\lim\limits_{x\to 0^+}\dfrac{x-4|x|}{x} = -3 \neq \lim\limits_{x\to 0^-}\dfrac{x-4|x|}{x} = 5$

$\therefore \lim\limits_{x\to 0}\dfrac{x-4|x|}{x}$ doesn't exist.

13. $\lim\limits_{x\to 5}\dfrac{5x-x^2}{x-5} = \lim\limits_{x\to 5}\dfrac{x(5-x)}{-(5-x)} = -\lim\limits_{x\to 5} x = -5$

14. $\lim\limits_{x\to 0^+}\dfrac{x^3}{|x|} = \lim\limits_{x\to 0^+}\dfrac{x^3}{x} = \lim\limits_{x\to 0^+} x^2 = 0$

and $\lim\limits_{x\to 0^-}\dfrac{x^3}{|x|} = \lim\limits_{x\to 0^-}\dfrac{x^3}{-x} = -\lim x^2 = 0$

$\therefore \lim\limits_{x\to 0}\dfrac{x^3}{|x|} = 0$

15. $\lim\limits_{x\to -2}\dfrac{x^2-5x-14}{x^2-4} = \lim\limits_{x\to -2}\dfrac{(x-7)(x+2)}{(x-2)(x+2)} = \lim\limits_{x\to -2}\dfrac{x-7}{x-2} = \dfrac{9}{4}$

16. The group work is designed to help students to understand or explore the basic limit theorems.

Answers to Exercise 2.6

1. It can be given as oral question to summarize one side limits

a.	5	b.	2	c.	doesn't exist
d.	4	e.	2.5	f.	doesn't exist

2.

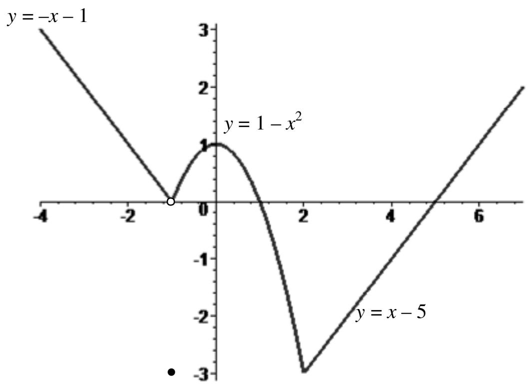


Figure 2.5

a. 0 b. –3 c. 0 d. –2

3. a. 3 b. $-\frac{149}{5}$ c. doesn't exist

4. This exercise can be given to students to work in groups.

a. $$\lim_{x\to 3}\frac{x-3}{\sqrt{x^2-6x+9}} = \lim_{x\to 3}\frac{x-3}{\sqrt{(x-3)^2}}$$

$$=\lim_{x\to 3}\frac{x-3}{|x-3|} \text{ doesn't exist.}$$

This can be shown by drawing the graph or by computing the one side limits.

b. $$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}\cdot\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$$

$$= \lim_{x\to 0}\frac{x^2+1-1}{x^2\left(\sqrt{x^2+1}+1\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+1}+1}=\frac{1}{2}$$

c. Doesn't exist

d. $$\lim_{x\to -2}\frac{x^3+8}{x+2}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{x+2} = \lim_{x\to -2}\left(x^2-2x+4\right)=12$$

e. $$\lim_{x\to 0^+}\frac{x^3}{|x|+x} = \lim_{x\to 0^+}\frac{x^3}{2x} = \lim_{x\to 0^+}\frac{x^2}{2} = 0$$

$$\lim_{x\to 0^-}\frac{x^3}{|x|+x} = \lim_{x\to 0^-}\frac{x^3}{x-x} \text{ which is undefined,}$$

$$\therefore \lim_{x\to 0^+}\frac{x^3}{|x|+x} \text{ doesn't exist}$$

f. $\lim\limits_{x\to -5} \dfrac{x^2+x-20}{x^2+4x-5} = \lim\limits_{x\to -5} \dfrac{(x+5)\,(x-4)}{(x+5)\,(x-1)}$

$= \lim\limits_{x\to -5} \dfrac{x-4}{x-1} = \dfrac{-9}{-6} = \dfrac{3}{2}$

g. Using direct substitution $\lim\limits_{x\to 0} \dfrac{\sin\ x\ +\ 1}{x+\cos\ x} = \dfrac{\sin\ 0\ +\ 1}{0\ +\ \cos\ 0} = 1$

h. $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2} = \lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}+\sqrt{2}\right)} = \lim\limits_{x\to 2} \dfrac{1}{\sqrt{x}+\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

i. $\lim\limits_{x\to 4} \dfrac{\sqrt{x-2\sqrt{x}+1}-1}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{\sqrt{\left(\sqrt{x}-1\right)^2}-1}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{\left|\sqrt{x}-1\right|-1}{\sqrt{x}-2}$

Consider the one side limits

$\lim\limits_{x\to 4^+} \dfrac{\left|\sqrt{x}-1\right|-1}{\sqrt{x}-2} = \lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{\sqrt{x}-2} = 1$ and $\lim\limits_{x\to 4^-} \dfrac{\left|\sqrt{x}-1\right|-1}{\sqrt{x}-2} = \lim\limits_{x\to 4^-} \dfrac{\sqrt{x}-2}{\sqrt{x}-2} = 1$

$\therefore \lim\limits_{x\to 4} \dfrac{\sqrt{x-2\sqrt{x}+1}-1}{\sqrt{x}-2} = 1$

j. First show that $\sqrt{x^2-1} = \sqrt{x-1}\,\sqrt{x\ +\ 1}$

Next, $x - 1 = \left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)$ for $x > 0$

Then, $\dfrac{\sqrt{x-1}\ +\ \sqrt{x}-1}{\sqrt{x^2-1}} = \dfrac{\sqrt{x-1}}{\sqrt{x^2-1}} + \dfrac{\sqrt{x}-1}{\sqrt{x^2-1}}$

$= \dfrac{\sqrt{x-1}}{\sqrt{x-1}\sqrt{x+1}} + \dfrac{\sqrt{x}-1}{\sqrt{x-1}\sqrt{x+1}}$

$= \dfrac{\sqrt{x-1}}{\sqrt{x-1}} \cdot \dfrac{1}{\sqrt{x+1}} + \dfrac{\left(\sqrt{x}-1\right)\sqrt{x-1}}{|x-1|\ \sqrt{x\ +\ 1}}$

$= \dfrac{\sqrt{x-1}}{\sqrt{x-1}} \cdot \dfrac{1}{\sqrt{x\ +\ 1}} + \dfrac{\left(\sqrt{x}-1\right)\sqrt{x-1}}{\left|\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)\right|\ \sqrt{x\ +1}}$

when we apply limits,

$\lim\limits_{x\to 1} = \dfrac{\sqrt{x-1}+\sqrt{x}-1}{\sqrt{x^2-1}} = \lim\limits_{x\to 1} \dfrac{\sqrt{x-1}}{\sqrt{x-1}} \cdot \dfrac{1}{\sqrt{x\ +\ 1}} + \lim\limits_{x\to 1} \dfrac{\left(\sqrt{x}-1\right)\sqrt{x-1}}{\left|\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)\right|\left(\sqrt{x\ +1}\right)}$

$= \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x\ +1}} + \lim\limits_{x\to 1} \dfrac{\pm\sqrt{x-1}}{\left(\sqrt{x}+1\right)\sqrt{x\ +\ 1}} = \dfrac{1}{\sqrt{2}}$

Answers to Activity 2.6

1. a. 0 b. 3 c. ∞
2. a. $\dfrac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$ b. 0 c. same

Answers to Activity 2.7

This activity extends students' knowledge to look for the application of limits in sketching graph of rational functions.

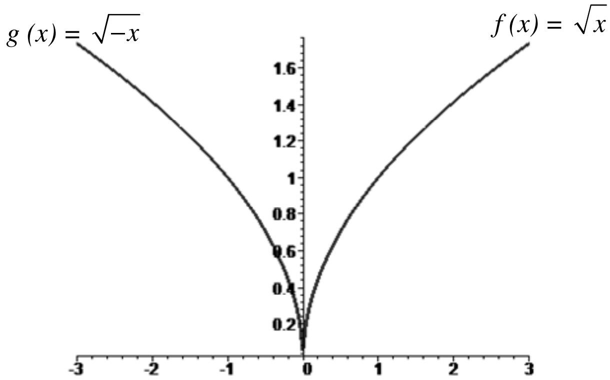


Figure 2.6

a. 0 b. doesn't exist c. doesn't exist d. 0

2. a. $\dfrac{4}{3}$ b. $\dfrac{4}{3}$ c. –1.5 d. –1.5

e. 0 f. 0 g. 2 h. 1

Answers to Activity 2.8

1. i. ∞ ii. ∞ iii. 3 iv. 4

Discuss the connection between these limits and the vertical and horizontal asymptotes of rational functions.

2. i. $\lim_{x\to a} f(x)$ exists and equals to the one side limits.

ii. $\lim_{x\to a} f(x)$ doesn't exist.

The one side limit is equal to L.

The purpose of Exercise 2.7 is to introduce students on continuity of a function at a point. In question 6, for high ability students, you can introduce the unfamiliar words: removable and non-removable discontinuity can be clearly seen at this level.

Answers to Exercise 2.7

1. $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{140x+25}{2x+3} = 70$

   As time gone the yearly product will be closer to 70 quintals per hectare.

2. $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} \frac{45x+35}{9x+2} = 5.$ The unemployment rate will decrease to 5%.

3. Except l, all are to be oral questions.

   a. 0 b. doesn't exist c. doesn't exist

   d. doesn’t exist e. doesn't exist f. 0

   g. ∞ h. $-\infty$ i. ∞

   j. ∞ k. $\frac{5}{3}$

   l. As $x \to 5^+$, $x^2 - 9 \to 16^+$ so that $4 - \sqrt{x^2-9} \to 0^-$.

   Therefore, $\lim_{x\to 5^+} \sqrt{4-\sqrt{x^2-9}}$ doesn't exist.

4. Help slow learners to answer orally.

   a. 1.6 b. 2 c. $-\infty$ d. 4 e. 0

   f. -2 g. -3 h. -2 i. 2

5. Extend the basic limit theorems to one side limits

   a. $\lim_{x\to 2^+} \left(f(x) + g(x)\right) = \lim_{x\to 2} \left(e-1\right) x + 3 + \frac{1}{x} = 2e + 1.5$

   b. $\lim_{x\to 2^-} \left(f(x) - g(x)\right) = \lim_{x\to 2} \left(e^x - \frac{1}{x}\right) = e^2 - 0.5$

   c. $e \times 1 = e$

   d. $\lim_{x\to -1^+} \frac{f(x)-g(x)}{f(x)\,g(x)} = \lim_{x\to -1} \frac{e^x - \left(x^2-x\right)}{e^x(x^2-x)} = \frac{1-2e}{2}$

6. The graphs in a and f have vertical asymptotes at the given points.

   The graphs in b, c, d and e have holes at the given points.

   The following are solutions to one side limtits.

   a. $\lim_{x\to -5^+} \frac{x}{x+5} = -\infty$; $\lim_{x\to -5^-} \frac{x}{x+5} = \infty$

   b. $\lim_{x\to 4^+} \frac{x^3+1}{x+1} = \lim_{x\to -1^+} \left(x^2 - x+1\right) = 3$; $\lim_{x\to -1^-} \left(x^2 - x+1\right) = 3$

   c. $\lim_{x\to 1^+} \frac{\left|x^2-1\right|}{x-1} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+}(x+1) = 2$

   $\lim_{x\to 1^-} \frac{\left|x^2-1\right|}{x-1} = \lim_{x\to 1^-} \frac{1-x^2}{x-1} = \lim_{x\to 1^-}\left(-(x+1)\right) = -2$

d. $\lim\limits_{x\to 3^+}\dfrac{(x-3)^3}{|x-3|}=\lim\limits_{x\to 3+}\dfrac{(x-3)^3}{x-3}=\lim\limits_{x\to 3^+}(x-3)^2=\ 0$

$\lim\limits_{x\to 3^-}\dfrac{(x-3)^3}{|x-3|}=\lim\limits_{x\to 3^-}\dfrac{(x-3)^3}{3-x}=-\lim\limits_{x\to 3^-}(x-3)^2=0$

e. $\lim\limits_{x\to 0^+}\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}=\lim\limits_{x\to 0^+}\dfrac{x+1}{x-1}=-1;\ \lim\limits_{x\to 0^+}\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}=\lim\limits_{x\to 0^-}\dfrac{x+1}{x-1}=-1$

f. $\lim\limits_{x\to \pi^+}\dfrac{x}{\sin x}=-\infty;\ \lim\limits_{x\to \pi^-}\dfrac{x}{\sin x}=\infty$

The graph of $f(x)=\dfrac{\sin x}{x}$ is drawn in the textbook. See the graph of $f(x)=\dfrac{x}{\sin x}$ which is the reciprocal of the graph of $f(x)=\dfrac{\sin x}{x}$.

2.3 CONTINUITY OF A FUNCTION

Periods Allotted: 5 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *define continuity of a function in an interval.*
- *describe the properties of a continuous functions.*
- *use properties of continuous functions to determine the continuity of various functions.*

Vocabulary: Continuity at a point, Continuity on an interval, Continuous functions, Minimum and maximum values, Extreme values, Bisection

Introduction

This sub unit defines continuity and its mathematical applications. The intermediate value theorem and the extreme value theorem are stated and explained with the aid of graphs and numerical examples.

Teaching Notes

In teaching this topic you may use the following suggested teaching strategies.

✓ Discuss one side limits and non-existence of limits.

✓ Show continuity and discontinuity graphically.

- ✓ Define "continuity of a function at a point x_o" and " continuity of a function over an open interval".
- ✓ Discuss one side continuity and continuity on a closed interval.
- ✓ Define a continuous function.
- ✓ Introduce essential properties of continuous functions.
- ✓ Introduce the "Intermediate value theorem".
- ✓ Introduce the concept "maximum /minimum".
- ✓ Discuss the extreme value theorem.

Assessment

Ask oral questions to redefine continuity of a function

- ✓ at a point x_o.
- ✓ over an interval I.
- ✓ at each point of the domain.
- ✓ Give different exercise problems on continuity of a function from the textbook or other related materials as class work or homework and check the solutions.
- ✓ Also, ask oral questions to assess whether or not the students are able to re-state:
  - the properties of continuous functions.
  - the intermediate value theorem.
  - the extreme value theorem.

Give various exercise problems on the application of the intermediate value theorem. For example, approximate the zeros of $f(x) = x^3 - x + 1$.

Give exercise problems to determine the maximum and minimum values of continuous functions on a closed interval.

For example, find the maximum and minimum values of

i. $f(x) = x^2 - 5x + 1$ on $[1, 4]$

Here the minimum value is $\frac{-21}{4}$ and the maximum value is -3.

ii. $f(x) = \begin{cases} \frac{1}{x}, if\ x \le 1 \\ x^2, if\ x > 1 \end{cases}$ on $[-0.5, 2]$

This doesn't have minimum or maximum value. Sketch the graph and explain to the class.

Additional exercise problems for high ability students

1. Redefine $f(x)=\dfrac{x^2-1}{x-1}$ so that *f* is a continuous function

2. Let $f(x)=\begin{cases} e^{ax}, \text{if } x\ge 0 \\ x+a, \text{if } x<0 \end{cases}$

   Determine the value of *a* so that *f* is a continuous function.

   Solution:

1. $g(x)=\begin{cases} f(x), \text{if } x\ne 1 \\ 2, \text{if } x=1 \end{cases}$

2. $a = 1$

2.3.1 Continuity of a Function at a Point

Based on the fact that students know the difference between functional value and limit value at a point, use Activity 2.9 to lead students to the definition of continuity of a function at a point.

Alternatively, you can start the lesson by asking the following question. Draw the graph of each of the following functions.

1. $f(x)=x^2$ 2. $f(x)=\dfrac{1}{x}$ 3. $f(x)=2^x$.

Which of the graphs is drawn without lifting the pencil from the paper?

Discuss with students which graphs are drawn continuously.

Answers to Activity 2.9

None of the graphs is connected and $f(x_o) = \lim\limits_{x\to x_o} f(x)$.

The graphs in a) and b) have holes

The graphs in c) and d) have gaps

Group work 2.1 is designed to assess how students associate continuity of a function with real life. Ask some students to demonstrate what they have done on the blackboard. Finally give Exercise 2.8 as class work or homework to assess the competencies in continuity of a function.

Answers to Group Work 2.1

1. Select students at random to sketch the graphs that satisfy the given condition and ask the rest of the students whether or not the given condition is satisfied.

2. i. max = $f(b)$, min = $f(a)$

   ii. max = $f(a)$, min = $f(b)$

3. This exercise is designed to show an application of the intermediate value theorem.

   Let $x > 0$, then $A(x) = x^2$ gives the area of a square of side length x.

   A is continuous on (0, 10].

   i. a. $0 < 11\sqrt{7} < 100$. Hence $\exists x \in (0, 10]$ such that $A(x) = 11\sqrt{7}$.

      b. Also, $\exists y \in (0, 10]$ such that $A(y) = 11\sqrt{17}$

   ii. Let $r > 0$, then $A(r) = \pi r^2$ gives the area of a circle of radius r. A is a continuous function, there exists $x \in (10, 20)$ such that $\pi x^2 = 628\ \text{cm}^2$ because $100\pi < 628 < 400\pi$.

   iii. Clearly height is continuous. If you are 1.8 m today, there was a year that you were 0.9 m tall.

Answers to Exercise 2.8

1. a. Every constant function is continuous. $\Rightarrow f(x) = 3$ is continuous at $x = 5$.

   b. Every polynomial is continuous. $\Rightarrow f(x) = 2x^2 - 5x + 3$ is continuous at $x = 1$.

   c. discontinuous at $x = 3$.

   d. continuous. Sketch the curve of f and explain.

   e. sketch the curve of f for further explanation and show that f is discontinuous at 0.

   f. $f(1) = 0,\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} |x| - 1 = 0,$

   $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} 1 - |x| = 0 \Rightarrow \lim\limits_{x\to 1} f(x) = f(1)$

   Also, $\lim\limits_{x\to -1} f(x) = f(-1)$

   Therefore, f is continuous at $x = \pm 1$

2. a. $\lim\limits_{x\to 2^+} f(x) = f(2)$ (since f is continuous at $x = 2$)

   $\Rightarrow 2a - 3 = 2(2) + 5 \Rightarrow a = 6$

b. $$\begin{cases}\lim\limits_{x\to 2^-} f\ (x)\ = f\ (2)\ \text{(since } f \text{ is continous at } x\ =\ 2)\\ \lim\limits_{x\to 3^+} f\ (x)\ = f\ (3)\ \text{(since } f \text{ is continous at } x\ =\ 3)\end{cases}$$

$$\Rightarrow \begin{cases}2a-b=a\ (2^2)+b(2)+1\\ 3b+4=a\ (3^2)\ +\ b\ (3)\ +\ 1\end{cases}$$

$$\Rightarrow\ a=\frac{1}{3},\ b=\ -\frac{5}{9}$$

c. $$\left\{\sqrt{\frac{1}{4}-1+a}\ =\ -\sqrt{-\frac{1}{4}+1-\frac{3}{4}}\right.$$

$$\Rightarrow\ \left\{\sqrt{a-\frac{3}{4}}\ =\ 0 \Rightarrow\ a=\ \frac{3}{4}\right.$$

d. $\lim\limits_{x\to 5}\ \dfrac{k\ (x-5)}{x^2-25}=5\ \Rightarrow\ \dfrac{k}{10}=5\Rightarrow k=50.$ But f is discontinuous at $x=-5$

e. $2^{|4-c|}\ =\ 8\ \Rightarrow\ |4-c|\ =\ 3\ \Rightarrow 4-c\ =\ -3$ or $4-c=3$

$\Rightarrow c = 1$ or $c = 7$

3. a. $\lim\limits_{x\to 2}\ \dfrac{x^2-4}{x-2}=\lim\limits_{x\to 2}\ x+2=4$ but $f(2)=8$

$\Rightarrow$ f is discontinuous at $x = 2$.

$\Rightarrow$ f is continuous on $(-\infty, 2)$ and on $(2, \infty)$

b. $(-\infty, \infty)$.

c. $\lim\limits_{x\to 1}\ 4\ \dfrac{|x^2-1|}{x-1}$ doesn't exist. f is continuous on $(-\infty, 1)$ and on $(1, \infty)$

d. $\left[-\dfrac{1}{2},\ \dfrac{1}{2}\right]$

e. $\left(-\dfrac{3}{2},\ \dfrac{3}{2}\right)$

f. $$\lim_{x\to -1}\ \frac{5\ (x^3+1)}{x+1}\ =\ \lim_{x\to -1}\ \frac{5\ (x\ +1)\ (x^2-x+1)}{x+1}$$

$$=\ 5\lim_{x\to -1}\ \left(x^2-x+1\right)=15$$

But $f(-1) = 10 \Rightarrow f$ is discontinuous at $x = -1$

Thus, f is continuous on $(-\infty, -1)$ and on $(-1, \infty)$

g. $5 - x^2 \geq 0$ and $2 - \sqrt{5-x^2} \geq 0$

$\Rightarrow |x| \leq \sqrt{5}$ and $4 \geq 5 - x^2$

$\Rightarrow |x| \leq \sqrt{5}$ and $x^2 \geq 1$

$\Rightarrow |x| \leq \sqrt{5}$ and $|x| \geq 1$

$\Rightarrow f$ is continuous on $\left[-\sqrt{5}, -1\right]$ and on $\left[1, \sqrt{5}\right]$

4. This problem can be given as a group work.

$$f(x) = \begin{cases} 900, \text{if } 0 \leq x \leq 10000 \\ 0.02x + b, \text{ if } 10000 < x \leq 15000 \\ 0.02x + b + 500, \text{ if } 15000 < x \leq 25000 \end{cases}$$

When $x = 10000$, $0.02x + b = 9000$

$\Rightarrow 0.02\,(10000) + b = 9000$

$\Rightarrow b = 700$

$$\Rightarrow f(x) = \begin{cases} 900, \text{ if } 0 \leq x \leq 10000 \\ 0.02x + 700, \text{ if } 10000 < x \leq 15000 \\ 0.02x + 1200, \text{ if } 15000 < x \leq 25000 \end{cases}$$

F is not continuous at $x = 15{,}000$

2.4 EXERCISES ON APPLICATION OF LIMITS

Periods Allotted: 3 Periods

Competency

At the end of this sub-unit, students will be able to:

- *consolidate what they have studied on limits.*

Introduction

In this sub-unit, the mathematical applications of limits are presented in a variety of situations in real life.

For example the area of a circle, continuous compound interest and life expectancy are some of such applications.

For example, $\lim\limits_{n\to\infty} 1-\left(\frac{1}{2}\right)^n = 1$

The amount of money invested in a bank at an interest rate of r% per-year compounded continuously.

$$A = \lim_{n\to\infty} P\left(1+\frac{r}{100n}\right)^{nt} = P\lim_{n\to\infty}\left(1+\frac{r}{100n}\right)^{nt} = Pe^t$$

Teaching Notes

The work of Activity 2.10 involves using a calculator to find the approximated value of $\frac{\sin x}{x}$ in question 1.

Using this result, students are expected to explore the limit values in questions 2 and 3.

State theorem 2.7 and discuss the examples with the class or group. Then discuss Activity 2.11 and the worked examples.

Additional exercise problems for high ability students

1. Draw the graph of each of the following functions:

   a. $f(x) = \dfrac{x}{\tan x}$ b. $g(x) = \dfrac{\tan x}{x}$ c. $h(x) = \tan\left(\dfrac{\pi}{x}\right)$

2. Using the graph in question 1 above ,demonstrate each of the following limits.

   a. $\lim\limits_{x\to 0}\dfrac{x}{\tan x}$ b. $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$ c. $\lim\limits_{x\to\infty}\tan\left(\dfrac{\pi}{x}\right)$

3. Evaluate : a. $\lim\limits_{x\to\infty}\left(2+\dfrac{1}{x}\right)^x$ b. $\lim\limits_{x\to\infty}\left(\dfrac{1}{2}+\dfrac{1}{x}\right)^x$

Solution:

1. a. b.

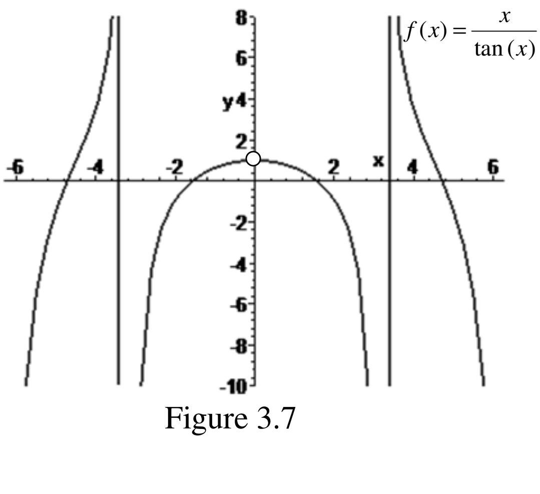


Figure 3.7

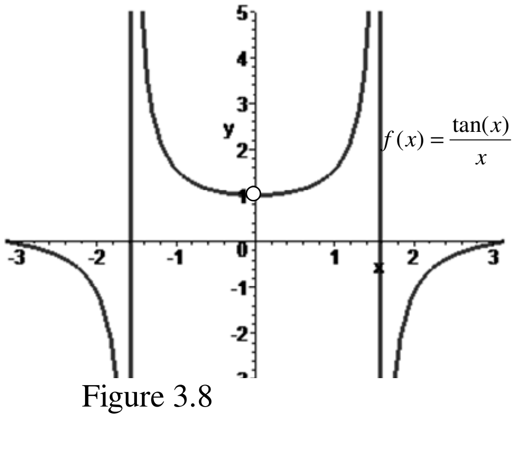


Figure 3.8

c.

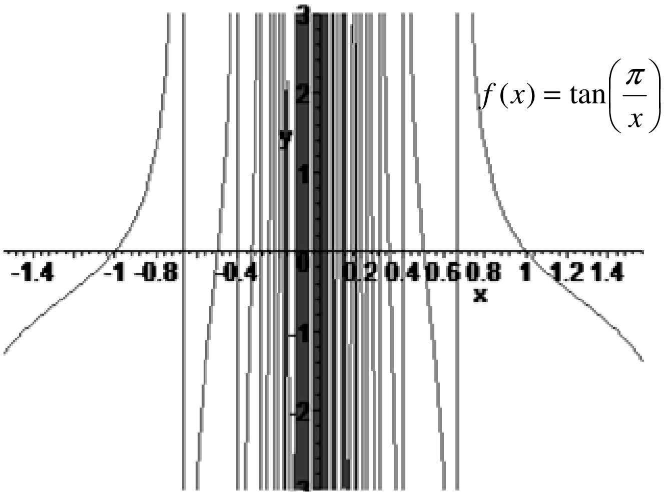


Figure 3.9

2. It can be seen from the graphs that,

   a. 1 b. 1 c. Does not exist

3. a. $\lim_{x\to\infty}\left(2+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(2^x\right)\left(1+\frac{1}{2x}\right)^x=\infty+e^{\frac{1}{2}}=\infty$

   b. $\lim_{x\to\infty}\left(\frac{1}{2}+\frac{1}{x}\right)^x=\lim_{x\to\infty}\left(\frac{1}{2}\right)^x\left(1+\frac{2}{x}\right)^x=0\times e^2=0$

Assessment

It is possible to assess understanding of the students by giving Exercise 2.9 as class work or homework or you can:

- ✓ ask them to show related problems with convergence and divergence of number sequences and sequences of partial sums.
- ✓ give exercise problems either from the textbook or other resources which are related to real life situations or related to other subjects.
- ✓ use the exercises in the textbook as class work or homework and check the solutions.

Answers to Activity 2.10

1. This table is designed to explore $\lim_{x\to 0}\frac{\sin x}{x}$.

2. From the table, you can conclude that $\lim_{x\to 0}\frac{\sin x}{x}$ = 1.

3. Help students to understand the graph of the function, $f(x)=\dfrac{\sin x}{x}$. From the graph, it is clear that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

Answers to Activity 2.11

1. a. Yes b. 6 c. $\left(1+\dfrac{1}{k}\right)^k$ increases

   d. $2<\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n<3$

2. a. $\mathbb{R}\setminus\{0\}$ b. Not because f is not defined at $x=0$

   c. $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to-\infty} f(x)=e$

Answers to Exercise 2.9

1. a. $$\lim_{x\to 0}\frac{\tan\ (4x)}{\tan\ (3x)}=\frac{\lim\limits_{x\to 0} 4\ \dfrac{\tan\ (4x)}{4x}}{\lim\limits_{x\to 0} 3\ \dfrac{\tan\ (3x)}{3x}}=\frac{4}{3}$$

   b. $$\lim_{x\to -2}\frac{\sin\ (x+2)}{x^3+2x^2+x+2}=\lim_{x\to -2}\frac{\sin\ (x+2)}{x^2(x+2)+(x+2)}$$
   $$=\lim_{x\to -2}\frac{\sin\ (x+2)}{(x+2)\ (x^2+1)}$$
   $$=\lim_{x\to -2}\frac{\sin\ (x+2)}{x+2}\ .\ \lim_{x\to -2}\frac{1}{x^2+1}=1\times\frac{1}{5}=\frac{1}{5}$$

   c. Let $y=x-\dfrac{\pi}{2}$, then $x=y+\dfrac{\pi}{2}$.

   When, cos $x\to\dfrac{\pi}{2}$, $y\to 0$

   $$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{x-\frac{\pi}{2}}{\cos\ x}=\lim_{y\to 0}\frac{y}{\cos\left(y+\dfrac{\pi}{2}\right)}$$
   $$=\lim_{y\to 0}\frac{y}{\cos\ y\cos\dfrac{\pi}{2}-\sin y\sin\dfrac{\pi}{2}}=\lim_{y\to 0}\frac{y}{-\sin\ y}=-1$$

d. $\lim_{x\to 0}\dfrac{\sin x}{x^3-x}=\lim_{x\to 0}\dfrac{\sin x}{x\,(x^2-1)}=\lim_{x\to 0}\dfrac{\sin x}{x}\times\lim_{x\to 0}\dfrac{1}{x^2-1} = -1$

e. $\lim_{x\to\pi^-}\dfrac{\sin x}{1-\cos x} = 0$

f. 0

g. $\dfrac{1}{\text{e}}$

h. $\lim_{x\to\infty}\left(\dfrac{x}{x+3}\right)^{8-5x} = \lim_{x\to\infty}\left(\dfrac{x+3}{x}\right)^{5x-8} = \lim_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{5x-8} = \text{e}^{15}$

i. $\lim_{x\to\infty}\left(\dfrac{x+4}{x-1}\right)^{3x-1} = \dfrac{\lim_{x\to\infty}\left(1+\dfrac{4}{x}\right)^{3x-1}}{\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{3x-1}} = \dfrac{e^{12}}{e^{-3}} = e^{15}$

j. $\lim_{x\to\infty}\left(\dfrac{2x+5}{2x-11}\right)^{x+1} = \dfrac{\lim_{x\to\infty}\left(1+\dfrac{5}{2x}\right)^{x+1}}{\lim_{x\to\infty}\left(1-\dfrac{11}{2x}\right)^{x+1}} = \dfrac{\text{e}^{\frac{5}{2}}}{\text{e}^{-\frac{11}{2}}} = \text{e}^{8}$

k. Let $y=\dfrac{1}{x}$, then $x=\dfrac{1}{y}$, $as\ x\to 0^+,\ y\to\infty.$

Hence, $\lim_{x\to 0^+}(5x+1)^{\frac{1}{x}} = \lim_{y\to\infty}\left(\dfrac{5}{y}+1\right)^{y} = e^5$.

In general $\lim_{x\to 0^+}(cx+1)^{\frac{1}{x}} = e^c \forall c\in\mathbb{R}$

l. $\lim_{x\to 0}\ \sin\left(\dfrac{1}{x}\right)$ doesn't exist.

m. $\lim_{x\to\infty}\tan\left(\dfrac{1}{x}\right)= 0$

2. The amount of money invested in a bank at an interest rate of r % per-year compounded continuously.

$$A=\lim_{n\to\infty}P\left(1+\frac{r}{100n}\right)^{nt} = P\lim_{n\to\infty}\left(1+\frac{r}{100n}\right)^{nt} = Pe^{nt}$$

3. $A=4500\times e^{(0.03)\times(10.25)}$

At the end, give the review exercise as class work or home work. Students of high ability should not be required to do every problem. The Review Exercises are designed to assess the entire topic.

Answers to Review Exercises on Unit 2

1. a. -1

b. $$\lim_{x\to -1}\frac{x+1}{x^2+7x+6} = \lim_{x\to -1}\frac{x+1}{(x+1)\,(x+6)}$$

$$= \lim_{x\to -1}\frac{1}{x+6} = \frac{1}{5}$$

c. $$\lim_{x\to 9}\frac{\sqrt{x}-3}{x^2-81} = \lim_{x\to 9}\frac{\sqrt{x}-3}{(x-9)\,(x+9)}$$

$$= \lim_{x\to 9}\frac{\sqrt{x}-3}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)(x+9)}$$

$$= \lim_{x\to 9}\frac{1}{\left(\sqrt{x}+3\right)(x+9)} = \frac{1}{108}$$

d. $$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0}\frac{\left(\sqrt{x+4}-2\right)}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x\to 0}\frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0}\frac{1}{\left(\sqrt{x+4}+2\right)} = \frac{1}{4}$$

e. $\lim\limits_{x\to 0}\dfrac{\cos x}{x}$ doesn't exist

2. $$\frac{x\,|x-5|}{x^2-25} = \begin{cases} \dfrac{x\,(x-5)}{x^2-25} = \dfrac{x}{x+5}, & if\ x > 5 \\[2ex] \dfrac{x\,(5-x)}{x^2-25} = \dfrac{-x}{x+5}, & if\ x < 5 \end{cases}$$

a. $$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\frac{x}{x+5} = \frac{1}{2}$$

b. $$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+}\frac{-x}{x+5} = \infty$$

c. $\lim\limits_{x\to 5^-} f(x) = \lim\limits_{x\to -5} \dfrac{-x}{x+5} = -\dfrac{1}{2}$

d. $\lim\limits_{x\to -5^-} f(x) = \lim\limits_{x\to -5^-} \dfrac{-x}{x+5} = -\infty$

3.

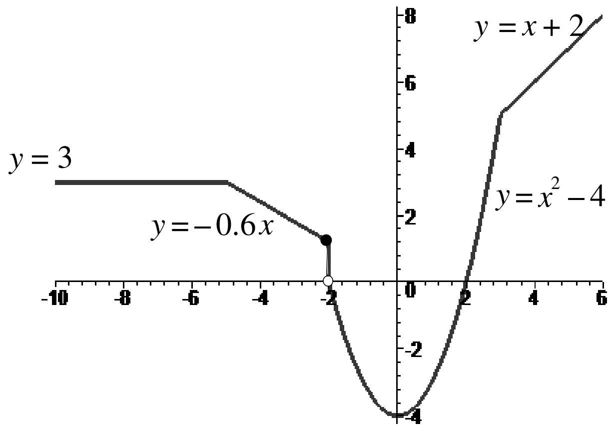


Figure 2.10

a. $\lim\limits_{x\to -5} f(x) = 3$

b. $\lim\limits_{x\to -2^+} f(x) = 0, \ \lim\limits_{x\to -2^-} f(x) = 1.2$

$\therefore \lim\limits_{x\to -2} f(x)$ doesn't exist.

c. $\lim\limits_{x\to 3} f(x) = 5$

4. a. –5. For every polynomial $p(x)$, $\lim\limits_{x\to c} p(x) = p(c)$

b. $\lim\limits_{x\to 2} \sqrt{x^2-5x}$ doesn't exist because $x^2 - 5x < 0$ near $x = 2$

c. $\lim\limits_{x\to -7} \dfrac{x^2-49}{x^2+6x-7} = \lim\limits_{x\to -7} \dfrac{(x-7)(x+7)}{(x+7)(x-1)}$

$$= \lim_{x\to -7} \frac{x-7}{x-1} = \frac{7}{4}$$

d. $\dfrac{3x-4|x|}{5x} = \begin{cases} -\dfrac{1}{5}, & if\ x > 0 \\ \dfrac{7}{5}, & if\ x < 0 \end{cases}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{3x-4|x|}{5x}$ doesn't exist

e. $$\lim_{x\to -5} \frac{x^3+125}{x+5} = \lim_{x\to -5} \frac{(x+5)\left(x^2-5x+25\right)}{x+5}$$

$$= \lim_{x\to -5} \left(x^2-5x+25\right) = 75$$

f. $$\lim_{x\to 1} \frac{\sin\ (x-1) + x^2-1}{x-1} = \lim_{x\to 1} \frac{\sin\ (x-1)}{x-1} + \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$= 1 + \lim_{x\to 1} \frac{(x-1)\ (x+1)}{x-1} = 3$$

g. Since sine function is continuous

$$\lim_{x\to\infty} \sin\left(\frac{\pi}{x}\right) = \sin\left(\lim_{x\to\infty} \frac{\pi}{x}\right) = \sin\ 0 = 0$$

h. Doesn't exist

i. 1

j. $$\lim_{x\to 0} \frac{\sin^3(5x)}{\sin\ \left(4x^3\right)} = \frac{\lim\limits_{x\to 0} 125\left(\dfrac{\sin\ 5x}{5x}\right)^3}{\lim\limits_{x\to 0} 4\ \dfrac{\sin\ (4x^3)}{4x^3}} = \frac{125}{4}$$

Problems 5 and 6 can be given as a group work. The continuity of the function may be verified using graphs.

5. a. not continuous

b. not continuous since $f\,(3)$ is undefined.

c. continuous

d. $f\left(\frac{1}{2}\right) = \frac{1}{4}$ and $\lim\limits_{x\to\frac{1}{2}} f\,(x) = \frac{1}{4}$

e. $f\,(0) = e^0 = 1,\ \lim\limits_{x\to 0} f\ (x) = \lim\limits_{x\to 0} \dfrac{\cos\ x}{e^x} = 1$

Thus, it is continuous at $x = 0$

6. a. $\lim\limits_{x\to 2^+} f\,(x) = f\ (2) \Rightarrow 2a - 1 = 2^2 + 3\ (2)$

$$\Rightarrow a = 5.5$$

b. $\lim\limits_{x\to a} f\ (x) = f\ (a)$

$$\Rightarrow \lim_{x \to a} \frac{x^2 - ax}{x - a} = 2$$

$$\Rightarrow \lim_{x \to a} \frac{x\,(x-a)}{x - a} = 2$$

$$\Rightarrow \lim_{x \to a} x = 2 \Rightarrow \quad a = 2$$

c. $\sin(k + 0) = 1$

$\Rightarrow \sin k = 1$

$\Rightarrow k = \dfrac{\pi}{2} + 2n;\ n \in \mathbb{Z}$

d. $\begin{cases} \lim\limits_{x \to a^-} f(x) = f(a) \\ \lim\limits_{x \to b^+} f(x) = f(b) \end{cases}$

$\Rightarrow \begin{cases} a^2 + 1 = 15 - 5a \\ 5b - 25 = 15 - 5b \end{cases}$

$\Rightarrow \ a = 2$ or -7 and $b = 4$

7. a. $$\lim_{x \to \infty} \frac{3x^3 + 5x^2 - 11}{2x^3 - 1} = \lim_{x \to \infty} \frac{\left(\dfrac{3x^3 + 5x^2 - 11}{x^3}\right)}{\left(\dfrac{2x^3 - 1}{x^3}\right)} = \frac{\lim\limits_{x \to \infty} \left(3 + \dfrac{5}{x} - \dfrac{11}{x^3}\right)}{\lim\limits_{x \to \infty} 2 - \dfrac{1}{x^3}} = \frac{3}{2}$$

b. $$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1} - 10}{\sqrt{x^2 + 1} + 9} = \lim_{x \to \infty} \frac{\dfrac{\sqrt{x^2 + 1} - 10}{x}}{\dfrac{\sqrt{x^2 + 1} + 9}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\dfrac{x^2 + 1}{x^2}} - \dfrac{10}{x}}{\sqrt{\dfrac{x^2 + 1}{x^2}} + \dfrac{9}{x}}$$

$$= \frac{\lim\limits_{x \to \infty} \sqrt{1 + \dfrac{1}{x^2}} - \dfrac{10}{x}}{\lim\limits_{x \to \infty} \sqrt{1 + \dfrac{1}{x^2}} + \dfrac{9}{x}} = 1$$

8. a. 0 b. doesn't exist c. doesn't exist

d. $-\infty$ e. ∞ f. doesn't exist

g. $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}} = \lim\limits_{x\to 0^+} \dfrac{\sin x}{x}\sqrt{x}$

$$= \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \sqrt{x} = 1\times 0 = 0$$

h. doesn't exist because as $x \to -5^-$, $25 - x^2 \to 0^-$

i. $\lim\limits_{x\to 7^-} \dfrac{x^2\left|x^2-49\right|}{x-7} = -\lim\limits_{x\to 7^-} \dfrac{x^2(x-7)\ (x+7)}{x-7}$

$$= -\lim_{x\to 7^-} x^2(x+7) = -686$$

9. These problems may be given as group work for class room discussion.

a. $\dfrac{1-x}{x} \geq 0 \Rightarrow 0 < x \leq 1 \Rightarrow f$ is continuous on (0, 1]

b. $x > 0$ and $\ln\sqrt{x} \geq 0 \Rightarrow x \geq 1 \Rightarrow f$ is continuous on $[1, \infty)$

c. $\dfrac{x}{e^x-1} > 0 \Rightarrow (x > 0 \wedge e^x - 1 > 0) \vee (x < 0 \wedge e^x - 1 < 0)$

$$\Rightarrow (x > 0 \wedge e^x > 1) \vee (x < 0 \wedge e^x < 1)$$

$$\Rightarrow (x > 0) \vee (x < 0)$$

$$\Rightarrow x \neq 0 \Rightarrow f \text{ is continuous on } (-\infty, 0) \cup (0, \infty)$$

d. $\dfrac{4x-3}{x-4} \geq 0 \wedge x \neq 4 \Rightarrow x \leq 3$ or $x > 4$

$\Rightarrow f$ is continuous on $\left(-\infty, \dfrac{3}{4}\right] \cup (4, \infty)$

10. Drawing the graphs of the functions on the given intervals is important to see the maximum and the minimum values.

	Minimum	Maximum
a	–4	11
b	–8	1
c	$-\frac{1}{4}$	12
d	No	No
e	$-\frac{9}{16}$	2.5
f	0	9

11. There are real zeros between.

a. –1 and 0; 1 and 2

b. 1 and 2

c. –2 and –1

d. –2 and –1; and at $x = 2$

e. –3 and –2, –2 and –1, 1 and 2, 2 and 3

12. a. $$\lim_{x\to 0}\frac{\sin\left(\frac{x}{\pi}\right)}{\tan\ x} = \frac{\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{x}{\pi}\right)}{\pi\left(\frac{x}{\pi}\right)}}{\lim\limits_{x\to 0}\dfrac{\tan\ x}{x}} = \frac{1}{\pi}$$

b. $$\lim_{x\to 0}\frac{\sin(x^3)}{x^3} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^3 = 1$$

c. Let $y = \frac{1}{x}$, then as $x \to \infty,\ y \to 0^+$.

$$\therefore\ \lim_{x\to\infty} x\ \tan\left(\frac{1}{x}\right) = \lim_{y\to 0^+}\frac{\tan\ (y)}{y} = 1$$

d. $\lim_{x\to 0} \dfrac{x-\tan x}{x} = \lim_{x\to 0} \dfrac{x}{x} - \lim_{x\to 0} \dfrac{\tan x}{x} = 1-1 = 0$

e. Let $\dfrac{1}{y} = \dfrac{1}{x+11}$. then

$x = y - 11$ and as $x \to \infty$, $y \to \infty$.

$\Rightarrow \lim_{x\to\infty}\left(1 + \dfrac{3}{x+11}\right)^{x+6} = \lim_{y\to\infty}\left(1 + \dfrac{3}{y}\right)^{y-5} = e^3$

13. The life expectancy of the next few years is

$f(1) = \dfrac{370}{7}$

$f(2) = 58$

$f(10) = \dfrac{1130}{17}$

As time goes on $\lim_{x\to\infty} \dfrac{210x+116}{3x+4} = 70$ the life expectancy of the country will be 70 years.

14. 60 words/ minute

UNIT 3 INTRODUCTION TO DIFFERENTIAL CALCULUS

INTRODUCTION

In this unit, there are several outcomes which are expected to be accomplished.

Among those the following two are the most important.

The first one is to enable students to understand the concept of differentiation.

The second one is to enable students to apply differential calculus in many fields of study as indicated in the introduction part of the textbook.

Therefore, students should be able to exercise the different rules and methods of differentiation.

In teaching this unit, graphical presentation is essential.

The activities are believed to introduce unfamiliar terms and create an opportunity for students to explore rules and methods of differentiation. The exercises are prepared on the basis of the assumptions of learning gaps among students.

The exercises are designed to be useful in providing a diagnosis of individual learning difficulties by indicating the rule which needs to be explained and represented to various students.

Unit Outcomes

After completing this unit, students will be able to:

- *describe the geometrical and mechanical meaning of derivative.*
- *determine the differentiability of a function at a point.*
- *find the derivatives of some selected functions over intervals.*
- *apply the sum, difference, product and quotient rule of differentiation of functions.*
- *find the derivatives of power functions, polynomial functions, rational functions, simple trigonometric functions, exponential and logarithmic functions.*

Suggested Teaching Aids in Unit 3

- ✓ Prepare charts containing the rules of differentiation
  - the sum and difference rules
  - the product rule
  - the quotient rule
  - the chain rule
- ✓ Prepare charts containing the derivatives of the standard functions.
- ✓ Prepare charts showing tangents to the graphs of standard functions at a point.

3.1 INTRODUCTION TO DERIVATIVES

Periods Allotted: 10 periods

Competencies

At the end of this sub-unit, students will be able to:

- *find the rate of change of one quantity with respect to another.*
- *sketch different straight lines and curved graphs and find out slopes at different points of each graph.*
- *define differentiability of a function at a point* x_o.
- *explain the geometrical and mechanical meaning of derivative.*
- *set up the equation of tangent line at the point of tangency, using the concept of derivative.*
- *find the derivative of elementary functions over an interval.*

Vocabulary: Slope, Gradient, Secant line, Tangent line, Time, Velocity, Displacement, Average rate of change, Instantaneous rate of change, Quotient, Difference, Differentiation, Derivatives

Introduction

The purpose of this topic is to demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability. The topic begins by showing the derivative of a function as the slope of the tangent line to the graph of the function. Some problems of physics that involve the rate of change of a function are used to demonstrate the interpretation of the derivatives as an instantaneous rate of change.

In order to cover the most important topics and save the precious class time, you may spend much time in deciding what and how to teach.

Like the other units, devote more time to **learner–centered approach** than **teacher – centered approach**. Accordingly, involve students in

- ✓ finding derivatives of functions.
- ✓ solving problems such as finding slopes and tangent lines.
- ✓ developing concepts and exploring rules.
- ✓ generating realistic and illustrative examples.

The following are some strategies in teaching this topic.

- Help students to make oral and written presentations in order to enable them develop their mathematics thinking.
- Help students to understand the main concepts in differentiation so that they can simplify expressions and summarize the rules. This will result in improving the efficiency of their memory, communicate mathematically and manage their time properly.
  - ✓ Identify the problems from the textbook or from other learning sources for those students who are fast learners and have exceptionally high abilities.
  - ✓ Provide additional problems to slow learners who face difficulty in understanding the concept.

Additional exercise problems for high ability students

Draw the graph of each of the following functions and their derivatives in the same coordinate system and demonstrate to the class.

1. $f(x) = \sqrt{x}$
2. $f(x) = \dfrac{1}{x}$
3. $f(x) = x + \sin x$
4. $f(x) = \dfrac{x}{x-1}$

Solution:

1. 2.

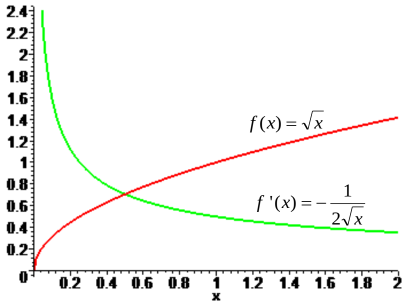


Figure 3.1

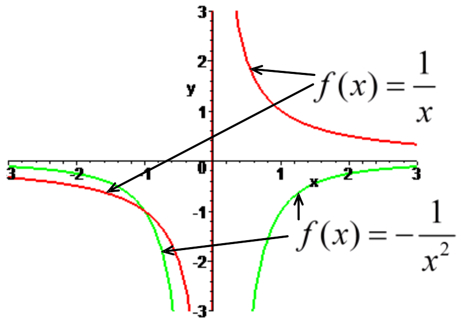


Figure 3.2

3. 4.

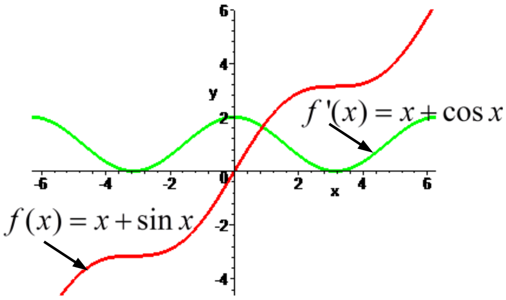


Figure 3.3

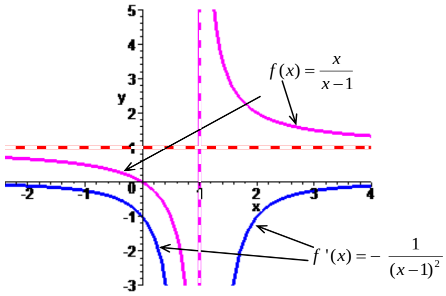


Figure 3.4

Assessment

Oral questions are especial important aspect of assessment during instruction.

For example

- ✓ ask students to determine the slope of a curved graph at a given point.

  Such as $f(x) = x^2$ at $x = 2$, $g(x) = x^3$ at $x = -1$
- ✓ ask students to restate differentiability of a function f at appoint x_o.

The exercises can also be used for classroom discussion, class work or home work.

Ask students to determine the equation of a line tangent to the graph of a function f at a point x_o.

When almost every student completes the exercises, you may give solutions to all exercises so that they could check their own work.

The exercises are designed to assess how well students mastered the topics and whether or not students are ready to start the next topic.

Also, encourage students to use calculators and computers to solve some problems and to sketch graphs.

3. 1.1 Understanding Rates of Change

Teaching Notes

Activity 3.1 is designed to associate and link derivatives and slopes in a meaningful way. The concept of slope occurs in many applications of mathematics. Some of them are shown in the activity exercises. Slope represents constant rate of change in linear functions and extends to non-constant rates of change in the coming topics.

Then, by forming different groups discuss Activity 3.1. As it is indicated in the introduction part, encourage student to see the relation between gradient (slope) and the speed of a particle.

Answer to Activity 3.1

1. a. $\Delta C = 2\pi\,(3\text{ cm} + 1\text{ cm}) - 2\pi(3\text{ cm}) = 2\pi\text{ cm}$

   b. $C = 2\pi\,(3\text{ cm} + 1\text{ cm}) = 2\pi(4\text{ cm}) = 8\pi\text{ cm}$, at t = 1

   $C = 2\pi(3\text{ cm} +2\text{ cm}) = 2\pi(5\text{ cm}) = 10\pi\text{ cm}$, at $t = 2$

   $C = 2\pi(3\text{ cm} +3\text{ cm}) = 2\pi(6\text{ cm}) = 12\pi\text{ cm}$, at $t = 3$

   c. $\dfrac{\Delta C}{\Delta t} = \dfrac{2\pi\left(\Delta r\right)}{\Delta t} = 2\pi\,\text{cm/s}$

   d. $\dfrac{\Delta C}{\Delta t} = \dfrac{2\pi\Delta r}{\Delta t} = 2\pi\,\text{cm/s}$

2. a. $t_1 = \dfrac{2\text{km}}{30\,\text{km/hr}} = \dfrac{1}{15}\text{hr}$

   $t_2 = \dfrac{2\text{km}}{120\,\text{km/hr}} = \dfrac{1}{60}\text{hr.}$

   Average speed = $\dfrac{4\text{km}}{\dfrac{1}{15}\text{hr} + \dfrac{1}{60}\text{hr}} = 48\,\text{km/hr}$

   b. Yes, considering a speed limit of $80 km/hr$.

   c. No

3. a.

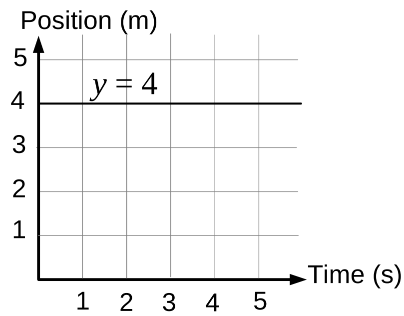


Figure 3.5

b. Slope = 0

c. $0\ \text{m/s}$, for each given interval

4. i. a.

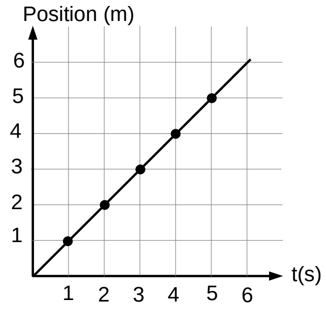


Figure 3.6

b. slope = 1.

c. $1\ \text{m/s}$ for each of the given intervals

ii. a.

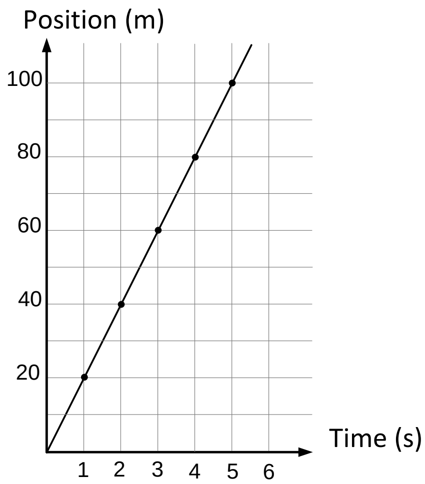


Figure 3.7

b. Slope = 20

c. $20\,\text{m/s}$ for each intervals given

5. The speed is represented by the slope of the line.

3.1.2 Graphical Definition of Derivatives

Teaching Notes

Before starting instruction formally, ask the students what they mean by secant line and tangent line. Obviously, most of them will give the definition of secant and tangent lines of a circle. Then, you can start the lesson by giving the definition for secant and tangent lines to a curve. Students should notice that a tangent line may cross a curve at some other point.

After discussing the worked examples, give Exercise 3.1 as class work and home work. The aim of the exercise is to assess the performance of students in understanding the definition of the derivative of a function at a point.

Answers to Exercise 3.1

1. a. $$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\left((3+h)^2-3-h+3-\left(3^2-3+3\right)\right)}{h}$$
$$= \lim_{h\to 0}\left(h+5\right)=5$$

b. $$\lim_{\delta x\to 0}\frac{\left(f(-1+\delta x)-f(-1)\right)}{\delta x} = \lim_{\delta x\to 0}\frac{1-2\delta x+\left(\delta x\right)^2+1-\delta x+3-\left(1+1+3\right)}{\delta x}$$
$$= \lim_{\delta x\to 0}\frac{\left(\delta x\right)^2-3\delta x}{\delta x} = \lim_{\delta x\to 0}\left(\delta x-3\right) = -3$$

c. $$\lim_{x\to 4}\frac{f(x)-f(4)}{x-4} = \lim_{x\to 4}\frac{x^2-x+3-\left(4^2-4+3\right)}{x-4}$$
$$= \lim_{x\to 4}\frac{x^2-x-12}{x-4} = \lim_{x\to 4}\left(x+3\right)=7$$

2. Problems a to d can be given to high ability students but i) might be appropriate only for top students.

a. –6 b. –6 c. –4 d. $\frac{1}{6}$

e. $$\lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)} = \lim_{x\to -3}\frac{\frac{3x-1}{5x-3}-\frac{-9-1}{-15-3}}{x+3}$$
$$= \lim_{x\to -3^+}\frac{\frac{3x-1}{5x-3}-\frac{5}{9}}{x+3}$$
$$= \lim_{x\to -3}\frac{27x-9-25x+15}{9(x+3)\,(5x-3)} = \lim_{x\to -3}\frac{2}{9(5x-3)} = \frac{-1}{81}$$

f. $\frac{1}{4}$

g. $$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x^2}{x} = 0$$
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = 1$$

$\Rightarrow$ The graph of *f* has no slope at $x = 0$.

h. $$\lim_{x\to -2^+} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x\to -2^+} \frac{x^2 + 2 - \left((-2)^2 + 2\right)}{x + 2}$$

$$= \lim_{x\to -2^+} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x\to -2^+} (x-2) = -4$$

$$\lim_{x\to -2^-} \frac{f(x) - f(-2)}{x - (2)} = \lim_{x\to -2^-} \frac{4x - 2 - (6)}{x + 2}$$

$$\Rightarrow \lim_{x\to -2^-} \frac{4(x - 2)}{x + 2} \text{ doesn't exist}$$

i. $$\lim_{x\to o^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\sqrt{1-x} - 1}{x - 0} = \frac{-1}{2}$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^2 + 1 - 1}{x} = \lim_{x\to 0^+} \frac{x^2}{x} = 0$$

Therefore $\lim_{x\to 0} \frac{\sqrt{1-x}}{x}$ doesn't exist

Hence, the slope of the graph of f at (0, 1) doesn't exist.

3.1.3 Differentiation of a Function at a Point

In the discussion of the worked examples and Exercise 3.1, students have seen that the slope of a function at a point is the limit of the quotient difference. This is defined as the derivative of a function at that point. Also, they realized that the slope of the tangent line is the slope of the curve of the graph of the function at that point. Hence, you can start the lesson by defining a tangent line.

After completely discussing the different notations of derivatives and the worked examples with the class or group give Exercise 3.2 as home work or class work.

Answers to Exercise 3.2

Problems 5 and 6 can be used for classroom discussion.

1. a. Slope, $m = 2$, equation: $y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$
   b. $m = -19$, equation $y - 17 = -19(x - (-2)) \Rightarrow y = -19x - 21$
   c. $m = 3$, equation: $y - 0 = 3(x + 1) \Rightarrow y = 3x + 3$
   d. No slope, equation: $x = 1$
   e. $m = -\frac{1}{2}$, $y - 1 = -\frac{1}{2}(x - 1) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}$
2. a. $y = -6$ b. $y = -4x - 10$ c. $y = 2x - 7$
   d. No tangent line e. $y = x$
3. Suppose the slope of the graph at $x = a$ is 0.

   Then, $\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = 0$

$$\Rightarrow \lim_{x\to a} \frac{x^3 - 3x - (a^3 - 3a)}{x - a} = 0$$

$$\Rightarrow \lim_{x\to a} \frac{x^3 - a^3 - (3x - 3a)}{x - a} = 0$$

$$\Rightarrow \lim_{x\to a} \frac{(x - a)(x^2 + ax + a^2) - 3(x - a)}{x - a} = 0$$

$$\Rightarrow \lim_{x\to a} x^2 + ax + a^2 - 3 = 0 \Rightarrow a^2 + a^2 + a^2 - 3 = 0$$

$$\Rightarrow 3a^2 - 3 = 0 \Rightarrow a = \pm 1$$

4. The slope of the graph of f at $x = a$ is $3a^2 + 2a - 1 > 0$

$$\Rightarrow (3a - 1)(a + 1) > 0 \Rightarrow a > \frac{1}{3} \text{ or } a < -1$$

5. $-2x = 3x^2 - 8 \Rightarrow 3x^2 + 2x - 8 = 0 \Rightarrow x = -2$ or $\frac{4}{3}$

6. Let m_a be the slope of f at $x = a$, then

$$m_a = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x\to a} \frac{x^3 - x^2 - x + 1 - (a^3 - a^2 - a + 1)}{x - a}$$

$$= \lim_{x\to a} \frac{(x - a)(x^2 + ax + a^2) - (x - a)(x + a) - (x - a)}{x - a}$$

$$= \lim_{x\to a} \left(x^2 + ax + a^2 - (x + a) - 1\right)$$

$$= a^2 + a^2 + a^2 - (2a) - 1 \quad = 3a^2 - 2a - 1$$

The graph of $f(x) = x^3 - x^2 - x + 1$ intersects the x-axis at $x = \pm 1$, and the y-axis at $y = 1$

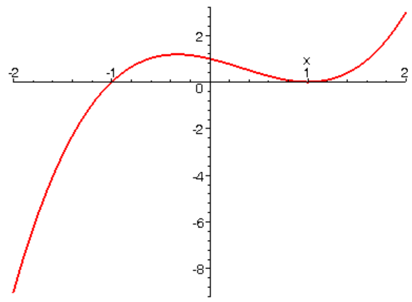


Figure 3.8

a. Clearly the x-axis is tangent to the graph of f at $x = 1$.
The slope of the tangent line at $x = -1$ is $m_{-1} = 3(-1)^2 - 2(-1) = 4$
$\Rightarrow$ The equation of the tangent line is $y - f(-1) = 4(x - (-1))$
$\Rightarrow y - 0 = 4(x + 1) \Rightarrow y = 4x + 4$

b. The graph crosses the y-axis at $(0, 1)$
Hence, $m_o = 3(0^2) - 2(0) - 1 = -1$

The equation of the tangent line at $x = 0$ is $y - f(0) = -1\,(x - 0)$

$\Rightarrow y - 1 = -x \quad \Rightarrow y = 1 - x$

c. $x^3 - x^2 - x + 1 = 1 - x^2 \Rightarrow x^3 - x = 0 \Rightarrow x = 0, \pm 1.$

Therefore, the required tangent lines are those found in (a) and (b) above.

The Derivative as a Function

The aim of Activity 3.2 is to define the derivative as a function at a point. Hence, you may start the lesson by discussing the activity with the class or group. Then discuss the worked examples using the different notations for derivatives.

Answers to Activity 3.2

1. $\mathbb{R}$
2. $\mathbb{R}\setminus\{0\}$ becuase $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ = doesn't exist
3. $\lim\limits_{x\to 2^+}\dfrac{f(x) - f(2)}{x-2} = \lim\limits_{x\to 2}\dfrac{4x - 4 - (8-4)}{x-2} = 4$

   $\lim\limits_{x\to 2^-}\dfrac{f(x) - f(2)}{x-2} = \lim\limits_{x\to 2}\dfrac{x^2 - 4}{x-2} = 4$. Therefore, $f'(x_o)$ exists $\forall x_o \in \mathbb{R}$.

Exercise 3.3 is concerned with the derivative of a function at a point. Questions 8 to 15 should be given as group work. Your role is to move around the groups and help those who are looking for you. Finally, ask some students to demonstrate the solutions on the black board.

The exercise is also designed to assess how well students have mastered the derivatives of a function and to decide whether or not they are ready to determine the intervals on which a function is differentiable.

Answers to Exercise 3.3

1. 1   2. 2   3. $2x + 4$
4. $\lim\limits_{t\to x}\dfrac{f(t) - f(x)}{t-x} = \lim\limits_{t\to x}\dfrac{-(t-x)}{tx\,(t-x)} = \lim\limits_{t\to x}\left(\dfrac{-1}{tx}\right) = \dfrac{-1}{x^2}$, $x^2 \neq 0$
5. $\dfrac{1}{2\sqrt{x}}$
6. $3x^2$
7. $$\lim_{t\to x}\frac{(3t-2)^2 - (3x-2)^2}{t-x} = \lim_{t\to x}\frac{(3t-2)-(3x-2))(3t-2+(3x-2))}{t-x}$$

   $$= \lim_{t\to x}\frac{(3t-3x)(3t+3x-4)}{t-x}$$

   $$= \lim_{t\to x} 3(3t+3x-4) = 18x - 12$$

8. $4x + 3x^2$

9. $-\dfrac{1}{3\sqrt[3]{x^2}}.$

10. $$\lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{\dfrac{t+2}{3-2t} - \dfrac{x+2}{3-2x}}{t - x}$$

$$= \lim_{t\to x} \frac{3(t-x)+4(t-x)}{(t-x)(3-2t)(3-2x)} = \frac{7}{(3-2x)^2}$$

11. $\left(x - \dfrac{3}{x}\right)^2 = x^2 - 6 + \dfrac{9}{x^2}$

From previous examples, its derivative is $2x - \dfrac{18}{x^3}$

12. –5

13. $2 + \dfrac{2}{7}x + 5x^4$

14. $\left(x + \dfrac{1}{x^2}\right)^3 = x^3 + 3 + \dfrac{3}{x^3} + \dfrac{1}{x^6}.$

By observing some of the above exercises, the derivative is found to be $3x^2 - \dfrac{9}{x^4} - \dfrac{6}{x^7}$

15. $$\lim_{t\to x} \frac{\sqrt[3]{t} + t - \dfrac{1}{\sqrt{t}} - \sqrt[3]{x} - x + \dfrac{1}{\sqrt{x}}}{t - x} = \lim_{t\to x} \frac{\sqrt[3]{t} - \sqrt[3]{x}}{t - x} + \lim_{t\to x} \frac{t-x}{t-x} + \lim_{t\to x} - \frac{\left(\dfrac{1}{\sqrt{t}} - \dfrac{1}{\sqrt{x}}\right)}{t-x}$$

$$= \lim_{t\to x} \frac{\sqrt[3]{t} - \sqrt[3]{x}}{\left(\sqrt[3]{t} - \sqrt[3]{x}\right)\left(\sqrt[3]{t}^2 + \sqrt[3]{tx} + \sqrt[3]{x^2}\right)} + 1 + \frac{1}{2x^{\frac{3}{2}}}$$

$$= \lim_{t\to x} \frac{1}{\sqrt[3]{t^2} + \sqrt[3]{tx} + \sqrt[3]{x^2}} + 1 + \frac{1}{2x^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x^2} + \sqrt[3]{x^2}} + 1 + \frac{1}{2x^{\frac{3}{2}}} = \frac{1}{3\sqrt[3]{x^2}} + 1 + \frac{1}{2x^{\frac{3}{2}}}$$

3.1.4 Differentiability on an Interval

Students already know how to find the derivative of a function at an arbitrary point. This sub-topic deals with determining the interval on which a function is differentiable. Discuss the worked examples with the class or group. Then give Exercise 3.4 as class work or homework. It is good if the students discuss the exercise in pairs.

Answers to Exercise 3.4

1. $(-\infty,\ \infty)$ 2. $(-\infty,\ \infty)$ 3. $\mathbb{R}\setminus\{0\}$

4. $f(x) = \sqrt{x-2} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x-2}}$. The domain of f' is $(2,\ \infty)$.

5. $$f'(x) = \lim_{t\to x} \frac{\sqrt{9-4t^2} - \sqrt{9-4x^2}}{t-x}$$

$$= \lim_{t\to x} \frac{9-4t^2-(9-4x^2)}{(t-x)\ \sqrt{9-4t^2}+\sqrt{9-4x^2}}$$

$$= \lim_{t\to x} \frac{-4\,(t^2-x^2)}{(t-x)\left(\sqrt{9-4t^2}+\sqrt{9-4x^2}\right)} = -4\lim_{t\to x}\frac{t+x}{\sqrt{9-4t^2}+\sqrt{9-4x^2}}$$

$$= -4\ \frac{(2x)}{2\sqrt{9-4x^2}} = -\frac{4x}{\sqrt{9-4x^2}}$$

The domain of f' is $\left(-\frac{3}{2},\ \frac{3}{2}\right)$

Draw the graphs for problems 6, 7, 8, 9 and 10 to clarify the given answer.

6. $(-\infty,\ 5]$; $[5,\ \infty)$. But $f'(5)$ doesn't exist.

7. $\left(-\infty,\ \frac{3}{2}\right]$ and $\left[\frac{3}{2},\ \infty\right)$. But $f'\left(\frac{3}{2}\right)$ doesn't exist.

8. $(-\infty,\ 0]$; $[0,\ 1]$; $[1,\ \infty)$. But $f'(0)$ and $f'(1)$ do not exist.

9. $$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x \leq 0 \end{cases}$$

f is differentiable on $(-\infty,\ \infty)$

10. $$f(x) = \begin{cases} x, & \text{if } x > 1 \\ 2-x^2, & \text{if } x \leq 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1, & \text{if } x > 1 \\ -2x, & \text{if } x < 1 \\ \nexists, & \text{if } x = 1. \end{cases}$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x-1}{x-1} = 1$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{2-x^2-(1)}{x-1} = -2$$

$\Rightarrow f'(1)$ doesn't exist, But f has derivatives on $(-\infty,\ 1]$; $[1,\ \infty)$

11. $(-\infty,\ 3)$; $(3,\ \infty)$

3.2 DERIVATIVES OF SOME FUNCTIONS

Periods allotted: 3 periods

Competency

At the end of this sub-unit, students will be able to:

- *find the derivatives of power, simple trigonometric, exponential and logarithmic functions.*

Vocabulary: Differentiation, Power functions, Trigonometric functions, Exponential functions, Logarithmic functions.

Introduction

This topic first deals with finding the derivatives of some functions at a point. Next, it extends to find a new function $f\,'(x)$, the derivative of $f\,(x)$ with respect to x in the domain of f for which the derivative is defined.

Teaching Notes

You may start the lesson by revising the concepts of power, polynomial, rational, trigonometric, exponential and logarithmic functions. After the revision, it is also possible to:

- ✓ use the class activities and the examples to discuss the differentiation of power, simple trigonometric, exponential and logarithmic functions.
- ✓ encourage students to explore the rules of differentiation such as the power rule.
- ✓ give as many problems as possible until most of the students can use the rules of differentiation automatically.

Assessment

To assess your students, asking oral questions on the revision of polynomial and rational functions is recommended. However, it is also possible to:

- ✓ use the examples to assess students during instruction and
- ✓ give various exercises as class work/homework.

Differentiation of Power Functions

You can start the lesson by discussing Activity 3.3. Students should be reminded that the limits in question 1 are the derivatives of some simple functions. In question 2, they should be guided to reach the conclusion that $(x^n)' = nx^{n-1}$. After discussing the worked examples with the class or group, give Exercise 3.5 as class work and homework.

Answers to Activity 3.3

Activity 3.3 is designed to enable students to derive the derivatives of the above mentioned functions using the definition of differentiation.

1. a. $3x^2$ b. $\frac{1}{3x^{\frac{2}{3}}}$ c. 0 d. 1 e. $\ln a$

2. a. 1 b. $2x$ c. $4x^3$ d. $-\frac{1}{x^2}$ e. $\frac{-5}{x^6}$

   f. $\frac{1}{2\sqrt{x}}$ g. $\frac{-3}{2x^{\frac{5}{2}}}$ h. $-\frac{1}{3x^{\frac{4}{3}}}$

Answers to Exercise 3.5

Consider this Exercise as Oral Questions

1. a. $3x^2$ b. $5x^4$ c. $11x^{10}$ d. $-7x^{-8}$

   e. $-10x^{-11}$ f. $\frac{3}{4}x^{-\frac{1}{4}}$ g. $\frac{-5}{3}x^{-\frac{8}{3}}$ h. $\frac{7}{2}x^{\frac{5}{2}}$

   i. $\frac{13}{6}x^{\frac{7}{6}}$ j. $(1-\pi)\,x^{-\pi}$

2. $f(x) = \sqrt[3]{x}\; x^2 = x^{\frac{1}{3}+2} = x^{\frac{7}{3}}$

   $\Rightarrow f'(x) = \frac{7}{3}x^{\frac{4}{3}}$

   a. $f'(0) = 0$ b. $f'(1) = \frac{7}{3}$ c. $f'(8) = \frac{112}{3}$

3. $f'(x) = -\frac{4}{5}x^{-\frac{9}{5}}$

   a. $-\frac{4}{5}a^{-\frac{9}{5}} = -\frac{4}{5} \Rightarrow a = 1$

   The equation of the tangent line is: $y - 1 = -\frac{4}{5}(x-1)$; $f'(1) = -\frac{4}{5}$

   $\Rightarrow y = -\frac{4}{5}x + \frac{9}{5}$

   b. $f'(a) = \infty$ or $f'(a) = -\infty \Rightarrow a = 0$

   $\Rightarrow$ The y-axis is the line tangent to the graph of f at $x = 0$.

Derivatives of Simple Trigonometric Functions

You can start the lesson by asking oral questions like the addition identity sin(x + h) and cos (x + h). Then give Exercise 3.6 to assess the competencies.

Answers to Exercise 3.6

1. a. $-\cos x$ b. $\sin\theta$ c. $\sec(x)\tan(x)$ d. $-\csc(t)\cot(t)$

2. a. $y = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$ b. $y = -x + \frac{\pi}{2}$

   c. $y = 2x - \frac{\pi}{2} + 1$

3. $y - f(a) = f'(a)(x - a)$

   $\Rightarrow y - \sin(a) = \cos(a)(x - a)$

   $\Rightarrow y = \cos(a)(x - a) + \sin(a)$

   The y-intercept is $\cos(a)(0 - a) + \sin(a) = -a\cos(a) + \sin(a) = \frac{\sqrt{3}}{6} - \frac{\pi}{3}$

   By inspection, $a = \frac{\pi}{3}$.

   $$\Rightarrow y = \cos\frac{\pi}{3}\left(x - \frac{\pi}{3}\right) + \sin\frac{\pi}{3}$$

   $$= \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \sin\frac{\pi}{3} = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

   The x- intercept is $\frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} = 0$

   $$\Rightarrow x = \frac{\pi}{3} - \sqrt{3}.$$

Derivatives of Exponential and Logarithmic Function

Like the proceeding functions, you can start the lesson by asking some oral questions such as the properties of exponential and logarithmic function. Then discuss the worked examples with the class and give Exercise 3.7 as class work and home work.

Answers to Exercise 3.7

In this exercise, you may ask question 1 orally.

1. a. $3^x \ln 3$ b. $\frac{1}{2}\sqrt{3^x}\ln 3$ c. $49^x \ln(49)$

d. $(\pi + 1)^x \ln(\pi + 1)$ e. $4e^{4x}$ f. $\frac{3}{2}\sqrt{e^{3x}}$

g. $36^x \ln 36$ h. $\frac{1}{x\ln 7}$ i. $\frac{1}{x}$

j. $-\frac{1}{x\ln 8}$ k. $\frac{3}{5x}$

2. $y - e = f'(1)(x - 1)$

$\Rightarrow y - e = e(x - 1) \Rightarrow y = ex$

3. $f(x) = \ln x \Rightarrow f'\left(\frac{1}{e^3}\right) = \frac{1}{\left(\frac{1}{e^3}\right)} = e^3$

The equation of the tangent line is:

$y - (-3) = e^3\left(x - \frac{1}{e^3}\right)$

$\Rightarrow y = e^3x - 4$

4. $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2.$

$\lim\limits_{x\to\infty} 2^x \ln 2 = \infty$ and $\lim\limits_{x\to-\infty} 2^x \ln 2 = 0$

5. $g(x) = \log_2 x \Rightarrow g'(x) = \frac{1}{x\ln 2}$

$\lim\limits_{x\to 0^+} \frac{1}{x\ln 2} = \lim\limits_{x\to\infty} \frac{x}{\ln 2} = \infty$

3.3 DERIVATIVES OF COMBINATIONS AND COMPOSITIONS OF FUNCTIONS

Periods allotted: 12 periods

Competencies

At the end of this sub-unit, students will be able to:

- *apply the sum and difference formulae of differentiation functions.*
- *apply the product and quotient formulae of differentiation functions.*
- *apply the chain rule.*
- *differentiate composition of functions.*
- *find the* 2^{nd} *and the* n^{th} *derivative of a function.*

Vocabulary: Combinations of functions, Sum rule, Difference rule, Product rule, Quotient rule, Chain rule

Introduction

The goal of this topic is to derive the derivatives of the combinations and compositions of functions. Thus, the activities and the examples are prepared to help students to understand, derive and use the rules of differentiation of combination of functions.

Furthermore, this topic aims at enabling students to understand the chain rule and applying it in the calculation of a variety of compositions of functions.

Higher derivatives are discussed at the end of the topic. Since $f'(x)$ is a function of x, the second derivative $f''(x)$ is found by differentiating $f'(x)$. Through sequential steps, the n^{th} derivative $f^{(n)}(x)$ will be found.

In this topic, the activity exercises are designed to develop the knowledge and skills of students in computing the derivatives of combinations and compositions of functions.

The exercises are designed to provide a review of the rule of differentiation of combinations and compositions of functions.

You can determine which exercises need further work and which are grasped successfully.

Teaching Notes

- ✓ Discuss the sum and difference of two functions by producing examples.
- ✓ Help students to explore the sum and difference rules.
- ✓ Discuss the applications of the sum and difference of several functions using appropriate examples such as, the derivatives of polynomial functions.
- ✓ Revise the product and quotient of two functions.
- ✓ Discuss the product and quotient rules.
- ✓ Discuss the applications of the product rule on the derivatives of the product of three or more functions.
- ✓ Revise the composition of functions using appropriate examples.
- ✓ Discuss the chain rule and demonstrate the applications using appropriate examples.
- ✓ Illustrate the differentiation of compositions of functions by making use of the chain rule with different examples.
- ✓ Introduce the second derivative $f''(x)$ and the n^{th} derivative $f^{(n)}(x)$ at $x = x_o$ on interval I.
- ✓ Discuss the derivatives of second or higher order polynomial, rational, power, trigonometric functions.

To start the lesson, you can use Activity 3.4 to ask oral questions. For example, ask students to answer question 1 orally. Ask students to do question 2 on the black board.

Give questions 3 to 6 as group work and check the solutions. Then discuss the worked examples with the class or group.

Answers to Activity 3.4

	$f(x) + g(x)$	$f(x) - g(x)$	$f(x)\,g(x)$	$\frac{f(x)}{g(x)}$
i.	$3x^2 + 7x + 2$	$-3x^2 - 3x$	$6x^3+13x^2+7x+1$	$\frac{2x+1}{3x^2+5x+1}$
ii.	$\frac{8x^2+16x^3+6x-1}{2+4x}$	$\frac{8x^2+16x^3+2x+3}{4x+2}$	$\frac{8x^3-4x^2+2x-1}{2(1+2x)}$	$\frac{2(4x^2+8x^3+2x+1)}{2x-1}$
iii.	$e^x + \sin x$	$e^x - \sin x$	$e^x \sin x$	$\frac{e^x}{\sin x}$
iv.	$\log(x^2+1) + \cos\left(\frac{1}{x}\right)$	$\log(x^2+1) - \cos\left(\frac{1}{x}\right)$	$\log(x^2+1)\cos\left(\frac{1}{x}\right)$	$\frac{\log(x^2+1)}{\cos\left(\frac{1}{x}\right)}$
v.	$3^{x^2+1} + \tan x$	$3^{x^2+1} - \tan x$	$3^{x^2+1}\tan x$	$\frac{3^{x^2+1}}{\tan x}$

2. a. 4 b. $8x + 3$ c. $\frac{1}{2\sqrt{x}} - \frac{1}{x^2}$ d. $\frac{10}{(4x+2)^2}$

3. a and b are true ; c and d are false. This is a mistake committed by some students. The derivative of the product (respectively the quotient) is not equal to the product of the derivatives (respectively the quotient of the derivatives).

4. a. $2x + 3^x \ln 3$ b. $\cos x - \frac{1}{x\ln 2}$

   c. $(2x)3^x + (3^x \ln 3)(x^2 - 1)$ d. $\frac{\cos x \log_2 x - \frac{\sin x}{x\ln 2}}{(\log_2 x)^2}$

5. $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$.

   This is to introduce to students that differentiability is stronger than continuity.

6. $f(x) = \begin{cases} x^2, & \text{if } x \le 2 \\ 8 - 2x, & \text{if } x > 2 \end{cases}$

   The graph has a sharp point at (2, 4)

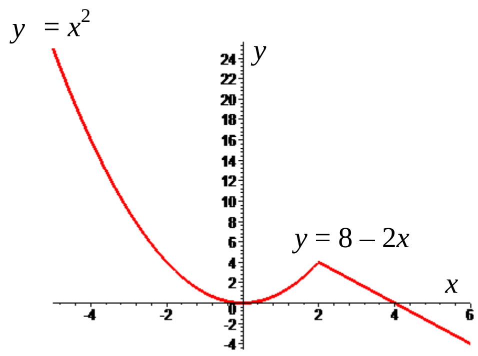


Figure 3.9

f is a continuous function.

It is differentiable on $\mathbb{R}\setminus\{2\}$.

3.3.1 Derivatives of a Sum or Difference of Two Functions

Before you state and prove the sum or difference rule, encourage students to express $(f+g)'(x_o)$ in terms of $f'(x_o)$ and $g'(x_o)$. After you state and prove the theorem formally, discuss the worked examples with the class or group.

3.3.2 Derivative of Product and Quotient of Functions

Encourage students to discuss Activity 3.5 and see that the derivative of the product is not equal to the product of the derivatives.

Answers to Activity 3.5

1. a. $$\lim_{h\to 0}\frac{(x+h)\,e^{x+h}-xe^x}{h}=\lim_{h\to 0}\frac{xe^{x+h}+he^{x+h}-xe^x}{h}$$

$$=\lim_{h\to 0}\left(\frac{xe^x\left(e^h-1\right)}{h}+\frac{he^{x+h}}{h}\right)$$

$$=xe^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}+\lim_{h\to 0}e^{x+h}$$

$$=(xe^x)\times 1+e^x=xe^x+e^x$$

b. $$\lim_{h\to 0}\frac{(x+h)\sin(x+h)-x\sin x}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)\sin x\cos h+(x+h)\cos x\sin h-x\sin x}{h}$$

$$= \lim_{h\to 0} x\sin x\frac{(\cos h - 1)}{h} + \lim_{h\to 0}\sin x\cos h + \lim_{h\to 0} x\cos x\frac{\sin h}{h} + \lim_{h\to 0}\cos x\sin h$$

$$= (x \sin x) \times 0 + \sin x(1) + x \cos x \times 1 + \cos x(0)$$

$$= \sin x + x \cos x$$

2. a. $xe^x + e^x$ b. $\sin x + x \cos x$

Then state and prove the product rule. After discussing the worked examples with the class or group discuss, activity 3.6 to introduce the quotient rule.

Here, the students should notice that the quotient rule will be easily proved using the product rule and the derivative of $\left(\frac{1}{g(x)}\right)$.

Answers to Activity 3.6

1. $f(x) = e^x \Rightarrow f'(x) = e^x$

   $g(x) = x \Rightarrow g'(x) = 1$

   a. $f'(x)\, g(x) = x\, e^x$ b. $g'(x)\, f(x) = e^x$

   c. $\frac{f'(x)}{g'(x)} = e^x$ d. $\frac{e^x(x) - 1.e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

2. a. $-\frac{1}{x^2}$ b. $-\frac{1}{2x^{\frac{3}{2}}}$ c. $-\frac{3}{(3x+1)^2}$

   d. $-\frac{1}{2(\sqrt{x}+1)^2\sqrt{x}}$ e. $-\frac{e^x}{(e^x+1)^2}$

After discussing the worked examples using the different notations give Exercise 3.8 as class work and homework. You should ask some selected problems to be demonstrated on the black board by students of different abilities. You should also encourage students to appreciate the product and the quotient rules.

Assessment

You can use any of the following to assess students' understanding.

- ✓ ask oral questions on combinations and compositions of functions.
- ✓ ask oral questions to restate the different rules of differentiation.

- ✓ use the class activities and the examples to assess students during instruction.
- ✓ give various exercise problems on the second and n[th] derivatives of functions.
- ✓ give the exercises as class work/home work.
- ✓ use the review exercises to assess the entire topic.

Answers to Exercise 3.8

1. a. $3x^2 - 2x - 1$ b. $\dfrac{7}{2\sqrt{x}} + e^x - \cos x$

   c. $h'(x) = \dfrac{x + 5 - x}{(x+5)^2} = \dfrac{5}{(x+5)^2}$ d. $1 + \cos x - e^x$

   e. $$k'(x) = \frac{(x \sin x)'\,(x - e^x) - (x - e^x)'\,(x \sin x)}{(x - e^x)^2}$$
   $$= \frac{-e^x \sin x + x^2 \cos x - xe^x \cos x + xe^x \sin x}{(x - e^x)^2}$$

   f. $$f'(x) = \frac{\frac{1}{2\sqrt{x}}\,(x \cos x) - (\cos x - x \sin x)\sqrt{x}}{x^2 \cos^2 x} = \frac{-\cos x + 2x \sin x}{2x\sqrt{x}\cos^2 x}$$

   g. $\dfrac{2\cos^2 x - 1}{-\sin^2 x \cos^2 x}$.

   h. $-\dfrac{1}{x^2} + \dfrac{2}{x^3} + \dfrac{x \sin x - 2\cos x}{x^3 \cos^2 x}$

   i. $\dfrac{4(x^2 + 1) - 2x(4x + 5)}{(x^2 + 1)^2} = \dfrac{-2(2x^2 + 5x - 2)}{(x^2 + 1)^2}$

   j. $2x \ln x + x$

2. a. $\dfrac{1}{x} + e^x$ b. $(2x - 2)e^x + (x^2 - 2x - 3)e^x$

   c. $$\frac{dy}{dx} = \frac{-\frac{1}{x}(x^2) - 2x(1 - \ln x)}{x^4} = \frac{-1 - 2(1 - \ln x)}{x^3}$$
   $$= -\frac{2 \ln x - 3}{x^3}$$

   d. $$\frac{dy}{dx} = \frac{2x \cos x - (-\sin x)\,(x^2 + 1)}{\cos^2 x}$$
   $$= \frac{2x \cos x + \sin x\,(x^2 + 1)}{\cos^2 x}$$

e. $\frac{dy}{dx} = \frac{\left(e^x + 1\right)\left(x+1\right) - \left(e^x + x - 1\right)}{\left(x+1\right)^2} = \frac{xe^x + 2}{\left(x+1\right)^2}$

f. $\frac{dy}{dx} = \frac{\cos x\left(1 - \cos x\right) - \left(\sin^2 x\right)}{\left(1 - \cos x\right)^2} = \frac{1}{\cos x - 1}$

g. $\frac{dy}{dx} = \frac{-cox\left(x + \cos x\right) - \left(1 - \sin x\right)^2}{\left(x + \cos x\right)^2} = \frac{2\sin x - x\cos x - 2}{\left(x + \cos x\right)^2}$

h. $\frac{dy}{dx} = \frac{\left(e^x \sin x\right)'\left(e^x + 1\right) - e^x\left(e^x \sin x\right)}{\left(e^x + 1\right)^2}$

$= \frac{e^x \sin x + e^{2x}\cos x + e^x \cos x}{\left(e^x + 1\right)^2}$

i. $\frac{dy}{dx} = \frac{2x\left(x + \ln x\right) - \left(1 + \frac{1}{x}\right)x^2}{\left(x + \ln x\right)^2} = \frac{x\left(x + 2\ln x - 1\right)}{\left(x + \ln x\right)^2}$

j. $\frac{dy}{dx} = \left(e^x\right)'\left(1 + x^2\right)\tan x + e^x\left(1 + x^2\right)'\tan x + e^x\left(1 + x^2\right)\left(\tan x\right)'$

$= e^x (1 + x^2) \tan x + e^x (2x) \tan x + e^x (1 + x^2) \sec^2 x$

k. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2 + 1}{x^2 - 1}\right) = \frac{2x\left(x^2 - 1\right) - 2x\left(x^2 + 1\right)}{\left(x^2 - 1\right)^2} = \frac{-4x}{\left(x^2 - 1\right)^2}$

l . The aim of this problem is to introduce the chain rule. Here, we consider it as the product of three functions.

$\frac{dy}{dx} = \left(e^x - \sqrt{x}\right)'\left(e^x - \sqrt{x}\right)^2 + \left(e^x - \sqrt{x}\right)\left(e^x - \sqrt{x}\right)'\left(e^x - \sqrt{x}\right)$
$+ \left(e^x - \sqrt{x}\right)^2\left(e^x - \sqrt{x}\right)' = 3\left(e^x - \frac{1}{2\sqrt{x}}\right)\left(e^x - \sqrt{x}\right)^2$

3. a. $f'(x) = \frac{2}{\left(x+1\right)^2} \Rightarrow f'(0) = 2$

The equation of the tangent line is:

$y - f(0) = 2(x - 0)$

$\Rightarrow y = 2x - 1$

b. $f'(x) = \dfrac{3\left(4 - x^2\right) - (-2x)\,(3x + 1)}{\left(4 - x^2\right)^2}$

$\Rightarrow f'(1) = \dfrac{17}{9}$

The equation of the tangent line is $y - \dfrac{4}{3} = \dfrac{17}{9}\,(x - 1) \Rightarrow\; y = \dfrac{17}{9}x - \dfrac{5}{9}$

c. $f'(x) = e^x \sin x + e^x \cos x \Rightarrow f'(0) = 1.$
The equation of the tangent line is, $y = x$

d. $f'(x) = \dfrac{(2x - 4)\,\left(e^{-x} + 1\right) - \left(-e^{-x}\right)\left(x^2 - 4x\right)}{\left(e^{-x} + 1\right)^2}$

$\Rightarrow f'(0) = -2$
$\Rightarrow$ The tangent line has an equation $y = -2x$

3.3.3 The Chain Rule

Start the lesson by discussing the introductory example in the student textbook.

In Activity 3.7, after completing question 1, allow students to explain how the derivatives of the powers could be determined without expanding the expressions.

In question 2, the students should apply the result of the discussion in question 1.

Answers to Activity 3.7

2. a. $4\,(2x^3 + 1)^3\,(6x^2)$ b. $11\,(2x^3 + 1)^{10}\,(6x^2)$
   c. $n\,(2x^3 + 1)^{n-1}\,(6x^2)$

3. a. $3\cos x + 1$ b. $f(h(x)) = 3\left(\dfrac{3x - 1}{x^2 + 1}\right) + 1$

$= \dfrac{x^2 + 9x - 2}{x^2 + 1}$

   c. $h(g(x)) = \dfrac{3\,\text{co}x - 1}{\cos^2 x + 1}$ d. $f'(x) = 3 \Rightarrow f'(g(x)) = 3$

   e. $g'(x) = -\sin x$
$\Rightarrow f'(g)(x)).\ g'(x) = -3\sin x$

   f. $h'(x) = \dfrac{3(x^2 + 1) - 2x(3x - 1)}{\left(x^2 + 1\right)^2} = \dfrac{-3x^2 + 2x + 3}{\left(x^2 + 1\right)^2}$

$\Rightarrow h'(g(x)).\ g'(x) = \left(\dfrac{-3\cos^2 x + 2\cos x + 3}{\left(\cos^2 x + 1\right)^2}\right)(-\sin x)$

3.3.4 Derivatives of Composite Functions

The activities give students the chance to explore the different rules for the derivatives of combinations of functions. Likewise, encourage students to explore the derivatives of compositions of functions.

Basically, question 3 of Activity 3.7 is designed to introduce the chain rule.

Ask some students from different groups to demonstrate the solutions on the board.

Some students may write the derivative of $(2x^3 + 1)^4$ to be simply $4(2x^3 + 1)^3$. You should give careful explanations to such misunderstandings. Having completed the activity, state and prove the chain rule. At last, students must be told to do the worked examples repeatedly until they master the rules.

Additional exercise problems for high ability students

Find the derivatives of each of the following functions with respect to x.

1. $f(x) = \dfrac{x^2}{e^x \sin x}$
2. $f(x) = \dfrac{e^x + \sin x}{x - e^x}$
3. $f(x) = \dfrac{\ln\sqrt{x^2+1}}{e^{\sqrt{x^2+1}}}$
4. $f(x) = \dfrac{1+\cos x}{x - \sin x}$

Solution:

1. $f'(x) = \dfrac{2x(e^x \sin x) - (e^x \sin x + e^x \cos x)x^2}{(x-e^x)^2} = \dfrac{xe^x(2\sin\ x - x\sin\ x + x\cos\ x)}{(x-e^x)^2}$

2. $f'(x) = \dfrac{(e^x + \cos x)(x - e^x) - (1-e^x)(e^x + \sin x)}{(x-e^x)^2}$

$$= \dfrac{xe^x + x\cos x - \cos x\, e^x - e^x - \sin x + e^x \sin x}{(x-e^x)^2}$$

3. Note that $\ln\sqrt{x^2+1} = \dfrac{1}{2}\ln(x^2+1)$

$$f'(x) = \dfrac{\dfrac{1}{2(x^2+1)}.2x\mathrm{e}^{\sqrt{x^2+1}} - \mathrm{e}^{\sqrt{x^2+1}}.\dfrac{x}{\sqrt{x^2+1}}\ln\left(\sqrt{x^2+1}\right)}{\mathrm{e}^{2\sqrt{x^2+1}}}$$

$$= -\dfrac{\dfrac{1}{2}x\left(-2\sqrt{x^2+1} + \ln(x^2+1)x^2 + \ln(x^2+1)\right)e^{-\sqrt{x^2+1}}}{(x^2+1)\left(\sqrt{x^2+1}\right)}$$

4. $f'(x) = \dfrac{-\sin(x - \sin x) - (1-\cos x)(1+\cos x)}{(x-\sin x)^2}$

$$= \dfrac{-x\sin x + \sin^2 x - (1-\cos^2 x)}{(x-\sin x)^2}$$

$$= \frac{-x\sin x+\sin^2 x+\cos^2 x-1}{(x-\sin x)^2} = \frac{-x\sin x}{(x-\sin x)^2}$$

Answers to Exercise 3.9

1. a. e^{x+6} b. $10(x+5)^9$ c. $48(4x+5)^{11}$

d. $3\cos(3x)$ e. $-2x\sin(x^2+1)$

f. $$f'(x) = \frac{e^{x+2}\left(xe^x - 1\right) - \left(xe^x + e^x\right)e^{x+2}}{\left(xe^x - 1\right)^2}$$

$$= -\frac{e^{x+2} + e^{2x+2}}{\left(xe^x - 1\right)^2}$$

g. $-5e^{-5x}\sin(4x^2+5x+1) + e^{-5x}(8x+5)\cos(4x^2+5x+1)$

h. $$\frac{x+1}{\sqrt{x^2+2x+3}}$$

i. $$\frac{2x}{\left(x^2+4\right)\ln 3}$$

j. $$f'(x) = \frac{2x\left(x + \ln\left(x^2+9\right)\right) - \left(1 + \frac{2x}{\left(x^2+9\right)}\right)x^2}{\left(x + \ln\left(x^2+9\right)\right)^2}$$

$$= \frac{x^4 + 9x^2 + 2x^3\ln\left(x^2+9\right) + 18x\ln\left(x^2+9\right) - 2x^3}{\left(x^2+9\right)\left(x + \ln\left(x^2+9\right)\right)^2}$$

k. $$f'(x) = \frac{\cos x\sqrt{2x+1} - \frac{\sin x}{\sqrt{2x+1}}}{2x+1} = \frac{\left(2x+1\right)\cos x - \sin x}{\left(2x+1\right)\sqrt{2x+1}}$$

l. $f(x) = \sin(x^2) + \cos(x^2)$

$\Rightarrow f'(x) = 2x\cos(x^2) - 2x\sin(x^2)$

m. $$f(x) = \ln\left(\frac{1}{x^2+1}\right) \Rightarrow f(x) = -\ln\left(x^2+1\right)$$

$$\Rightarrow f'(x) = \frac{-2x}{x^2+1}$$

n. $$f(x) = \ln\sqrt{x^2+1} \Rightarrow f(x) = \frac{1}{2}\ln\left(x^2+1\right)$$

$$\Rightarrow f'(x) = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

o. $f'(x) = \cos\sqrt{\ln\left(x^2 + 7\right)}.\ \dfrac{1}{2\sqrt{\ln\ (x^2 + 7)}}.\ \dfrac{1}{x^2 + 7}.\ 2x$

$$= \frac{x\cos\ \sqrt{\ln\left(x^2 + 7\right)}}{\left(x^2 + 7\right)\ \sqrt{\ln\ \left(x^2 + 7\right)}}$$

p. $\ln\left(\dfrac{10x + 3}{5x^2 + x + 3}\right) = \ln\left(10x + 3\right) - \ln\left(5x^2 + x + 3\right)$

$\Rightarrow f'(x) = \dfrac{10}{\left(10x + 3\right)} - \dfrac{10x + 1}{5x^2 + x + 3}$

q. $f'(x) = \left(e^{-\sqrt{x^2 + 1}}\right)'.\sin\left(\sqrt{x^2 + 1}\right) + e^{-\sqrt{x^2 + 1}}.\ \left(\sin\sqrt{x^2 + 1}\ \right)'$

$$= \frac{-x}{\sqrt{x^2 + 1}}\ e^{-\sqrt{x^2+1}}\sin\sqrt{x^2+1} + e^{-\sqrt{x^2+1}}.\cos\sqrt{x^2+1}.\frac{x}{\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2 + 1}}\ e^{\sqrt{x^2 + 1}}\ \left(-\sin\sqrt{x^2+1} + \cos\sqrt{x^2 + 1}\ \right)$$

r. $f(x) = \ln\ \sqrt{\cos\left(x^2 + 3\right)} = \dfrac{1}{2}\ln\ \left(\cos\ \left(x^2 + 3\right)\right)$

$\Rightarrow f'(x) = \dfrac{-x\ \sin\left(x^2 + 3\right)}{\cos\ \left(x^2 + 3\right)} = -x\tan\left(x^2+3\right)$

2. a. $f(x) = xe^{-\sqrt{x+1}} \Rightarrow f'(x) = e^{-\sqrt{x+1}} + xe^{-\sqrt{x+1}}.\ \dfrac{1}{-2\sqrt{x+1}}$

$\Rightarrow f'(0) = \dfrac{1}{e}$

The equation of the tangent line is $y = \dfrac{1}{e}x$

b. $f(x) = e^{x-x^2} \Rightarrow f'(x) = -2x\ e^{2-x^2}$

$\Rightarrow f'(1) = -2e$

$\Rightarrow$ The tangent line is $y - e = -2e(x - 1)$

$\Rightarrow y = -2ex + 3e$

c. $f(x) = \ln\left(\dfrac{x + 1}{\cos x}\right)$

$\Rightarrow f'(x) = \dfrac{\cos x}{x + 1}\ \dfrac{\left(\cos x + \left(\sin\ x\right)\ \left(x + 1\right)\right)}{\cos^2 x} = \dfrac{\cos x + \sin x + x\sin x}{(x+1)\ \cos x}$

$\Rightarrow f'(0) = 1$

$\Rightarrow$ The tangent line is $y - f(0) = f'(0)\ (x - 0)$

$\Rightarrow y - 0 = x \Rightarrow y = x$

d. $f(x) = \dfrac{e^{3x+2}}{1-2x} \Rightarrow f'(x) = \dfrac{3e^{3x+2}(1-2x) + 2e^{3x+2}}{(1-2x)^2}$

$\Rightarrow f'(-1) = \dfrac{3e^{-1}(3) + 2e^{-1}}{9} = \dfrac{11}{9e}$

$\Rightarrow$ The tangent line is $y - \dfrac{1}{3e} = \dfrac{11}{9e}(x+1)$

$\Rightarrow y = \dfrac{11}{9e}x + \dfrac{14}{9e}$

e. $f(x) = (8 - x^3)\sqrt{2-x}$

$\Rightarrow f'(x) = -3x^2\sqrt{2-x} - \dfrac{(8-x^3)}{2\sqrt{2-x}}$

$\Rightarrow f'(-2) = -28$

$\Rightarrow$ The tangent line is $y - 32 = -28(x+2) \Rightarrow y = -28x - 56 + 32$

$\Rightarrow y = -28x - 24$

3. a. $\dfrac{d}{dx}\sqrt{1+x^6} = \dfrac{1}{2\sqrt{1+x^6}}.\dfrac{d}{dx}(x^6+1) = \dfrac{6x^5}{2\sqrt{1+x^6}} = \dfrac{3x^5}{\sqrt{1+x^6}}$

b. $\dfrac{d}{dx}\left(\sqrt{1+3x^2}\,e^x\right) = \dfrac{6x}{2\sqrt{1+3x^2}}e^x + e^x\sqrt{1+3x^2}$

$= \dfrac{3xe^x + e^x(1+3x^2)}{\sqrt{1+3x^2}} = \dfrac{e^x(3x^2+3x+1)}{\sqrt{1+3x^2}}$

c. $\dfrac{6x^2\sqrt{1+x^4} - \dfrac{4x^3(2x^3)}{2\sqrt{1+x^4}}}{\sqrt{1+x^4}^2} = \dfrac{6x^2\sqrt{1+x^4}^2 - 4x^6}{(1+x^4)\sqrt{1+x^4}} = \dfrac{2x^2(3+x^4)}{(1+x^4)\sqrt{1+x^4}}$

d. $\dfrac{dy}{dx} = 2\sqrt{\dfrac{x^3+1}{x^2}}\left(\dfrac{2x(x^3+1) - 3x^2(x^2)}{(x^3+1)^2}\right)$

$= \dfrac{1}{2}\sqrt{\dfrac{x^3+1}{x^2}}\,\dfrac{(2x^4+2x-3x^4)}{(x^3+1)^2} = \dfrac{2x-x^4}{2(x^3+1)^2}\sqrt{\dfrac{x^3+1}{x^2}}$

e. $9\left(\dfrac{2x-1}{3-4x}\right)^8.\dfrac{d}{dx}\left(\dfrac{2x-1}{3-4x}\right) = 9\left(\dfrac{2x-1}{3-4x}\right)^8\dfrac{(2(3-4x)+4(2x-1))}{(3-4x)^2}$

$= 18\dfrac{(2x-1)^8}{(3-4x)^{10}}$

f. $\cos\left(\ln\sqrt{e^x}\right)=\cos\left(\frac{1}{2}\ln\left(e^x\right)\right)=\cos\left(\frac{1}{2}x\right)$

$$\Rightarrow \frac{d}{dx}\cos\left(\ln\sqrt{e^x}\right)=\frac{d}{dx}\cos\left(\frac{1}{2}x\right)=-\frac{1}{2}\sin\left(\frac{1}{2}x\right)$$

g. $\frac{d}{dx}\left(ax+b\right)^r = ra\left(ax+b\right)^{r-1}$

3.3.5 Higher Order Derivatives of a Function

To start the lesson on the second derivatives, ask question 1 of Activity 3.8 orally. Then give questions 1 to 3 as group work and assign a few group representatives to explain the solutions on the black board.

Students should realize that the second derivative is the derivative of the function f' and the third derivative is the derivative of the function f''. Likewise $f^{(n)}$ is the derivative of $f^{(n-1)}$.

Additional Exercise Problems for high ability students

Find the second derivative of each of the following functions.

a. $y=\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$ b. $y=\ln\left(\frac{1+\tan\left(\frac{1}{2}x\right)}{1-\tan\left(\frac{1}{2}x\right)}\right)$.

Solution:

a. $$\frac{dy}{dx}=\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\times\sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\times\frac{1}{2}$$

$$=\frac{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}\times\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$=\frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)2\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$=\frac{1}{\sin\left(2\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)} = \frac{1}{\cos x}=\sec x$$

$$\Rightarrow\frac{d^2y}{dx^2} = \frac{\sin x}{\cos^2 x}=\sin x\sec^2 x.$$

2. $$\frac{dy}{dx} = \frac{1-\tan\left(\frac{1}{2}x\right)}{1+\tan\left(\frac{1}{2}x\right)} \times \frac{\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)\left(1-\tan\left(\frac{1}{2}x\right)\right)+\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)\left(1+\tan\left(\frac{1}{2}x\right)\right)}{\left(1-\tan\left(\frac{1}{2}x\right)\right)^2}$$

$$= \frac{\sec^2\left(\frac{1}{2}x\right)}{1-\tan^2\left(\frac{1}{2}x\right)} = \frac{1}{\cos^2\left(\frac{1}{2}x\right)} \times \frac{\cos^2\left(\frac{1}{2}x\right)}{\cos^2\left(\frac{1}{2}x\right)-\sin^2\left(\frac{1}{2}x\right)} = \frac{1}{\cos x} = \sec x.$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(\sin x) = \sin x \sec^2 x$$

3. Find a formula for the n^{th} derivative of $y = \ln(x+1)$.

Solution:

$$\frac{dy}{dx} = \frac{1}{x+1},\ \frac{d^2y}{dx^2} = -\frac{1}{(x+1)^2},\ \frac{d^3y}{dx^2} = \frac{2}{(x+1)^3},\ \frac{d^4y}{dx^4} = -\frac{6}{(x+1)^4}$$

Inductively it can be shown that $\dfrac{d^ny}{dx^n} = \dfrac{(-1)^{n-1}(n-1)!}{(x+1)^n}$.

Answers to Activity 3.8

1. $f(x) = x^3 + 4x + 5$

   a. $f'(x) = 3x^2 + 4$ b. $\left(f'(x)\right)' = \left(3x^2 + 4\right)' = 6x$

2. $f(x) = \begin{cases} x^2, & \text{if } x < 3 \\ 6x - 9, & \text{if } x \geq 3 \end{cases}$

   a. $f'(x) = \begin{cases} 2x, & \text{if } x < 3 \\ 6, & \text{if } x \geq 3 \end{cases}$

   $f'(3) = 6$

   b.

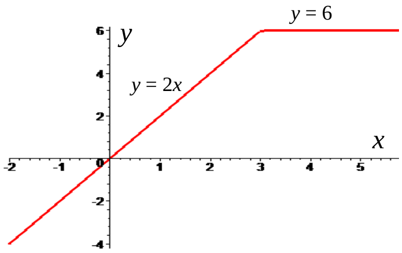


Figure 3.10

$$\left(f'(x)\right)' = \begin{cases} 2, & \text{if } x < 3 \\ 0, & \text{if } x \geq 3 \end{cases}$$

f' is not differentiable at $x = 3$

3. $f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$

4. $f(x) = |x|x$

$$\Rightarrow f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} \Rightarrow (f'(x))' = \begin{cases} 2, & \text{if } x \geq 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow (f'(x))' \text{ does not exist}$$

Discuss the worked examples with the class or group. Encourage students to demonstrate some of the examples on the blackboard.

Make sure that students are able to determine higher derivatives without the aid of additional activities.

A variety of exercise problems are given in Exercise 3.10. Inform the students that they are going to attempt these problems for themselves.

Allocate each question to the class and the solutions should be checked and displayed on the black board.

Answers to Exercise 3.10

1. a. $f'(x) = 3$, $f''(x) = 0$

   b. $f'(x) = 12x^2 - 12x + 7$, $f''(x) = 24x - 12$

   c. $f'(x) = \dfrac{1}{2\sqrt{x}} + \cos x$, $f''(x) = -\dfrac{1}{4x\sqrt{x}} - \sin x$

   d. $f'(x) = \dfrac{3}{2}\sqrt{x} + \cos x$, $f''(x) = \dfrac{3}{4\sqrt{x}} - \sin x$

   e. $f'(x) = \dfrac{\cos x\,(x+1) - \sin x}{(x+1)^2}$

   $$f''(x) = \frac{\left(-\sin x(x+1) + \cos x - \cos x\right)(x+1)^2 - 2(x+1)\left(\cos x(x+1) - \sin x\right)}{(x+1)^4}$$

   $$= \frac{\left(-(x+1)\sin x\right)(x+1)^2 - 2(x+1)\left((x+1)\cos x - \sin x\right)}{(x+1)^4}$$

   $$= -\frac{\sin x\,(x+1)^2 - 2\left((x+1)\cos x - \sin x\right)}{(x+1)^3}$$

   f. $f'(x) = \dfrac{2x}{x^2+1}$, $f''(x) = \dfrac{2\left(x^2+1\right) - 2x(2x)}{\left(x^2+1\right)^2}$

   g. $f'(x) = \dfrac{2x\,(x+1) - \left(x^2 - 4\right)}{(x+1)^2} = \dfrac{x^2 + 2x + 4}{(x+1)^2}$

$$f''(x) = \frac{(2x + 2)(x+1)^2 - 2(x+1)(x^2 + 2x + 4)}{(x+1)^4}$$

$$= \frac{(2x+2)(x+1) - 2(x^2 + 2x + 4)}{(x+1)^3} = -\frac{6}{(x+1)^3}$$

h. $f'(x) = \sec x \tan x$

$f''(x) = \sec x \tan^3 x + 2\sec^3 x \tan^3 x + 3\sec^3 x \sin x = \sec x \tan^2 x + \sec^3 x$

i. $$f'(x) = \frac{2x\sqrt{4 - x^2} + \frac{x}{\sqrt{4-x^2}}.x^2}{4 - x^2}$$

$$= \frac{2x(4 - x^2) + x^3}{(4 - x^2)\sqrt{4 - x^2}} = \frac{8x - x^3}{(4-x^2)\sqrt{4-x^2}}$$

$$f''(x) = \frac{(8x - x^3)(4-x^2)^{\frac{3}{2}} - (8x - x^3)\left((4-x^2)^{\frac{3}{2}}\right)}{(4-x^2)^3}$$

$$= \frac{(8-3x^2)(4-x^2)^{\frac{3}{2}} + 3x(8x - x^3)(4-x^2)^{\frac{1}{2}}}{(4-x^2)^3} = \frac{4(8+x^2)}{(x^2-4)^2\sqrt{4-x^2}}$$

2. a. $\frac{dy}{dx} = 3e^{3x+2}$; $\frac{d^2y}{dx^2} = 9e^{3x+2}$

b. $$\frac{dy}{dx} = \frac{d}{dx}\log_3{}^{\sqrt{x+1}} = \frac{1}{2}\frac{d}{dx}\log_3{}^{(x+1)} = \frac{1}{2(x+1)\ln 3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2\ln 3}\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{1}{2\ln 3(x+1)^2}$$

c. $$y = \ln\left(\frac{1}{x^2+1}\right) = -\ln(x^2+1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{x^2+1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2(x^2+1) - 2x(-2x)}{(x^2+1)^2} = \frac{2x^2 - 2}{(x^2+1)^2}$$

d. $$\frac{dy}{dx} = 2\cos(2x+1)(-\sin(2x+1)) \times 2$$

$= -2(2\cos(2x+1)\sin(2x+1)) = -2\sin(4x+2)$

$$\Rightarrow \frac{d^2y}{dx^2} = -8\cos(4x+1)$$

e. $\dfrac{dy}{dx} = 3(\ln x)^2\left(\dfrac{1}{x}\right) \Rightarrow \dfrac{d^2y}{dx^2} = 6\ln x\left(\dfrac{1}{x}\right)\left(\dfrac{1}{x}\right) - 3\dfrac{(\ln x)^2}{x^2} = \dfrac{3\ln x}{x^2}\ (2 - \ln x)$

f. $\dfrac{dy}{dx} = \dfrac{-3x^2}{1 - x^3} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-6x(1-x^3)-(-3x^2)(-3x^2)}{(1 - x^3)^2} = \dfrac{-3x^4 - 6x}{(1 - x^3)^2}$

g. $y = \ln\left(\dfrac{x}{\sqrt{x+2}}\right) = \ln x - \dfrac{1}{2}\ln(x + 2)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{2(x + 2)}$.

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2} + \dfrac{1}{2(x + 2)^2} = -\dfrac{x^2 + 8x + 8}{2x^2\ (x + 2)^2}$

h. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}\ \sin\sqrt{x} + e^{\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\ \cos\ \sqrt{x}\right)$

$= \dfrac{1}{2}\ \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\ (\sin\ \sqrt{x} + \cos\sqrt{x})$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{1}{2}\dfrac{\left(\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}.\sqrt{x} - \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}\right)}{x}(\sin\sqrt{x} + \cos\sqrt{x})$

$+\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\cos\sqrt{x} - \dfrac{1}{2\sqrt{x}}\sin\sqrt{x}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{2}e^{\sqrt{x}}\left(\dfrac{\sqrt{x}-1}{x\sqrt{x}}\right)(\sin\ \sqrt{x} + \cos\sqrt{x}) + \dfrac{e^{\sqrt{x}}}{4x}\ (\cos\sqrt{x} - \sin\sqrt{x})\right)$

$= \dfrac{e^{\sqrt{x}}}{4x\sqrt{x}}\ (2\sqrt{x}\ \cos\sqrt{x} - \sin\sqrt{x}\ - \cos\sqrt{x})$

i. $\dfrac{dy}{dx} = \cos\ (2x\ \cos x).\ (2\ \cos x - 2x\ \sin x)$

$\dfrac{d^2y}{dx^2} = -\sin\ (2x\ \cos x)\ (2\cos x - 2x\ \sin x)\ (2\cos x - 2x\ \sin x)$

$+ \cos(2x\cos x)\ (-2\sin x - 2\sin x - 2x\cos x)$

$= -4\sin(2x\cos x)\ (\cos x - x\sin x)^2 + \cos(2x\cos x)\ (-4\sin x - 2x\cos x)$

j. $\dfrac{dy}{dx} = \dfrac{d}{dx}\ \ln\ (\ln(x)) = \dfrac{1}{\ln x}\ \dfrac{d}{dx}\ \ln(x) = \dfrac{1}{x\ln x}.$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-\dfrac{d}{dx}\ (x\ln x)}{(x\ln x)^2} = \dfrac{-\left(\ln x + x.\ \dfrac{1}{x}\right)}{(x\ln x)^2}. = -\dfrac{(\ln x + 1)}{(x\ln x)^2}.$

k. $\frac{dy}{dx} = \sqrt{x^2 + 1} + \frac{(x + 1)x}{\sqrt{x^2 + 1}}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x}{\sqrt{x^2 + 1}} + \frac{(2x+1)\sqrt{x^2+1} - \frac{x}{\sqrt{x^2 + 1}}(x^2 + x)}{x^2 + 1}$$

$$= \frac{2x^3 + 3x + 1}{(x^2 + 1)\sqrt{x^2 + 1}}.$$

3. a. $f(x) = e^{3x+1} \Rightarrow f'(x) = 3e^{3x+1}$

$f''(x) = 3^2\,(e^{3x+1})$

$f^{(3)}(x) = 3^3\,(e^{3x+1})$

Inductively, it can be shown that $f^{(n)}(x) = 3^n e^{3x+1}$

b. $f(x) = e^{x^2} \Rightarrow f'(x) = 2xe^{x^2}$

$$\Rightarrow f''(x) = 2e^{x^2} + 4x^2e^{x^2} = e^{x^2}\,(4x^2 + 2)$$

Similarly $f^{(3)}(x) = e^{x^2}\left(8x^3 + 12x\right)$

$f^{(4)}(x) = e^{x^2}\left(16x^4 + 48x^2 + 12\right)$

$f^{(5)}(x) = e^{x^2}\left(64x^6 + 480x^4 + 720x^2 + 120\right)$

This problem can be given as a project work.

c. $f'(x) = \frac{-2x}{x^2 + 1}$

$f''(x) = \frac{2(x^2 - 1)}{(x^2 + 1)^2}$

$f^{(3)}(x) = \frac{-4x\,(x^2 - 3)}{(x^2 + 1)^3}$

$f^{(4)}(x) = \frac{12\,(x^4 - 6x^2 + 1)}{(x^2 + 1)^4}$

d. $f'(x) = e^{-x^2 + 7x - 3}(-2x + 7)$

$f''(x) = e^{-x^2 + 7x - 3}\left((-2x + 7))^2 - 2\right)$

$f^{(3)}(x) = e^{-x^2 + 7x - 3}\left((7 - 2x)^3 - 6\,(7 - 2x)\right)$

$f^{(4)} = e^{-x^2 + 7x - 3}\left((7 - 2x)^4 - 12\,(7 - 2x)^2 + 12\right)$

There are a variety of exercise problems on the Review Exercises. You can select the problems for the students based on the level of their ability range.

Assessment

To assess whether students have captured the intended objectives, you can:

- ✓ use the class activities and the examples to assess students during instruction.
- ✓ give various exercise problems on the second and n^{th} derivatives of functions.
- ✓ give the exercises as class work/home work.
- ✓ use the review exercises to assess the entire topic.

In addition, the review exercises are also designed to assess how well students have mastered differentiation in order to start applications of differential calculus.

Answers to Review Exercises on Unit 3

1. $\dfrac{4x+3-(-8+3)}{x+2}=\dfrac{4x+8}{x+2}=\dfrac{4(x+2)}{x+2}$

2. $\dfrac{2x^2+1-(2+1)}{x+1}=\dfrac{2x^2-2}{x+1}=\dfrac{2(x-1)(x+1)}{x+1}$

3. $\dfrac{\dfrac{x+1}{x-2}-\dfrac{1}{4}}{x-(-2)}=\dfrac{4x+4-x+2}{x+2}=\dfrac{3(x+2)}{4(x-2)(x+2)}$

4. $\dfrac{\dfrac{x+1}{x^2}-2}{x-\left(-\dfrac{1}{2}\right)}=\dfrac{-2(2x^2-x-1)}{x^2(2x+1)}=\dfrac{-2(2x+1)(x-1)}{x^2(2x+1)}$

5. $\dfrac{|x+4|}{x+4}$

6. $\dfrac{\sqrt{x}+5-\left(\sqrt{\dfrac{9}{4}}+5\right)}{x-\dfrac{9}{4}}=\dfrac{2(2\sqrt{x}-3)}{4x-9}$

7. $\dfrac{2^x-2^0}{x^0}=\dfrac{2^x-1}{x}$

8. $\dfrac{\sqrt{1-3x^2}-\dfrac{\sqrt{7}}{4}}{x-\dfrac{\sqrt{3}}{4}}=\dfrac{4\sqrt{1-3x^2}-\sqrt{7}}{4x-\sqrt{3}}$

9. 0

10. 0

11. $2x-3$

12. $8x-8$

13 $4x^3-21x^2$

14. $6x-11$

15. $2(x-3)$

16. $75x^2-250x-1$

17. $12x^2 - \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{2\sqrt{x}}$

18. $3^{x-2}\ln 3 + \frac{1}{2\sqrt{x}} + 10x + \frac{1}{x^2}$

19. $e^x - e^{-x}$

20. $4\cos(4x)$

21. $-2x\sin(x^2+4)$

22. $6\sec^2(6x-1)$

23. $\frac{7}{7x+3}$

24. $1 - \frac{4}{x^2}$

25. $1 - 4(x+1)$

26. $-\frac{1}{x^2} + \frac{15}{x^4} - \frac{12}{x^5}$

27. $10x + 5$

28. $-x^{-2} - 2x^{-3} - 6x^{-4}$

29. $\frac{(x-1)'\,x\sqrt{x} - \left(x\sqrt{x}\right)'\,(x-1)}{\left(x\sqrt{x}\right)^2} = \frac{3-x}{2x^2\sqrt{x}}$

30. $\frac{1}{2\sqrt{1-3x^2}} \times \left(1-3x^2\right)' = \frac{-3x}{\sqrt{1-3x^2}}$

31. $\frac{\left(x^2+1\right)'\log_2^x - \left(x^2+1\right)\left(\log_2^x\right)'}{\left(\log_2^x\right)^2} = \frac{2x\,\log_2^x - \frac{\left(x^2+1\right)}{x\ln 2}}{\left(\log_2^x\right)^2} = \frac{2x\ln 2}{\ln x} - \frac{\left(x^2+1\right)\ln 2}{x\left(\ln(x)\right)^2}$

32. $-2x\,e^{4-x^2}$

33. $\frac{d}{dx}\left(x^{\frac{1}{3}} + x^{\frac{2}{3}} + x^{\frac{3}{4}}\right) = \frac{1}{3}x^{\frac{-2}{3}} + \frac{2}{3}x^{\frac{-1}{3}} + \frac{3}{4}x^{\frac{-1}{4}}$

34. $e^{1-x}(1-x)$

35. $\left(x^{-2}\right)'\left(e^x+1\right) + x^{-2}\left(e^x+1\right)' = \frac{-2e^{-x} - 2 + xe^x}{x^3}$

36. $\frac{1}{x}\left(x^2+1\right) + 2x\,\ln x$

37. $8(2x+1)^3$

38. $\frac{3(x-1)^2\,\sqrt{x} - \frac{1}{2\sqrt{x}}\,(x-1)^3}{\left(\sqrt{x}\right)^2} = \frac{(x-1)^2\,(5x+1)}{2x\sqrt{x}}$

39. $\frac{d}{dx}\left(x\ln x - x\right) = \ln x + x.\frac{d}{dx}\ln x - 1 = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$

40. $\left(\log\sqrt{x^2+2}\right)' = \frac{1}{\sqrt{x^2+2}\,\ln 10}.\left(\sqrt{x^2+1}\right)'$

$$= \frac{1}{\ln 10\sqrt{x^2+2}}.\ \frac{1}{2\sqrt{x^2+2}}.2x$$

$$= \frac{x}{\ln 10\left(x^2 + 2\right)} = \frac{x}{\left(x^2+2\right)\ln 10}$$

41. $\dfrac{1}{2x\sqrt{\ln x}} + \dfrac{\sqrt{e^x}}{2} + 2^x \ln 2$

42. $\dfrac{1}{4\sqrt{\ln\sqrt{x}}\,\ln 10}$

43. $e^x(\cos x - \sin x)$

44. $\dfrac{5}{3}\left(\dfrac{1}{x\sin x}\right)^{\frac{2}{3}}\left(\dfrac{1}{x\sin x}\right)' = \dfrac{5}{3}\left(\dfrac{1}{x\sin x}\right)^{\frac{2}{3}}\left(\dfrac{-(x\sin x)'}{(x\sin x)^2}\right)$

$$= \frac{5}{3}\left(\frac{1}{x\sin x}\right)^{\frac{2}{3}}\left(\frac{-\sin x - x\cos x}{(x\sin x)^2}\right) = \frac{5}{3}\left(\frac{1}{x\sin x}\right)^{\frac{8}{5}}(-\sin x - x\cos x)$$

45. $-\sin\sqrt{\ln\left(x^2+1\right)}\,.\,\dfrac{2x}{2\left(x^2+1\right)\sqrt{\ln\left(x^2+1\right)}} = \dfrac{-x\sin\sqrt{\ln\left(x^2+1\right)}}{\left(x^2+1\right)\sqrt{\ln\left(x^2+1\right)}}$

46. $\sec^2\left(\dfrac{x^2-1}{x}\right)\left(\dfrac{x^2+1}{x^2}\right)$

47. $2\sec^3(x+3)\sin(x+3)$

48. $\left(x^{-2}\right)'\sin\left(x^2\right) + \left(x^{-2}\right)\left(2x\cos\left(x^2\right)\right) = -\dfrac{2}{x^3}\sin\left(x^2\right) + \dfrac{2}{x}\cos(x^2)$

49. $\dfrac{x^4+7x^2-10x-4}{\left(x^2+1\right)^2}$

50. $\dfrac{\left(e^x\sin x\right)'\ln x - (\ln x)'\left(e^x\sin x\right)}{(\ln x)^2} = \dfrac{\left(e^x\sin x + e^x\cos x\right)\ln x - \dfrac{e^x\sin x}{x}}{(\ln x)^2}$

$$= \frac{xe^x(\sin x + \cos x)\ln x - e^x\sin x}{x(\ln x)^2}$$

51. $\sqrt{1-(2+x)^{\frac{3}{2}}} - \dfrac{3x\sqrt{2+x}}{4\sqrt{1-(2+x)^{\frac{3}{2}}}}$

52. $\dfrac{e^{\sin\sqrt{x+3}}.\ \cos\sqrt{x+3}}{2\sqrt{x+3}}$

53. $\dfrac{d}{dx}\left(e^x\sin x\right) = e^x\sin x + e^x\cos x = e^x(\sin x + \cos x)$

54. a. $f'(x) = \begin{cases} 3x^2, \text{ if } x \geq 0 \\ 2x, \text{ if } x < 0 \end{cases}$ b. $f'(x) = \begin{cases} \dfrac{-2^x \ln 2}{\left(2^x + 1\right)^2}, \text{ if } x < 1 \\ \text{doesn't exist, if } x = 1 \\ -\dfrac{1}{\left(x+2\right)^2}, \text{ if } x > 1 \end{cases}$

c. $f'(x) = \begin{cases} \dfrac{-2x}{\left(x^2+1\right)\ln 10}, \text{if } x < 3 \\ \text{doesn't exist if } x = 1 \\ -\dfrac{1}{\left(x+3\right)\ln 10}, \text{ if } x > 3 \end{cases}$

55. -3

56. 1

57. $\dfrac{2}{9}$

58. $\dfrac{19\sqrt{2}}{4}$

59. 0

60. 9

61. 20

62. $\sqrt{2}$

63. $\dfrac{e^2\sqrt{3}}{2}$

64. $\dfrac{\sqrt{2}}{2}$

65. 0

66. doesn't exist

67. $f'(x) = \begin{cases} 3x^2, \text{ if } x \leq -1 \\ 3, \text{ if } x > -1 \end{cases}$

$\Rightarrow f'(-1) = 3$

68. $y - 6 = -4(x + 1)$

69. $y - 1 = \dfrac{3}{2}\left(x-1\right)$

70. $y - \dfrac{2}{5} = \dfrac{3}{25}\left(x - 2\right)$

71. $y - 4 = \dfrac{-3}{8}\left(x + 4\right)$

72. $y - \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}\left(x - \dfrac{\pi}{3}\right)$

73. No

74. $y - \sqrt{5} = \dfrac{-2}{5}\sqrt{5}\left(x - 2\right)$

75. $y - 1 = \dfrac{1}{10\ln 10}\left(x - 7\right)$

76. $y - \dfrac{1}{e} = \dfrac{1}{e}\left(x + 2\right)$

77. $y - \dfrac{1}{e} = 0\,(x - c) \Rightarrow y = \dfrac{1}{e}$

78. $y - 1 = 2(x - 1)$

79. $y - 0 = \dfrac{1}{2}\left(x - 0\right)$

80. The curve $y = x^2 - 5x + 1$ crosses the line $y = 7$

$\Rightarrow x^2 - 5x + 1 = 7$

$\Rightarrow x = -1, 6$

$\dfrac{dy}{dx} = 2x - 5 \Rightarrow$ The slope of the curve at

a. $x = -1$ is -7

$\Rightarrow$ The equation of the tangent line at $x = -1$ is $y - 7 = -7\,(x + 1)$

$\Rightarrow y = -7x$

b. $x = 6$ is 7

$\Rightarrow y - 7 = -7\,(x - 6)$

$\Rightarrow y = 7x - 35$

81. $y = \dfrac{1}{x} + x^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{x^2} + 2x$

$\Rightarrow 2x^3 + 3x^2 - 1 = 0$

$\Rightarrow x = -1, \dfrac{1}{2}$

The equation of the tangent line:

a. at $x = -1$ is; $y = -3\,(x + 1)$

b. at $x = \dfrac{1}{2}$ is; $y - \dfrac{9}{4} = -3\left(x + \dfrac{1}{2}\right)$

$\Rightarrow y = -3x + \dfrac{3}{4}$

82. $f'(x) = 6x + 4$

The slope of the graph of f at $x = a$ is $6a + 4$

$\Rightarrow 6a + 4 = -8$

$\Rightarrow 6a = -12 \Rightarrow a = -2$

$\Rightarrow y - 0 = -8\,(x + 2)$ is the equation of the tangent line $a + x = -2$

$\Rightarrow y = -8x - 16 \Rightarrow k = -16$

83. 2

84. $72x^7$

85. $14(x^2 + 5)^5\,(13x^2 + 5)$

86. e^{1-x}

87. $\dfrac{21}{2}x^5 - 12x + 2$

88. $\dfrac{3}{4x^2\sqrt{x}}$

89. $\dfrac{-2\left(x^2 - 1\right)}{\left(x^2 + 1\right)^2}$

90. $-\sin(x - \cos x)\,(1 + \sin x)^2 + \cos(x - \cos x)\cos x$

91. $-2e^x \sin x$

92. $3e^{-2x}\cos x + 4e^{-2x}\sin x$

93. $\dfrac{-48}{(2x + 3)^3}$

94. $\dfrac{2x^3 + 3x^2 + 3x + 8}{(x + 1)^3}$

95. $\dfrac{5\left(8\sqrt{x+3} - 5\right)\left(\sqrt{x+3} + 5\right)^8}{2\,(x + 3)\sqrt{x + 3}}$

96. $-14\sin(4x + 5) + 16x\cos(4x + 5) - 16\sin(4x + 5)\,x^2$

97. $e^{-x}\,(6 - 18x + 9x^2 - x^3)$

98. $\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = \left. e^x \ln x + \dfrac{2e^x}{x} - \dfrac{e^x}{x^2}\right|_{x=1} = e$

UNIT 4 APPLICATIONS OF DIFFERENTIAL CALCULUS

INTRODUCTION

The main task of this unit is to examine the use of calculus in general; and differential calculus in particular in finding extreme values, curve sketching and other applications.

You have already discussed some of the applications of derivatives. But now, students know the differentiation rules and they are in a better position to pursue the applications of differentiation in greater depth. You may therefore help them learn how derivatives affect the shape of the graph of a function and, in particular, to locate maximum and minimum values of functions. You may help students do practical problems that require them to minimize or maximize. For example, everybody wants to minimize costs and maximize profits. You may also ask them what they want to maximize or minimize in their everyday life.

Unit Outcomes

After completing this unit, students will be able to:

- *find local maximum or local minimum of a function in a given interval.*
- *find absolute maximum or absolute minimum of a function.*
- *apply mean-value theorem.*
- *solve simple problems in which the studied theorems, formulae and procedures of differential calculus are applied.*
- *solve application problems.*

Suggested Teaching Aids in Unit 4

You know that students learn in a variety of different ways. Some are visually oriented and more prone to acquire information from photographs or videos. Others do best when they hear instructions rather than read them. Teachers use teaching aids to provide these different ways of learning. It is recommended that you may use:

- Models of graphs.
- Solid figures such as cuboids, sphere, cone, cylinders and computers.
- You can also use other types of materials as long as they help the learners to get the skills required.

This unit begins with an opening problem which may motivate students to follow attentively the unit and see the applications of differential calculus. Therefore, before passing on to any sub-topic of this unit, make students discuss the opening problem.

Answer to Opening Problem

The square sheet has area of 12 m^2

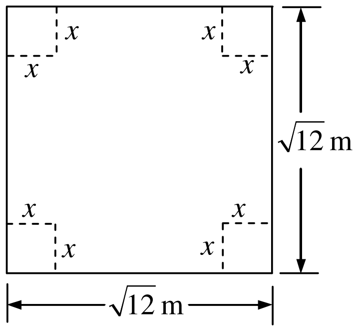


Figure 4.1

The box has to be constructed $\left(\sqrt{12}-2x\right)\left(\sqrt{12}-2x\right)$ square meters and height x meters.

Thus volume of the box is:

$$V(x)=\left(\sqrt{12}-2x\right)^2.x=\left(12-4\sqrt{12}x+4x^2\right).x$$

$$=4x^3-4\sqrt{12}x^2+12x$$

$$V'(x)=12x^2-8\sqrt{12}x+12$$

$$\Rightarrow x^2-\frac{8}{\sqrt{12}}x+1=0$$

$$\Rightarrow x=\frac{1}{\sqrt{3}}\text{ m or } x=\sqrt{3}\text{ m}$$

$$V''(x)=24x-8\sqrt{12}$$

$V''\left(\frac{1}{\sqrt{3}}\right)=8\sqrt{3}-16\sqrt{3}<0$ and $V''\left(\sqrt{3}\right)=24\sqrt{3}-16\sqrt{3}>0$

Since $V''(x)$ is negative for $x=\frac{1}{\sqrt{3}}$ and positive for $x=\sqrt{3}$, the maximum is located at $x=\frac{1}{\sqrt{3}}$ m.

Hence, each of the squares to be cut should have size $\frac{1}{3}\text{m}^2$.

4.1 EXTREME VALUES OF A FUNCTION

Periods Allotted: 13 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *consolidate the concept zero(s) of a function.*
- *find critical numbers and maximum and minimum values of a function on a closed interval.*
- *explain the geometric interpretations of Rolle's theorem and mean value theorem.*
- *find numbers that satisfy the conclusions of mean value theorem and Rolle's theorem.*

Vocabulary: Zeros of a function, Critical number, Minimum value, Maximum value, Extreme value, Relative maximum value, Relative minimum value, Absolute maximum value, Absolute minimum value

Introduction

Under this topic, there are a lot of technical terms which are very important in determining the extreme values of a function whenever they exist. Before defining each of the technical terms, you may use activities which you may think motivate students towards defining the terms.

Teaching Notes

In each of the sub-topics a hint is given how to start the lessons but your creativity in the classroom is very crucial. The intention of the teaching note is to provide you information to use activities, opening problems and group-works to motivate and guide students rather than lecturing. Now, you may start this sub-topic by revising zero(s) of functions because this may help students in finding critical numbers in the foregoing discussion. You may motivate students and assist them to compute zeros of different types of functions such as polynomial functions, rational functions, functions involving radicals, trigonometric functions, etc as given in the students' textbook in revision exercise on zeros of functions.

Answers to Revision Exercises

1. a. $\frac{2}{3}$ b. 2 c. -2

   d. 1 e. 1 f. $7, \frac{2}{7}$ g. 1, 7

2. a. $x = \frac{3}{2}$ b. $x = 1$ c. $x = 1$

   d. $x = -2$ and $x = 2$ e. $x = -3$ and $x = 2$

   f. No x-intercept g. No x-intercept

3. For polynomials, use factor theorem (factorization method).

After having discussed zeros of functions, you immediately go to the discussion of extreme values of a function. In the process, discuss the meaning of extreme values and conditions under which extreme values of a function exists, critical numbers, critical values. To do this, use Activity 4.1 which may motivate the students and help them to discover those conditions under which a function has extreme values.

Answers to Activity 4.1

1. a. S' = {3, 5, 7, 9, 11, 13}

   b. the largest element of S' is 13

   c. the smallest element of S' is 3.

2 a. S' = (3, 13) b. We cannot list all the elements of S'

   c. S' has no largest element d. S' has no smallest element

3. a. S' = [3, 13] b. We cannot list all the elements of S'

   c. 13 is the largest element of *S*'. d. 3 is the smallest element of S'

4. a. No largest and no smallest element.

   b. $\frac{1}{5}$ is the smallest element and has no largest element.

c. $\frac{1}{5}$ is the smallest value and there is no largest element.

d. No largest and no smallest element.

As a summary of the discussion on Activity 4.1, give the definitions of maximum value, minimum value on a given set. State an extreme-value theorem, and discuss the conditions under which the extreme value exists. You may illustrate the theorem by using graphs of functions as given in the students' textbook.

Assessment

Use revision exercise and Activity 4.1 to assess the background of students. You should give homework, class-works and assess students by checking their exercise books.

It is better to use performance assessment such as engaging students in: debate on activities, presentation of assignments and home-works individually or in groups.

The intention of the assessment is to provide a tool for the identification of students who are experiencing major difficulty. Such an assessment is effective for planning a remedial program. So the assessment in Revision Exercise and Activity 4.1 provide you with a valuable profile of the strengths and weaknesses of each student, and thus enable you to direct your teaching at the identified weaknesses.

Answers to Activity 4.2

Just by looking at the graph which is given in students' textbook, we can take those points on the x-axis which give minimum or maximum value.

a. The maximum occurs at $\boldsymbol{a}$ and the minimum occurs at $\boldsymbol{c_1}$.

b. The maximum occurs at $\boldsymbol{c_4}$ and the minimum occurs at $\boldsymbol{c_3}$.

c. The maximum occurs at $\boldsymbol{c_4}$ and the minimum occurs at $\boldsymbol{c_3}$.

d. The maximum occurs at $\boldsymbol{c_4}$ and the minimum occurs at $\boldsymbol{b}$.

At this stage, you may discuss what happens at valleys and peaks as given in the student' textbook and give definitions of relative extreme values, critical values and critical numbers and discuss the importance of critical numbers in finding absolute maximum and absolute minimum and then illustrate it using examples.

Assessment

Use Activity 4.2 to assess the background of students. You should give homework, class-works and assess students by checking their exercise books. For this purpose, you can use Exercise 4.1 or any other similar exercises.

Answers to Exercise 4.1

No	Critical number(s)	Absolute maximum value	Absolute minimum value
1	0	1	–8
2	−1, 0, 1	11	2
3	0, 1	7	0
4	$\frac{\pi}{6}, \frac{5}{6}\pi$	$2+2\pi$	$\frac{5}{6}\pi-\sqrt{3}$
5	0, 2	0	–4
6	−2, 0, 2	64	–64

Activity 4.3 is very important for the understanding of the concepts in Rolle's Theorem and its proof. Just by looking at the graph, which is given in students' textbook, we see that the horizontal tangents are given in **figure 4.3**. And using these, you can help or guide students to answer the questions that follow. Thus, discuss Activity 4.3 with the students, state the theorem and prove it by using the Activity.

Answers to Activity 4.3

1. i. $(x, f(x))\ x\in[a, b]$

   ii. $(c, f(c))$

   iii. $(c_1, f(c_1))$, $(c_2, f(c_2))$ and $(c_3, f(c_3))$

   iv. $(c, f(c))$

2. 0

3. Slope at $(x = c) = f'(c)$, if it exists

Assessment

Use Activity 4.4 to assess the background of students on the relation between slope and derivatives. You should give homework, class-works and assess students by checking their exercise books.

After having discussed Activity 4.4, state and prove the mean value theorem. There are important results that can be followed from the mean value theorem such as increasing and decreasing tests for continuous functions, and first derivative test for extreme values of a function. You may use Activity 4.5 and 4.6 for this purpose to motivate students.

Answers to Activity 4.4

a. Slope of the secant line $AB = \dfrac{f(b)-f(a)}{b-a}$

b. Equation of secant line AB : $y = \dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)$

c. Yes, because given a line and a point not on the line, we can draw exactly one line which is parallel to the given line through the given point.

d. Yes, because given a line and a point not on the line, we can draw exactly one line which is parallel to the given line through the given point.

e. Yes; it is equal to the derivative of the function f at the points of tangency.

f. Since two non-vertical parallel lines have the same slope, we can conclude that the slope of the tangent line is equal to the slope of the secant line; which is $\dfrac{f(b)-f(a)}{b-a}$

g. Yes, it is the difference of value of a function f and a linear function which describes the equation of the secant line.

Answers to Exercise 4.2

1. a. $f(x)=x^2-4x+1$ is a polynomial, hence continuous on [0, 4] and differentiable on (0, 4). Moreover, $f(0)=1=f(4)$. Therefore it satisfies the conditions of Rolle's theorem. By Rolle's Theorem there is a number c in (0, 4) such that $f'(c)=0$; but then $f'(x)=2x-4$ implies that: $2c-4=0$.

   Therefore, $c=2$ satisfies the conclusion of Rolle's Theorem.

   b. $f(x)=x^3-3x^2+2x+5$ is a polynomial, it is continuous on [0, 2] and differentiable on (0, 2).

   $f(0) = 5 = f(2)$. Hence it satisfies the Roll's theorem.

   Thus, by Roll's Theorem, there is a number c in (0, 2) such that $f'(c) = 0$. But $f'(x) = 3x^2 - 6x + 2$ implies that

   $3c^2 - 6c + 2 = 0$

   $\Rightarrow c = 1 \pm \dfrac{\sqrt{3}}{3}$ satisfies the conclusion of Roll's Theorem.

c. $f(x) = \sin 2\pi x$ is continuous on $[-1, 1]$, differentiable on (–1, 1); $f(-1) = \sin(-2\pi) = 0 = \sin(2\pi) = f(1)$.

Therefore, it satisfies the conditions of Rolle's Theorem. By Rolle's Theorem, there is a number c in (–1, 1) such that $f'(c) = 0$; but then

$f'(x) = 2\pi \cos 2\pi x$ implies that: $2\pi \cos 2\pi c = 0$. Therefore $c = \frac{1}{4}$ and

$c = -\frac{1}{4}$ satisfy the conclusion of Rolle's Theorem.

d. $f(x) = x\sqrt{x+6}$ is continuous on $[-6, 0]$, differentiable on (–6, 0);

$f(-6) = 0 = f(0)$. Therefore, it satisfies the conditions of Rolle's Theorem.

By Rolle's Theorem, there is a number c in (–6, 0) such that $f'(c) = 0$;

$c = -4$ satisfies the conclusion of Rolle's Theorem.

2. It does not contradict Rolle's theorem since the function is not differentiable at 0 which is in the interior of the given interval and hence does not satisfy the requirements of Rolle's theorem.

3. It does not contradict Rolle's theorem since the function is not differentiable at 1 which is in the interior of the given interval and hence does not satisfy the requirements of Rolle's theorem.

4. a. $f(x) = 3x^2 + 2x + 5$ is continuous on $[-1, 1]$, differentiable on (–1, 1);

Therefore, it satisfies the conditions of mean value theorem. By mean value theorem, there is a number c in (–1, 1) such that $f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$;

Therefore, $c = 0$ satisfies the conclusion of the mean value theorem.

b. Since f is a polynomial, it is continuous and differentiable for all $x \in \mathbb{R}$, it is continuous and differentiable on [0, 2]. By the mean value theorem there is a number C in (0, 2] such that

$$f'(c) = \frac{f(2) - f(0)}{2-0}$$

$f(2) = 9, f(0) = -1$

$f'(c) = 3c^2 + 1$

$\Rightarrow 5 = 3c^2 + 1 \Rightarrow c = \pm\frac{2\sqrt{3}}{3}$

Since $c \in (0, 2)$ we choose $c = \dfrac{2\sqrt{3}}{3}$.

Hence, $c = \dfrac{2\sqrt{3}}{3}$ satisfies the mean value theorem.

c. f is continuous and differentiable on [0, 1]. Then by the mean-value theorem, there is a number c in (0, 1) such that

$$\frac{f(1) - f(0)}{1 - 0} = f'(\text{c})$$

$$f(1) = 1,\ f(0) = 0,\ f'(c) = \frac{1}{3}c^{\frac{-2}{3}}$$

$$\Rightarrow \frac{1}{3}c^{\frac{-2}{3}} = 1 \Rightarrow c = \frac{1}{3\sqrt{3}}.$$

Hence, $c = \dfrac{1}{3\sqrt{3}}$ satisfies the mean value theorem.

d. Since f is continuous and differentiable on [1, 4], there is a number c in (1, 4] such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$f(4) = \frac{2}{3},\ f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c) = \frac{2}{(c+2)^2}$$

$$\Rightarrow \frac{1}{9} = \frac{2}{(c + 2)^2} \Rightarrow c = -2 \pm 3\sqrt{2}.$$

Since $c \in [1, 4]$, we choose $c = -2 + 3\sqrt{2}$. Therefore, $c = -2 + 3\sqrt{2}$ satisfies the mean value theorem.

5. It does not contradict the mean value theorem since the function is not differentiable at 1 which is in the interior of the given interval and hence does not satisfy the requirements of the mean value theorem.

6. Let $f(x) = x^5 + 10x + 3, f(-1) - 8 < 0, f(1) = 14 > 0$.

   Since f is a polynomial, it is continuous. By the intermediate value theorem, there is a number c between $(-1, 1)$ such that $f'(c) = 0$.

Answers to Activity 4.5

1. a. It rises in some intervals and it falls in some other intervals.

   b. It rises on the intervals [*a*, *p*] and [*r*, *b*] it falls on [*p*, *q*]

   c. on [*q*, *r*]

2. a. Yes

   b. i. False ii. True

   c. i. False ii. False iii. True

   d. $f(x_1) < f(x_2)$

Answers to Activity 4.6

1. The tangent line at any point in the interval (*a*, *d*) rises from left to right as it can be seen from the graph and hence $f'(x) > 0$ on the interval (*a*, *c*) as it represents slope.

2. The tangent line at any point in the interval (*c*, *d*) falls from left to right as it can be seen from the graph and hence $f'(x) < 0$ on the interval (*c*, *d*) as it represents slope. You may discuss with students and answer by giving similar arguments on the remaining intervals.

Answers to Exercise 4.3

1. Strictly increasing on the intervals $(-\infty,-2]$ and $[2,\infty)$ but strictly decreasing on the interval [–2, 2].

2. $f(x) = x - 2\sin x$ on $[0, 2\pi]$, $f'(x) = 1 - 2\cos x$

$$f'(x) = 0 \Rightarrow 1 - 2\cos x = 0$$

$$\Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{3} \text{ or } x = \frac{5}{3}\pi \text{ on the interval } [0, 2\pi].$$

Thus, $f'(x) > 0 \Rightarrow 1 - 2\cos x > 0$

$\Rightarrow \cos x < \frac{1}{2}$ on $[0, 2\pi] \Rightarrow x \in [\frac{\pi}{3}, \frac{5\pi}{3}]$

Hence, *f* is strictly increasing on $[\frac{\pi}{3}, \frac{5\pi}{3}]$.

Likewise; $f'(x) < 0 \Rightarrow 1 - 2\cos x < 0$

$\Rightarrow \cos x > \frac{1}{2}$ on $[0, 2\pi]$

$\Rightarrow x \in [0, \frac{\pi}{3}]$ and $x \in [\frac{5\pi}{3}, 2\pi]$

Hence, f is strictly decreasing on $[0, \frac{\pi}{3}]$ and $[\frac{5\pi}{3}, 2\pi]$.

3. Strictly increasing on the intervals $(-\infty, 0]$ and $[2, \infty)$ but strictly decreasing on the interval [0, 2].

4. Strictly increasing on the intervals $(-\infty, 0]$ and $[1, \infty)$ but strictly decreasing on the interval [0, 1].

5. Strictly increasing on the intervals $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$; strictly decreasing on the intervals $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$.

6. Strictly increasing on the interval $\left[0, \sqrt[3]{\frac{4}{9}}\right]$ and strictly decreasing on $(-\infty, 0]$ and $\left[\sqrt[3]{\frac{4}{9}}, \infty\right]$.

7. Strictly increasing on the interval $(-\infty, \infty)$.

8. Strictly increasing on the interval $[-\frac{2}{3}, \infty)$; but strictly decreasing on the interval $(-1, -\frac{2}{3}]$

9. $f(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$

$$f'(x) = \frac{x+1}{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$$

Since 0, −3 and −1 are critical numbers. f is strictly increasing on [−3, −1] and [0, ∞) and strictly decreasing on (−∞, −3) and [−1, 0]

Strictly increasing on the intervals $(-\infty, -3]$ and $(-1, \infty]$; but strictly decreasing on the interval $[-3, -1]$.

10. Strictly increasing on the intervals $\left(-\infty, 0\right]$ and $[1,\ \infty)$; but strictly decreasing on the interval [0, 1]

    Since 0 and 1 are critical numbers, *f* is strictly increasing on the $(-\infty, 0]$ and $[1, \infty)$ and strictly decreasing on [0, 1].

11. Strictly increasing on the interval $[0, \infty)$; but strictly decreasing on the interval $(-\infty, 0]$

12. Strictly increasing on the interval $[-2, 4]$; but strictly decreasing on the intervals $\left(-\infty, -2\right]$ and $[4,\ \infty)$.

13. Strictly increasing on the interval $[-1, \infty)$; but strictly decreasing on the interval $(-\infty, -1]$

14. Strictly increasing on the interval $[0,\ \infty)$; but strictly decreasing on the interval $(-\infty, 0]$

15. Strictly increasing on the intervals $\left(-\infty, 0\right]$ and $[8,\ \infty)$; but strictly decreasing on the interval [0, 4) and (4, 8].

16. Strictly increasing on the interval $[3,\ \infty)$; but strictly decreasing on the interval $(-\infty, 3]$.

17. Strictly increasing on $(-\infty,\ \infty)$

18. Strictly decreasing on $(-\infty, \frac{3}{2})$

19. Strictly increasing on $[0,\ \infty)$ and strictly decreasing on $(-\infty,\ 0]$

20. Strictly increasing on $[1,\ \infty)$ and strictly decreasing on $(0,\ 1)$

Note: Use questions 17, 18, 19 and 20 to assess more able students and make them present their work in the class.

After having discussed Exercise 4.3, you need to discuss local extreme values of a function on its entire domain. You have already discussed critical numbers. But now the issue is whether all critical numbers give rise to maximum or minimum value. As it has already been discussed not every critical number gives rise to a maximum or a minimum. You therefore need to test whether or not *f* has a local maximum or minimum or neither at a critical number. So, you can use **Figure 4.8** and **Figure 4.9** in the student textbook. This leads you to the discussion of first derivative test for local extreme values of a function. Use Example 10 in student textbook for illustration purpose and give Exercise 4.4 as homework or assignment.

Answers to Exercise 4.4

1. It has only local minimum value which is –35.
2. It has both local maximum and local minimum: its local maximum is 5 and its local minimum is 1.
3. It has both local maximum and local minimum: its local maximum is $\frac{5}{3}\pi + \sqrt{3}$ and its local minimum is $\frac{\pi}{3} - \sqrt{3}$.
4. It has only local maximum value which is $\frac{1}{4}$.
5. It has both local maximum and local minimum: its local maximum is 0 and its local minimum is $-\frac{1}{2}$.
6. It has both local maximum and local minimum: its local maximum is 16 and its local minimum is –16.
7. It has both local maximum and local minimum: its local maximum is $\sqrt[3]{16}$ and its local minimum is 0.
8. It has neither local maximum nor local minimum value.
9. It has neither local maximum nor local minimum value.
10. $f'(x) = 2 - 2x^{\frac{-1}{3}}$

    $f'(x) = 0 \Rightarrow x = 1$

    $f'(x)$ doesn't exist $\Rightarrow x = 0$

    $\Rightarrow x = 0$ and $x = 1$ are critical numbers

    It has both local maximum and local minimum: its local maximum is 0 and its local minimum is -1.

After having discussed Exercise 4.4, you need to discuss concavity and inflection points. You may start this subtopic by posing questions like "what does f'' say about f ?" Let them observe the graph which is given under this topic to distinguish the behavior of the two curves by drawing tangent lines at each point in between a and b as given in students' textbook.

Answers to Activity 4.7

After you have discussed Activity 4.7 with the students, you may conclude the following points about the two curves given as f and g.

Conclusion:

- tangent to the curve f always lies below the curve f on (a, b).
- tangent to the curve g always lies above the curve g on (a, b).

Assessment

Use Activity 4.7 to assess the background of students. You should give homework, class-works and assess students by checking their exercise books.

After having discussed Activity 4.7, you need to give definition of concavity and then discuss concavity tests using second derivative and how the second derivative also helps in determining whether a local extreme point gives a local maximum or minimum. Using the examples given in the student's textbook, illustrate all these concepts as they are important to sketch the graph of a function. Here, examples are given in students' textbook on how to sketch graphs of functions. Summarize the important points that are important in sketching graphs of functions and how it becomes simpler using the concept of derivatives. Use Exercise 4.5 for homework or assignment. You may group students so that able students may help the less able students in sketching graphs of functions in this exercise.

Answers to Exercise 4.5

1. Given $f(x) = x^3 - 12x$ which is a polynomial function and hence its domain is the set of all real numbers.
   a. y-intercept is 0;
      x-intercept are: 0, $2\sqrt{3}$, $-2\sqrt{3}$
   b. No asymptotes
   c. Critical numbers are: -2 and 2
      f is strictly increasing on $(-\infty, -2]$ and $[2, \infty)$; but strictly decreasing on $[-2, 2]$
   d. Using first derivative test you can determine that $f(-2) = 16$ is the local maximum value and $f(2) = -16$ is the local minimum value.
   e. Using 2nd derivative test you can determine that the graph of f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$
   f. Inflection point (0, 0)

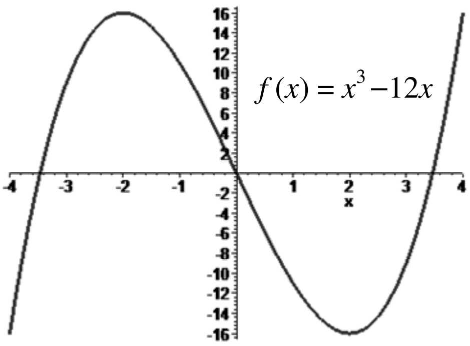


Figure 4.2

2. Given $f(x) = e^x$ which is exponential function and hence its domain is the set of all real numbers
   a. y-intercept is 1 and it has no x-intercept
   b. $y = 0$ is the horizontal asymptote.
   c. No critical number
   d. Using first derivative test $f'(x) = e^x > 0\ \forall\, x$ and hence $f(x) = e^x$ is strictly increasing on $(-\infty, \infty)$
   e. Using 2nd derivative test $f''(x) = e^x > 0\ \forall\, x$ and hence the graph is concave up on $(-\infty, \infty)$
   f. No inflection point.

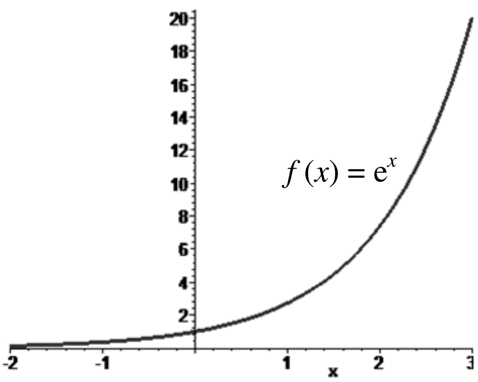


Figure 4.3

3. $f(x) = \ln x$ which is logarithmic function and hence its domain is all positive real numbers: $(0, \infty)$
   a. No y-intercept and its x-intercept is 1.
   b. $x = 0$ is its vertical asymptote
   c. no critical number

   Using first derivative test $f'(x) = \dfrac{1}{x} > 0$ for all positive values of x and hence f is strictly increasing on $(0, \infty)$.

d. No local maximum and no local minimum value

e. Using 2[nd] derivative test $f''(x) = -\dfrac{1}{x^2} < 0$ for all x in the domain, and hence the graph is concave down ward on $(0, \infty)$

f. No inflection point

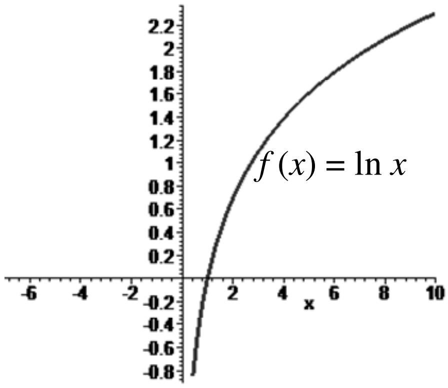


Figure 4.4

4. $f(x) = \dfrac{4}{1+x^2}$, Its domain is the set of all real numbers

a. y-intercept is 4 and it has no x-intercept.

b. $y = 0$ is its horizontal asymptote

c. 0 is the only critical number and using the first derivative test f is strictly decreasing on $[0, \infty)$

d. $f(0) = 4$ is its local maximum value and it has no local minimum value.

e. Using 2[nd] derivative test you can see that the graph of f is concave upward on $(-\infty, 0)$ and $(0, \infty)$.

f. $(0, 4)$ is the inflection point.

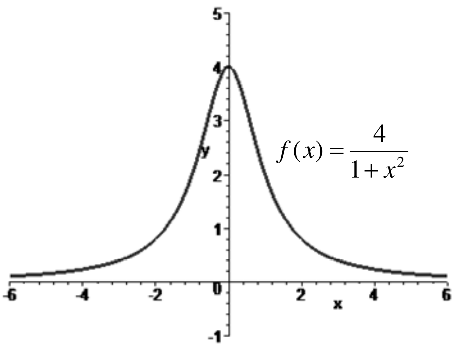


Figure 4.5

5. Given $f(x) = \dfrac{1}{4}x^4 - 2x^2$, we want to sketch the graph by indicating all the pertinent information which is necessary in sketching graphs.

a. The given function is polynomial and hence its domain is the set of all real numbers.

b. The y- intercept is 0; this means that the graph crosses the y-axis at $(0,0)$; and the x-intercepts are: $-2\sqrt{2}, 0,$ and $2\sqrt{2}$ this means the graph crosses the x-axis at $(-2\sqrt{2},0)$, $(0,0)$ and $(2\sqrt{2},0)$.

c. No asymptotes.

d. Critical numbers are: –2, 0, 2; using sign charts you can determine intervals of monotonicity: as f is strictly decreasing on $(-\infty,-2]$ and $[0,2]$ and strictly increasing on $[-2,0]$ and $[2,\infty)$.

e. Just from the same sign chart, by using first derivative test you can determine that the graph has valleys at $(-2,-4)$ and $(2,-4)$. This means –4 is the local minimum value; and a peak at $(0, 0)$. This means 0 is the local maximum value.

f. For intervals of concavity and inflection points, we use the second derivative test. Thus, equating the second derivative of the given function to zero, we get $x=-2\frac{\sqrt{3}}{3}$ and $x=2\frac{\sqrt{3}}{3}$; using sign chart for the second derivatives, you can determine intervals of concavity as: the graph is concave upward on the intervals $\left(-\infty, -2\frac{\sqrt{3}}{3}\right)$ and $\left(2\frac{\sqrt{3}}{3}, \infty\right)$; and concave downward on the interval $\left(-2\frac{\sqrt{3}}{3}, 2\frac{\sqrt{3}}{3}\right)$.

g. Inflection points are: $\left(-2\frac{\sqrt{3}}{3}, -\frac{20}{9}\right)$ and $\left(2\frac{\sqrt{3}}{3}, -\frac{20}{9}\right)$

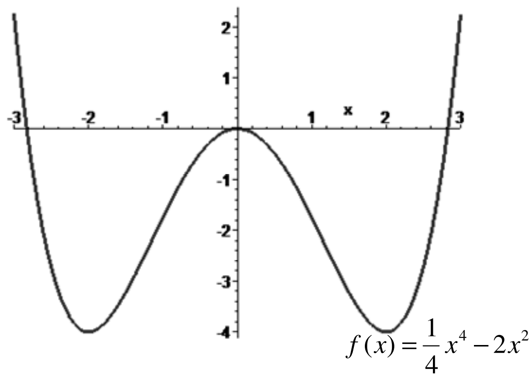


Figure 4.6

6. $f(x) = \frac{2x-6}{x^2-9}$, Domain = $\mathbb{R} \setminus \{-3, 3)$

$f(x) = \frac{2(x-3)}{(x+3)(x-3)} = \frac{2}{x+3}$

a. y-intercept is $\frac{2}{3}$ and its has no x-intercept.

b. $y = 0$ is its horizontal asymptote
$x = -3$ is its vertical asymptote
The graph is discontinuous as $x = 3$

c. $f'(x) = \frac{-2}{(x+3)^2}$ and hence f has no critical numbers.

$f'(x) < 0$ for every x in the domain, and hence

d. f is strictly decreasing on $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$.
No local maximum and minimum value

e. $f''(x) > 0$ for $x > -3$ and hence the graph of f is concave upward on $(-3, 3)$ and $(3, \infty)$
$f''(x) < 0$ for $x < -3$ and hence the graph of f is concave downward on $(-\infty, -3)$

f. It has no inflection points.

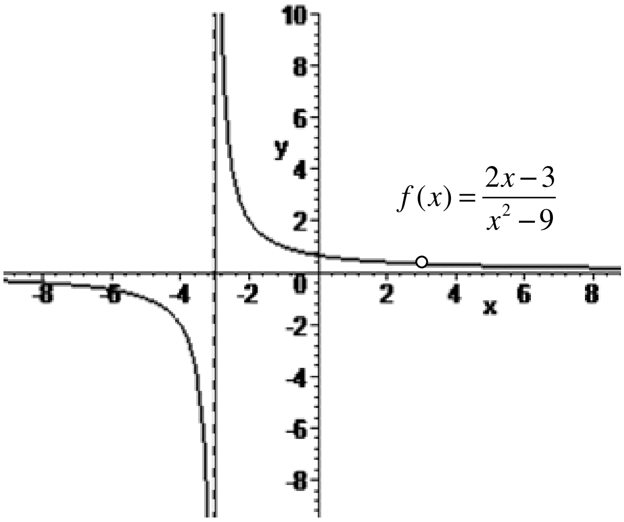


Figure 4.7

7. $f(x) = x^3 - \frac{3}{2}x^2 + 6x$

Domain is the set of all real numbers.

a. y-intercept is 0 and its x-intercept is also 0.
b. No asymptotes
c. No critical number

d. $f'(x) = 3x^2 - 3x + 6 > 0$ for an x and hence f is strictly increasing on $(-\infty, \infty)$

e. The graph of f is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$

f. $\left(\frac{1}{2}, \frac{11}{4}\right)$ is its inflection points.

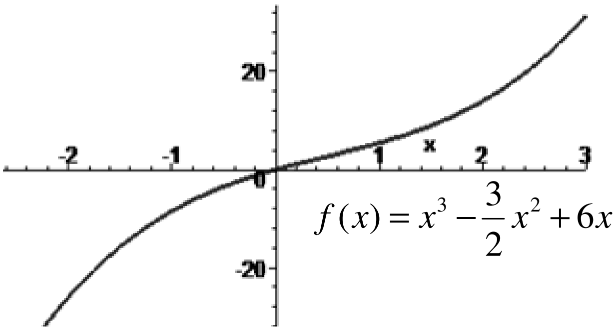


Figure 4.8

8. $f(x) = \dfrac{1}{2^x - 1}$, Domain = $\mathbb{R}\setminus\{0\}$

a. No y-intercept and x-intercept

b. $y = 0$ is its horizontal asymptote and $x = 0$ is its vertical asymptote

c. No critical numbers

The graph is strictly decreasing on $(-\infty, 0)$ and $(0, \infty)$

d. No local maximum and minimum value

e. The graph is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$

f. No inflection point.

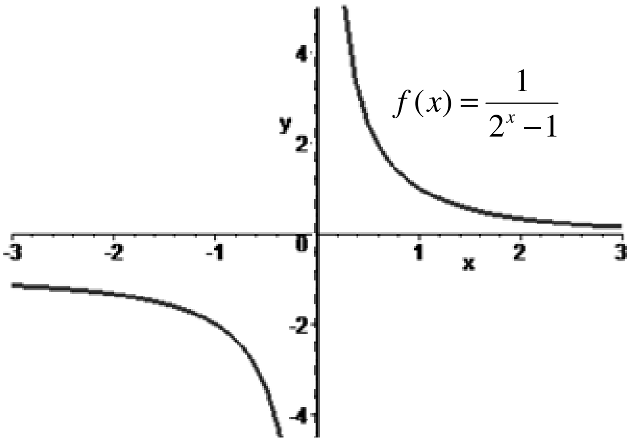


Figure 4.9

4.2 MINIMIZATION AND MAXIMIZATION PROBLEMS

Periods Allotted: 6 Periods

Competency

At the end of this sub-unit, students will be able to:

- *solve problems on application of differential calculus.*

Vocabulary: Minimization, Maximization

Introduction

Now at this stage, we have the basic tools which are necessary to find absolute maximum and minimum value. Therefore, this sub-unit is intended for illustrating the applications of derivatives on how to find maximum and minimum value of functions.

Teaching Notes

You may start this section by summarizing the basic tools which are necessary to find the absolute maximum and absolute minimum. Then extend the notion to practical problems which may be encountered in our daily life. You may encourage students to think about the problems and be able to convert into mathematical language as illustrated by examples in the students' textbook. After having illustrated them thoroughly using the examples given in the students' textbook, you can summarize how the students convert problems into mathematical language and solve them. You can form groups of students so that the more able students may help the less able ones and present their work under this sub-topic.

Assessment

You should give homework, class-works and assess students by checking their exercise books. Engaging students in debate on: activities, presentation of assignments and home-works individually or in groups.

Answers to Exercise 4.6

1. Let x be the first number and y be the second number. Then $xy = 288$ and $y + 2x$ is to be minimized. $xy = 288 \Rightarrow y = \frac{288}{x}$, thus minimize: $f(x) = \frac{288}{x} + 2x$.

   Applying derivative tests, we get $x = 12$ and $y = 24$.

2. a. $f(x) = x^2 - 4$

   Any point on the graph of $f(x) = x^2 - 4$ is of the form $(x, x^2 - 4)$.

Distance $d = \sqrt{(x-0)^2 + ((x^2-4)-2)^2}$

Thus, minimize $g(x) = x^4 - 11x^2 + 36$

$g'(x) = 4x^3 - 22x$

$= 2x(2x^2 - 11)$

$\Rightarrow x = 0$, or $x = \pm\sqrt{\frac{11}{2}}$

Hence $0, -\sqrt{\frac{11}{2}}, \sqrt{\frac{11}{2}}$ are critical numbers.

To check whether these numbers give a minimum distance, we use the second derivative test.

$g''(x) = 12x^2 - 22 > 0$ for $x = -\sqrt{\frac{11}{2}}$ or $x = \sqrt{\frac{11}{2}}$.

Thus, $g\left(-\sqrt{\frac{11}{2}}\right) = \frac{-265}{4} = g\left(\sqrt{\frac{11}{2}}\right)$.

Hence $\left(-\sqrt{\frac{11}{2}}, \frac{3}{2}\right)$ and $\left(\sqrt{\frac{11}{2}}, \frac{3}{2}\right)$ are the closest points to the point with coordinates (0, 2).

b. $f(x) = x^2 + 1$

Any point on the graph of $f(x) = x^2 + 1$ is of the form $(x, x^2 + 1)$

Distance $d = \sqrt{(x-0)^2 + (x^2+1-4)^2}$

Thus minimizing $g(x) = x^4 - 5x^2 + 9$.

$g'(x) = 4x^3 - 10x$

$= 2x(2x^2 - 5)$

$\Rightarrow x = 0$ or $x = \pm\sqrt{\frac{5}{2}}$ are critical numbers.

To check the numbers give a minimize distance, we use the second derivative test.

$g''(x) = 12x^2 - 10 > 0$ for $x = \pm\sqrt{\frac{5}{2}}$.

Therefore, $\left(-\sqrt{\frac{5}{2}}, \frac{7}{2}\right)$ and $\left(\sqrt{\frac{5}{2}}, \frac{7}{2}\right)$ are the closest points to (0, 4).

c. $f(x) = x^2$ on (0, 1)

$d = \sqrt{x^2 + (x^2-1)^2}$

$$g(x) = x^4 - x^2 + 1$$
$$g'(x) = 4x^3 - 2x$$
$$= 2x(2x^2 - 1)$$
$$\Rightarrow x = 0 \text{ or } x = \pm\frac{\sqrt{2}}{2}.$$

$g''(x) = 12x^2 - 2 > 0$ for $x = \frac{\sqrt{2}}{2}$ or $x = -\frac{\sqrt{2}}{2}$.

Thus, $g\left(-\frac{\sqrt{2}}{2}\right) = \frac{3}{4} = g\left(\frac{\sqrt{2}}{2}\right)$

Therefore, $\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ are the closest points to the point with coordinates (2, 1).

3. Let x be any positive number. It is required to minimize $f(x) = \frac{1}{x} + x$; so the minimum value is 2 which is obtained at $x = 1$.
4. Let x and y be the length and width of the rectangular region. Then $2(x + y) = 100 \Rightarrow y = 50 - x$; it is required to maximize $A(x) = xy = 50x - x^2$. Thus $x = 25$ and $y = 25$ maximizes the area.
5. $x = 100$ and $y = 100$: yield the maximum area.
6. $x = 300$ and $y = 600$: yield the smallest amount of fencing.
7. Can be done in a similar way as (4), (5) and (6).
8. Let r be the radius of the circle and x be the length of the sides of a square. Then, $2\pi r + 4x = 16 \Rightarrow r = \frac{16 - 4x}{2\pi}$ and we want to minimize the sum of the area of the circle and area of the square: A$= \pi r^2 + x^2 \Rightarrow A(x) = \pi\left(\frac{16 - 4x}{2\pi}\right)^2 + x^2$;

   $x = \frac{32}{2\pi + 8}$: yields the minimum area.
9. a. Let x be the length of the wire. Then, for the square, each side has length of $\frac{x}{4}$ and for the circle, the length is $10 - x$.

   So, to express the sum of areas,

$$A_{\text{square}} = \left(\frac{x}{4}\right)^2 \text{ and}$$

$$A_{\text{circle}} = \left(\frac{10-x}{2\pi}\right)^2 \text{ is the sum of the areas.}$$

b. The domain of A(x) is $10 - x > 0$.

$$\Rightarrow x < 10.$$

c. In order to get how much wire is used for the square and the circle, we use first derivative test.

$$A'(x) = 2\left(\frac{x}{4}\right)\left(\frac{1}{4}\right) + 2\left(\frac{10-x}{2\pi}\right)\left(\frac{-1}{2\pi}\right) = 0$$

$$\Rightarrow \frac{x}{8} - \left(\frac{10-x}{2\pi^2}\right) = 0$$

$$\Rightarrow x\frac{40}{\pi^2+4} = 2.884$$

$$\text{So } A_{\text{square}} = \left(\frac{2.884}{4}\right)^2 = 0.52\,\text{m}$$

$$A_{\text{circle}} = \left(\frac{10-2.884}{2\pi}\right)^2 = 1.282\,\text{m}$$

10. $R(x) = -x^3 + 450x^2 + 52{,}500\,x$

$R'(x) = -3x^2 + 900x + 52{,}500$

$R'(x) = 0 \Rightarrow -3x^2 + 900x + 52{,}500 = 0$

$\Rightarrow x = -50$ or $x = 350$.

To check whether these numbers give maximum revenue, we use the second derivative test.

$R''(x) = -6x + 900 < 0$ for $x = 350$.

So, $R(350) = -(350)^3 + 450\,(350)^2 + 52{,}500(350)$

$= -42{,}875{,}000 + 55{,}125{,}000 + 18{,}375{,}000$

$= 30{,}625{,}000.$

Therefore, 30,625,000 yield the maximum revenue.

11. Since the marginal cost is the derivative of the cost function,

$C'(x) = 0.0009x^2 - 0.08x + 0.5 = 0$ is the marginal cost.

To find the minimum cost, we use the marginal cost function.

$\Rightarrow C'(x) = 0.0009x^2 - 0.08x + 0.5 = 0$

$\Rightarrow x =$ 6.765 or $x = 82.12$

To check whether these numbers give minimum cost, we use second derivative test.

$\Rightarrow C''(x) = 0.0018x - 0.08 > 0$ for $x = 82.12$

Thus, C (82.12) = $0.0003(82.12)^3 - 0.04(82.12)^2 + 0.5(82.12) + 3700$

$= 166.1376 - 269.74 + 41.06 + 3700 = 3637.45$

Therefore, the minimum cost is Birr 3637.45.

12. X $(t) = \sin 2t + \sqrt{3}\cos 2t$

X′ $(t) = 2\cos 2t + \sqrt{3}\,(-\sin 2t).\,2$

X′ $(t) = 2\cos 2t - 2\sqrt{3}\sin 2t$

$\Rightarrow$ X ' $(t) = 0 \Rightarrow 2\cos 2t - 2\sqrt{3}\sin 2t = 0$

$\Rightarrow \tan 2t = \dfrac{\sqrt{3}}{3}$.

$\Rightarrow t = 15°$ or $t = 345°$.

To check whether these values give a maximum distance, we use the second derivative test.

X′ $(t) = -4\sin 2t - 4\sqrt{3}\cos 2t < 0$ for $t = 15°$, and $t = 345°$.

Thus, X ' (15°) = 2 and X (345°) = 1

Therefore, the maximum distance of the mass from the origin is 2 units.

13. From the given function, we can get C(3) = 39 °C and $C'(3) = -\dfrac{1}{4}$ °C/hr; these show that after 3 hrs the body temperature is decreased to 39 °C and it is decreasing at the rate of $-\dfrac{1}{4}$ °C/hr.

4.3 RATE OF CHANGE

Periods Allotted: 6 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *interpret and apply differential calculus on problems involving rate of change.*
- *consolidate what has been learnt in this unit.*

Vocabulary: Instantaneous rate of change, Average rate of change

Introduction

In the previous units, you have taught derivatives as rates of change. In this section you are going to show the students that rate of changes have many real-life applications. For example, you may show the students that velocity, acceleration, population growth rate, unemployment rate, production rate and water flow rates need the concept of derivatives. Thus the main task of this section is to give due emphasis on rates of changes and to distinguish average rates of change from instantaneous rates of change.

Teaching Notes

You may ask students how to find the slope of the tangent line at a point and the slope of the secant line. So, the distinction between the two rates of change is comparable to the distinction between the two slopes. Illustrate this by using the graphs given in the students' textbook**.**

Assessment

You should give homework, class-works and assess students by checking their exercise books. For this purpose, you may use Exercise 4.7.

Answers to Exercise 4.7

1. i. a. $-16\text{m}/\text{sec}$ b. $-48\text{m}/\text{sec}$

   c. $-80\text{m}/\text{sec}$ d. $-96\text{m}/\text{sec}$

   ii. a. $-16\text{m}/\text{sec}$ b. $-32\text{m}/\text{sec}$

   c. $-48\text{m}/\text{sec}$ d. $-64\text{m}/\text{sec}$

2. a. $P(0)=10{,}000$, $P(5)=10{,}810$, $P(8)=11{,}824$, $P(10)=12{,}720$

   As the time in years increases, the population also increases.

   b. $\dfrac{dP}{dt}=44t+52$

c. The rates of population growth at $t=0$, $t=5$, $t=8$ and $t=10$ are: 52 persons/year, 272 persons/year, 404 persons/year and 492 persons/year respectively.

After having discussed Exercise 4.7, introduce related rates. You may probably start with the following examples:

Example:

If the volume of spherical balloon is changing with time, then one can ask if there is any relation between changes in volume and changes in radius.

The volume V of sphere with radius r is given by $V(r)=\frac{4}{3}\pi r^3$. Thus, the rate of change V with respect to time is given by $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$.

Therefore, the two rates are related by the above equation and such rates are called related rates. Or you may start by using those examples which are given in the students' textbook.

Answers to Exercise 4.8

1. a. $48\pi\,\text{cm}^2/\text{min}$ b. $72\pi\,\text{cm}^2/\text{min}$
2. a. $48\pi\,\text{cm}^3/\text{min}$ b. $108\pi\,\text{cm}^3/\text{min}$
3. a. $-\frac{3}{16}\text{m}/\text{sec}$ b. $-\frac{1}{3}\text{m}/\text{sec}$ c. $-\frac{9\sqrt{19}}{76}\text{m}/\text{sec}$
4. a. $\frac{dy}{dx}=-\frac{x}{y}$ and $\frac{dx}{dy}=-\frac{y}{x}$

   b. $\frac{dy}{dx}=\frac{2xy-3y-y^2}{3x+2xy-x^2}$ and $\frac{dx}{dy}=\frac{3x+2xy-x^2}{2xy-3y-y^2}$

   c. $\frac{dy}{dx}=\frac{y-1-y^2}{2xy-1-x}$ and $\frac{dx}{dy}=\frac{2xy-1-x}{y-1-y^2}$

   d. $\frac{dy}{dx}=\frac{3x^2y^3-y-2xy^2}{x+2x^2y-3x^3y^2}$ and $\frac{dx}{dy}=\frac{x+2x^2y-3x^3y^2}{3x^2y^3-y-2xy^2}$

   e. $\frac{dy}{dx}=\frac{y+y\sin x-\sin y}{x\cos y+\cos x-x}$ and $\frac{dx}{dy}=\frac{x\cos y+\cos x-x}{y+y\sin x-\sin y}$
5. $V(r)=\frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

   $\Rightarrow \frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$

   a. $\frac{5}{\pi}\text{cm}/\text{min}$ b. $\frac{5}{4\pi}\text{cm}/\text{min}$ c. $\frac{5}{9\pi}\text{cm}/\text{min}$

6. $V(r) = \frac{1}{3}\pi r^2 h$ (Volume of right circular cone)

But $h = 3r \Rightarrow V(r) = \pi r^3$

$\Rightarrow \frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

a. $54\pi \text{cm}^3/\text{min}$ b. $216\text{cm}^3/\text{min}$

Answers to Review Exercises on Unit 4

1. a. Critical numbers: –2, 0, 2

f is strictly increasing on $[-2, 0]$ and $[2, \infty)$

f is strictly decreasing on $(-\infty, -2]$ and $[0, 2]$

$f(-2) = -10 = f(2)$ is a local minimum value.

$f(0) = 6$ is a local maximum value.

The graph of f is concave upward on $\left(-\infty, -2\frac{\sqrt{3}}{3}\right)$ and $\left(2\frac{\sqrt{3}}{3}, \infty\right)$

The graph of f is concave downward on $(-2\frac{\sqrt{3}}{3}, 2\frac{\sqrt{3}}{3})$

Inflection points: $\left(-2\frac{\sqrt{3}}{3}, -\frac{26}{9}\right)$ and $\left(2\frac{\sqrt{3}}{3}, -\frac{26}{9}\right)$

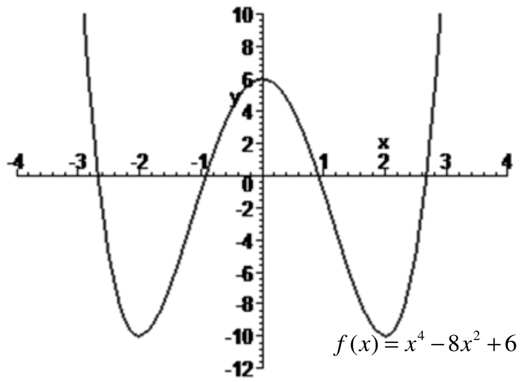


Figure 4.10

b. Critical numbers: –3, 1

f is strictly increasing on $(-\infty, -3]$ and $[1, \infty)$

f is strictly decreasing on $[-3, 1]$

0 is the local minimum value.

32 is the local maximum value.

The graph of f is concave upward on $(-1, \infty)$

The graph of f is concave downward on $(-\infty, -1)$

Inflection point: $(-1, 16)$

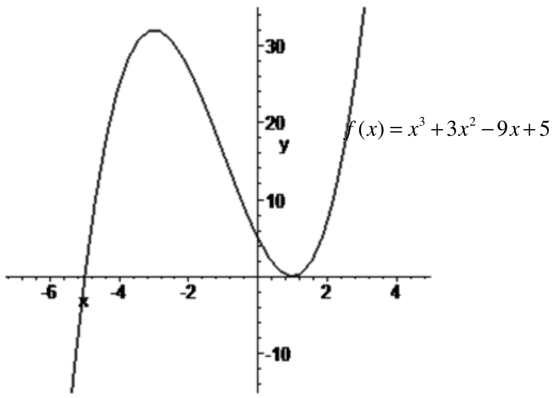


Figure 4.11

c. Critical numbers –1 and 1 f is strictly increasing on $[-1, 1]$, f is strictly decreasing on $(-\infty, -1]$ and $[1, \infty)$, $f(-1) = -1$ is the local minimum and $f(1) = 1$ is the local maximum.

The graph of f is concave upward on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$ and concave downward on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$ inflection points: (0, 0), $\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$ and $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$

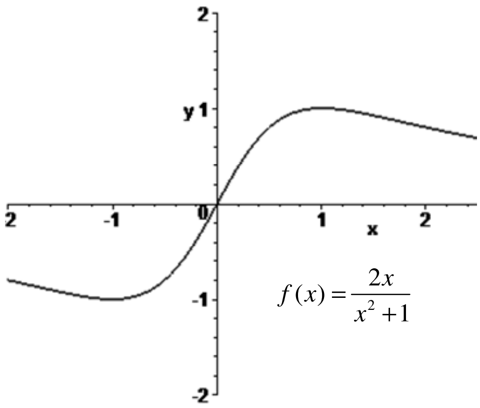


Figure 4.12

d. Critical numbers: 0, 4

f is strictly increasing on $(-\infty, 0]$ and $[4, \infty)$

f is strictly decreasing on $[0, 2)$ and $(2, 4]$

6 is the local minimum value.

–2 is the local maximum value.

The graph of f is concave upward on $(2, \infty)$

The graph of f is concave downward on $(-\infty, 2)$

No inflection point

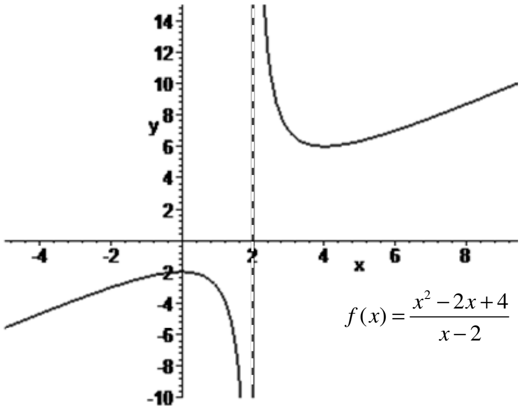


Figure 4.13

2. a. Absolute maximum value is 15

   Absolute minimum value is -10

   b. Absolute maximum value is 27

   Absolute minimum value is 7

   c. Absolute maximum value is 1

   Absolute minimum value is $\dfrac{4}{5}$

   d. Absolute maximum value is –2

   Absolute minimum value is $\dfrac{-19}{5}$

3. The length of the square base of the box is 4 m and its height is 2m.

4. $\left(\dfrac{\sqrt{2}}{2}, \dfrac{3}{2}\right)$, $\left(\dfrac{-\sqrt{2}}{2}, \dfrac{3}{2}\right)$(Hint: use the same method as in example 6 page 197 students' textbook)

5. The minimum value is -1 which is obtained at $\dfrac{\pi}{2}$ and $\dfrac{3}{2}\pi$; and the maximum value is 1.5 which is obtained at $\dfrac{7\pi}{6}$ and $\dfrac{11}{6}\pi$

6. Maximize: $A = 2hr + \dfrac{1}{2}\pi r^2$ using the condition $12 = 2h + 2r + \pi r$. From this condition $h = 6 - r - r\dfrac{\pi}{2}$. Thus apply derivatives on $A(r) = 12r - 2r^2 - \dfrac{1}{2}\pi r^2$ and then you get $r = \dfrac{12}{4+\pi}$ and $h = 6 - \dfrac{12+6\pi}{4+\pi}$, therefore, *h* and 2*r* is the required dimension of the bottom of the window.

7. The dimension of the rectangle should be $\frac{9\sqrt{2}}{4}$ and $\frac{\sqrt{153}}{2}$

8. Hint: the volume V of a canonical vase is given by $V = \frac{1}{3}\pi h r^2$.

Use similar triangles to find the relationship between r and h as follows: $\frac{r}{h} = \frac{15}{25} = \frac{3}{5}$.

From this equation, you get $r = \frac{3}{5}h$. Then substituting into the formula of V

we get, $V(h) = \frac{3}{25}\pi h^3$.

Now differentiating both sides of the equation with respect to time t,

$$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \frac{dh}{dt} \Rightarrow 18 = \frac{9}{25}\pi (20)^2 \frac{dh}{dt}$$

Therefore, the water level is rising at the rate $\frac{dh}{dt} = \frac{1}{8\pi}$ $cm/\sec$ when the depth of water is 20 cm.

9. Hint: you can let x be the distance from the 6m pole to the point at which the wire is staked to the ground, and then the distance from the point to the 15 m tall pole is $20 - x$. Now, let L_1 be the length of the hypogenous formed from the 6 meter tall pole to the point and L_2 to be the length of the hypogenous formed from the 15 meter tall pole to the point. Thus, total length of the wire is $L_1 + L_2$ and you need to determine the value of x that will minimize L where

$$L = \sqrt{36 + x^2} + \sqrt{625 - 40x + x^2}$$

10. Hint: have the relation $x^2 + y^2 = z^2$ where x is the distance the truck travelled and y is the distance of the car from the intersection and z to be the hypotenuse relating x and y. Then differentiate both sides with respect to time t to get

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

Substituting $x = 40$ and $y = 9$, so that $z = \sqrt{1681}$ you get $\frac{dz}{dt}$ which tells you how the distance between the car and truck changing.

UNIT 5

INTRODUCTION TO INTEGRAL CALCULUS

INTRODUCTION

Integral calculus is widely applied in the current science and technology.

In this unit, there are three major outcomes which are expected to be accomplished. The first is to enable students to understand the indefinite integrals.

The second is to enable students to understand the definite integrals.

The third one is to enable students to use the definite integrals in solving real world problems and problems in other subjects. Therefore, students should be able to exercise the properties and techniques of integration.

They should know when and what technique of integration to be used.

The activities are designed to introduce the different techniques of integration.

The examples are expected to be sufficient to practice each and every techniques i.e. presented for this introductory part of integration.

There are several graphical illustrations in particular in the definite integral, area and volume.

The exercises are designed to assess each topic of the unit.

Unit Outcomes

After completing this unit, students will be able to:

- *understand the concept of definite integrals.*
- *integrate different polynomial functions, simple trigonometric functions, exponential and logarithmic functions.*
- *use the various techniques of integration to evaluate a given integral.*
- *use the fundamental theorem of calculus for computing definite integrals.*
- *apply the knowledge of integral calculus to solve real life mathematical problems.*

Suggested Teaching Aids in Unit 5

- **Graphing calculators**
- **Application of software on integrals**
  - ✓ Charts containing the derivatives and anti-derivatives of standard functions parallel graphs.
  - ✓ Charts containing different techniques of integration.
  - ✓ Charts containing
    - Area under a curve
    - Area bounded by graphs
  - ✓ Charts on
    - Volume of solid of revolution
    - Solid material showing a solid formed by revolution

5.1 INTEGRATION AS A REVERSE PROCESS OF DIFFERENTIATION

Periods Allotted: 7 periods

Competencies

At the end of this sub-unit, students will be able to:

- *differentiate between the concepts differentiation and integration.*
- *use properties of indefinite integrals in solving problems of integration.*
- *integrate some simple functions.*

Vocabulary: Anti-derivative, Integration, Indefinite integral

Introduction

This topic deals with the derivation of formulas for finding anti derivatives. The term indefinite integral refers to the notion of anti-derivatives. It is important to inform students that the interval [a, b] over which a function is being integrated is called the domain of integration. The indefinite integral doesn't have domain of integration. In section 5.3, students will see that the definite integral has domain of integration.

Teaching Notes

In teaching this sub unit, use the activities to introduce integration as a reverse process of differentiation.

The activities are designed to give more time for the learner – centered instruction than teacher centered instruction. Thus, in teaching this sub topic, involve students in finding anti-derivatives of different simple functions.

It is clear that students who master differentiation will gain the advantage of exploring the integrals of various functions.

For example, the formulas for $\int x^r\,dx$, $\int \frac{1}{x}dx$, $\int \frac{1}{x+1}\,dx$, $\int e^x\,dx$, $\int a^x dx$, and determination of integrals of simple trigonometric functions are expected from students.

Give as many problems as possible until most of the students can find the integrals of such functions automatically.

The examples are designed to assess students during instruction. The exercises could be given as class work or homework. You may use some of the examples for class room discussion. For example, discuss the family of parallel curves $y = x^2 + c$ in connection with the indefinite integral of $f(x) = 2x$.

Additional exercise problems for high ability students

1. Draw the graphs of $F(x) = \int \sin(2x)\,dx$ i.e $F(x) = \frac{-\cos(2x)}{2} + c$

   Show that they have the same slope at $x = a$. By varying the values of a, like $0, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, etc. demonstrate the parallel curves on a wall chart or on the board using colored chalk. Take $c = -3, -2, -1, 0, 1, 2, 3$

2. Repeat question 1, if

   a. $F(x) = \int (3x^2 + 2x + 1)dx$

      For $c = -6, -3, 0, 1, 4, 8$ at $x = 1, -1$.

   b. $F(x) = \int e^{3x} dx$

      For $c = -6, -13, 0, 3, 8, 14$ at $x = -\frac{1}{2}, 0.1$

Solution:

1.

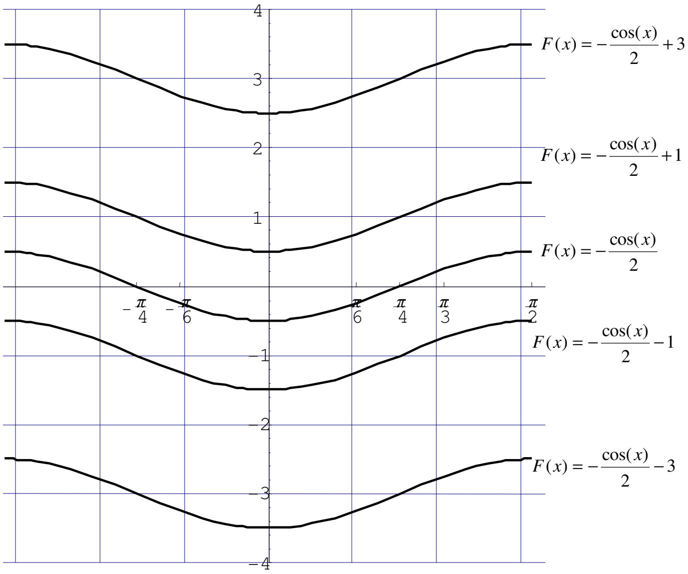


Figure 5.1

2. a.

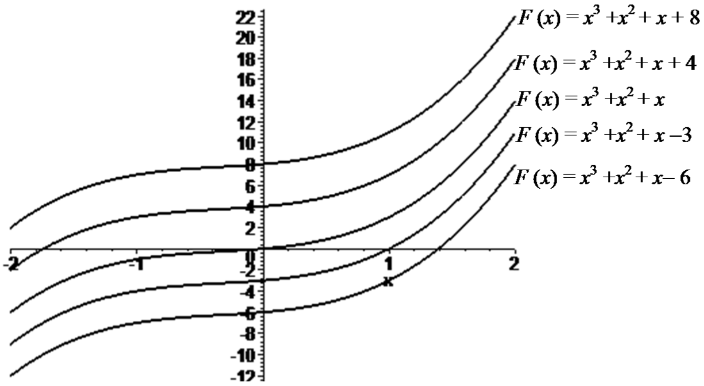


Figure 5.2

b.

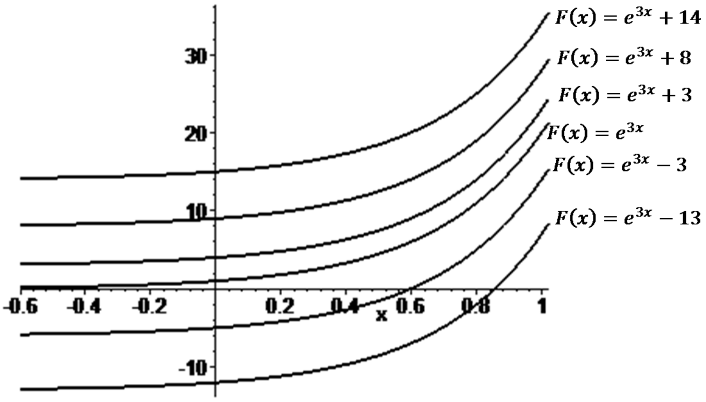


Figure 5.3

Assessment

- ✓ ask oral questions on the definition and properties of the indefinite integral.
- ✓ give various exercise problems on the applications of the standard formulas of integration such as evaluating

$$\int \frac{5}{7x^{\frac{3}{4}}}\,dx,\quad \int \frac{(x+1)^2}{\sqrt{x}}\,dx,\quad \int \left(2x-3\cos x+e^x\right)dx$$

5.1.1 The Concept of Indefinite Integral

Teaching Notes

Before you start the formal instruction, make sure that all students are capable of finding the derivative of simple functions.

Activity 5.1 deals with introducing the concept of indefinite integral. It should be discussed with students before giving the definition of anti-derivatives.

Students should do question 3 in groups. Ask the groups to explain the differences and similarities between the graphs of the functions. Encourage students to reach to the conclusion that the difference between any two of the functions is a constant. Eventually, they will see that this is the constant of integration.

Answers to Activity 5.1

A constant is added in the indefinite integral to indicate that there is a family of anti-derivatives for a single function that differs only by a constant.

1. x^2, x^2+1, x^2-3, if $f'(x) = 2x$ and $g'(x) = 2x$, then $f(x) - g(x)$ is a constant.
2. $\{f(x) : f'(x) = 2x\} = \{x^2 + c;\ c \in \mathbb{R}\}$
3.

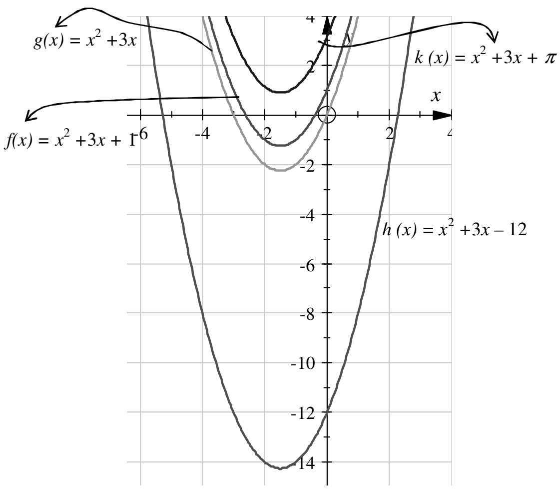


Figure 5.4

The work of Activity 5.2 is designed to associate the rules of the derivatives of the simple functions in exploring the rules for their integrals. This can be assessed by using oral questions or by allowing students to write on the black board.

Encourage students to explore or to investigate the integrals of some simple functions such as $\int x^n dx;\ n \neq -1$.

Answers to Activity 5.2

1.

$f(x)$	4	x	x^2	x^3	x^{10}	x^n	$\sin x$	$\cos x$	$\tan x$	$\cot x$	e^x	4^x	$\ln x$	$\log x$
$f'(x)$	0	1	$2x$	$3x^2$	$10x^9$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	$-\csc^2 x$	e^x	$4^x \ln 4$	$\frac{1}{x}$	$\frac{1}{x\ln 10}$

2. a. $\int x^4\, dx = \frac{x^5}{5} + c$ b. $\int \sin x\, dx = -\cos x + c$

c. $\int \cos x\, dx = \sin x + c$ d. $\int \sec^2 x\, dx = \tan x + c$

e. $\int \csc^2 x\, dx = -\cot x + c$ f. $\int e^x\, dx = e^x + c$

g. $\int 4^x\, dx = \frac{4^x}{\ln 4} + c$ h. $\int \frac{1}{x}\, dx = \ln|x| + c$

i. $\int \frac{1}{x\ln 10} = \log x + c$

Discuss the worked examples with the class or group and then give Exercise 5.1 as class work or home work.

Allow students exercise the power rule and check the solutions by differentiation.

Answers to Exercise 5.1

1. $\int x^3\, dx = \frac{x^4}{4} + c$ 2. $\int 2x^4\, dx = \frac{2x^5}{5} + c$

3. $\int x^{-3}\, dx = \frac{-x^{-2}}{2} + c$ 4. $\int x^{\frac{2}{5}}\, dx = \frac{5}{7}x^{\frac{7}{5}} + c$

5. $\int \frac{4}{x^{1.5}}\, dx = \int 4x^{-1.5}\, dx$

$$= \frac{4x^{-0.5}}{-0.5} + c = -\frac{8}{\sqrt{x}} + c$$

6. $\int 6x^2\ \sqrt{x}\, dx = \int 6\, x^{\frac{5}{2}}\ dx = \dfrac{6(x)^{\frac{5}{2}+1}}{\frac{5}{2}+1} + c = \dfrac{12}{7}\ x^3\sqrt{x} + c$

7. $\int \dfrac{1}{8\sqrt[3]{x}}\ dx = \dfrac{1}{8}\int x^{-\frac{1}{3}}\ dx = \dfrac{1}{8}\left(\dfrac{x^{\frac{2}{3}}}{\frac{2}{3}}\right) + c = \dfrac{3}{16}x^{\frac{2}{3}} + c$

8. Let $u = 3x - 1$, then $du = 3dx$

$$\Rightarrow\ \frac{1}{3}\ du = dx$$

$$\Rightarrow\ \int (3x-1)^6\ dx = \frac{1}{3}\int u^6\ du = \frac{1}{21}\ u^7 + c = \frac{1}{21}\ (3x-1)^7 + c$$

9. $u = 1 - 2x\ \Rightarrow\ -\dfrac{1}{2}\ du = dx$

$$\Rightarrow\ \int \sqrt[3]{1-2x}\ dx = -\frac{1}{2}\int u^{\frac{1}{3}}\ du\ =\ -\frac{1}{2}\left(\frac{3}{4}u^{\frac{4}{3}}\right) + c$$

$$= \frac{-3}{8}u\sqrt[3]{u} + c = \frac{-3}{8}(1-2x)\sqrt[3]{1-2x} + c$$

$$= \frac{-3}{8}(1-2x)^{\frac{4}{3}}$$

10. $u = 4 - 3x\ \Rightarrow -\dfrac{1}{3}du = dx$

$$\int 8\ \sqrt[4]{4-3x}^{\,3}\ dx = \frac{-8}{3}\int u^{\frac{3}{4}}\ du$$

$$= \frac{-8}{3}\ \frac{u^{\frac{3}{4}+1}}{\frac{3}{4}+1} + c\ = -\frac{32}{21}\ u^{\frac{7}{4}} + c$$

$$= -\frac{32}{21}\ (4-3x)^{\frac{7}{4}} + c$$

11. $u = 4 - 5x\ \Rightarrow\ -\dfrac{1}{5}\ du = dx$

$$\Rightarrow \int \frac{3}{\sqrt[4]{4-5x}}\ dx = -\frac{1}{5}\int \frac{3}{\sqrt[4]{u}}\ du$$

$$= -\frac{3}{5}\int u^{-\frac{1}{4}}\ du = \frac{-3}{5}\left(\frac{u^{\frac{3}{4}}}{\frac{3}{4}}\right) + c = -\frac{4}{5}\ (4-5x)^{\frac{3}{4}} + c$$

12. $u = 2x - 3 \Rightarrow \frac{1}{2}du = dx$

$$\int (2x-3)^{\frac{1}{2}}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{\frac{1}{2}\left(u^{\frac{1}{2}+1}\right)}{\frac{1}{2}+1} + c = \frac{1}{2}\left(\frac{2}{3}\right)\left(u^{\frac{3}{2}}\right) + c$$

$$= \frac{1}{3}\left((2x-3)^{\frac{3}{2}}\right) + c$$

13. $u = 4x - \pi \Rightarrow \frac{1}{4}\,du = dx$

$$\Rightarrow \int (4x-\pi)^{\sqrt{2}}\,dx = \frac{1}{4}\int u^{\sqrt{2}}\,dx = \frac{1}{4\left(\sqrt{2}+1\right)}\,u^{1+\sqrt{2}} + c$$

$$= \frac{1}{4\left(\sqrt{2}+1\right)}\,(4x-\pi)^{1+\sqrt{2}} + c$$

Answers to Exercise 5.2

Consider $\int e^{ax+b}\,dx$ where $a \neq 0$, let $u = ax + b$, then $\frac{1}{a}\,du = dx$

$$\Rightarrow \int e^{ax+b}\,dx = \frac{1}{a}\int e^{u}\,du = \frac{1}{a}\,e^{u} + c = \frac{1}{a}\,e^{ax+b} + c$$

Also, $\int a^{cx+d} = \frac{a}{c\ln a}\,a^{cx+d}$ where $c \neq 0$, $a > 0$

Using this result, we have the following values of the integral.

1. $\frac{1}{3}\,e^{3x} + c$ 2. $-\frac{1}{5}e^{-5x} + c$ 3. $\frac{5^{x+1}}{\ln\ 5} + c$

4. $\frac{-2^{4-x}}{\ln\ 4} + c$ 5. $-\frac{1}{3}\,\mathrm{e}^{2-3x} + c$ 6. $-2e^{-1-2x} + c$

7. $\frac{-5}{e^{\pi+x}} + c$ 8. $\frac{2}{\ln 3}\left(\sqrt{3}\right)^{x+5} + c$

9. $$\int \frac{4e^{4}}{e^{4x+1}}\,dx = 4\int e^{4-(4x+1)}dx$$

$$= 4\int e^{3-4x}dx = -e^{3-4x} + c$$

10. $$\int \sqrt{e^{2x}}\,dx = \int e^{x}dx = e^{x} + c$$

11. $\dfrac{4^{3x-5}}{3\ln 4}+c$

12. $\displaystyle\int \frac{2^{1-3x}}{3^{x+1}}\,dx=\int \frac{2\left(8^{-x}\right)}{3\left(3^{x}\right)}\,dx=\int \frac{2}{3}\left(\frac{1}{24^{x}}\right)dx$

$$=\frac{-\frac{2}{3}\left(\frac{1}{24^{x}}\right)}{\ln 24}+c=-\frac{2}{3\ln 24\left(24^{x}\right)}+c$$

13. $\displaystyle\int 2^{x+3}\times 3^{4-2x}\,dx=\int \frac{2^{x}\times 2^{3}\times 3^{4}}{9^{x}}\,dx=\int\left(\frac{2}{9}\right)^{x}\times 8\times 81\,dx$

$$=\int 648\left(\frac{2}{9}\right)^{x}dx=648\int\left(\frac{2}{9}\right)^{x}dx=\frac{648\left(\frac{2}{9}\right)^{x}}{\ln\frac{2}{9}}+c$$

Answers to Exercise 5.3

In these exercises we use

i. $\displaystyle\int \frac{1}{ax}\,dx=\frac{1}{a}\ln|x|+c$

ii. $\displaystyle\int \frac{k}{ax+b}\,dx=k\int \frac{1}{ax+b}\,dx$

$u = ax + b \Rightarrow \dfrac{1}{a}du=dx$

$$\Rightarrow\ k\int \frac{1}{ax+b}\,dx=\frac{k}{a}\int du$$

$$=\frac{k}{a}\ln|u|+c=\frac{k}{a}\ln|ax+b|+c$$

1. $\dfrac{1}{3}\ln|x|+c$
2. $\dfrac{5}{2}\ln|x|+c$
3. $2\ln|x+1|+c$
4. $\dfrac{3}{2}\ln|2x-1|+c$
5. $\dfrac{\sqrt{2}}{-3}\ln|1-3x|+c$
6. $\dfrac{4}{\pi}\ln|\pi x-1|+c$
7. $\dfrac{3}{2}\ln|2x-5|+c$
8. $2\ln\left|\dfrac{1}{2}x+1\right|+c$
9. $\displaystyle\int \frac{1}{x(x+1)}\,dx=\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\ln|x|-\ln|x+1|+c=\ln\left|\frac{x}{x+1}\right|+c$

5.1.2 Properties of Indefinite Integrals

Teaching Notes

The purpose of this activity is to explore the properties of the indefinite integral. Discuss question 1 by dividing the class into two groups, group 1 and group 2.
Give question 1 a, c and e to group 1 and b, d and f to group 2.
Ask some students from each group to demonstrate their works.
Ask some other students to generalize what they have seen.

Answers to Activity 5.3

1. a. $\frac{x^3}{3}+\frac{2}{3}x\sqrt{x}-e^x+c$ b. $\frac{x^3}{3}+\frac{2}{3}x\sqrt{x}-e^x+c$

   c. $\frac{4}{3}x^3-2x^2+x-\frac{1}{6}+c$ d. $\frac{4}{3}x^3-2x^2+x+c$

   e. $2x\sqrt{x}+\frac{2}{5}x^2\sqrt{x}+2\sqrt{x}+e^{-x}+c$ f. $2x\sqrt{x}+\frac{2}{5}x^2\sqrt{x}+2\sqrt{x}+e^{-x}+c$

2. Yes, as shown in problem (1) above.
3. A family of parallel curves of $y = x^2 + c$.

After discussing the worked examples, give exercise 5.4 as class work or home work.

Questions 1, 2 and 3 may be suitable for low ability students. The other students need not do these problems.

Answers to Exercise 5.4

1. $\int \frac{d}{dx}x^3dx=\int 3x^2dx=\frac{3x^3}{3}+c=x^3+c$ 2. $\frac{d}{dx}\int x^3\,dx=\frac{d}{dx}\left(\frac{x^4}{4}+c\right)=x^3$

Using the properties of the indefinite integrals we have, integral of the sum is equal to the sum of the integrals. Therefore,

3. $\frac{x^7}{7}+\frac{3}{4}x^{\frac{4}{3}}+\frac{1}{3x^3}-\frac{2}{\sqrt{x}}+c$ 4. $\frac{2}{3}x\sqrt{x}-\frac{3}{4}x^4-\frac{1}{x}+2x+c$

5. $-\frac{1}{3x^3}-\frac{1}{2x^2}-\frac{1}{x}+\ln|x|+c$ 6. $\frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$

7. $\frac{1}{3}z^3+\frac{1}{2}z^2-2z-\frac{1}{z}+\ln|z|+c$ 8. $\frac{x^4}{4}-x+c$

9. $\frac{t^2}{2}-3t+4\ln|t|+c$ 10. $\ln|x|-\frac{1}{x}+c$

11. $e^x + e^{-x} + \ln|x| + c$

12. $\frac{2}{3}e^{\frac{3}{2}x} - 6\sqrt{e^x} - \frac{4}{\sqrt{e^x}} + c$

13. $\frac{x^4}{2} + \frac{e^{2x}}{2} - \frac{1}{2}\ln|x| + c$

14. $e^x - e^{2x} + \frac{e^{3x}}{3} + c$

15. $\frac{-3^{1-2x}}{\ln 9} + \frac{-2}{\sqrt{2^x}\ln 2} - \frac{1}{2e^{2x}} + c$

After discussing the integrals of the simple trigonometric functions with the class or group, give Exercise 5.5 as class work or homework. All students are expected to answer questions 1 to 5. Questions 6, 7 and 8 need further simplification. One purpose of Exercise 5.5 is to introduce the elementary substitution for the next subunit.

Answers to Exercise 5.5

1. $-3\cos x + c$

2. $u = 2x \Rightarrow \frac{1}{2}u = dx$

$$\Rightarrow \int \cos(2x)dx = \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin(2x) + c$$

3. $u = 4x - 1 \Rightarrow dx = \frac{1}{4}\, du$

$$\Rightarrow \int \sin(4x-1)\, dx = \frac{1}{4}\int \sin u\, du = -\frac{1}{4}\cos u + c = -\frac{1}{4}\cos(4x-1) + c$$

4. $u = 4x + \frac{\pi}{3} \Rightarrow dx = \frac{1}{4}du$

$$\Rightarrow \int 3\cos\left(4x + \frac{\pi}{3}\right)dx = \frac{3}{4}\int \cos u du = \frac{3}{4}\sin u + c = \frac{3}{4}\sin\left(4x + \frac{\pi}{3}\right) + c$$

5. $\int(\sin(3x) + \cos(4x))dx = \int \sin(3x)dx + \int \cos(4x)dx = -\frac{1}{3}\cos(3x) + \frac{1}{4}\sin(4x) + c$

6. $\int \sec^2(2x+1)dx = \frac{1}{2}\int \sec^2 u du = \frac{1}{2}\tan u + c = \frac{1}{2}\tan(2x+1) + c$

7. $\int \csc(2x)\cot(2x)dx = \frac{1}{2}\int \csc(u)\cot(u)du = \frac{-1}{2}\csc(u) + c = \frac{-1}{2}\csc(2x) + c$

8. $\int \sec\left(2x - \frac{\pi}{4}\right)\tan\left(2x - \frac{\pi}{4}\right)dx = \frac{1}{2}\int \sec(u)\tan(u)du$

$$= \frac{1}{2}\sec(u) + c = \frac{1}{2}\sec\left(2x - \frac{\pi}{4}\right) + c$$

5.2 TECHNIQUES OF INTEGRATION

Periods Allotted: 9 periods

Competency

At the end of this sub-unit, students will be able to:

- *use different techniques of integration for computation of integrals .*

Vocabulary: Integration by substitution, Partial fractions, Integration by parts

Introduction

This topic involves manipulating integrals by transforming into familiar forms. This may help students to think creatively and to suggest a technique of integration.

Students already know the integrals of functions in standard form. Teachers must be able to help students learn how to obtain the integrals of certain functions that are not in standard form. Therefore, students need to know some techniques to transform the given functions to the standard forms.

The techniques that they are expected to use are as follows:

- Integration by substitution
- Integration by partial fractions
- Integration by parts

The activities are designed to introduce the above techniques of integration.

Moreover, there are appropriate and sufficient examples.

In addition to the above, teachers must be able to teach students how to relate techniques of integration to the corresponding rules of differentiation.

For example, the chain rule with the method of substitution.

The product rule with integration by parts

Teachers should also work with students on developing the most efficient techniques for determining integrals.

Once students have learned the techniques, it is easier for them to generate their own examples and perform the reverse operations.
Ask oral questions to define terms, to clarify terms in the techniques.
Discuss the need for other techniques of integration to compute some indefinite integrals.

For example, discuss the need for a technique to evaluate $\int x\left(x^2+5x+6\right)^5 dx$.

Assessment

Give the exercises as class work or home work to apply the techniques of integration.

- ✓ The exercises are designed to assess how well students master the techniques and also to assess whether or not they are ready to start the next topic.
- ✓ Give a sufficient number of exercise problems to apply each technique of integration.

Additional exercise problems for high ability students

1. $\int \cos^3 x\,dx$ 2. $\int \sin^4 x\,dx$ 3. $\int \cos^5 x\,dx$

Solution:

1. $$\int \cos^3 x\,dx = \int \cos^2 x \cos x\, dx$$
$$= \int (1-\sin^2 x)\cos x\, dx$$
$$= \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \frac{\sin^3 x}{3} + c$$

2. $$\int \sin^4 x\, dx = \int \sin^2 x \times \sin^2 x\, dx$$
$$= \int (1-\cos^2 x).\ \sin^2 x\, dx$$
$$= \int \left(\sin^2 x - \sin^2 x \cos^2 x\right) dx$$
$$= \int \sin^2 x\,dx - \int \sin^2 x \cos^2 x\, dx$$
$$= \int \frac{1-\cos(2x)}{2}\, dx - \int (\sin x \cos x)^2\, dx$$
$$= \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos(2x)\,dx - \int \left(\frac{\sin(2x)}{2}\right)^2 dx$$
$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) - \frac{1}{4}\int \sin^2(2x)\, dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) - \frac{1}{4}\int \frac{1-\cos(4x)}{2}\, dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) - \frac{1}{4}\int \left(\frac{1}{2} - \frac{\cos(4x)}{2}\right) dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) - \frac{1}{4}\left[\int \frac{1}{2}\, dx - \frac{1}{2}\int \cos(4x)\, dx\right]$$
$$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x - \frac{1}{4}\left[\frac{1}{2}x - \frac{1}{2}\times\frac{1}{4}\sin(4x)\right]$$
$$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x - \frac{1}{8}x + \frac{1}{32}\sin(4x)$$
$$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x - \frac{1}{8}x + \frac{1}{16}\sin(2x)\cos(2x)$$
$$= -\frac{1}{2}\sin x \cos x + \frac{1}{16}\sin(2x)\cos(2x) + \frac{3}{8}x + c$$

3. $\int \cos^5 x\,dx = \int \cos^2 x \cos^2 x \cos x\,dx$

$$= \int \left(1-\sin^2 x\right)\cos^2 x \cos x\,dx$$

$$= \int (\cos^2 x - \sin^2 x \cos^2 x)\cos x\,dx$$

$$= \int (\cos^3 x - \sin^2 x \cos^3 x)\,dx$$

$$= \int (\cos^3 x - \cos x \sin^2 x \cos^2 x)\,dx$$

$$= \int (\cos^3 x - \cos x \sin^2 x(1-\sin^2 x))\,dx$$

$$= \int \cos^3 x - \cos x \sin^2 x + \cos x \sin^4 x\,dx$$

$$= \int \cos^3 x\,dx - \int \sin^2 x \cos x\,dx + \int \sin^4 x \cos x\,dx$$

$$= \sin x - \frac{\sin^3 x}{3} - \frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + c = \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + c$$

5.2.1 Integration by Substitution

Teaching Notes

You can start the lesson by giving the solutions of Exercise 5.5 by using elementary substitution as an alternative method of finding the integrals.

Then discuss the worked examples with the class or group and give Exercise 5.6 as class work or home work. There are sufficient exercise problems to be done by students of all abilities. If there are problems which are difficult you can rearrange the groups. Students should quickly find the integrals in questions 1 and 2.

Answers to Exercise 5.6

1. a. $u = x^2 + 1 \Rightarrow du = 2x\,dx$

$$\Rightarrow \int 2x\left(x^2+1\right)^3 dx = \int u^3 du = \frac{u^4}{4} + c = \frac{\left(x^2+1\right)^4}{4} + c$$

b. $u = x^2 + 4 \Rightarrow x\,dx = \frac{1}{2}\,du$

$$\Rightarrow \int x\sqrt{x^2+4}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}u\sqrt{u} + c = \frac{1}{2}(x^2+4)\left(\sqrt{x^2+4}\right) + c$$

c. $u = x^3 + 1 \Rightarrow x^2 dx = \frac{1}{3}du$

$$\Rightarrow \int x\sqrt{x^3+1}\,dx = \frac{1}{3}\int \sqrt{u}\,du = \frac{2}{9}u\sqrt{u} + c = \frac{2}{9}(x^3+1)\left(\sqrt{x^3+1}\right) + c$$

d. $u = x^2 + x + 9 \Rightarrow du = \left(2x+1\right)dx$

$$\Rightarrow \int \left(2x+1\right)\sqrt{x^2+x+9}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u\sqrt{u} + c$$

$$= \frac{2}{3}\left(x^2+x+9\right)\sqrt{x^2+x+9}+c$$

e. $\int \sin x \cos x dx = \int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x)+c$

f. $u = x^2 + 3x + 4 \quad \Rightarrow \quad du = (2x + 3)\,dx$

$\Rightarrow \int (2x+3)e^{x^2+3x+4}dx = \int e^u du = e^u + c = e^{x^2+3x+4}+c$

g. $u = \cos x \;\Rightarrow\; -du = \sin x\,dx$

$\Rightarrow \int \sin x e^{\cos x}dx = -\int e^u du = -e^u + c = -e^{\cos x}+c$

h. $u = x - 3 \;\Rightarrow du = dx$

Also, $x = u + 3 \;\Rightarrow x + 2 = u + 5$

$\Rightarrow \int (x+2)\sqrt{x-3}\,dx = \int (u+5)\sqrt{u}\,du$

$$= \int \left(u\sqrt{u}+5\sqrt{u}\right)du = \frac{2}{5}u^2\sqrt{u}+\frac{10}{3}u\sqrt{u}+c$$

$$= \frac{2}{5}(x-3)^2\sqrt{x-3}+\frac{10}{3}(x-3)\sqrt{x-3}+c$$

2. Using the above techniques together with the suggested substitution, we give the integrals as follows.

a. $\frac{2}{9}(3x-2)^{\frac{3}{2}}+c$

b. $-\frac{1}{15}\left(1-5x^2\right)^{\frac{3}{2}}+c$

c. $-\frac{1}{2}\cos(2x)+c$

d. $\frac{10}{3}\sqrt{x+1}^3-\frac{8}{5}(x+1)^2\sqrt{x+1}+c$

e. $\frac{\left(x^2-3\right)^6}{12}+c$

f. $\frac{1}{18}\left(3x^3+2\right)+c$

g. $u = 1 + e^x \;\Rightarrow\; du = e^x dx$

$\Rightarrow \int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}du = \frac{2}{3}u\sqrt{u}+c = \frac{2}{3}\left(1+e^x\right)\sqrt{1+e^x}+c$

h. $u = \cos x \;\Rightarrow\; -du = \sin x\,dx$

$\Rightarrow \int \sin x \cos^{10} x\,dx = -\int u^{10}du = \frac{-u^{11}}{11}+c = \frac{-\cos^{11}(x)}{11}+c$

i. $\frac{1}{6}\sqrt{4x-3}^3+c$

j. $-\frac{3}{2}\sqrt{1-x}^3+c$

k. $\frac{7}{6}(3x-7)^{\frac{2}{7}}+c$

l. $\frac{5}{3\ln 3}\left(3^{0.6x+\pi}\right)+c$

m. $-\sin(3 - x) + c$

n. $u = x^2 + 7 \;\Rightarrow\; \frac{1}{2}du = xdx$

$$\Rightarrow \int x\sin\left(x^2+7\right)dx = \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c$$

$$= -\frac{1}{2}\cos\left(x^2+7\right)+c$$

o. $u = 2x^2 - 5x + 4 \Rightarrow du = (4x - 5)\, dx$

$$\Rightarrow \int \frac{4x-5}{2x^2-5x+4}dx = \int \frac{du}{u} = \ln|u| + c = \ln\left|2x^2-5x+4\right| + c$$

p. $u = x + 3 \Rightarrow du = dx$. Also, $x = u - 3 \Rightarrow x + 1 = u - 2$

$$\Rightarrow \int \frac{x+1}{\sqrt{x+3}}du = \int \frac{u-2}{\sqrt{u}}du = \int \sqrt{u} - \frac{2}{\sqrt{u}}du$$

$$= \frac{2}{3}u\sqrt{u} - 4\sqrt{u} + c = \frac{2}{3}(x+3)\sqrt{x+3} - 4\sqrt{x+3} + c$$

q. $\frac{1}{26}(3+2x)^{13} + c$

r. $\frac{1}{2}\tan^2 x + c$

s. $-\frac{1}{2}\cos(2x+\pi) + c$

t. $u = x\sqrt{x} \Rightarrow du = \frac{3}{2}\sqrt{x}\, dx \Rightarrow \frac{2}{3}du = \sqrt{x}\, dx$

$$\Rightarrow \int 5^{x\sqrt{x}}\sqrt{x}dx = \frac{2}{3}\int 5^u du = \frac{2\left(5^u\right)}{3\ln 5} + c = \frac{2\left(5^{x\sqrt{x}}\right)}{3\ln 5} + c$$

u. $\sqrt{x^2+5} + c$

v. $\frac{4}{5}(x+3)^2\sqrt{x+3} - 6\sqrt{x+3}^3 + c$

3. a. $\frac{1}{8}x^8 + \frac{5}{4}x^4 + c$

b. $\frac{x^2}{2} + \frac{1}{2x^2} + c$

c. $\frac{2^{x^2}}{2\ln 2} + c$

d. $\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \int \frac{\frac{d}{dx}(\sin x)}{\sin x}dx = \ln|\sin x| + c$

e. $u = 1 - \cos x \Rightarrow du = \sin x\, dx$

$$\Rightarrow \int \sin x\sqrt{1-\cos x}\, dx = \int \sqrt{u}\, dx = \frac{2}{3}u\sqrt{u} + c = \frac{2}{3}(1-\cos x)\sqrt{1-\cos x} + c$$

f. $\frac{2}{3}\left(4+e^x\right)^{\frac{3}{2}} + c$

g. $\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$

h. $\dfrac{\sin(4x+3)}{4}+c$

i. $u = 1 - 3^{x+1} \Rightarrow du = -\ln 3\ (3^{x+1})\ dx$

$\Rightarrow\ du = -3\ln 3\ (3^{x})\ dx$

$\Rightarrow\ -\dfrac{1}{3\ln 3}du = 3^{x}\,dx$

$\Rightarrow\ \int 3^{x}\left(1-3^{x+1}\right)^{9} dx = -\dfrac{1}{3\ln 3}\int u^{9}du = -\dfrac{1}{30\ln 3}u^{10}+c$

$= -\dfrac{1}{30\ln 3}\left(1-3^{x+1}\right)^{10}+c$

j. $\ln|4x-2| + c$

k. $\dfrac{\ln|ax+b|}{a} + c$

l. If $n = 1$, $\dfrac{1}{a}\ln|ax+b|+c$

If $n \neq 1$, $\dfrac{1}{a(1-n)(ax+b)^{n-1}+c}$

m. $\sqrt{x^{2}+1}+c$

n. $\dfrac{1}{6}x^{6}-\dfrac{8}{3}x^{3}+c$

o. $2e^{\sqrt{t}}+c$

p. $\dfrac{2}{\ln 2}2^{\sqrt{y}}+c = \dfrac{2^{\sqrt{y}+1}}{\ln 2} + c$

q. $u = 3 + 5x \ \Rightarrow\ \dfrac{1}{5}du = dx$

Also, $x = \dfrac{u-3}{5}$

$\Rightarrow \int x\sqrt{3+5x}\ \mathrm{d}x = \dfrac{1}{5}\int \dfrac{u-3}{5}\sqrt{u}\,du$

$= \dfrac{1}{25}\int u\sqrt{u}-3\sqrt{u}\,du$

$=\dfrac{1}{25}\left(\dfrac{2}{5}u^{2}\sqrt{u}-2u\sqrt{u}\right)+c$

$= \dfrac{2}{125}(3+5x)^{2}\sqrt{3+5x}-\dfrac{2}{25}(3+5x)\sqrt{3+5x}+c$

r. $u = 3 + \cos t \Rightarrow -du = \sin t\, dt$

$$\Rightarrow \int \frac{\sin t}{\sqrt{3+\cos t}}dt = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u} + c = -2\sqrt{3+\cos t} + c$$

s. $u = 1-\cos(2t) \Rightarrow du = 2\sin(2t)\,dt \Rightarrow \frac{1}{2}du = \sin(2t)\,dt$

$$\Rightarrow \int \frac{\sin(2t)}{1-\cos(2t)}dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|1-\cos(2t)| + c$$

t. $\frac{e^{x^2+7}}{2} + c$

u. $\frac{1}{2}\ln|1-2x+4x^2| + c$

v. $\frac{1}{14}(3x^2+2x+5)^7 + c$

w. $\frac{3}{10}(x^2-2x+5)^{\frac{5}{3}} + c$

x. $\frac{1}{11}\sin^{11}(x) + c$

5.2.2 Integration by Partial Fractions

Teaching Notes

Move around individual groups and assist those who need help.

The first exercise in this activity is to revise the techniques of decompositions into partial fractions. The 2nd and 3rd exercises are designed to introduce the method of integration using partial fractions along with the method of substitution.

Answers to Activity 5.4

1. a. $\frac{1}{x} - \frac{1}{x+1}$

   b. $\frac{2}{x-2} - \frac{1}{x-1}$

   c. $\frac{2}{x-1} - \frac{1}{(x-1)^2}$

   d. $x + 4 - \frac{1}{2(x-1)} + \frac{27}{2(x-3)}$

   e. $\frac{5}{9(x-3)} - \frac{2}{3x^2} - \frac{5}{9x}$

   f. $\frac{11}{12}\left(\frac{1}{(x-2)}\right) - \frac{2}{3}\left(\frac{1}{x+1}\right) + \frac{3}{4}\left(\frac{1}{x+2}\right)$

   g. $\frac{-2}{(1+x)^2} + \frac{3}{x+1} - \frac{3}{x+2}$

2. a. $\int \frac{x+2}{x+3}dx = \int\left(1 - \frac{1}{x+3}\right)dx = x - \ln|x+3| + c$

   b. $\int \frac{x+2}{4x-3}dx = \int\left(\frac{1}{4} + \frac{11}{(16x-12)}\right)dx = \frac{1}{4}x + \frac{11}{16}\ln|16x-12| + c$

   c. $\frac{1}{4}\ln|4x+1| + c$

d. $\int \frac{4x-5}{5x-4}dx = \int\left(\frac{4}{5}-\frac{9}{25x-20}\right)dx = \frac{4}{5}x-\frac{9}{25}\ln|25x-20|+c$

e. $\frac{-1}{6(2x-1)^3}+c$

f. $x-\frac{32}{(x-3)^2}-\frac{48}{x-3}+12\ln|x-3|+c$

3. The expression should be decomposed into partial fractions.

Answers to Exercise 5.7

1. $\int \frac{x}{x+5}dx = \int 1-\frac{5}{x+5}dx = x-5\ln|x+5|+c$

2. $\int \frac{4x+1}{x^2-3x+2}dx = \int \frac{4x+1}{(x-2)(x-1)}dx = \int \frac{A}{x-2}+\frac{B}{x-1}dx$

$$= A\ln|x-2|+B\ln|x-1|+c$$
$$= 9\ln|x-2|-5\ln|x-1|+c$$

3. $\int \frac{x^2-x-2}{x^2+x-2}dx = \int\left(1-\frac{2x}{x^2+x-2}\right)dx$

$$= \int\left(1-\frac{2x}{(x+2)(x-1)}\right)dx$$
$$= x+\int\frac{A}{x+2}dx+\int\frac{B}{x-1}dx$$
$$= x+A\ln|x+2|+B\ln|x-1|+c$$
$$= x-\frac{4}{3}\ln|x+2|-\frac{2}{3}\ln|x-1|+c$$

4. $\int \frac{x^2+4}{x^2-1}dx = \int\left(1+\frac{5}{x^2-1}\right)dx$

$$= \int\left(1+\frac{5}{(x-1)(x+1)}\right)dx$$
$$= x+\int\frac{A}{x-1}dx+\int\frac{B}{x+1}dx$$
$$= x+A\ln|x-1|+B\ln|x+1|+c$$
$$= x+\frac{5}{2}\ln|x-1|-\frac{5}{2}\ln|x+1|+c = x+\frac{5}{2}\ln\left|\frac{x-1}{x+1}\right|+c$$

5. $3x-\ln|x+2|+c$

6. $$\int \frac{x}{x^2-2x-8}dx = \int \frac{x}{(x-4)(x+2)}dx$$
$$= \int \frac{A}{x-4} + \frac{B}{x+2}dx$$
$$= A\ln|x-4| + B\ln|x+2| + c = \frac{2}{3}\ln|x-4| + \frac{1}{3}\ln|x+2| + c$$

7. First decompose $\frac{x}{\left(x^2+3x+2\right)^2}$ into partial fractions.

$$\frac{x}{\left(x^2+3x+2\right)^2} = \frac{x}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+1)^2} + \frac{D}{(x+2)^2}$$
$$\Rightarrow \int \frac{x}{\left(x^2+3x+2\right)^2}dx = A\ln|x+1| + B\ln|x+2| - \frac{C}{x+1} - \frac{D}{x+2}$$
$$= 3\ln|x+1| - 3\ln|x+2| + \frac{1}{x+1} + \frac{2}{x+2} + c = 3\ln\left|\frac{x+1}{x+2}\right| + \frac{1}{x+1} + \frac{2}{x+2} + c$$

Using the above techniques we can get the integrals of the others.

8. $x + \frac{1}{x+1} + 4\ln|x+1| - 8\ln|x+2| + c$

9. $-\frac{1}{x+2} + c$

10. $$\int \frac{x^2+2x-3}{x^2(x^2-5x+6)}dx = \int\left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x-3}\right)dx$$
$$= A\ln|x| - \frac{B}{x} + C\ln|x-2| + D\ln|x-3| + c$$

where $A = -\frac{1}{12},\ B = -\frac{1}{2},\ C = \frac{-5}{4},\ D = \frac{4}{3}$

5.2.3 Integration by Parts

Integration by parts is more difficult than integration by substitution. Hence some students may omit doing this part. In this case, it might be a good idea to group students by similar abilities. Then monitoring the progress of each group, discuss the worked examples.

Answers to Activity 5.5

Using problem 1 and 2 help students to explore the technique of integration by parts

1. a. $\ln x$ b. xe^x c. $\cos x - x\sin x + \sin x$
   d. $2e^x \cos x$ e. $2x\ln x - x$

2. We perform the reverse operations of the above derivatives.

a. $x\ln x - x + c$ b. $xe^{x} - e^{x} + c$ c. $\sin x - x\cos x + c$

d. $\frac{1}{2}e^{x}(\sin x - \cos x) + c$ e. $\frac{1}{2}x^{2}\ln x - \frac{1}{4}x^{2} + c$

3. Students are expected to indicate a technique that is similar to the one used in problem 2.

Answers to Exercise 5.8

1. $\int xe^{1-x}dx = e\int xe^{-x}dx$

$u = x \text{ and } dv = e^{-x}$

$\Rightarrow du = dx \text{ and } v = -e^{-x}$

$$\Rightarrow \int xe^{1-x}dx = e\int xe^{-x}dx$$

$$= e\left(-xe^{-x} - \int -e^{-x}dx\right)$$

$$= -xe^{1-x} + e\int e^{-x}dx$$

$$= -xe^{1-x} - e^{1-x} + c = -e^{1-x}(1+x) + c$$

2. $\cos x + x\sin x$

3. $\int xe^{3x+1}dx$

$u = x \Rightarrow du = dx$

$dv = e^{3x+1} \Rightarrow v = \frac{1}{3}e^{3x+1}$

$$\Rightarrow \int xe^{3x+1}dx = \frac{x}{3}e^{3x+1} - \frac{1}{3}\int e^{3x+1}dx = \frac{x}{3}e^{3x+1} - \frac{1}{9}e^{3x+1} + c$$

4. $\int x^{2}e^{x}dx$

$u = x^{2} \Rightarrow du = 2x\,dx$

$dv = e^{x} \Rightarrow v = e^{x}$

$$\Rightarrow \int x^{2}e^{x}dx = x^{2}e^{x} - 2\int xe^{x}dx$$

$$= x^{2}e^{x} - 2\left(xe^{x} - e^{x}\right) + c = x^{2}e^{x} - 2xe^{x} + 2e^{x} + c$$

5. $4\sin x - 4x\cos x$

6. Here, we apply integration by parts twice.

$\int e^x \cos(2x)\,dx$

$u = e^x \Rightarrow du = e^x dx$

$dv = \cos(2x) \Rightarrow v = \frac{1}{2}\sin(2x)$

$\Rightarrow \int e^x \cos x\,dx = \frac{1}{2}e^x \sin(2x) - \int \frac{1}{2}\sin(2x) e^x dx$

Also, by the same step, $\int \sin(2x) e^x dx$

$$\Rightarrow \int e^x \cos(2x) = \frac{1}{2}e^x \sin(2x) - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)e^x - \int -\frac{1}{2}\cos(2x)e^x dx\right)$$

$$= \frac{1}{2}e^x \sin(2x) + \frac{1}{4}\cos(2x)e^x - \int \frac{1}{4}\cos(2x)e^x dx$$

$\Rightarrow \frac{5}{4}\int e^x \cos(2x)dx = \frac{1}{2}e^x \sin(2x) + \frac{1}{4}\cos(2x)e^x + c$

$\Rightarrow \int \cos(2x)e^x dx = \frac{4}{5}\left(\frac{1}{2}e^x\left(\sin(2x) + \frac{1}{2}\cos(2x)\right)\right) + c$

$$= \frac{1}{5}e^x\left(\cos(2x) + 2\sin(2x)\right) + c$$

7. $-\frac{1}{10}e^{3x}(\cos x - 3\sin x) + c$

8. $-\frac{1}{5}e^{-x}\left(2\cos(2x) + \sin(2x)\right) + c$

9. $x\ln(4x) - x + c$

10. $\frac{1}{4}x^4 \ln x - \frac{x^4}{16} + c$

11. $e^x(x+1) + c$

12. $2\cos x - x^2\cos x + 2x\sin x + c$

13. $\frac{1}{3}x^3\ln(2x) - \frac{1}{9}x^3 + c$

14. $\frac{1}{2}x^2\ln(nx) - \frac{1}{4}x^2 + c$

15. $u = x \Rightarrow du = dx$

$du = \sin(nx) \Rightarrow v = -\frac{\cos(nx)}{n}$

$\Rightarrow \int x \sin\,(\mathrm{n}x)dx = \frac{-x\cos(nx)}{n} - \int -\frac{\cos(nx)}{n}dx = \frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + c$

5.3 DEFINITE INTEGRALS, AREA AND THE FUNDAMENTAL THEOREM OF CALCULUS

Periods Allotted: 8 periods

Competencies

At the end of this sub-unit, students will be able to:

- *compute area under a curve.*
- *use the concept of definite integral to calculate the area under a curve.*
- *state fundamental theorem of calculus.*
- *apply fundamental theorem of calculus to solve integration problems.*
- *state properties of definite integrals.*
- *apply the properties of definite integrals for computations of integration.*

Vocabulary: Area under a curve, Partition, Definite integral, Lower limit of integration Upper limit of integration.

Introduction

This topic begins by approximating the area of a region under a curve. The limiting value of the sum of the areas as discussed in the textbook is called the definite integral. Furthermore, the addition of a constant in the definite integral is irrelevant.

Using the definite integral, there is a variety of applications such as area between curves, volume of revolution, work done, displacement, velocity and acceleration.

Teaching Notes

In teaching this unit you can,

- ✓ use the class activities or any other appropriate examples, to illustrate how to compute areas of regions with curved boundaries.
- ✓ introduce the concept of definite integral as the limit of a sum appropriate example. the first example can be a constant function like $f(x) = 2$.
- ✓ discuss the relationships between integration and area bounded by a curve and the x – axis from x = a to $x = b$.
- ✓ discuss how to evaluate definite integral with the help of appropriate examples.
- ✓ For example, evaluate $\int_1^2 x^2\,dx$
- ✓ introduce the concept of the fundamental theorem of calculus with the help of appropriate examples.

- ✓ discuss how the fundamental theorem relates differentiation and integration.
- ✓ discuss the properties of definite integrals by using the examples given in the student textbook.

Assessment

Assessing students' understanding is vital. In order to ensure formative assessment you can:

- ✓ use the activities and the example to assess students on computation of areas under given curves.
- ✓ give exercises on problems of approximations of areas under a curve say $y = x + 1$ from $x = 0$ to $x = 1$ by subdividing the given interval into equal lengths.
- ✓ give the exercises as class work or home work.
- ✓ ask oral questions such as: state the fundamental theorem of calculus, state the properties of definite integrals.
- ✓ give the exercises on the applications of the fundamental theorem of calculus as class work or home work.

The opening problem is designed to give a picture of the subunit. Here students are expected only to give the answer in terms of the area of the region, like equal, less or greater. Don't try to give a numeric value. After teaching area of a region under a curve, you can ask students to give the actual area of a region.

Answer to Opening Problem

After teaching area of a region under a curve, you can ask students to give the actual area of a region which is calculated as:

$$A = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(5 - 2x^2 + 4x^3 - x^5\right) dx = 5x - \frac{2}{3}x^3 + x^4 - \frac{1}{6}x^6 \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{29}{6}$$

5.3.1 The Area of a Region under a Curve

Discuss activity 5.6 to start instruction formally. Ask three students to present question 1 on the black board. In the same way proceed to questions 2.

Give questions 3 and 4 for different groups. After few minutes ask some students to demonstrate their findings on the black board.

Answers to Activity 5.6

1. Encourage students to give as many partitions as possible.
2. a. $\left[0, \frac{1}{5}\right], \left[\frac{1}{5}, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, 1\right]$

b. Length, $\Delta x = \frac{5-3}{10} = 0.2$

Thus, the intervals are: $[3,\ 3.2],[3.2,3.4],[3.4,3.6],...,[4.8,5]$

c. $\left[0,\ \frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\left[\frac{2}{n},\frac{3}{n}\right],...,\left[\frac{n-1}{n},\frac{n}{n}\right]$

3. Let A_i be the area of the rectangle, then $A_i = \frac{1}{n}\left(\frac{i-1}{n}+1\right) = \frac{i-1}{n^2}+\frac{1}{n}$

The sum of the areas of the n-rectangle will be

$$A = \sum_{i=1}^{n}\left(\frac{i-1}{n^2}+\frac{1}{n}\right)$$

$$= \frac{1}{n^2}\sum_{i=l}^{n}(i-l)+\sum_{i=l}^{n}\frac{1}{n} = \frac{1}{n^2}\frac{(n-1)(n)}{2}+\frac{n}{n} = \frac{n^2-n}{2n^2}+1$$

i. a. $n=3 \Rightarrow A = \frac{3^2-3}{2\times 3^2}+1 = \frac{4}{3}$ b. $n=5 \Rightarrow A = \frac{5^2-5}{2\times 5^2}+1 = \frac{7}{5}$

c. $n=10 \Rightarrow A = \frac{10^2-10}{2\times 10^2}+1 = \frac{29}{20}$

ii. $\lim_{n\to\infty}\left(\frac{n^2-n}{2n^2}+1\right) = \frac{1}{2}+1 = \frac{3}{2}$

4. i. $A_i = \frac{1}{n}\left(\frac{i}{n}+1\right) = \frac{i}{n^2}+\frac{1}{n}$

$$\sum_{i=l}^{n}A_i = \sum_{i=l}^{n}\frac{i}{n^2}+\frac{1}{n} = \frac{n(n+1)}{2n^2}+1$$

a. $n=3 \Rightarrow A = \frac{5}{3}$ b. $n=5 \Rightarrow A = \frac{8}{5}$ c. $n=10 \Rightarrow A = \frac{31}{20}$

ii. $\lim_{n\to\infty}\left(\frac{n(n+1)}{2n^2}+1\right) = \lim_{n\to\infty}\frac{n^2+n}{2n^2}+1 = \frac{3}{2}$

Answers to Exercise 5.9

1. a. 4.48 b. 4.11

c. $\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^3.\frac{1}{n} = \frac{15n^2+14n+3}{4n^2}$

2. a. 10 b. $\frac{15}{4}$ c. $\frac{32}{3}$ d. $\frac{2}{3}$

3. a. It exists, because $f(x) = \tan x$ is continuous on $\left(\frac{-\pi}{4},\frac{\pi}{4}\right)$.

b. exists c. exists

d. doesn't exist. because f is not continuous at $x = 1$ and $x = -1$

e. doesn't exist. because f is not continuous at $x = -3$ and $x = 3$

f. exists

4. a. 16 b. $\frac{9}{2}$ c. $\frac{8}{3}$

   d. doesn't exist e. 0

5. 0 (It represents a point)

6. Since $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta x\ ,\ z_k \in [x_k, x_{k-1}]$

$$= \lim_{x\to\infty} \sum_{k=1}^{n} f(z_k)\Delta s, z_k \in [s_k, s_{k-1}] = \int_a^b f(s)\,ds$$

$$\Rightarrow \int_a^b f(x)\ dx = \int_a^b f(s)\ ds, s \in [a, b]$$

7. Since $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, then $A_1 = \int_a^c f(x)\,dx$ and $A_2 = \int_c^b f(x)\,dx$

   Therefore $A = A_1 + A_2$

5.3.2 Fundamental Theorem of Calculus

Teaching Notes

To start the lesson, discuss the existence of an alternative method to evaluate $\int_0^1 x^2 dx$ using the indefinite integral $\int x^2 dx$. Then, state the fundamental theorem of calculus and discuss the worked examples. Next, let the students discuss activity 5.7 and explore the properties of the definite integral by themselves.

Ask some students to give a short demonstration of the properties they found.

Answers to Activity 5.7

1. $\int x^2 dx = \frac{x^3}{3} + c$ *and* $\int \left(1 - \frac{1}{x}\right) dx = x - \ln|x| + c$

   a. $\int_1^3 f(x) + g(x)\,dx = \frac{x^3}{3} + x - \ln|x| \Big|_1^3 = \frac{32}{3} - \ln 3$

   b. $\int_3^3 f(x)\,dx = \frac{x^3}{3}\Big|_3^3 = 0$

   c. The same as (a)

d. $\int_{-2}^{3} 4f(x) = 4\left(\frac{x^3}{3}\right)\Big|_{-2}^{3} = \frac{140}{3}$

e. The same as (d)

f. $\int_1^4 g(x)dx + \int_4^{10} g(x)dx - \int_1^{10} g(x)dx = x\ln|x|\Big|_1^4 + x\ln|x|\Big|_4^{10} - x\ln x\Big|_1^{10}$

$$= 4\ln 4 + 10\ln 10 - 4\ln 4 - 10\ln 10 = 0$$

2. a. 0

b. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

c. Yes, look at problem (1).

d. Yes

e. Encourage students to produce such examples

The questions in activity 5.8 have already been discussed in the method of substitution. Let the students discuss how to determine the limits of integration.

Answers to Activity 5.8

1. a. $-4 \le u \le 5$ b. $0 \le u \le \sqrt{3}$

c. $e^2 \le u \le e^5$ d. $1 - 2\sqrt{3} \le u \le 1$

e. $u = 0$

2. $u = x^2 + 4 \Rightarrow a^2 + 4 \le u \le b^2 + 4 \Rightarrow c = a^2 + 4$ and d = b^2 + 4

3. $u = x^2 + x - 2 \Rightarrow du = (2x+1)dx$

$$\Rightarrow \int (2x+1)\sqrt{x^2+x-2}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u\sqrt{u} + c$$

$$= \frac{2}{3}(x^2+x-2)\sqrt{x^2+x-2} + c$$

$$\Rightarrow \int_1^2 (2x+1)\sqrt{x^2+x-2}\, dx = \frac{2}{3}(x^2+x-2)\sqrt{x^2+x-2}\Big|_1^2 = \frac{16}{3}$$

Answers to Exercise 5.10

1. 15 2. $b - a$ 3. –12

4. 10.5

5. $\frac{b^{n+1} - a^{n+1}}{n+1}$ 6. $\frac{-173}{12}$ 7. $\sqrt{3} - \frac{1}{3}$

8. $\frac{4}{\ln 2}$ 9. $\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}x - \frac{\cos x \sin x}{2}\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}$

10. $\int_2^3 \frac{1}{x}dx = \ln|x|\Big|_2^3 = \ln 3 - \ln 2$ 11. $\frac{9}{4}\ln 5$

12. $\int_0^{\sqrt{\pi}} x\sin\left(x^2+3\right)dx$

Let $u = x^2 + 3$, then $\frac{1}{2}du = xdx \; and \; 3 \le u \le \pi + 3$

$$\Rightarrow \int_0^{\sqrt{\pi}} x\sin\left(x^2+3\right)dx = \frac{1}{2}\int_3^{\pi+3} \sin u \; du$$

$$= \frac{-\cos u}{2}\Bigg|_3^{\pi+3} \sin u \, du$$

$$= -\frac{1}{2}\left(\cos\left(\pi+3\right) - \cos 3\right) = \cos 3$$

13. $\int_{\frac{\pi}{2}}^{\pi} x\sin x \, dx = \sin x - x\cos x\Bigg|_{\frac{\pi}{2}}^{\pi} = \pi - 1$

14. $\int_e^{e^3} x\ln x \, dx = \frac{1}{2}x^2\ln x - \frac{x^2}{4}\Bigg|_e^{e^3} = \frac{5e^6 - e^2}{4}$

15. $\int_1^3 \frac{1}{2x^2+x-1}dx = \int_1^3\left(\frac{A}{2x-1} + \frac{B}{x+1}\right)dx$

$$= \frac{A}{2}\ln|2x-1| + B\ln|x+1|\,\Bigg|_1^3$$

$$= \frac{\ln|2x-1|}{3} - \frac{\ln|x+1|}{3}\Bigg|_1^3 = \frac{\ln 5 - \ln 2}{3}$$

16. $\int_{-1}^1 \frac{2x+3}{\left(x^2+3x+4\right)^6}\,dx = + \frac{-1}{5\left(x^2+3x+4\right)^5}\Bigg|_{-1}^1 = \frac{1}{5}\left(\frac{1}{8^5} - \frac{1}{2^5}\right) = \frac{1023}{163840}$

17. $\int_{-1}^{\frac{1}{2}} (4x+3)^{10}\,dx = \frac{(4x+3)^{11}}{44}\Bigg|_{-1}^{\frac{1}{2}} = \frac{5^{11}+1}{44}$

18. $\int_{\sqrt{2}}^3 x\sqrt{x^2+7}dx = \frac{1}{3}\left(x^2+7\right)^{\frac{3}{2}}\Bigg|_{\sqrt{2}}^3 = \frac{37}{3}$

19. $\frac{1729}{2}$

20. $\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}\Bigg|_4^9 = 2e^2\left(e-1\right)$

21. $\int_0^2 (x-2)\sqrt{x+1}\;dx = \frac{2}{5}(x+1)^{\frac{5}{2}} - 2(1+x)^{\frac{3}{2}}\Bigg|_0^2 = \frac{8-12\sqrt{3}}{5}$

22. $\int_{-1}^{0} \frac{x+4}{3x+1} dx$ undefined because $\frac{x+4}{3x+1}$ is not continuous on $[-1, 0]$

23. $\int_{-4}^{3} \frac{x+1}{x^2-x-6}$ is undefined because $\frac{x+1}{x^2-x-6}$ is not continuous on $[-4, 3]$

24. $\int_{-1}^{1} \frac{3t^2-1}{e^{t^3-t}} dt = \frac{1}{e^{t^3-t}}\Big|_{-1}^{1} = \frac{1}{e^o} - \frac{1}{e^o} = 0$

25. $\int_{-1}^{1} \frac{e^x}{1+e^x} dx = \ln\left(e^x+1\right)\Big|_{-1}^{1} = \ln\left(e+1\right) - \ln\left(\frac{1}{e}+1\right)$

$$= \ln\left(e+1\right) - \ln\left(\frac{e+1}{e}\right)$$

$$= \ln\left(e+1\right) - \ln\left(e+1\right) + \ln e = \ln e \ = 1$$

26. $f(x) = \frac{1}{x^2}$ is discontinuous at $x = 0$.

$\Rightarrow \int_{-2}^{2} \frac{1}{x^2}\, dx$ is undefined

27. If f is an even function, then $\int_0^a f(x)\,dx = \int_{-a}^0 f(x)\,dx$

$\Rightarrow \int_{-a}^{a} f(x) = 2\int_0^a f(x)$

For example, $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos x\,dx = \sin x \Big|_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \sin\frac{\pi}{4} - \left(-\sin\frac{\pi}{4}\right) = 2\sin\frac{\pi}{4} = \sqrt{2}$

Also, $2\int_0^{\frac{\pi}{4}} \cos x\ dx = 2\sin x\Big|_0^{\frac{\pi}{4}} = \sqrt{2}$

28. If f is an odd function, then $\int_{-a}^{0} f(x) = -\int_0^a f(x)$ so that

$$\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x) + \int_0^a f(x) = 0$$

a. $\int_{-3}^{3} \left(x^3+x\right) dx = \frac{x^4}{4} + \frac{x^2}{2}\Big|_{-3}^{3} = \left(\frac{(-3)^4}{4} + \frac{(-3)^2}{4}\right) = 0$

b. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = 0$

c. $\int_{-1}^{1} \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln\left(x^2+1\right)\Big|_{-1}^{1} = \frac{1}{2}\left[\ln 2 - \ln(2)\right] = 0$

5.4 APPLICATIONS OF INTEGRAL CALCULUS

Periods Allotted: 6 periods

Competency

At the end of this sub-unit students will be able to:

- *apply the knowledge on integral calculus to solve problems.*

Vocabulary: Area between curves, Volume, Solid of revolution

Introduction

Among the several applications of integral calculus, some which are designed for this sub-unit are presented in two sections.

In section 5.4.1, the area of a region that is bounded by the graphs of continuous function on an interval will be discussed.

The second section i.e, section 5.4.2 is about the volume of revolution. Also, the application of integral calculus in physics, such as work, displacement, velocity are presented.

5.4.1 The Area Between two Curves

Teaching Notes

Let students discuss Activity 5.9 for themselves since they already know how to determine the area of a shaded part in plane geometry.

Answers to Activity 5.9

1. a. $\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$ b. $\int_0^1 \left(x^2+1\right)dx = \left.\frac{x^3}{3}+x\right|_0^1 = \frac{4}{3}$

2.

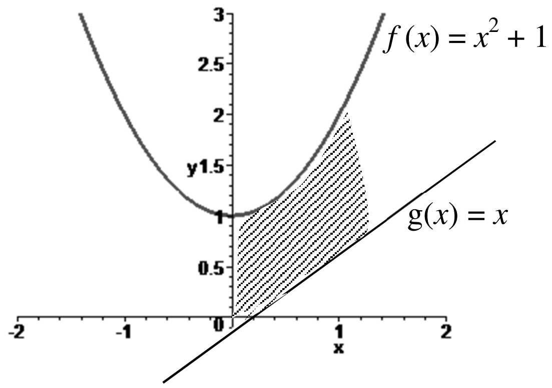


Figure 5.5

$$A = \int_0^1 \left(x^2+1\right)dx - \int_0^1 x\,dx$$

$$= \left.\frac{x^3}{3}+x\right|_0^1 - \left.\frac{x^2}{2}\right|_0^1$$

$$= \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$

Discuss the worked examples with the class or group.

Encourage the students to find a formula for an area of a region bounded by two curves while they are doing activity 5.10.

Answers to Activity 5.10

1. $\int_1^3 (x^2+4-1)\,dx = \dfrac{x^3}{3}+3x\Big|_1^3 = 9+9-\left(\dfrac{1}{3}+3\right)=\dfrac{44}{3}$

2. $A_1=\int\limits_a^0 f(x)dx,\ A_2=\int\limits_0^b f(x)dx.\ A=A_1+A_2$

Additional exercise problems for high ability students

1. Find the area of the region bounded by the curves $y = \left|x^2-1\right|$ and $y=3-x^2$.

Solution: The curves meet at $x = \pm\sqrt{2}$.

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} (3-x^2)-\left|x^2-1\right| dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (3-x^2)\,dx - \int_{-\sqrt{2}}^{\sqrt{2}} \left|x^2-1\right| dx$$

$$= \frac{14}{3}\sqrt{2} - \int_{-\sqrt{2}}^{\sqrt{2}} \left|x^2-1\right| dx$$

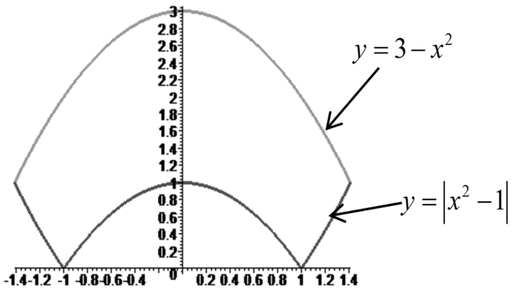


Figure 5.6

But $\int_{-\sqrt{2}}^{\sqrt{2}} \left|x^2-1\right| = \int_{-\sqrt{2}}^{-1}\left(x^2-1\right)dx + \int_{-1}^{1}(1-x^2)\,dx + \int_{1}^{\sqrt{2}}(x^2-1)\,dx$

$$= \left(\frac{2}{3}-\frac{1}{3}\sqrt{2}\right)+\left(\frac{4}{3}\right)+\frac{2}{3}-\frac{1}{3}\sqrt{2}$$

$$= \frac{8}{3}-\frac{2}{3}\sqrt{2}$$

$$\Rightarrow\ A=\frac{14}{3}\sqrt{2}-\frac{8}{3}+\frac{2}{3}\sqrt{2}=\frac{16}{3}\sqrt{2}-\frac{8}{3}$$

2. Draw the region bounded by the graphs of $f(x) = \tan x$, $g(x) = \cot x$ and $h(x) = -1$ from $x=-\dfrac{\pi}{4}\ to\ x=\dfrac{3\pi}{4}$ on a wall chart. Calculate its area and demonstrate to the class.

Solution:

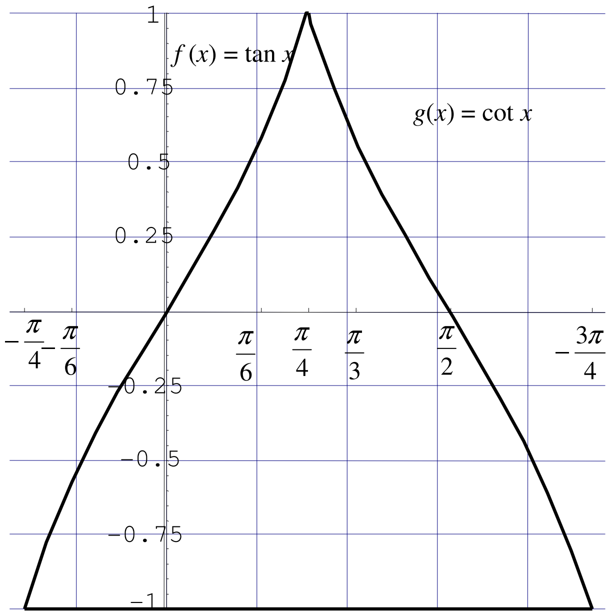


Figure 5.7

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\tan x + 1)\, dx + \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (\cot x + 1)\, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Assessment

You can assess your students by giving them chance to illustrate the application of integral calculus in solving problems on

- ✓ area
- ✓ volume
- ✓ displacement
- ✓ work by using appropriate examples

Use the examples in the student textbook or you may give additional examples.

- ➢ Use the examples on the application problems to assess students during instruction.
- ➢ Use the exercise in the topic as class work or home work.
- ➢ You may select some of problems for class room discussion or project work.

Answers to Exercise 5.11

1. a. $-\int_{-3}^{0} x\, dx + \int_{0}^{2} x\, dx = -\left(\frac{x^2}{2}\right)\Big|_{-3}^{0} + \frac{x^2}{2}\Big|_{0}^{2} = \frac{9}{2} + 2 = 6.5$

b.

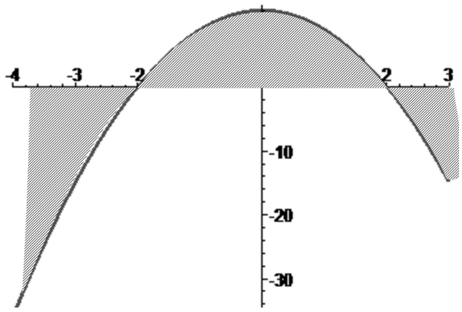


Figure 5.8

$$A=-\int_{-4}^{-2}\left(12-3x^2\right)\,dx+\int_{-2}^{2}12-3x^2dx-\int_{2}^{3}12-3x^2dx$$

$$=-\left(12x-x^3\right)\Big|_{-4}^{-2}+12x-x^3\Big|_{-2}^{2}-\left(12x-x^3\right)\Big|_{2}^{3}=32+32+7=71$$

$$A=\int_{-1}^{1}x^3dx=2\int_{0}^{1}x^3dx=\frac{x^4}{2}\Big|_{0}^{1}=\frac{1}{2}$$

d. $A=\int_{-1}^{4}2^x dx=\frac{2^x}{\ln 2}\Big|_{-1}^{4}=\frac{31}{2\ln 2}$

e. $A=\int_{1}^{e^2}\ln x\;dx-\int_{\frac{1}{e}}^{1}\ln x\,dx$

$$=x\ln x-x\Big|_{1}^{e^2}-\left(x\ln x-x\Big|_{\frac{1}{e}}^{1}\right)$$

$$=e^2+1+\frac{e-2}{e}\quad=e^2+2-\frac{2}{e}$$

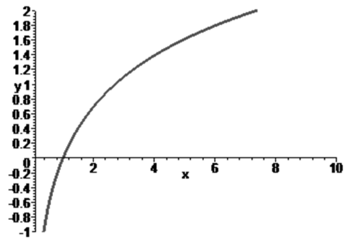


Figure 5.9

f. $A=\int_{\frac{\pi}{4}}^{\pi}\sin x\,dx-\int_{\pi}^{\frac{7}{4}\pi}\sin x\,dx\;=1+\frac{\sqrt{2}}{2}-\left(-1-\frac{\sqrt{2}}{2}\right)=2+\sqrt{2}$

g. $\int_{\frac{1}{10}}^{1}\frac{1}{x}\;dx=\ln|x|\Big|_{\frac{1}{10}}^{1}=\ln 10$

h. $\int_{-1}^{\frac{1}{2}}\left(x^2+4\right)\;dx=\frac{x^3}{3}+4x\Big|_{-1}^{\frac{1}{2}}=\frac{49}{24}-\left(-\frac{13}{3}\right)=\frac{51}{8}$

2. a. $x^2 = \sqrt{x} \Rightarrow x = 0,1$

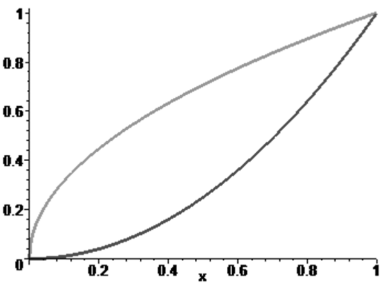


Figure 5.10

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \frac{2}{3}x\sqrt{x} - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

b. $x^2 = |x| \;\Rightarrow\; x = 0, \pm 1$

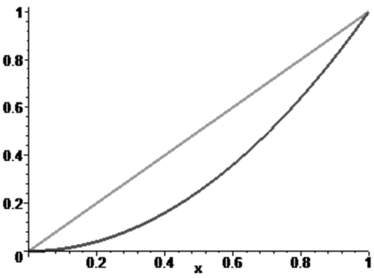


Figure 5.11

$$A = 2\int_0^1 x - x^2 dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

c. $3x^2 - 4 = 2x^2 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

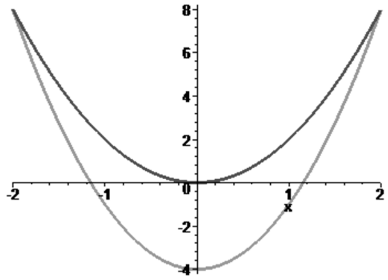


Figure 5.12

$$A = \int_{-2}^{2} \left(2x^2 - \left(3x^2 - 4\right)\right) dx$$

$$= \int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= 4x - \frac{x^3}{3}\Big|_{-2}^{2} = \frac{32}{3}$$

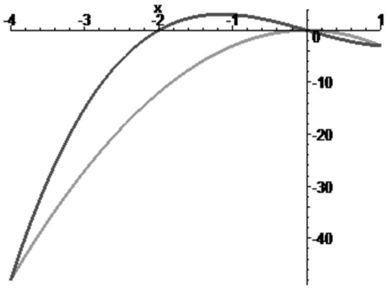
d. $x^3 - 4x = -3x^2$

$\Rightarrow x^3 + 3x^2 - 4x = 0$

$\Rightarrow x\left(x^2 + 3x - 4\right) = 0$

$\Rightarrow x\left(x+4\right)\left(x-1\right) = 0$

$\Rightarrow x = 0, 1, -4$



$$A = \int_{-4}^{0} \left(x^3 - 4x - (-3x^2) \right) dx + \int_{1}^{0} \left(-3x^2 - (x^3 - 4x) \right) dx$$

$$= \frac{x^4}{4} + x^3 - 2x^2 \Big|_{-4}^{0} + \left(-x^3 - \frac{x^4}{4} + 2x^2 \right) \Big|_{0}^{1}$$

$$= 32 + 0.75 = 32.75$$

Figure 5.13

5..1 Volume of Revolution

Teaching Notes

The opening problem is designed to give a picture of the subtopic, don't try to give a solution at this stage. Allow students to discuss only interms of the volume of revolution of the bowl. Like half or one third of the volume of the bowl, etc.

The focus of Activity 5.11 is to introduce students how to generate a solid of revolution. The activity is best if students work in groups. It needs to encourage the groups to construct practical models generated by functions such as $y = x$, $y = x^2$, etc. on specific intervals.

Answers to opening Problem

After teaching the volume of solid of revolution ask students to give the volume of the water in hemispherical bawl.

$$V = \pi \int_{-5}^{-4} \left(\sqrt{25 - y^2} \right)^2 dy$$

$$= \pi \left(25y - \frac{y^3}{3} \right) \Big|_{-5}^{-4} = \frac{14}{3} \pi$$

Additional exercise problems for high ability students

Find the volume of the solid of revolution generated by revolving the region bounded by the graph of $f(x) = 2x - 1 + 2^x$ and the x-axis between $x = -2$ and $x = 2$.

Solution: $f(0) = 0 \Rightarrow$ the graph passes through the origin.

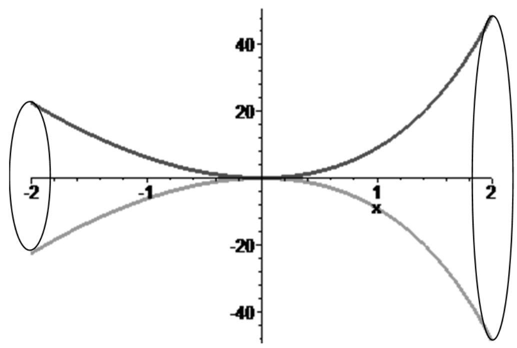


Figure 5.14

$$V = \pi\int_{-2}^{2}(2x-1+2^x)^2\,dx$$

$$= \pi\int_{-2}^{2}\left(4x^2-4x+2^{x+2}x+1-2^{x+1}+4^x\right)dx$$

$$= \pi\left(\frac{4}{x}x^3-2x^2+\frac{4x^2 2^x x}{\ln 2}-\frac{2\times 2^x}{\ln 2}-\frac{4\times 2^x}{(\ln 2)^2}+\frac{4^x}{\ln 4}\Bigg|_{-2}^{2}\right)$$

$$= \frac{\pi}{96\,(\ln(2))^2}\,(3309\ln 2+2432\,(\ln 2)^2-1440)$$

Answers to Activity 5.11

1. Cylinder
2. Cone
3. Sphere
4. Frustum
5. $y=\sqrt{4-x^2} \Rightarrow y^2+x^2=4$ and $0\le y\le\sqrt{3}$

Answers to Exercise 5.12

1. a. $\pi\int_0^1 (2x)^2 dx = \dfrac{4x^3\pi}{3}\Bigg|_0^1 = \dfrac{4}{3}\pi$

   b. $\pi\int_{-1}^{2}\left(x^2+1\right)^2 dx = \dfrac{78}{5}\pi$

   c. $\pi\int_1^2\left(e^x\right)^2 dx = \dfrac{e^2\left(e^2-1\right)}{2}$

   d. $\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin^2 x dx = \pi\left(-\dfrac{1}{2}\sin x\cos x+\dfrac{x}{2}\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right)$

$$=\left(\frac{\pi}{4}+\frac{1}{2}\left(\frac{1}{2}.\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)\right)\pi=\left(\frac{\pi}{12}+\frac{\sqrt{3}}{8}\right)\pi$$

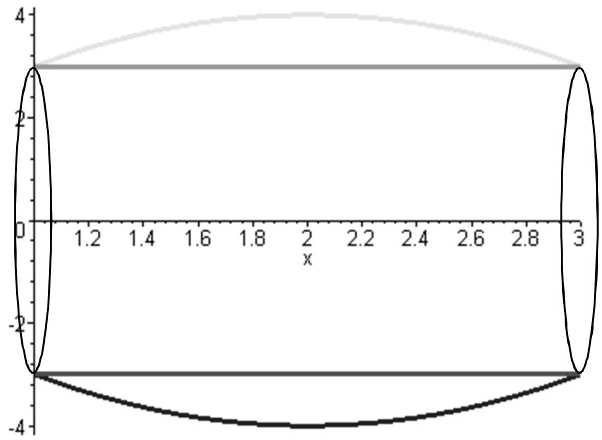
e. $\pi\int_{-3}^{1}\left|x\right|^{2}dx=\pi\int_{-3}^{1}x^{2}dx=\left.\frac{x^{3}\pi}{3}\right|_{-3}^{1}=\frac{28}{3}\pi$

f. $\pi\int_{-2}^{3}\left(2^{x}\right)^{2}dx=\pi\int_{-2}^{3}4^{x}dx=\left.\frac{\pi 4^{x}}{\ln 4}\right|_{-2}^{3}=\frac{1023}{32\ln 2}\pi$

g. $\pi\int_{-1}^{2}\left(x^{3}\right)^{2}dx=\left.\frac{x^{7}\pi}{7}\right|_{-1}^{2}=\frac{129\pi}{7}$

2. a. $4x-x^{2}=3\Rightarrow x^{2}-4x+3=0\Rightarrow x=1,3$

$\Rightarrow$ The graphs intersect at x = 1, 3



$$V=\pi\int_{1}^{3}\left(\left(4x-x^{2}\right)^{2}-9\right)dx$$

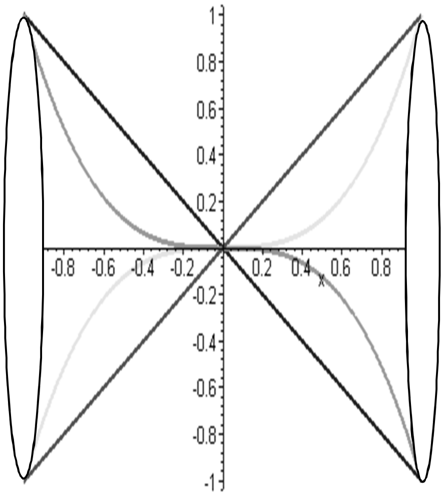
$$=\pi\left.\left(\frac{x^{5}}{5}-2x^{4}+\frac{16x^{3}}{3}-9x\right)\right|_{1}^{3}=\frac{136}{15}\pi$$

Figure 5.15

b. $x^{3}=x\Rightarrow x^{3}-x=0\Rightarrow x\left(x^{2}-1\right)=0$

$\Rightarrow x=0,\pm 1$

$\Rightarrow$ The graphs meet at x = 0, ± 1

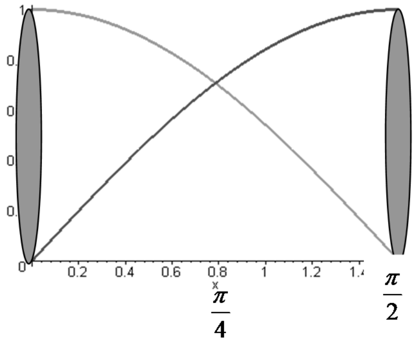


$$V=2\pi\int_{0}^{1}\left(x^{2}-x^{6}\right)dx$$

$$=2\pi\left.\left(\frac{x^{3}}{3}-\frac{x^{7}}{7}\right)\right|_{0}^{1}=\frac{8\pi}{21}$$

Figure 5.16

c. $\sin x = \cos x \ and \ 0 \le x \le \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4}$



$$V = \pi\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\,dx + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x - \cos^2 x\ dx$$

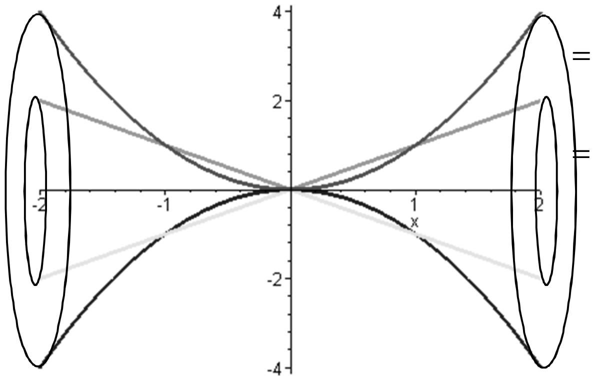
$$= \pi\int_0^{\frac{\pi}{4}} \cos(2x)\,dx + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -\cos(2x)\,dx$$

$$= \frac{\pi}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} - \frac{\pi}{2}\sin(2x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}(1) - \frac{\pi}{2}(0-1) = \pi$$

Figure 5.17

d.



$$V = 2\pi\int_0^1 \left(x^2 - x^4\right)dx + 2\pi\int_1^2 \left(x^4 - x^2\right)dx$$

$$= 2\pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 + 2\pi\left(\frac{x^5}{5} - \frac{x^3}{3}\right)\Bigg|_1^2$$

$$= \frac{4\pi}{15} + 2\left(\frac{58\pi}{15}\right) = \frac{120}{15}\pi = 8\pi$$

Figure 5.18

3. This can be given as group work. Consider the line passing through (0, *r*) and (h, R).

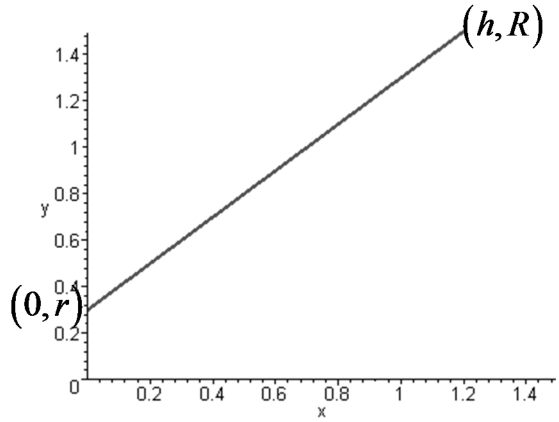


Figure 5.19

The equation of the line is $y = mx + r$ where $m = \dfrac{R-r}{h}$

$$\Rightarrow\ y = \frac{R-r}{h}x + r$$

$$\Rightarrow\ V = \pi\int_0^h \left(\left(\frac{R-r}{h}\right)x + r\right)^2 dx$$

$$= \pi\int_0^h \left(\left(\frac{R-r}{h}\right)^2 x^2 + 2rx\left(\frac{R-r}{h}\right) + r^2\right) dx$$

$$= \pi\left(\left(\frac{R-r}{h}\right)^2\left(\frac{x^3}{3}\right) + rx^2\left(\frac{R-r}{h}\right) + r^2x\right)\Bigg|_0^h = \frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$$

4. a. $a = -1.5t^2\ m/s^2$

$\Rightarrow$ when t = 10s,

$a = -1.5\ (10)^2$ m/s^2 = -150 m/s^2

b. $v = \int -1.5t^2\, dt = \frac{-t^3}{2} + c$

But when $t = 0$, $v = 4$

$\Rightarrow\ \frac{-0^3}{2} + c = 4 \Rightarrow\ c = 4 \Rightarrow\ v = \frac{-t^3}{2}\ 74$

$\Rightarrow$ when $t = 10s$,

$v = \left(\frac{-1000}{2} + 4\right) m/s = -496$ m/s

c. $s = \int v dt = \int\left(\frac{-t^2}{2} + 4\right) dz = \frac{-t^4}{8} + 4t + c$

But when $t = 0$, $s = 0$

$\Rightarrow\ c = 0 \Rightarrow\ s = \frac{-t^4}{8} + 4t \Rightarrow$ when $t = 10s$, $s = \left(\frac{-10^4}{8} + 4(10)\right)m = -1210$m

Answers to Review Exercises on Unit 5

In solving the Review Exercises, students should be encouraged to form their own groups with your assistance for checking the solutions.

1. $\frac{x^2}{4} + c$

2. $x^2 + 5x + c$

3. $\frac{x^3}{3} - \frac{3}{2}x^2 + 2x + c$

4. $\frac{1}{2}x^6 + c$

5. $\frac{x^8}{8} + c$

6. $\frac{2}{5}x^{\frac{5}{2}} + c$

7. $\frac{3}{4}x^{\frac{4}{3}} + c$

8. $\frac{3}{4}x^{\frac{4}{3}} + \frac{2}{5}x^{\frac{5}{2}} - \frac{1}{2}\ln x + \frac{x^{\pi+1}}{\pi+1} + c$

9. $-\frac{1}{x^2}+c$

10. $\frac{5}{3}x^{\frac{3}{5}}+c$

11. $-\cos(x+2)+c$

12. $\frac{4^x}{\ln 4}+\cos x+c$

13. $u=3x-4 \Rightarrow \frac{1}{3}du = \mathrm{d}x$

$$\Rightarrow \int \tan(3x-4)\,dx = \frac{1}{3}\int \tan u\,du = -\frac{1}{3}\ln|\cos u|+c$$

$$= -\frac{1}{3}\ln\ \cos(3x-4)+c$$

14. $\frac{1}{4}x^4+\frac{1}{2}x^2+c$

15. $\frac{1}{3}(2x+7)\sqrt{2x+7}$

16. $u=3x+5 \Rightarrow \frac{1}{3}du = dx$

$$\Rightarrow \int (3x+5)^{13}dx = \frac{1}{3}\int u^{13}du = \frac{u^{14}}{42}+c = \frac{(3x+5)^{14}}{42}+c$$

17. $\frac{x^7}{7}-\frac{3}{5}\,x^5+x^3-x+c$

18. $u=x^2+2x+5 \Rightarrow du=(2x+2)\,dx \ \Rightarrow \frac{1}{2}du=(x+1)\,dx$

$$\Rightarrow \int (x+1)\left(x^2+2x+5\right)^{10}dx = \frac{1}{2}\int u^{10}du = \frac{1}{22}u^{11}+c = \frac{\left(x^2+2x+5\right)^{11}}{22}+c$$

19. $\int \frac{x}{x+1}dx = \int 1-\frac{1}{x+1}dx = x-\ln|x+1|+c$

20. $\int \frac{1}{x^2-16}dx = \int \frac{1}{8(x-4)}-\frac{1}{8(x+4)}dx$

$$= \frac{1}{8}\left(\ln|x-4|-\ln|x+4|\right)+c$$

21. $u=x^2+4 \Rightarrow \frac{1}{2}du = xdx$

$$\Rightarrow \int x\sqrt{x^2+4}^{\,5}dx = \frac{1}{2}\int \sqrt{u}^{\,5}du = \frac{1}{7}u^{\frac{7}{2}}+c$$

$$= \frac{1}{7}\left(x^2+4\right)^{\frac{7}{2}}+c$$

22. $\int \frac{x}{x^2-2x-3}dx = \int \frac{1}{4(x+1)} + \frac{3}{4(x-3)}dx$

$$= \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-3| + c$$

23. $\frac{2^{4x+3}}{4\ln 2} + c$

24. $u = x^2+1 \Rightarrow \frac{1}{2}du = xdx$

$$\Rightarrow \int x\log\sqrt{x^2+1}\ dx = \frac{1}{2}\int x\log\left(x^2+1\right)dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\int \log u\,du = \frac{1}{4}\frac{u\ \log u - u}{\ln 10} + c$$

$$= \frac{1}{4\ln 10}\left(\left(x^2+1\right)\ln\left(x^2+1\right) - x^2 - 1\right) + c$$

25. $\frac{\sin^{(n+1)}(x)}{n+1} + c$

26. $\frac{3^{x+1}625^x}{5(\ln 3 + 4\ln 5)} + c$

27. $\frac{1}{2}(\ln(x))^2 + c$

28. $-\frac{1}{6}\cos\left(3x^2\right)$

29. $\int x|x|\ dx = \begin{cases} \int x^2dx, if\ x \ge 0 \\ \int -x^2dx, if\ x < 0 \end{cases} = \begin{cases} \frac{x^3}{3} + c,\ if\ x \ge 0 \\ \frac{-x^3}{3} + c, if x < 0 \end{cases}$

$$= \frac{1}{3}x^2|x| + c$$

30. $u = 6 + x \Rightarrow du = dx$

$$\int \sqrt{6+x}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u\sqrt{u} + c$$

$$= \frac{2}{3}(x+6)\sqrt{x+6} + c$$

31. $\frac{\sqrt{x}+x}{x\sqrt[3]{x}} = \left(x^{\frac{1}{2}} + x\right)\left(x^{-\frac{4}{3}}\right) = x^{\frac{-5}{6}} + x^{-\frac{1}{3}}$

$$\Rightarrow \int x^{-\frac{5}{6}} + x^{-\frac{1}{3}}\ dx = x^{\frac{1}{6}} + x^{\frac{2}{3}} + c$$

32. $\frac{4^x + 2^{x+1} + 2x\ln 2}{2\ln 2} + c$

33. $\frac{2}{\ln 2}2^{\sqrt{x}} + c$

34. $-\dfrac{1}{\ln 4\left(4^{x^2+x-1}\right)}+c$

35. $u = 1+2^x \Rightarrow du = 2^x \ln 2\, dx$

$$\Rightarrow \frac{1}{\ln 2}\, du = 2^x\, dx$$

$$\Rightarrow \int 2^x \sqrt{1+2^x}\, dx = \frac{1}{\ln 2}\int \sqrt{u}\, du$$

$$= \frac{2}{3\ln 2}\, u\sqrt{u} + 3$$

$$= \frac{2}{3\ln 2}\left(1+2^x\right)\sqrt{1+2^x} + c$$

36. Write the expression with a common exponent.

$$\frac{972e^{x+3}8^{-x}3^x}{-3\ln 2+\ln 3+1}+c$$

37. $\ln\left(3+\sin x\right)+c$

38. $-\sin\left(\dfrac{1}{x}\right)+c$

39. $2\tan\sqrt{x}+c$

40. $\dfrac{1}{6\cos^6 x}+c$

41. $\dfrac{1}{2}e^{x^2}+c$

42. $\dfrac{1}{2e^{x^2}}+c$

43. $-e^{\frac{1}{x}}+c$

44. $u = x^2+2x+4 \Rightarrow du = \left(2x+2\right)dx$

$$\Rightarrow \frac{1}{2}\, du = \left(x+1\right)dx$$

$$\int \frac{x+1}{x^2+2x+4}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\left|x^2+2x+4\right| + c$$

45. $-\dfrac{4}{x+3}$

46. $\displaystyle\int \frac{x^2}{x+3}dx = \int x-3+\frac{9}{x+3}\, dx$

$$= \frac{x^2}{2} - 3x + 9\ln|x+3| + c$$

47. $u = x^2 + x + 3 \Rightarrow du = (2x+1)\ dx$

$$\Rightarrow \int (2x+1)(x^2+x+3)^{10}\ dx = \int u^{10} du\ = \frac{u^{11}}{11} + c = \frac{(x^2+x+3)^{11}}{11} + c$$

48. $u = 9 + x \Rightarrow x = u - 9\ \ and\ \ du = dx$

$$\Rightarrow \int x\sqrt{9+x}^3\ dx = \int (u-9)\sqrt{u}^3\ du$$

$$= \int u\sqrt{u}^3 - 9\sqrt{u}^3\ du$$

$$= \frac{2}{7}u^{\frac{7}{2}} - \frac{18}{5}u^{\frac{5}{2}} + c = \frac{2}{7}(9+x)^{\frac{7}{2}} - \frac{18}{5}(9+x)^{\frac{5}{2}} + c$$

49. $e^{\sin x} + c$

50. $\sqrt{x^2+5} + c$

51. $2e^{2x} + 4e^x + x + c$

52. $\int \sin^3\left(\frac{x}{3}\right)dx = \int \sin\left(\frac{x}{3}\right)\left(1 - \cos^2\left(\frac{x}{3}\right)\right)dx$

$$= \int \sin\frac{x}{3}dx - \int \sin\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right)dx = -3\cos\left(\frac{x}{3}\right) + \cos^3\left(\frac{x}{3}\right) + c$$

53. $\frac{3}{2}\ln\left|x^2 - 1\right| + c$

54. $3x + \frac{9}{2}\ln|x-3| - \frac{9}{2}\ln|x+3| + c$

55. $-\frac{1}{3(x+1)} - \frac{2}{9}\ln|x+1| + \frac{2}{9}\ln|x-2| + c$

56. $\frac{7}{x+3} + 3\ln|x+3| + c$

57. $\frac{-4}{x} - 8\ln|x| - \frac{4}{x+1} + 8\ln|x+1| + c$

58. $\frac{19}{4}\ln|x+3| - \frac{3}{2(x+1)} - \frac{11}{4}\ln|x+1| + c$

59. $x + \frac{1}{4x} - \frac{\ln x}{16} + \frac{65}{16}\ \ln\ |x-4| + c$

60. $\frac{1}{4}\ln|x-1| - \frac{1}{2(x+1)} - \frac{1}{4}\ln|x+1| + c$

61. $b - a$

62. 8

63. $\frac{-5}{2}$

64. $\frac{45}{2}$

65. e – 1

66. $\frac{14}{3}$

67. $\frac{27-3^{\sqrt{2}}}{\ln 3}$

68. $\frac{45}{4}$

69. $\frac{4}{3}$

70. $e^4 - e^2$

71. $\frac{972}{\ln 3}$

72. $\frac{4-\sqrt{2}}{6\ln 2}$

73. ln 2

74. $\frac{2e^4-2}{e^2}$

75. $\frac{e^n - e}{n}$

76. 1 – 15 ln 2 + 5 ln 7

77. $\frac{10}{3}\sqrt{10}-\frac{1}{3}$

78. $2e\left(e^2-1\right)$

79. $-\frac{1}{5}$

80. $\frac{1}{24}$

81. $\frac{3}{4\ln 2}$

82. ln 5 – ln3

83. $\frac{8}{5}\sqrt{3}$

84. $\frac{-102943}{14}$

85. 0

86. $\int_{-1}^{2} 4dx = 4x\Big|_{-1}^{2} = 8-(-4) = 12$

87. $-\int_{-3}^{-1} 3xdx = -\frac{3x^2}{2}\Big|_{-3}^{-1} = -\frac{3}{2}(1-9) = -(-12) = 12$

88. $\int_{0}^{3} (3x+1)\,dx = \frac{3x^2}{2}+x\Big|_{0}^{3} = \frac{3}{2}(9)+3 = 16.5$

89. $\int_{0}^{3} 2x^2+1\,dx = \frac{2x^3}{3}+x\Big|_{0}^{3} = \frac{2}{3}(27)+3 = 21$

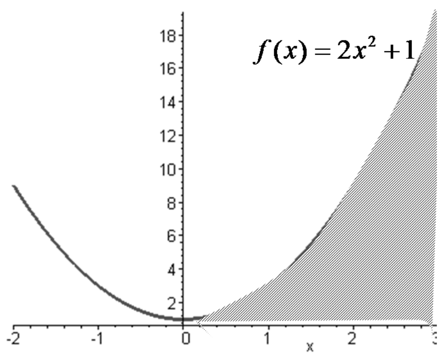


Figure 5.20

90. $\int_{-1}^{1} 1-4x^2dx = \left|-\int_{-1}^{-\frac{1}{2}}\left(1-4x^2\right)\right|dx + \int_{\frac{-1}{2}}^{\frac{1}{2}}\left(1-4x^2\right)dx + \left|\int_{\frac{1}{2}}^{1}\left(1-4x^2\right)dx\right|$

$$= \left|x-\frac{4x^3}{3}\right|_{-1}^{-\frac{1}{2}} + \left|x-\frac{4x^3}{3}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}\right| + \left|x-\frac{4x^3}{3}\Big|_{\frac{1}{2}}^{1}\right|$$

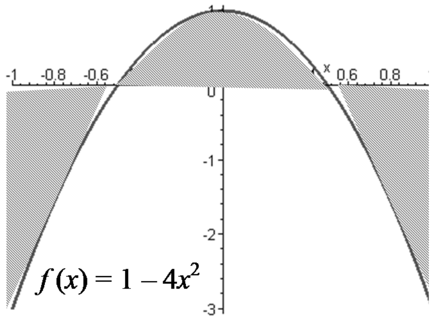


Figure 5.21

But $A_1 = \left|x-\frac{4}{3}x^3\Big|_{-1}^{-\frac{1}{2}}\right| = \left|-\frac{1}{2}-\left(\frac{4}{3}\right)\left(-\frac{1}{8}\right)-\left(-1-\left(\frac{4}{3}\right)(-1)\right)\right|$

$$= \left|-\frac{1}{2}+\frac{1}{6}-\left(-1+\frac{4}{3}\right)\right| = \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$A_2 = x-\frac{4}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2}-\frac{4}{3}\left(\frac{1}{8}\right)-\left(-\frac{1}{2}-\frac{4}{3}\left(-\frac{1}{8}\right)\right) = \frac{2}{3}$$

$$A_3 = \left|x-\frac{4}{3}x^3\Big|_{\frac{1}{2}}^{1}\right| = \left|1-\frac{4}{3}-\left(\frac{1}{2}-\frac{4}{3}\left(\frac{1}{8}\right)\right)\right| = \left|1-\frac{4}{3}-\left(\frac{1}{2}-\frac{1}{6}\right)\right| = \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$\therefore A = A_1 + A_2 + A_3 = \frac{2}{3}+\frac{2}{3}+\frac{2}{3} = 2$$

91. $A = -\int_{-\frac{1}{2}}^{0} x^3dx + \int_{0}^{2} x^3dx$

$$= -\frac{x^4}{4}\Big|_{\frac{-1}{2}}^{0} + \frac{x^4}{4}\Big|_{0}^{2} = \frac{1}{4}\left(0-\frac{1}{16}\right)+4 = \frac{257}{64}$$

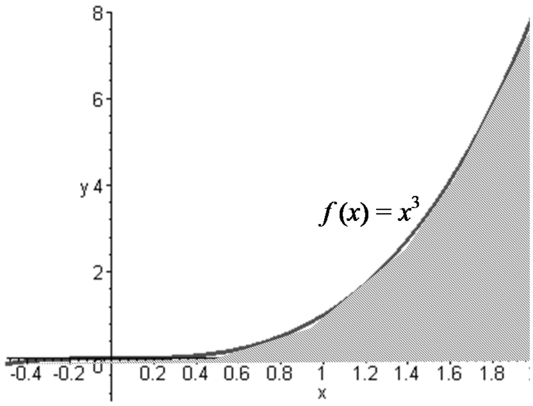


Figure 5.22

92. $\int_{-1}^{4} e^x dx = e^x \Big|_{-1}^{4} = e^4 - \left(\frac{1}{e}\right) = \frac{e^5 - 1}{e}$

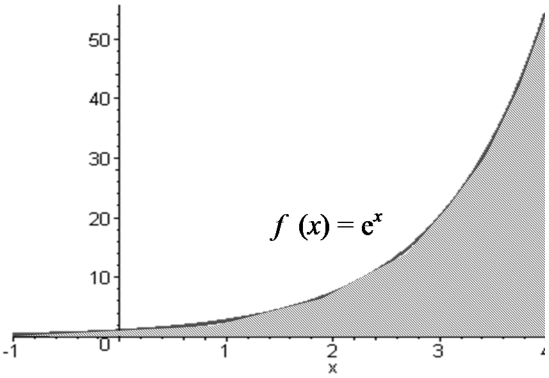
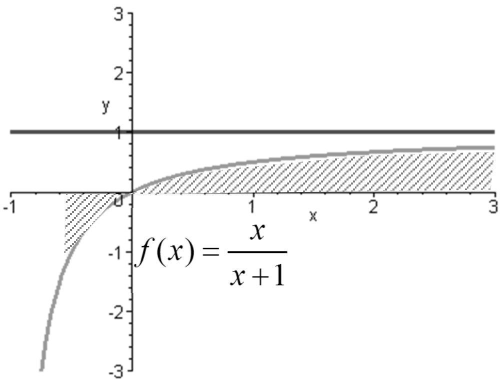


Figure 5.23

93.



$$A = -\int_{-\frac{1}{2}}^{0} \frac{x}{x+1}\, dx + \int_{0}^{3} \frac{x}{x+1} dx$$

$$= -\int_{-\frac{1}{2}}^{0} \left(1 - \frac{1}{x+1}\right) dx + \int_{0}^{3} 1 - \frac{1}{x+1} dx$$

$$= -\left(x - \ln\left|(x+1)\right|\right)\Big|_{-\frac{1}{2}}^{0} + x - \ln|x+1|\Big|_{0}^{3}$$

$$= -\left(0 - \ln 1 - \left(-\frac{1}{2} - \ln\frac{1}{2}\right)\right) + 3 - \ln|4| - \left(0 - \ln|1|\right)$$

Figure 5.24

$$= -\left(0 - 0 + \frac{1}{2} - \ln 2\right) + 3 - \ln|4| = 2.5 - \ln 2$$

94. $\int_{\frac{1}{4}}^{4}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\,dx=\frac{2}{3}x\sqrt{x}+2\sqrt{x}\Big|_{\frac{1}{4}}^{4}$

$$=\frac{2}{3}\times 4\sqrt{4}+2\sqrt{4}-\frac{1}{12}-1=\frac{33}{4}$$

95. $-\int_{\frac{1}{e}}^{1}\ln x\,dx+\int_{1}^{e}\ln x\;dx=-(x\ln x-x)\Big|_{\frac{1}{e}}^{1}+(x\ln x-x)\Big|_{1}^{e}$

$$=-\int_{\frac{1}{e}}^{1}\ln x\,dx+\int_{1}^{e}\ln x\;dx=-(x\ln x-x)\Big|_{\frac{1}{e}}^{1}+(x\ln x-x)\Big|_{1}^{e}$$

$$=\frac{e-2}{e}+1=\frac{2e-2}{e}$$

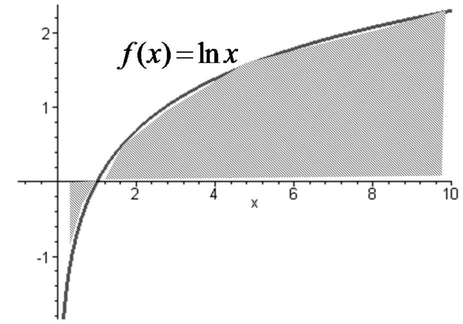


Figure 5.25

96. $x^3-2x^2-5x+6=0\Rightarrow x=1,3,-2$

$$A=\int_{-2}^{1}\left(x^3-2x^2-5x+6\right)dx-\int_{1}^{3}\left(x^3-2x^2-5x+6\right)dx$$

$$=\frac{x^4}{4}-\frac{2x^3}{3}-\frac{5x^2}{2}+6x\Big|_{-2}^{1}-\left(\frac{x^4}{4}-\frac{2x^3}{3}-\frac{5x^2}{2}+6x\right)\Big|_{1}^{3}$$

$$=\frac{63}{4}-\left(-\frac{16}{3}\right)=\frac{253}{12}$$

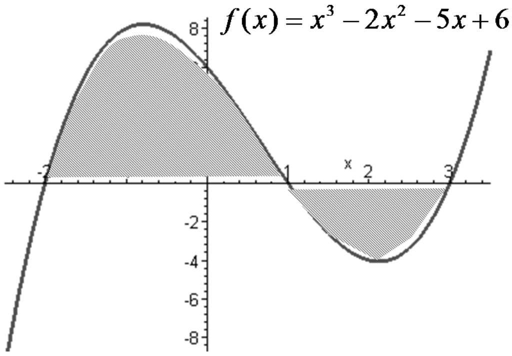
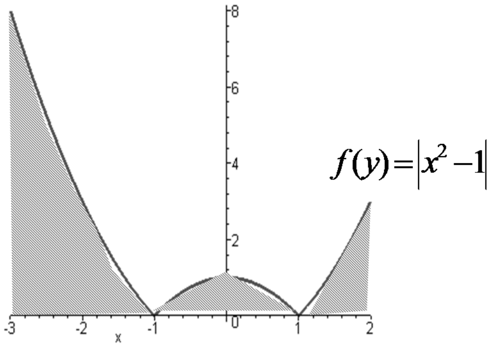


Figure 5.26

97. $\left|x^2-1\right|=0 \Rightarrow x=\pm 1$



$$A=\int_{-3}^{-1}(x^2-1)\,dx+\int_{-1}^{1}(1-x^2)\,dx+\int_{1}^{2}(x^2-1)\,dx$$

$$=\frac{x^3}{3}-x\Big|_{-3}^{-1}+x-\frac{x^3}{3}\Big|_{-1}^{1}+\frac{x^3}{3}-x\Big|_{1}^{2}$$

$$=\frac{20}{3}+\frac{4}{3}+\frac{4}{3}=\frac{28}{3}$$

Figure 5.27

98. a. $A=2\int_0^1 \frac{x}{2}-\frac{x^3}{x^2+1}\,dx=-\frac{1}{2}+\ln 2$

b. The triangle is enclosed by the lines $y=\frac{1}{4}x,\ y=\frac{4}{5}x$ and $y=\frac{5}{9}(x-5)+4$

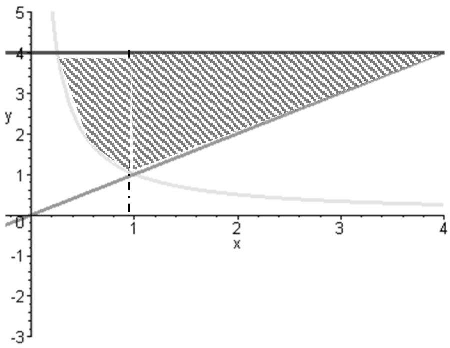
$$\Rightarrow A=\int_{-4}^{0}\left(\frac{5}{9}(x-5)+4-\frac{1}{4}x\right)dx+\int_{0}^{5}\left(\frac{5}{9}(x-5)+4-\frac{4}{5}x\right)dx$$

$$=\frac{11}{72}x^2+\frac{11}{9}x\Big|_{-4}^{0}-\frac{11}{90}x^2+\frac{11}{9}x\Big|_{0}^{5}=\frac{22}{9}+\frac{55}{18}=5.5$$

c. $A=\int_{-2}^{2}12-\left(x^4-x^3\right)\,dx$

$$=12x-\frac{x^5}{5}+\frac{x^3}{3}\Big|_{-2}^{2}=\frac{608}{15}$$

99. a. The region is enclosed by three functions.



$$A=\int_{\frac{1}{4}}^{1}4-\frac{1}{x}\,dx+\int_{1}^{4}4-x\,dx$$

$$=4x-\ln|x|\Big|_{\frac{1}{4}}^{1}+4x-\frac{x^2}{2}\Big|_{1}^{4}$$

$$=3-2\ln 2+\frac{9}{2}$$

$$=7.5-2\ln 2$$

Figure 5.28

b. $4-x^2=3x \Rightarrow \; x^2+3x-4=0 \Rightarrow x=-4,1$

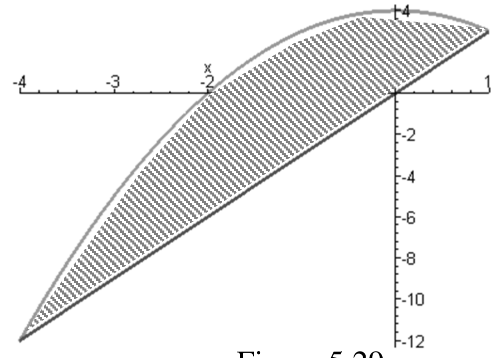
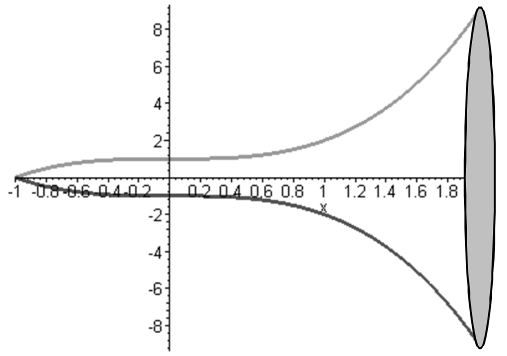


Figure 5.29

$$A=\int_{-4}^{1}\left(4-x^2\right)-3x\ dx$$
$$=4x-\frac{x^3}{3}-\frac{3x^2}{2}\bigg|_{-4}^{1}=\frac{125}{6}$$

100. a. $V=\pi\int_0^4\left(4x-x^2\right)^2 dx=\pi\left(\frac{x^5}{5}-2x^4+\frac{16}{3}x^3\right)\bigg|_0^4=\frac{512}{15}\pi$

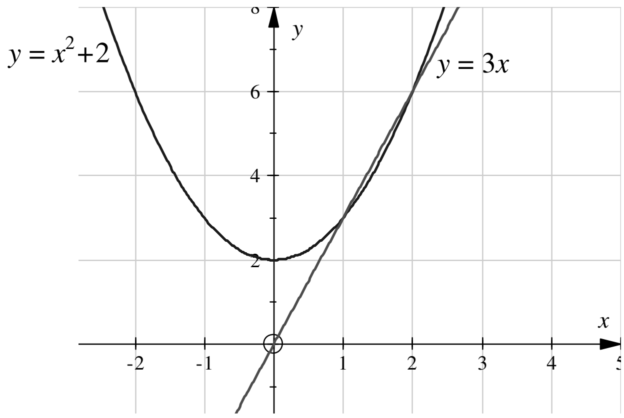
b. $V=\pi\int_{-1}^{2}\left(x^3+1\right)^2 dx=\pi\int_{-1}^{2}x^6+2x^3+1\ dx$



$$=\pi\left(\frac{x^7}{7}+\frac{x^4}{2}+x\right)\bigg|_{-1}^{2}$$
$$=\frac{405}{14}\pi$$

Figure 5.30

101. $x^2+2=3x \Rightarrow \; x^2-3x+2=0 \Rightarrow x=1,2$



$$V=\pi\int_1^2(3x)^2-\left(x^2+2\right)^2\ dx$$
$$=\pi\int_1^2 9x^2-x^4-4x^2-4\ dx$$
$$=\pi\left(3x^3-\frac{x^5}{5}-\frac{4}{3}x^3-4x\right)\bigg|_1^2$$
$$=\frac{22}{15}\pi$$

Figure 5.31

102.

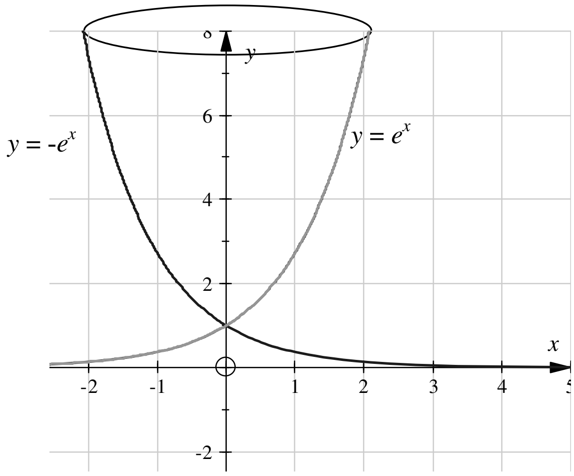


Figure 5.32

$$y = e^x \Rightarrow x = \ln y$$

The region is rotated about the y – axis.

$$v = \pi\int_1^e (\ln y)^2\, dy$$

The major task is to evaluate $\int (\ln y)^2 dy$. We use integration by parts.

$$u = \ln y \Rightarrow du = \frac{1}{y}\, dy$$

$$dv = \ln y \Rightarrow v = y\ln y - y$$

$$\Rightarrow \int (\ln y)^2 dy = \ln y\left(y\ln y - y\right) - \int \left(y\ln y - y\right)\frac{1}{y}dy$$

$$= y\left(\ln y\right)^2 - y\ln y - \int \ln y - 1\, dy$$

$$= y\left(\ln y\right)^2 - y\ln y - \left(y\ln y - 2y\right) + c$$

$$V = \pi\left(\left(\ln y\right)^2 - 2y\ln y + 2y\right)\Big|_1^e = (e-2)\pi$$

UNIT 6 THREE DIMENSIONAL GEOMETRY AND VECTORS IN SPACE

INTRODUCTION

This unit has two main tasks. The first task is to enable students setup a coordinate system in space and locate or determine the coordinates of a point using ordered triple of numbers. The second task is to enable students identify basic facts about coordinates and their use in determining geometric concepts in space and to solve some practical problems in real life.

In this unit, students will learn how to locate a point in space using three axes, how to calculate the distance between two points using the coordinates of the points including midpoint of a line segment, how to calculate the magnitude of a vector using the coordinates of the initial and terminal points of the vector, the angle between two vectors, the dot product of vectors and how to write equation of a sphere using the coordinates of its center and its radius.

Unit Outcomes

After completing this unit, students will be able to:

- *know methods and procedures in setting up coordinate system in space.*
- *know basic facts about coordinates and their use in determining geometric concepts in space.*
- *apply facts and principles about coordinates in space to solve related problems.*
- *know specific facts about vectors in space.*

Suggested Teaching Aids in Unit 6

In addition to the student's text book and the teacher's guide, it is necessary to bring to the class a ruler and set square which is big enough for black board use and a model of a rectangular box made of harder paper.

The coordinatization of the space is an extension of the coordinatization of the plane, and hence each topic in this unit begins with a brief revision of the respective topic on the coordinate plane.

In teaching this unit, you should use a method that allows students to actively participate. Using real objects such as the corner of a room, the floor and adjacent walls intersecting the floor of that room and encouraging students to hold three pens or pencils together in such a way that each is perpendicular to the other at the same point, are some of the aspects that make the learning practical and active.

6.1 COORDINATE AXES AND COORDINATE PLANES IN SPACE

Periods allotted: 2 periods

Competencies

At the end of this sub-unit, students will be able to:

- *construct the coordinate axes in space.*
- *identify planes determined by the axes in space.*
- *identify the octants determined by the planes and axes.*

Vocabulary: Axes, Coordinate plane, Octant

Introduction

The main task of this sub-unit is, to introduce to the students the concept of coordinate axes and coordinate planes in space as an extension of the rectangular coordinate system on a plane.

Teaching Notes

Start the lesson just by assessing the students' previous knowledge about locating a point on a plane. This may be done by encouraging them to do activity 6.1. Here note that the second question of Activity 6.1 is aimed at introducing "Z" axis on a single plane. This may make the transition from two axes to three axes smooth.

As mentioned in the introduction, this lesson is an extension of the coordinate system on a plane. Therefore, the lesson should start with the revision of the coordinate system on a plane. Then, the idea of three perpendicular lines at a point may be introduced to the students by asking them to hold their pen perpendicular to their notebook which is on their desk, at the origin of the coordinate system they have already drawn on their note book. That is, holding their pen perpendicular to both the x-axis and the y-axis at the same point which is the origin. Then, taking a corner on the floor in the class room as the origin, the three edges that are intersections of the two walls and the intersection of the floor with the two walls can be used to strengthen the idea. Here, it is important to note that the class room serves as a model for the 1st octant only. Finally, the students themselves can be encouraged to hold three pens in such a way that each pen is perpendicular to the other two at the same point.

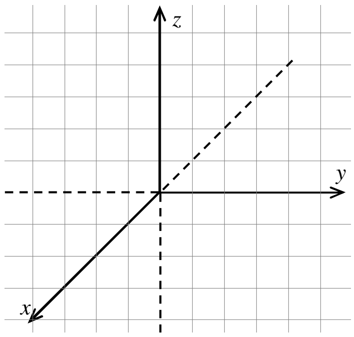


Figure 6.1

The naming of these lines is a matter of convention. The one shown above is the most common, and it is known as the ***right hand rule*** which is indicated in the students text book. That is holding the z-axis, when we curl our fingers from the x-axis to the y-axis, our thumb points in the positive direction of the z-axis.

The students then can be asked to discuss the different planes determined by any two of the three lines which are now going to be used as reference to locate points on the space.

After the students have identified the three planes, namely the xy-plane, the xz-plane and the yz-plane, which are also known as ***coordinate planes***, the different parts of the space determined by the three planes which are known as the ***octants*** will be discussed.

At this stage, the student may be assigned in groups to discuss and identify the coordinate planes and the octants and also they can be assigned to prepare different models with the necessary guide.

The diagram below may be used to show to the students how the three coordinate planes and the three axes intersect one another. However encouraging the students to prepare a model using harder paper is preferable.

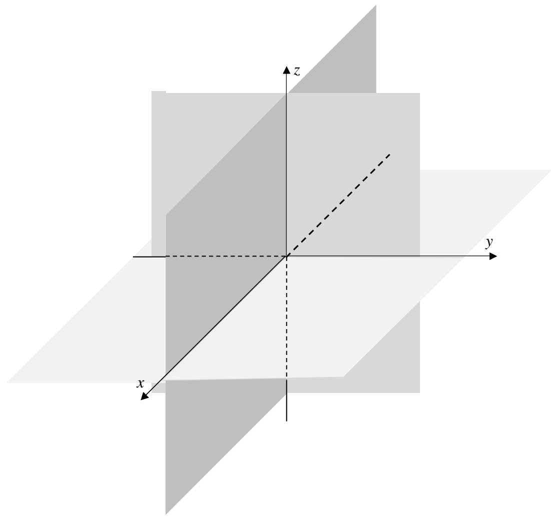
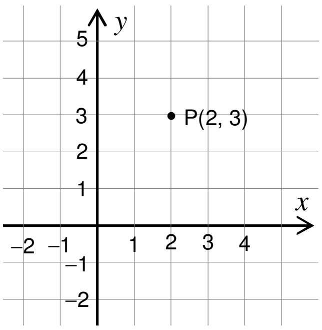


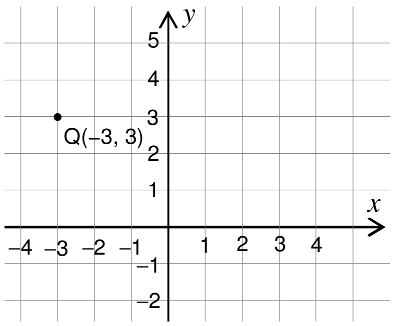
Figure 6.2

Answers to Activity 6.1

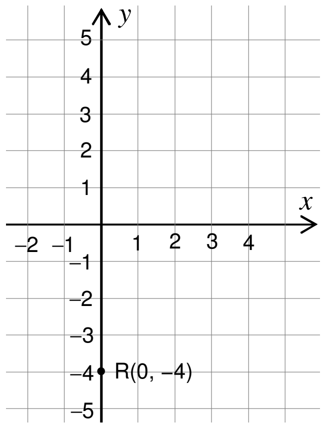
1. a.



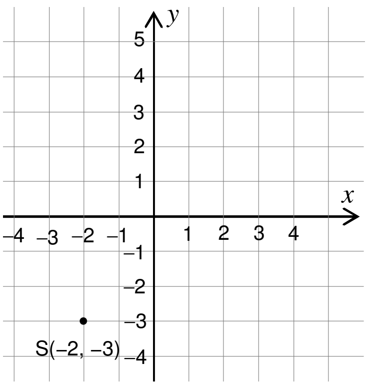
b.



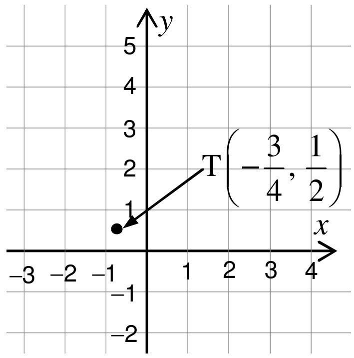
c.



d.



e.



f.

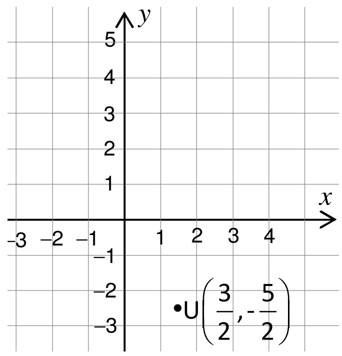


Figure 6.3

2. a.

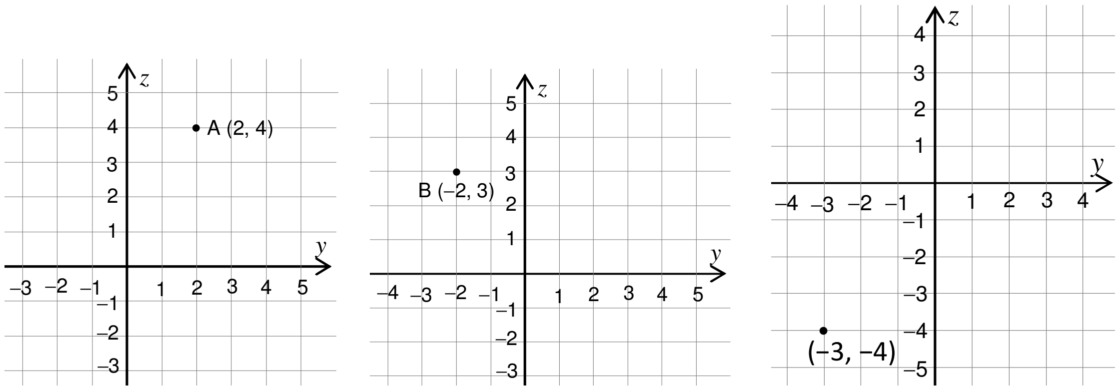


Figure 6.4

6.2 COORDINATES OF A POINT IN SPACE

Periods allotted: 2 periods

Competencies

At the end of this sub-unit, students will be able to:

- *read the coordinates of a point in space.*
- *describe how to locate a point in space.*
- *plot a point whose coordinates are given.*
- *give the equations for the planes determined by the axes.*

Vocabulary: Ordered triple, Directed distance coordinates.

Introduction

In this sub-unit, the students will learn how a point in space is located using ordered triples of real numbers and the coordinate axes introduced above. The subunit will also aim at enabling students to determine the coordinates of a point in space with reference to the axes and to notice the one-to-one correspondence between the set of all points in space and the set of all ordered triples of real numbers that can be established with the help of the axes.

Teaching Notes

Because locating a point in space is done in exactly the same way as locating a point on a plane, start the lesson with a brief revision of assigning coordinates to a point on the coordinate plane which the students are already familiar with.

Check if the students recall that the x and the y-coordinates of a point on the plane represent the distance of the point from the y and the x-axis respectively. Then, ask the students to plot the points in Activity 6.2 using the three coordinate axes introduced above. Because locating a point in space is confusing at the beginning, the points in activity 6.2 are deliberately chosen to be on the coordinate planes. That is in each case one of the three coordinates is taken to be zero. This might make the transition from plane to space smooth again. Whenever it is appropriate, you can also encourage the students in groups or individually, to locate points on the axes by giving them ordered triples where two of the three coordinates are zero.

Answers to Activity 6.2

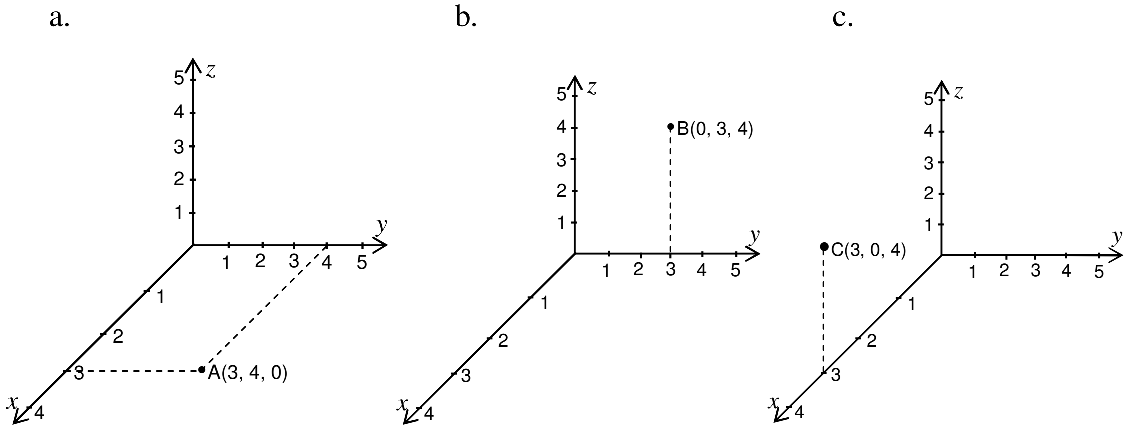


Figure 6.5

Finally, you can give them Exercise 6.1 to do it as a home work or class work as convenient. Particularly, questions 3, 4 and 5 are convenient for group discussion and students should be advised to use tools like ruler, set square and the like in dealing with such constructions.

Once the three reference axes are fixed in such a way that they intersect at some point O which is now the origin, proceed in a similar way to assign a point p in space with numbers taken from each axis. The ordered triples are the real numbers that represent the three directed distances of the point from the respective coordinate planes as discussed in the students' textbook. It is better to begin with points taken from octant 1 where all the three coordinates are positive. At this stage, you may give a concrete example by taking a point in the class room and ask the students to think of its distances from two walls and from the floor. If a corner on the floor is taken as the origin, the distance of the point from the floor will be its z-coordinate and its distances from the two walls will be its x and y coordinates. Since drawing a three dimensional figure is challenging, students should be encouraged to exercise drawing parallel lines and drawing a rectangular box in advance. Then guide students to plot a point whose coordinates are given, and also encourage them to observe that a single drawing can also be used to locate seven other points which are the vertices of the resulting rectangular box. After plotting (locating) as many points as possible, encourage the students to come to the conclusion that there is a one -to-one correspondence between the set of all points of the space and the set of all ordered triples of real numbers, which can be established with the help of the three axes.

Answers to Exercise 6.1

1. a.

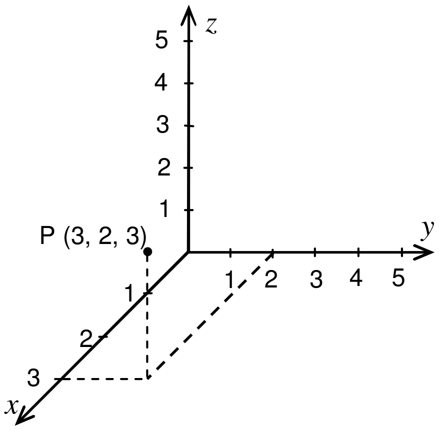


Figure 6.6

b.

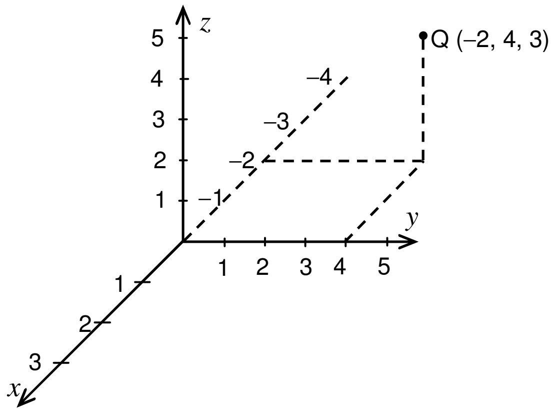


Figure 6.7

c.

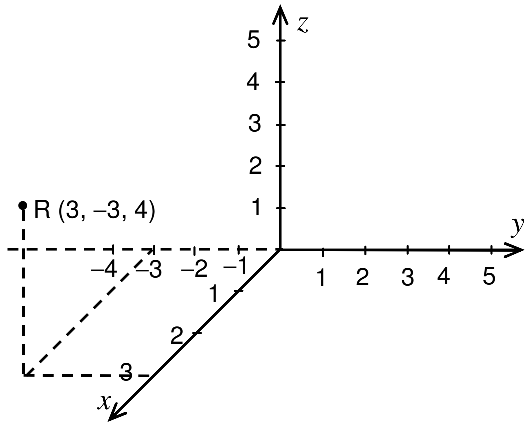


Figure 6.8

d.

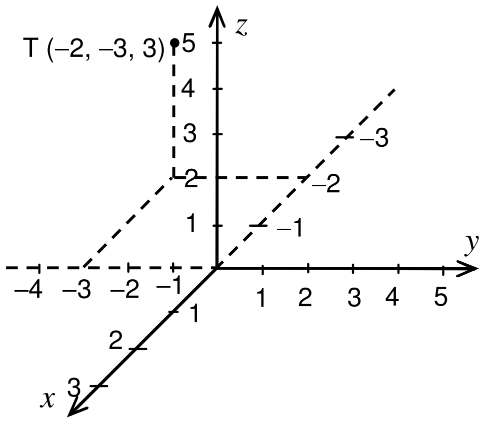


Figure 6.9

e.

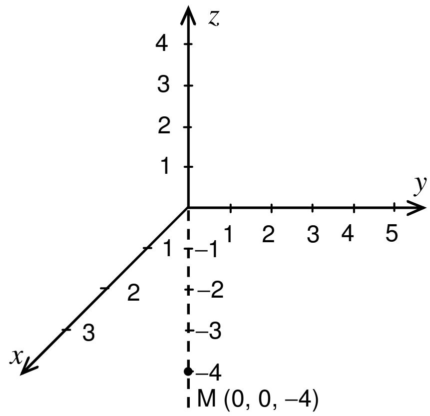


Figure 6.10

f.

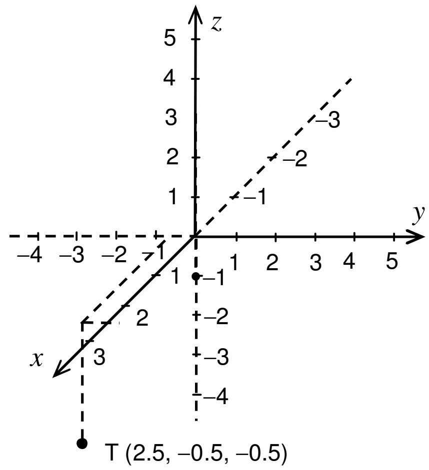


Figure 6.11

g.

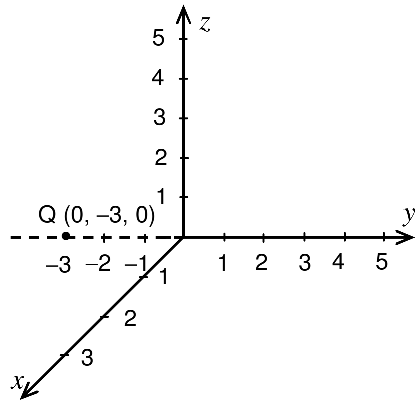


Figure 6.12

2. a. y-axis = $\{(x, y, z)\}: x, y, z \in \mathbb{R}$ and $x = z = 0\}$

   b. z-axis = $\{(x, y, z)\}: x, y, z \in \mathbb{R}$ and $y = x = 0\}$

   c. xy-plane = $\{(x, y, z)\}: x, y, z \in \mathbb{R}$ and $z = 0\}$

   d. xz-plane = $\{(x, y, z)\}: x, y, z \in \mathbb{R}$ and $y = 0\}$

   e. yz-plane = $\{(x, y, z)\}: x, y, z \in \mathbb{R}$ and $x = 0\}$

3. Here you can assist and advise the students to use their knowledge of geometry, particularly the drawing of parallel and perpendicular lines.

4. You can assist the students to do this activity in the same way, individually or in groups.

5. Students can be made to discuss in groups and come out with the required conclusion.

6.3 DISTANCE BETWEEN TWO POINTS IN SPACE

Periods Allotted: 2Periods

Competencies

At the end of this sub-unit, students will be able to:

- *show graphically how to find the distance between two points in space.*
- *compute distance between two given points in space.*

Vocabulary: Distance

Introduction

The task of this sub-unit is to enable students to calculate distance between two points in space using their coordinates.

Teaching Notes

Ask the students whether they recall the distance formula on a plane. Guide them to develop the distance formula which is $d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$ developed using the Pythagoras theorem. Make them exercise this formula by inviting them to do question 2 of Activity 6.3. Notice that each pair of points lies on the same plane though the points are in space. Then guide them to think of distance formula in space using the same method as discussed in the students' textbook.

Answers to Activity 6.3

1. $(PQ)^2 = (PR)^2 + (QR)^2$

   $(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

   $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. a. $AB = \sqrt{(1-3)^2 + (5-4)^2 + (0-0)^2} = \sqrt{5}$

   b. $CD = \sqrt{(0-0)^2 + (1-3)^2 + (2-4)^2} = 2\sqrt{2}$

   c. $EF = \sqrt{(1-4)^2 + (0-0)^2 + (1-5)^2} = 5$

This lesson requires a repeated application of the Pythagoras theorem. The lesson can be started by revising distance between two points on a plane and with active participation of the students. Distance between two points in space can be considered using models such as a rectangular box or the classroom itself considering opposite vertices such as one on the floor, and the opposite vertex on the ceiling.

Answers to Exercise 6.2

1. a. $\sqrt{14}$ units   b. $\sqrt{78}$ units   c. $7\sqrt{2}$ units

   d. $\sqrt{69}$ units   e. $\frac{\sqrt{157}}{4}$ units   f. $\sqrt{482}$ units

   g. $2\sqrt{37}$ units

2. Solution to the opening problem.

Consider a coordinate space whose origin is the take off point at the airport. Let the positive y- axis direction be North and the positive x-axis direction be East. Then, when we locate the positions of the planes just after one hour,

i. The distance covered by the plane heading east will be its x-coordinate and its y-coordinate will be zero.
ii. The distance covered by the plane heading north will be its y-coordinate and its x-coordinate will be zero.
iii. Their flight levels will be their z-coordinates respectively.

Therefore, just after one hour, the coordinates of the position of the plane heading east will be (700, 0, 12) and that of the plane heading north will be (0, 600, 10).

Thus, the direct distance between the two planes will be:

$$d = \sqrt{(700-0)^2 + (0-600)^2 + (12-10)^2} = \sqrt{490,000 + 360,000 + 4}$$
$$= \sqrt{850,004} = 921.96\,\text{km}$$

6.4 MIDPOINT OF A LINE SEGMENT IN SPACE

Periods Allotted: 1 Period

Competency

At the end of this sub-unit, students will be able to:

- *determine the coordinates of the midpoint of a line segment in space.*

Vocabulary: Midpoint, Line segment

Introduction

This sub-unit deals with finding the midpoint of a given line segment in space in terms of the coordinates of its end points. The formula for midpoint of a line segment in space is exactly the same as that of a line segment on a plane except that the ordered pairs are replaced by ordered triples of real numbers.

Teaching Notes

Assist the students to recall the midpoint formula on a plane. That is; when P (x_1, y_1) and Q(x_2, y_2) are the endpoints of the line segment PQ, its mid-point

M$(x_o, y_o) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. You may also ask them to do Activity 6.4 which helps them to recall and apply the midpoint formula. Notice that the points in the second question of Activity 6.4 all lie on the *xy*-plane, or their *z*-coordinates are all chosen to be zero; as this may simplify the transition from plane to space.

Answers to Activity 6.4

1. a. (1, 3) b. (1, 1) c. $\left(\frac{1}{2}, 0\right)$

2. a. (1, 3, 0) b. (1, 1, 0) c. $\left(\frac{1}{2}, 0, 0\right)$

As discussed in the student's textbook, the formula for the midpoint of a line segment in space is an extension of the formula for the midpoint of a line segment on the plane. Therefore, the lesson should start by revising the midpoint of a line segment on the plane. Then with active participation of the students, the formula for the midpoint of a line segment in space may be derived.

Assessment

Based on the minimum learning competencies that are expected from the students at the end of each subunit, use different assessment techniques to get feedback about their understanding of each lesson.

Give them exercise problems on:

- ✓ setting a coordinate system in space.
- ✓ locating a point in space given its coordinates.
- ✓ calculating distance between two points using the coordinates of the points.
- ✓ finding the midpoint of a line segment in terms of the coordinates of their end points.

The questions may be asked orally or be given to be done in groups or to be done as a homework or may be asked in a quiz or test form.

Answers to Exercise 6.3

1. a. (2, 2, 3) b. (–2, 0, 3) c. $\left(\frac{5}{8},\ 1,\ \frac{1}{2}\right)$

   d. $\left(0,\ \frac{9}{2},\ 4\right)$ e. $\left(-\frac{3}{2}, -2, -6\right)$ f. (3, –2, –1)

   g. $\left(0,\ \frac{1}{6}, -\frac{3}{8}\right)$

2. The other end point is at (7, 8, –10).

6.5 EQUATION OF A SPHERE

Periods Allotted: 2 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *describe the equation of a sphere.*
- *derive the equation of a sphere.*
- *solve problems related to a sphere.*

Vocabulary: Sphere, Centre, Radius, Equation of a sphere

Introduction

This sub-unit deals with the derivation of equation of a sphere. The derivation of the equation is very simple provided that the students know the definition of the sphere and properly apply their knowledge of distance between two points in space.

Teaching Notes

Start the lesson by asking the students to do Activity 6.5 which is helpful to lead the students to the main lesson. You may ask what they know about a sphere from earlier grades. Using a model and drawing, help students to identify the center and the radius of a sphere. Then based on the definition of a sphere, that is, a set of all points in space that are equidistant from a fixed point, and the distance between two points in space discussed in Section 6.3 above, guide the students to derive the equation of a sphere. You may discuss the examples given with active participation of the students to maximize their understanding. Finally, if O is the center and r is the radius of a sphere, then you can discuss the following three situations, namely,

i. when the distance of a given point p from O is less than r,

ii. when the distance of a given point p from O is equal to r,

iii. when the distance of a given point p from O is greater than r.

Answers to Activity 6.5

1. A sphere is a set of all points in space that are equidistant from a fixed point. The fixed point is called the center and the common distance of all the points of the sphere from the fixed paint (centre) is called the radius of the sphere.
2. a. The equation of such a sphere is $x^2 + y^2 + z^2 = 4$

   b. The distance of P (x, y, z) from the center is $d = \sqrt{x^2 + y^2 + z^2} = \sqrt{4} = 2$
3. The distance of P (3, 4, 0) from the centre is $\sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5$ which is the radius of the sphere!

This activity is designed to help students to recall their previous knowledge about equation of a circle and the definition of a sphere. So that using their knowledge of distance between two points in space, they can transfer to the derivation of equation of a sphere. Therefore, encourage them to do the activities.

Answers to Exercise 6.4

1. $(x - 3)^2 + y^2 + (z - 5)^2 = 16$
2. Given $x^2 + y^2 + z^2 - 6x - 4y - 10z = -22$

   $\Rightarrow x^2 - 6x + y^2 - 4y + z^2 - 10z = -22$

   $\Rightarrow x^2 - 6x + 9 - 9 + y^2 - 4y + 4 - 4 + z^2 - 10z + 25 - 25 = -22$

   $\Rightarrow (x - 3)^2 + (y - 2)^2 + (z - 5)^2 = 16$

   $\Rightarrow \sqrt{(x-3)^2 + (y-2)^2 + (z-5)^2} = 4$

   Hence the centre of the sphere is at (3, 2, 5) and its radius is 4 units.
3. The centre of the sphere will be the midpoint C(2, 3, 0) of the diameter $\overline{AB}$ and the radius is $\frac{1}{2}$AB. Therefore, the equation of the sphere will be

   $(x - 2)^2 + (y - 3)^2 + z^2 = 13$
4. The centre of the sphere is O (1, −2, 0)

   The distance between the centre of the sphere and point P (3, −1, 2) is

   $d = \sqrt{(3-1)^2 + (-1-(-2)^2 + (2-0)^2} = \sqrt{4+1+4} = 3$

   Hence the distance between the sphere and point P is $3 - 1 = 2$

5. Point A(2, 1, 2) is inside the sphere.

   Point B(–3, 2, 4) is inside the sphere.

   Point C(5, 8, 6) is outside the sphere.

   Point D(0, 8, 6) is on the sphere.

   Point E(–8, –6, 0) is also on the sphere.

6. Using completing the square you can write the equation of the sphere,

   $x^2 + y^2 + z^2 + 2x - y + z = 0$ as:

   $(x + 1)^2 + \left(y - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1 + \frac{1}{4} + \frac{1}{4} = 1.5$

   $\Rightarrow$ The centre of the sphere is $\left(-1, \frac{1}{2}, -\frac{1}{2}\right)$ and its radius is $\sqrt{1.5}$

The distance between each point from the centre of the sphere is calculated as:

a. $\sqrt{(-1-0)^2 + \left(\frac{1}{2} - 0\right)^2 + \left(-\frac{1}{2} - 0\right)^2} = \sqrt{1.5}$

   $\Rightarrow$ The point O (0, 0, 0) is on the sphere.

b. $\sqrt{(-1-(-1))^2 + \left(0 - \frac{1}{2}\right)^2 + \left(1 - \left(-\frac{1}{2}\right)\right)^2} = \sqrt{2.5}$

   $\Rightarrow$The point P (–1, 0, 1) is outside the sphere

c. $\sqrt{(0-(-1))^2 + \left(\frac{1}{2} - \frac{1}{2}\right)^2 + \left(0 - \left(-\frac{1}{2}\right)\right)^2} = \sqrt{1.25}$

   $\Rightarrow$ The point Q $\left(0, \frac{1}{2}, 0\right)$ is inside the sphere.

7. a. If a point P is on the z-axis, the x and y coordinates will be zero. Thus, any point P on the z-axis will have the form P (0, 0, z) for any real number z.

   b. Let Q (x, 0, 0) be on the x-axis whose distance from P (–1, –1, 2) is $\sqrt{12}$. Then, by the distance formula we get: $\sqrt{(x+1)^2 + (0+1)^2 + (0-2)^2} = \sqrt{12}$

   $\Rightarrow (x + 1)^2 + 1 + 4 = 12$

   $\Leftrightarrow x^2 + 2x + 1 = 7 \Leftrightarrow x^2 + 2x - 6 = 0 \Leftrightarrow x = \dfrac{-2 \pm \sqrt{28}}{2}$

   $\Leftrightarrow x = -1 \pm \sqrt{7}$

Therefore, the two points are $(-1-\sqrt{7}, 0, 0)$ and $(-1+\sqrt{7}, 0, 0)$.

Assessment

Ask them to

- ✓ define a sphere
- ✓ write equation of a sphere given the center and radius
- ✓ solve problems related to equation of a sphere

6.6 VECTORS IN SPACE

Periods Allotted: 8 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *describe vectors in space.*
- *use the unit vectors* **i, j** *and* **k** *while representing a vector in space.*
- *add, subtract vectors and multiply by a scalar in space.*
- *describe the properties of addition to solve exercise problems.*
- *show the closure property on their own.*
- *find the length of a vector in space.*
- *find the scalar product of two vectors in space.*
- *evaluate and show the angle between two vectors in space.*

Vocabulary: Vector quantity, Scalar quantity, Magnitude of a vector, Scalar (dot) product, Angle between two vectors.

Introduction

This sub-unit deals with vectors in space. It begins with revision of vectors on a plane and extends the idea of vector to a three dimensional set or space. Then, it discusses: the notion of vectors in space, addition and subtraction of vectors in space together with the properties of vector addition. Then, magnitude of a vector and scalar (dot) product of two vectors will be discussed.

Teaching Notes

Assist the students to recall their knowledge about vectors on a plane. It is important to encourage the students to do Activity 6.6 so that they can easily transfer to vectors in space.

Emphasis should be given to the representation of vectors using arrows, the direction of vectors and the use of coordinates in operating with vectors and also the use of the standard unit vectors **i** and **j**.

Answers to Activity 6.6

1. A vector on a plane is represented using an arrow.
2. The magnitude of a vector can be represented by the length of the arrow.
3. The direction of a vector can be shown using the arrow head.
4. The vector $\overrightarrow{OP}$ can be expressed as $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$.

Start the lesson by first revising physical quantities that are vectors and that are scalar quantities from their previous studies. Then, emphasize on vectors on a plane by considering their representations, operations and the angle between them using specific examples. Discuss with the students how a vector on a plane is expressed as a sum of its horizontal and vertical components giving a particular emphasis to the unit vectors, using the examples in the student text or other examples. Help the students to notice the role of the coordinates of the head of the vector in vector operations and in the calculation of magnitude of a vector.

The idea of vectors in space can easily be established using the coordinatization of space discussed earlier. Just as vectors on a plane were handled using the coordinates of the head (terminal point) and the two unit vectors **i** and **j** in the positive x and positive

y-axis direction respectively, vectors in space can be handled in terms of the coordinates (x, y, z) of the head (terminal point) of the vector and the three unit vectors **i, j** and **k** in the positive x, positive y and positive z - axis directions respectively.

Here, it is important to note that a good understanding of the coordinatization of the space, namely, locating (plotting) a given point in space and finding the distance of a given point from the origin are decisive in handling vectors and vector operations in space.

Addition and subtraction of Vectors

Vectors are added or subtracted using the coordinates of their terminal points that are real numbers. Therefore, the properties of vector addition are based on the properties of addition and multiplication of real numbers and they are verified very easily as discussed in the student textbook.

Magnitude of a Vector

When we agreed to represent a vector by an arrow, we have also agreed to represent the magnitude of the vector by the length of the arrow which is now the distance between the initial point and the terminal point of the vector in the coordinate space. How to find the distance between two points in space using their coordinates was already discussed in Section 6.3 above. Therefore, for a vector $\mathbf{v}$ if

i. the initial point is at the origin O and the terminal point is at

$\mathrm{P}(x, y, z)$, then $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$

ii. the initial point is at $A(x_1, y_1, z_1)$ and the terminal point is at $B(x_2, y_2, z_2)$,

then $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Scalar (dot) product and angle between two vectors in space

The scalar or dot product of two vectors in space is defined in exactly the same way as the scalar product of two vectors on a plane. The only difference is that, instead of ordered pairs, we use ordered triples. That is, for vectors $\mathbf{a}$ and $\mathbf{b}$ in space, their dot product is defined as:

$\mathbf{ab} = |\mathbf{a}|.|\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. When we solve for cos θ from this equation, we get that:

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|.\ |\mathbf{b}|}$$

When you teach this lesson, you should encourage the students to participate in the routine calculation of $\mathbf{a}.\mathbf{b}$ and the norms of the vectors $\mathbf{a}$ and $\mathbf{b}$. It is also important to note that the norm of the unit vectors is 1 and the dot product of perpendicular vectors is zero as a result of $\cos 90^0 = 0$.

To show that $\mathbf{a}.\mathbf{b}$ is a real number, you may take $\mathbf{a} = (x_1, y_1, z_1) = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = (x_2, y_2, z_2) = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and using the distributive property of multiplication over addition, obtain

$$\mathbf{a}.\mathbf{b} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$$

$$= x_1.x_2(\mathbf{i}.\mathbf{i}) + x_1y_2(\mathbf{i}.\mathbf{j}) + x_1z_2(\mathbf{i}.\mathbf{k}) + y_1x_2(\mathbf{j}.\mathbf{i}) + y_1y_2(\mathbf{j}.\mathbf{j}) + y_1z_2(\mathbf{j}.\mathbf{k}) + z_1x_2(\mathbf{k}.\mathbf{i}) + z_1y_2(\mathbf{k}.\mathbf{j}) + z_1z_2(\mathbf{k}.\mathbf{k})$$

$= x_1x_2 + y_1y_2 + z_1z_2$ because $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$

Answers to Exercise 6.5

1. a. $\sqrt{(-1)^2 + 3^2 + 0^2} = \sqrt{10}$ b. $\sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{11}$

c. $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{4}{5}\right)^2}=\sqrt{\frac{157}{50}}$

2. a. 2(1) + (–3) (0) + 1(4) = 6 b. –5(1) + 0(–3) + (1) (–2) = –7

c. –2(0) + 2(0) + 0(–1) = 0 d. 0(0) + 0(0) + 3(3) = 9

Answers to Exercise 6.6

1. a. $\mathbf{a}+\mathbf{b} = (1, 0, 6)$ b. $\mathbf{b}+\mathbf{a} = (1, 0, 6)$

c. $\mathbf{a}-\mathbf{b} = (1, 6, -2)$ d. $\mathbf{b}-\mathbf{a} = (-1, -6, 2)$

e. $\mathbf{a}+\mathbf{b}+\mathbf{v} = (-3, 3, 4)$ f. $\mathbf{b}+\mathbf{v}-\mathbf{u} = \left(\frac{-9}{2}, 0, 5\right)$

g. $\mathbf{a}+\mathbf{b}+\mathbf{v}+\mathbf{u} = \left(\frac{-5}{2}, 3, 1\right)$

2. a. $3\mathbf{a} = (3, 9, 6)$ b. $-4\mathbf{b} = (0, 12, -16)$

c. $2\mathbf{a}+3\mathbf{b} = (2, -3, 16)$ d. $3\mathbf{b}-\frac{1}{2}\mathbf{a}+2\mathbf{v} = \left(\frac{-17}{2}, \frac{-9}{2}, 7\right)$

3 Let $\mathbf{a}=(x_1, y_1, z_1)$, $\mathbf{b}=(x_2, y_2, z_2)$ and $\mathbf{c}=(x_3, y_3, z_3)$

Then $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(x_1, y_1, z_1)+((x_2, y_2, z_2)+(x_3, y_3, z_3))$

$$= (x_1, y_1, z_1) + ((x_2+x_3, y_2+y_3, z_2+z_3))$$

$$= (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$

On the other hand

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=((x_1, y_1, z_1)+(x_2, y_2, z_2))+(x_3, y_3, z_3)$$

$$= ((x_1+x_2, y_1+y_2, z_1+z_2)) + (x_3, y_3, z_3)$$

$$= (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$$

Therefore, $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$

4. Again let $\mathbf{a}=(x_1, y_1, z_1)$, $\mathbf{b}=(x_2, y_2, z_2)$ and let k be any real number. Then because multiplication of real numbers is distributive over addition of real numbers, we get:

$$k(\mathbf{a}+\mathbf{b}) = k\left((x_1,\ y_1,\ z_1)+(x_2,\ y_2,\ z_2)\right) = k\left(x_1+x_2,\ y_1+y_2,\ z_1+\ z_2\right)$$

$$= \left(k\left(x_1\ +\ x_2\right),\ k\left(y_1\ +\ y_2\right),\ k\left(z_1\ +\ z_2\right)\right)$$

$$= \left(kx_1 +\ kx_2, ky_1\ + ky_2,\ kz_1 + kz_2\right)$$

$$= \left(kx_1,\ ky_1,\ kz_1\right)\ +\ (kx_2,\ ky_2,\ kz_2)$$

$$= k(x_1, y_1, z_1) + k(x_2, y_2, z_2)$$

$$=\ k\mathbf{a} +\ k\mathbf{b}$$

5. a. $-4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ b. $\mathbf{i} - 3\mathbf{j} + \sqrt{2}\,\mathbf{k}$

c. $3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$ d. $0\mathbf{i} + 0\mathbf{j} + 3\mathbf{k}$

6. a. b.

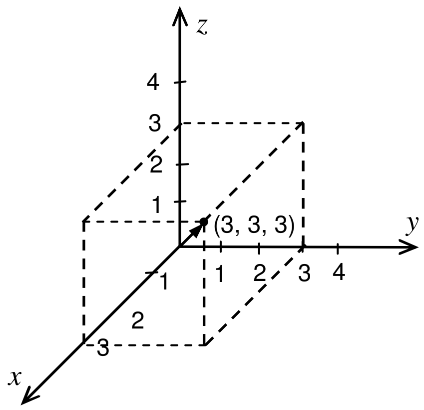


Figure 6.13

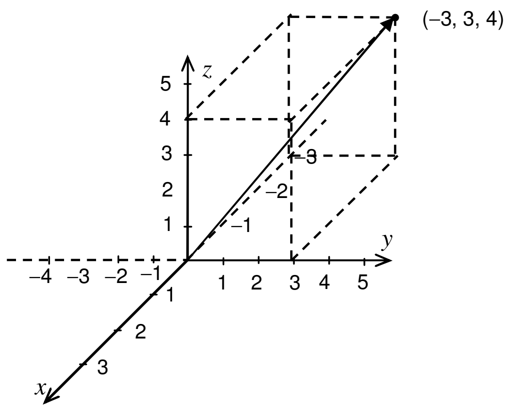


Figure 6.14

c.

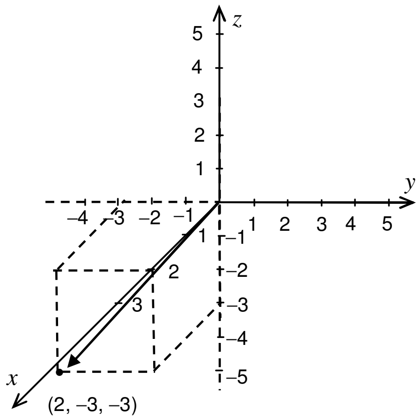


Figure 6.15

7. a. $|\mathbf{a}| = 3\sqrt{3}$ b. $|\mathbf{b}| = \sqrt{34}$ c. $|\mathbf{c}| = \sqrt{22}$

8. a. 2 b. –9 c. $\frac{-7}{4}$

   d. –1 e. 0 f. –6

9. Let us first name the two vectors by **a** and **b** respectively.
   Then,

   a. $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|.|\mathbf{b}|}$. But $\mathbf{a}.\mathbf{b} = (2, 0, 1).\ (0, -1, 0) = 0$

   $|\mathbf{a}| = \sqrt{5}$ and $|\mathbf{b}| = 1$

   Therefore, $\cos\theta = \frac{0}{(\sqrt{5})(1)} = 0 \Rightarrow \theta = 90^0$

   Notice that the vector (2, 0, 1) is on the *xz*-plane while the vector (0, –1, 0) is along the *y*-axis. Because the *y* - axis is perpendicular to the *xz*-plane, the two vectors are perpendicular to each other.

   b. $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|.|\mathbf{b}|}$

   But $\mathbf{a}.\mathbf{b} = (1, 1, 1).\ (1, 0, 1) = 1(1) + 1(0) + 1(1) = 2$

   $|\mathbf{a}| = \sqrt{3}$ and $|\mathbf{b}| = \sqrt{2}$

   Therefore, $\cos\theta = \frac{2}{\sqrt{3}\ \sqrt{2}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$

   $\Rightarrow \theta \cong 35^0 16'$

   c. $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

   But $\mathbf{a}.\mathbf{b} = (-1, 1, 1) . (2, 2, 2) = 2,\ |\mathbf{a}| = \sqrt{3}$ and $|\mathbf{b}| = 2\sqrt{3}$

   Therefore, $\cos\theta = \frac{2}{\sqrt{3}.2\sqrt{3}} = \frac{2}{2.3} = \frac{1}{3}$

   $\Rightarrow \theta = \cong 70.53^o$

Assessment

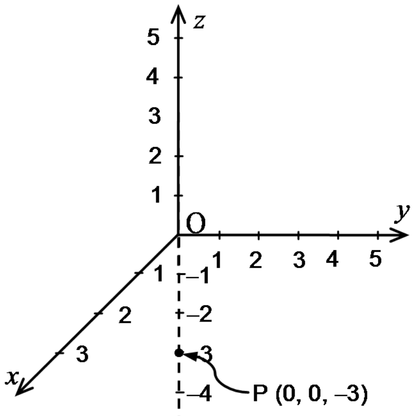
Based on the competencies that are expected, use different assessment techniques to get feedback on their level of understanding of the subunit.

- ✓ Ask them to define a vector, express a given vector as a sum of its components using the unit vectors.
- ✓ Give them exercise problems on addition and subtraction of given vectors, angle between two vectors magnitude of a vector, etc, in groups or as a homework.

Finally when you ask the students to try the review exercises, you may assign questions 11 and 12 for high ability learners if there are such students in the class or you may give hint for all students to attempt.

Answers to Review Exercises on Unit 6

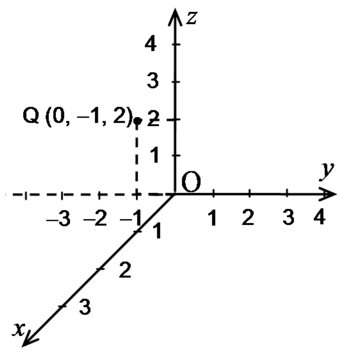
1. a.



$$OP = \sqrt{0^2 + 0^2\ )\ (-3)^2} = 3$$

Figure 6.16

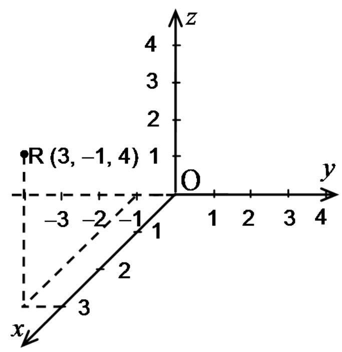
b.



$$OQ = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$$

Figure 6.17

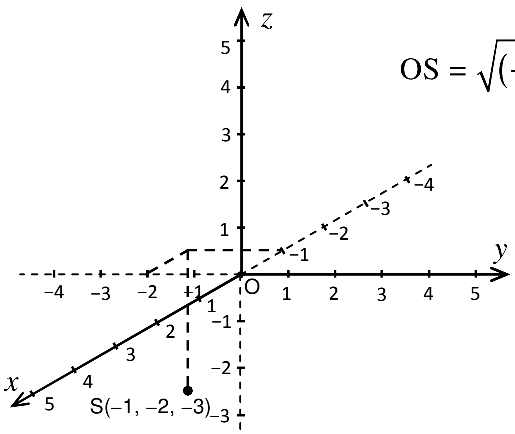
c.



$$OR = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$$

Figure 6.18

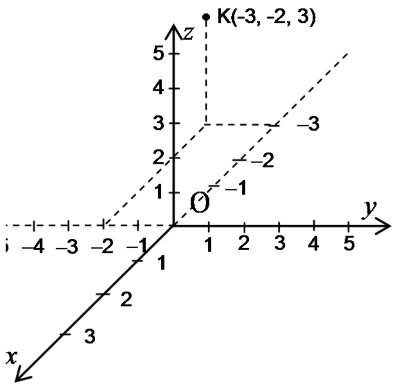
d.



$$OS = \sqrt{(-1)^2 + (-2)^2 + (-3)^2} = \sqrt{14}$$

Figure 6.19

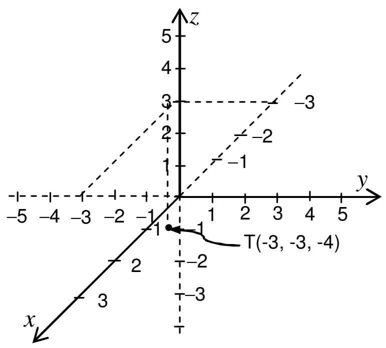
e.



$$OK = \sqrt{(-3)^2 + (-2)^2 + (3)^2} = \sqrt{22}$$

Figure 6.20

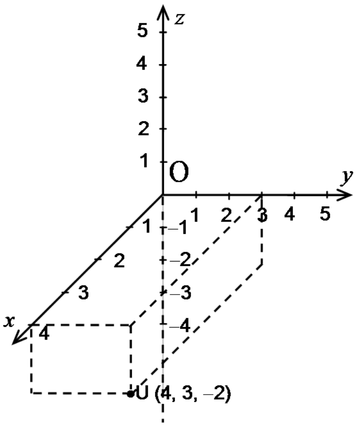
f.



$$OT = \sqrt{(-3)^2 + (-3)^2 + (-4)^2} = \sqrt{34}$$

Figure 6.21

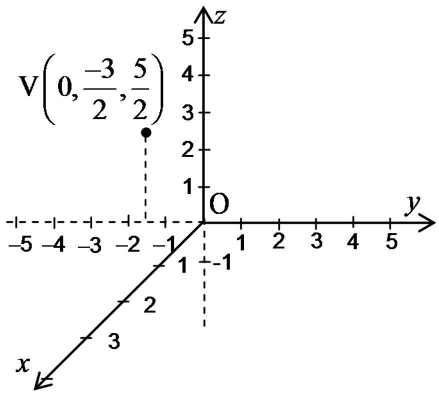
g.



$$OU = \sqrt{(4)^2 + (3)^2 + (-2)^2} = \sqrt{29}$$

Figure 6.22

h.



$$OV = \sqrt{(0)^2 + \left(\frac{-3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{17}{2}}$$

Figure 6.23

2. a. $5\sqrt{2}$ units b. 5 units c. 3 units

d. 4 units e. $\sqrt{34}$ units f. $\sqrt{41}$ units

g. 5 units

3. a. $AB = \sqrt{14}$ and midpoint of $\overline{AB}$ is at $\left(-\frac{1}{2}, \frac{1}{2}, 2\right)$

b. $AB = \sqrt{17}$ and midpoint of $\overline{AB}$ is at $\left(1, \frac{-1}{2}, 1\right)$

c. $AB = \sqrt{37}$ and midpoint of $\overline{AB}$ is at $\left(2, \frac{-1}{2}, 0\right)$

d. $AB = 4\sqrt{2}$ and midpoint of $\overline{AB}$ is at $(2, 0, -2)$

e. $AB = \sqrt{41}$ and midpoint of $\overline{AB}$ is at $\left(0, \frac{1}{2}, -1\right)$

f. $AB = \frac{9\sqrt{11}}{2}$ and midpoint of $\overline{AB}$ is at $\left(\frac{21}{4}, \frac{1}{4}, \frac{21}{4}\right)$

g. $AB = \frac{\sqrt{101}}{2}$ and midpoint of $\overline{AB}$ is at $\left(\frac{\sqrt{2}}{2}, \frac{11}{4}, \frac{\sqrt{3}}{2}\right)$

h. $AB = 7$ and midpoint of $\overline{AB}$ is at $\left(0, 0, \frac{3}{2}\right)$

4. a. (2, 2, 2) b. (0, 0, 0) c. (3, 2, 0) d. $\left(0, -2, \frac{-5}{2}\right)$

5. Using distance formula, we see that:

$AB = 3$, $BC = 3$ and $AC = \sqrt{2}$.

Therefore, ΔABC is isosceles as $AB = BC$.

6. a. $AB = \sqrt{14}$, $BC = \sqrt{14}$ and $AC = \sqrt{38}$.

Therefore, ΔABC is an isosceles triangle.

b. $AB = \sqrt{72}$, $BC = \sqrt{414}$ and $AC = \sqrt{342}$

Therefore, ΔABC is scalene triangle.

c. $AB = \sqrt{14}$, $BC = \sqrt{56}$ and $AC = \sqrt{126}$.

Therefore ΔABC is a scalene triangle.

d. Using distance formula we get that : $AB = \sqrt{158}$, $BC = \sqrt{129}$ and $AC = \sqrt{29}$ and we notice that $(BC)^2 + (AC)^2 = (AB)^2$.

Therefore, the triangle is right angled with hypotenuse $\overline{AB}$.

7. a.

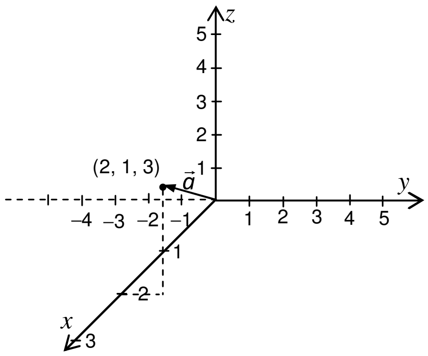


Figure 6.24

b.

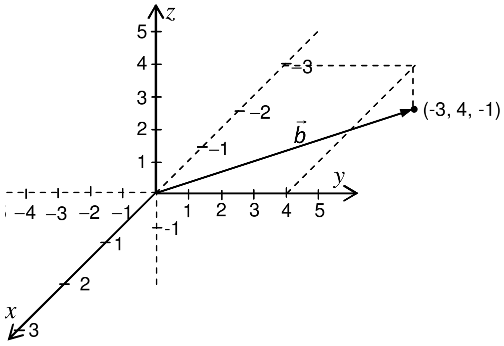


Figure 6.25

c.

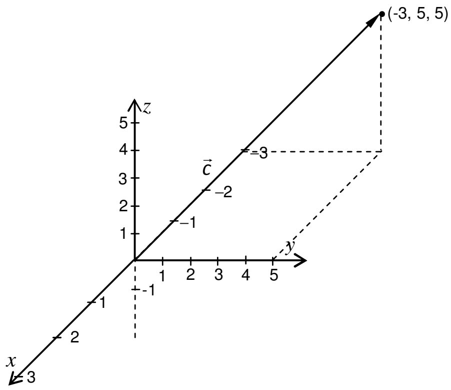


Figure 6.26

d.

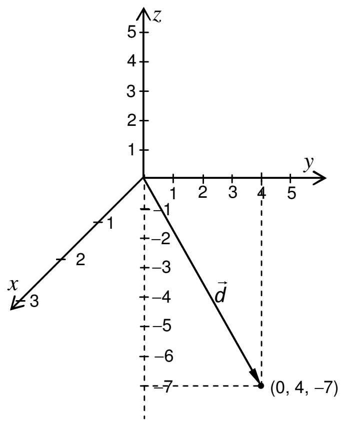


Figure 6.27

8. a. $\vec{\mathbf{a}}+\vec{\mathbf{b}} = -\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ b. $2\vec{\mathbf{a}}-\vec{\mathbf{c}} = 7\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

c. $\vec{\mathbf{b}} + \vec{\mathbf{c}} + \vec{\mathbf{d}} = -6\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}$ d. $2\vec{\mathbf{a}} - 3\vec{\mathbf{b}} + \vec{\mathbf{c}} = 10\mathbf{i} - 5\mathbf{j} + 14\mathbf{k}$

9. a. $|\vec{\mathbf{a}}| = \sqrt{14}$ b. $|\vec{\mathbf{b}}| = \sqrt{26}$

c. $|\vec{\mathbf{c}}| = \sqrt{59}$ d. $|\vec{\mathbf{d}}| = \sqrt{65}$

10. The radius will be the length of $\overline{\mathrm{AC}}$ which is $\sqrt{3} = r$. Using midpoint formula the coordinates of B are found to be (0, 3, 5).

The equation of the sphere will be $(x + 1)^2 + (y - 2)^2 + (z - 4)^2 = 3$

11. a. $x^2 + y^2 + z^2 - 2y = 4$
$\Rightarrow x^2 + (y - 1)^2 + z^2 = 5$
This is an equation of a sphere with centre O (0, 1, 0) and radius $\sqrt{5}$.

b. $x^2 + y^2 + z^2 - x + 2y - 3z + 4 = 0$

$$\left(x-\frac{1}{2}\right)^2 + (y+1)^2 + \left(z-\frac{3}{2}\right)^2 = -4 + \frac{1}{4} + 1 + \frac{9}{4} = -0.5$$

Since this equation is never true for any triple of real numbers x, y, z, it is not equation of a sphere.

c. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$

$\Rightarrow (x-1)^2 + (y+2)^2 + (z-3)^2 = -13 + 1 + 4 + 9 = 1$

This is an equation of a sphere with centre at O (1, –2, 3) and radius 1

12. a. $\vec{\mathbf{a}}.\vec{\mathbf{b}} = -23$ b. $\vec{\mathbf{c}}.\vec{\mathbf{d}} = 13$

c. $\vec{\mathbf{p}}.\vec{\mathbf{q}} = 0$ e. $\vec{\mathbf{a}}.\vec{\mathbf{b}} = 8$

13. a. $\cos\theta = \dfrac{-23}{\sqrt{29}.\sqrt{62}} \cong -0.542$ c. $\cos\theta = 0$

b. $\cos\theta = \dfrac{13}{\sqrt{62}.\sqrt{110}} \cong 0.157$ d. $\cos\theta = \dfrac{8}{\sqrt{194}.\sqrt{131}} \cong 0.05$

Additional Exercise for high ability students

1. If equation of a sphere is $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$, find its centre and its radius.

Solution:

Completing the square on each of the three variables we get:

$$x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 6z + 9 = 11 + 14 = 25$$

$$\Rightarrow (x-1)^2 + (y+2)^2 + (z-3)^2 = 5^2$$

Therefore, the centre is at (1, –2, 3) and its radius is 5.

2. If P (–2, 0, 4) and Q (4, 6, –4) are endpoints of a diameter of a sphere, find the equation of the sphere.

Solution:

The midpoint M of $\overline{PQ}$ will be the center and its coordinate are found to be at

(1, 3, 0) using the midpoint formula. Then, using distance formula, we get the radius (length of $\overline{MP}$) to be $\sqrt{34}$.

Therefore, the equation of the sphere will be $(x-1)^2 + (y-3)^2 + z^2 = 34$.

UNIT 7 MATHEMATICAL PROOFS

INTRODUCTION

A proof is the explanation of why a statement is true. In mathematics, we solve problems. But what makes one accept that the solution we gave and the assertions we made are correct?

Proof plays just this role – guaranteeing that the solutions we gave are correct.

There are different kinds of proofs – direct method, proof by contradiction, to name just two of them. In this section, students will be introduced to some of these proof methods.

Unit Outcomes

After completing this unit, students will be able to:

- *develop the knowledge of logic and logical statements.*
- *understand the use of quantifiers and use them properly.*
- *determine the validity of arguments.*
- *apply the principle of mathematical induction for a problem that needs to be proved inductively.*
- *realize the rule of inference.*

Suggested Teaching Aids in Unit 7

- ✓ Chart giving rules of connectives to be posted on the wall
- ✓ Chart showing inference rules to be posted on the wall
- ✓ Chart showing basic equivalences like

  $p \Rightarrow q \equiv \neg p \vee q$; $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$, etc

7.1 REVISIONS ON LOGIC

Periods Allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *recall what they have studied about statements and logical connectives in the previous grade.*
- *revise open statements.*
- *understand the concept of quantifiers.*
- *determine truth values of statements with quantifiers.*
- *define argument and validity.*
- *check the validity of a given argument.*
- *use rules of inference to demonstrate the validity of a given argument.*

Vocabulary: Proposition, Connective, Open statement, Quantifier, Argument, Validity of an argument, Premise, Conclusion.

Introduction

The main task of this sub-unit is to briefly review Mathematical logic in order to pave the way for studying different methods of proofs in Mathematics. It revises propositions, connectives with their rules, open propositions and the quantifiers and then it deals with revision of arguments and validity of arguments where different examples and the rules of inference are discussed.

Teaching Notes

The students have already discussed propositions, open propositions and quantifiers in Unit 4 of Grade 11. This section helps to revise what they learned there. Thus, we suggest that you give the students a pre – test or an assignment or ask them to do

Activities 7.1, 7.2 and 7.3 to see how much they know about propositions, open propositions, rules of connectives, equivalent propositions, quantifiers, arguments and validity and rules of inference. Based on the outcome, you can revise the topic to fill the gap of knowledge they have on these concepts.

Assessment

Based on the listed competences:

- ✓ Ask the students oral questions on statements and the connectives.
- ✓ Give them exercise problems either from the student text or from other sources, on the determination of truth values of statements involving quantifiers.
- ✓ Give them exercise problems on arguments and checking validity of arguments.
- ✓ Check the solutions given by students and give them appropriate corrections.

Answers to Activity 7.1

The purpose of this activity is to revise the main concepts on logic.

1. A statement is a sentence which is either True or False but not both at the same time.
2. The propositional connectives are: Not ($\neg$) , and ($\wedge$), or ($\vee$), if … ,then… ($\Rightarrow$) & if and only if ($\Leftrightarrow$).
3. A compound (complex) statement is a statement that is formed by using one or more connectives.
4.

p	**q**	**$\neg$ p**	**p $\wedge$ q**	**p $\vee$ q**	**p $\Rightarrow$ q**	**p $\Leftrightarrow$ q**
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

5. a. T b. F c. T d. F e. F

6. a.

p	**q**	**$\neg$p**	**$\neg$p$\vee$q**
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

b.

p	**q**	**$\neg$p**	**p$\Rightarrow$q**	**(P $\Rightarrow$ q) $\Leftrightarrow$ $\neg$p**
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

c.

p	q	r	p ∧q	(p∧q)⇒r
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T
T	T	F	T	F
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

d.

p	q	r	p⇒q	¬r	¬(p⇒q)	¬(p⇒q)∨¬r
T	T	T	T	F	F	F
T	F	T	F	F	T	T
F	T	T	T	F	F	F
F	F	T	T	F	F	F
T	T	F	T	T	F	T
T	F	F	F	T	T	T
F	T	F	T	T	F	T
F	F	F	T	T	F	T

Answers to Activity 7.2

This activity will help the students to revise about open statements and quantifiers. After completing this activity, the students should able to determine whether a sentence is an open statement or not and the use of quantifiers.

1. It is an open statement.
2. It is a statement with truth value T.
3. It is an open statement.
4. It is a statement with truth value F, since 2 is a prime number which is not odd.
5. It is a statement with truth value T, since 29 is a number between 15 and 30 which is prime.
6. It is a statement with truth value F.

Answers to Exercise 7.1

1. $\forall (x)\ P(x) \equiv$ Every student has studied geometry.

   $(\exists x)\ P(x) \equiv$ There is at least one student who has studied geometry.

2. a. F b. T c. T d. F e. T
3. a. F b. F c. T d. T e. T
4. a. Let $P(x)$: x is a student in this class

   $Q(x)$: x has visited Gondar

   Hence the symbolic form is $(\exists x)\ (P(x) \wedge Q(x))$

   b. Let $P(x)$ and $Q(x)$ be as given in a) and let $R(x)$: x has visited Hawassa.

   Hence the symbolic form is $(\forall\ x)\ \ (P(x) \Rightarrow (Q(x) \vee R(x)))$

Answers to Activity 7.3

1. a. Let p = the day is cloudy
   q = It rains

   If p ⇒ q is true, then q ⇒ p may not be true. For example, if p is F and q is T, then p ⇒ q is T but q ⇒ p is F.
   Therefore, the conclusion is not meaningful.

   b. For $x = 2$ and $y = 9$, x is prime and y is composite. But $x + y = 11$ is prime
   Hence, the conclusion is false.

2.

p	q	r	¬q	p⇒q	¬q ⇒ r
T	T	T	F	T	T
T	F	T	T	F	T
F	T	T	F	T	T
F	F	T	T	T	T
T	T	F	F	T	T
T	F	F	T	F	F
F	T	F	F	T	T
F	F	F	T	T	F

In rows 3, 4 and 7, you have p ⇒ q and ¬q ⇒ r are true while p is false.

Answers to Exercise 7.2

1. a. A statement (T) b. A statement (F)
   c. Open statement d. A proposition (T)
   e. Neither a proposition, nor open statement
2. a. i. $p\vee\neg q$ ii. $p\Rightarrow q$ iii. $\neg p\Leftrightarrow q$ iv. $q\Rightarrow\neg p$
   b. i. 5 + 3 = 9 and today is not sunny.
   ii. If 5 + 3 ≠ 9 then today is sunny.
   iii. If either 5 + 3 = 9 or today is sunny, then it is not sunny today.
3. a.

p	q	¬p	¬q	¬p∨q	¬q⇒¬p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Hence $\neg p\vee q\equiv\neg q\Rightarrow\neg p$

b.

p	q	$\neg p$	$\neg q$	$\neg p \Leftrightarrow \neg q$	$p \Leftrightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\therefore\ p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$

c.

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \Leftrightarrow q$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	F	T	T	F	F	F
T	F	F	T	F	T	T	T
F	T	T	T	F	T	T	T
F	F	T	F	F	T	F	F

$\neg p \Leftrightarrow q \equiv (p \vee q) \wedge \neg(p \wedge q)$

4. a.

p	q	$p \wedge q$	$p \Rightarrow q$	$(p \wedge q) \Rightarrow (p \Rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Thus the argument is valid.

b.

p	q	$\neg q$	$q \Rightarrow p$	$p \wedge \neg q$	$(p \wedge \neg q) \Rightarrow (q \Rightarrow p)$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	T	F	T

The argument is valid.

c.

p	q	$p \wedge q$	$p \Rightarrow q$	$p \vee q$	$[(p \wedge q) \wedge (p \Rightarrow q)] \Rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	T	T	T
F	F	F	T	F	T

The argument is valid.

5. a. Let p: You have troubles

   q: You are married

   Then the argument is

$$\begin{array}{l} q \Rightarrow p \\ \underline{\neg p} \\ \neg q \end{array}$$

1. $q \Rightarrow p$ is true (premise)
2. $\neg p$ is truth (premise)
3. p is false (definition of $\neg$)
4. q is false (definition of $\Rightarrow$ and 1)
5. $\neg q$ is true (definition of $\neg$)

   $\therefore$ The argument is valid.

An alternative method is to show that $\left[(q \Rightarrow p) \wedge \neg p\right] \Rightarrow \neg q$ is *a* tautology.

b. Let p: Legesse drinks beer

   q: Legesse is at least 18 years old

The argument is

$$\begin{array}{l} p \Rightarrow q \\ \underline{\neg p} \\ \neg q \end{array}$$

The argument is a fallacy, because if p is false and q is true, we have that

$p \Rightarrow q$ is T and $\neg p$ is T but $\neg q$ is F.

6. a.
   1. $p \Rightarrow (q \vee r)$ is true (premise)
   2. $\neg q \wedge \neg r$ is true (premise)
   3. $\neg (q \vee r)$ is true (Demorgan's 2)
   4. $q \vee r$ is false (rule of $\neg$)
   5. p is false (rule of $\Rightarrow$, 1, 4)
   6. $\neg p$ is true (rule of $\neg$)

   $\therefore$ The argument is valid.

b. Let p: I study

   q: I will fail mathematics

   r: I watch TV frequently

   Thus the argument is

$$\begin{array}{l} p \Rightarrow \neg q \\ \neg r \Rightarrow p \\ \underline{q} \\ r \end{array}$$

1. $p \Rightarrow \neg q$ is true (premise)
2. $\neg r \Rightarrow p$ is true (premise)
3. q is true (premise)
4. $\neg r \Rightarrow \neg q$ is true (syllogism 1, 2)
5. $\neg\, q$ is F (from 3)
6. $\neg r$ is False (rule of implication & 4)
7 r is True (rule of negation)

$\therefore$ The argument is valid.

7 a.

p	q	r	¬p	¬q	$p\wedge\neg q$	$r\vee\neg p$	$q\Rightarrow r$	$[(p\wedge\neg q)\wedge(r\vee\neg p)]\Rightarrow(q\Rightarrow r)$
T	T	T	F	F	F	T	T	T
T	T	F	F	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	F	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

So, the argument is valid.

b.

p	q	¬q	$p\wedge\neg q$	$(p\wedge\neg q)\wedge\neg q$	$[(p\wedge\neg q)\wedge\neg q]\Rightarrow p$
T	T	F	F	F	T
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	F	F	T

$\therefore$ The argument is valid.

c.

p	q	r	$\neg r$	$p\vee\neg r$	$p\vee q$	$(p\vee\neg r)\wedge(p\vee q)$	$q\vee r$	$[(p\vee\neg r)\wedge(p\vee q)]\Rightarrow(q\vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	T	T	T	T	F	F
F	T	T	F	F	T	F	T	T
F	T	F	T	T	T	T	T	T
F	F	T	F	F	F	F	T	T
F	F	F	T	T	F	F	F	T

So, the argument is invalid (not valid).

8. a. 1. $p \Rightarrow \neg q$ is true (premise)

2. r is true (Premise)

3. $r \Rightarrow$q is T (Premise)

4. q is true (Modes ponens 2,3)

5. $\neg p$ is T (Modes Tolens, 1,4). So, the argument is valid.

b. Let p: Hailu's books are on the desk.

q: Hailu's books are on the shelf.

The argument is:

$$\begin{array}{l} p \vee q \\ \underline{\neg q} \\ p \end{array}$$

Hence 1. $p \vee q$ (premise)

2. $\neg q$ (Premise)

3. p (Disjunctive syllogism, 1,2).

So the argument is valid.

c. Let p: 5 is even

q: 2 is prime

r: 4 is positive

The argument now becomes:

$$\begin{array}{l} p \Rightarrow q \\ q \Leftrightarrow r \\ \underline{\neg r} \\ \neg p \end{array}$$

Thus, 1. $p \Rightarrow q$ (premise)

2. $q \Leftrightarrow r$ (Premise)

3. $p \Rightarrow r$ (syllogism, 1,2)

4. $\neg r$ (Premise)

5. $\neg p$ (Modes Tolens, 3, 4)

$\therefore$ The argument is valid.

7.2 DIFFERENT TYPES OF PROOFS

Periods Allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to;

- *apply the principle of mathematical induction for proving.*
- *identify a problem and determine whether it could be proved using principle of mathematical induction or not.*

Vocabulary: Proof, Direct proof, Exhaustion, Indirect proof, Proof by contradiction, Counter-example.

Introduction

This sub-unit discusses some very common proof techniques such as direct and indirect proofs, proof by exhaustion, proof by contradiction and disproving by counter example by considering an appropriate example for each method.

Teaching Notes

As you teach this section, you need to stress the fact that examples, no matter how many, are not enough to establish a mathematical truth since they cannot cover the infinite number of instances that the assertion may cover. It would be good if you start the section by discussing postulates, definitions and theorems, etc. It is also good to note that in a proof, the argument derives its conclusions from:

- ✓ *the premises of the statement*
- ✓ *other theorems*
- ✓ *definitions*
- ✓ *the postulates of the mathematical system*

Note also that memorizing proofs is a difficult task and may even make students turn away from mathematics. So, to avoid such danger, let students consider particular cases and arrive to a conjecture and then let them try to convince you and their classmates why the assertion is true.

In doing so, with your demand for clear, logical and systematic presentation, they might arrive at a proof or a counter example of their own. They will also easily understand and appreciate the proof presented by others.

Note: Students may have difficulty in proving what an equation holds?. Here are some suggestions:

1. It is better to reduce the most complicated side and reduce it to the other side
2. to show $f = g$, notice that
   a. $f - g = 0 \Leftrightarrow f = g$
   b. $f \le g \wedge g \le f \Leftrightarrow f = g$
   c. $f = h$ and $h = g \Leftrightarrow f = g$

In addition, encourage the students to do Activity 7.4 which helps in developing their skill of proving a given statement. They can be assigned to do this activity in groups and you could go around and encourage them to participate in the activity. In the process you can assess their level of understanding of the topic so that you could plan the next lesson appropriately or fill the gap early enough. In all the discussions of the examples, encourage the students to participate actively.

Answers to Activity 7.4

1. Proof; if x and y are even integers, then there exist integers m and n such that

   $x = 2n + 1$ and $y = 2m + 1$.

   $\Rightarrow x + y = (2n+1)+(2m+1)$

   $= (2n+2m)+(1+1)$ (*Addition of integers is both commutative and associative*)

   $= 2(n+m)+2$

   $= 2(n+m+1)$ (2 *is a common factor*)

   $= 2k$, where $k = n + m + 1$ which is an integer

   $\Rightarrow x + y = 2k$, which is an even integer.

   Therefore, $x + y$ is an even integer.

2. Proof: $n > m$ (*Given*)

   $\Rightarrow 5n > 5m$ (*multiplication property of order*)

   $\Rightarrow 5n + mn > 5\text{m} + mn$ (*Addition property of order*)

   $\Rightarrow n\,(5 + m) > m\,(5 + n)$ (*Distributive property*)

   $\Rightarrow \dfrac{m+5}{n+5} > \dfrac{m}{n}$ (*Dividing by positive numbers*)

Answers to Activity 7.5

The purpose of this activity is to show the students about the relation between the two quantifiers.

1. $(\exists x)$ $(x^2 \leq 0$, where x is a real number)
2. $(\forall x)$ $(2x$ is not a prime number)
3. $(\exists x)$ $(\forall y)$ $(x \neq y^2 + 1$, where x and y are real numbers)

Answers to Exercise 7.3

1. (use direct proof)

   Let n be odd

   $\Rightarrow n = 2k + 1$ for some $k \in \mathbb{Z}$

   $\Rightarrow n + 2 = 2k + 3 = 2(k + 1) + 1$ is odd $\Rightarrow n + (n + 2) = 2k + 1 + 2k + 3$

   $= 4k + 4 = 4\,(k + 1)$

   Therefore $n + (n + 2)$ is a multiple of 4.

2. (use direct proof)

   $a < b \Rightarrow a + a < a + b$

   $\Rightarrow 2a < a + b$

   $\Rightarrow a < \dfrac{a+b}{2}$ - - - - - (1)

   Again $a < b \Rightarrow a + b < b + b$

   $\Rightarrow a + b < 2b$

   $\Rightarrow \dfrac{a+b}{2} < b$ - - - - (2)

   From (1) and (2) $a < \dfrac{a+b}{2} < b$

   Let $c = \dfrac{a+b}{2}$, $c \in \mathbb{Q}$ because it is a sum of rational numbers. Thus, the assertion is true.

3. (use proof by contrapositive)

   $a < 20$ and $b < 20$

   $\Rightarrow 20 - a > 0$ and $20 - b > 0$

   Since a sum of the two positive numbers is positive,

   $(20 - a) + (20 - b) > 0$

   $\Rightarrow 40 - (a + b) > 0$

   $\Rightarrow a + b < 40$

   Hence by the rule of contrapositive, the assertion follows.

4. (Proof by cases)

When we divide an integer n by 3, the remainders are either 0, 1, or 2.

i.e $n = 3m$, for some $m \in \mathbb{Z}$

or $n = 3m + 1$, for some $m \in \mathbb{Z}$

or $n = 3m + 2$, for some $m \in \mathbb{Z}$

Case 1 $n = 3m$

$\Rightarrow n^2 = 9m^2 = 3(3m^2)$

Let $k = 3m^2 \Rightarrow n^2 = 3k$

Case 2 $n = 3m + 1$

$\Rightarrow n^2 = (3m + 1)^2 = 9m^2 + 6m + 1$

$= 3\ (3m^2 + 2m) + 1$

Let $k = 3m^2 + 2m \Rightarrow\ n^2 = 3k + 1$

Case 3 $n = 3m + 2$

$\Rightarrow n^2 = (3m + 2)^2 = 9m^2 + 12m + 4 = (9m^2 + 12m + 3) + 1$

$= 3\ (3m^2 + 4m + 1) + 1$

Let $k = 3m^2 + 4m + 1 \Rightarrow n^2 = 3k + 1$

$\therefore\ \forall n \in \mathbb{Z}$, $n^2 = 3k$ or $n^2 = 3k + 1$, for some $k \in \mathbb{Z}$

5. (Proof by contrapositive)

Suppose m and n are perfect squares

$\Rightarrow\ \exists p, \ell \in \mathbb{Z}$ such that $m = p^2$ and $n = \ell^2$

$\Rightarrow mn = p^2\ell^2 = (p\ell)^2$

$\Rightarrow mn$ is a perfect square

$\therefore$ The assertion follows by taking the contrapositive.

6. (proof by contradiction)

Suppose $\exists a, b \in \mathbb{Z}$ Such that a and b are relatively prime and $\dfrac{a}{b} = \sqrt{5}$

$\Rightarrow a^2 = 5b^2 \Rightarrow a^2$ is a multiple of 5

$\Rightarrow a$ is a multiple of 5 $\Rightarrow a = 5\ell$

$\Rightarrow a^2 = 5b^2 = 25\ell^2$

$\Rightarrow b^2 = 5\ell^2 \Rightarrow b^2$ is a multiple of 5

$\Rightarrow b$ is a multiple of 5

$\Rightarrow a$ and b are both multiple of 5 and hence they are not relatively prime.

Contradiction!

$\therefore\ \sqrt{5}$ is not rational.

7. (Proof by contradiction)

   Suppose on the contrary $\sqrt{x^2 + y^2} = x + y$

   $\Rightarrow x^2 + y^2 = (x + y)^2 = x^2 + y^2 + 2xy$

   $\Rightarrow xy = 0$

   $x = 0$ or $y = 0$

   Contradicting the fact that x and y are positive.

8. a. If $x \in (A \cap B)$, then $= x \in A$ and $x \in B \Rightarrow x \in (A \cup B)$

   Thus, $(A \cap B) \subseteq (A \cup B)$ is True.

   b. The assertion fails, when $n = 2$. Because, $2^2 - 1 = 3$ is prime.

9. a. Suppose x and y are even

   $\Rightarrow x = 2k$ and $y = 2\ell$, for some $k, \ell \in \mathbb{Z}$

   $\Rightarrow xy = 2k2\ell = 2(2k\ell) = 2m$, where $m = 2k\ell \in \mathbb{Z}$

   $\Rightarrow xy$ is even

   Therefore, the statement "x and y are even $\Rightarrow xy$ is even" is proved.

   b. Suppose n is not odd

   $\Rightarrow n$ is even

   $\Rightarrow n = 2k$, for some $k \in \mathbb{Z}$

   $\Rightarrow 3n + 2 = 3(2k) + 2 = 2(3k + 1)$

   $\Rightarrow 3n + 2$ is even

   $\therefore 3n + 2$ odd $\Rightarrow n$ is odd (taking the contrapostive)

   c. The assertion is not true. As a counter example, take

   $n = 6$

   Then $6! = 720$ and $6^3 = 216$

   Thus $6! > 6^3$

   $\therefore \forall n\left(n! < n^3\right)$ is false

   d. Counter example. Take $n = 1$

Assessment

Order the students to do the exercises in the textbook in group and check their answers. If possible, give them additional exercises which require them to employ different methods of proofs they had studied and check their answers.

7.3 PRINCIPLE OF MATHEMATICAL INDUCTION AND ITS APPLICATION

Periods Allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *apply the principle of mathematical induction for proving.*
- *identify a problem and determine whether it could be proved using principle of mathematical induction or not.*

Vocabulary: Mathematical induction

Introduction

This subunit deals with the common and powerful method of proving assertions involving natural numbers. It begins with simple and interesting examples that motivate the students to think about the importance of the principle of mathematical induction. Then, the principle of mathematical induction will be stated and sufficient examples will be considered in its application.

Teaching Notes

Because the principle is based on the nature of natural numbers, it will be very important to review the way natural numbers are constructed. That is, each natural number after the first is obtained by adding 1 on the preceding natural number. Then before stating the principle of mathematical induction, you may show to the students how important it is by carefully leading the discussion based on the three examples in the student textbook. The discussion can be conducted in groups or with all the students together. It is very important that the students go through the examples carefully and participate actively so that they will realize the powerfulness and the importance of the principle of mathematical induction in proving assertions that involve natural numbers.

Answers to Exercise 7.4

1. When $n = 1$, $1 = \dfrac{1(1+1)}{2}$ is true, suppose it is true for k

   i.e $1 + 2 + \ldots + k = \dfrac{k(k+1)}{2}$

Now adding $(k + 1)$ on both sides we get: $1 + 2 + \ldots + k + (k + 1) = \dfrac{k(k+1)}{2} + (k+1)$

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Hence, it is true for $k + 1$. Thus, the assertion is true for each $n \in \mathbb{N}$.

2. To show that $2 + 4 + 6 + \ldots + 2n = n(n + 1)$ * for all $n \in$ N

   when $n = 1$, $2n = 2 = 1(1 + 1)$, it is true.

   Suppose it is true for $n = k$

   i.e $2 + 4 + 6 + \ldots + 2k = k(k + 1)$ (**)

   Now adding the next even natural number 2 (k +1) on left hand sides of (**) we get:

   $$2 + 4 + 6 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$$
   $$= (k + 1)(k + 2)$$
   $$= (k + 1)[(k + 1) + 1]$$

   But this is the original formula (*) itself when n is replaced by $k + 1$.

   Therefore, the formula (*) holds for any natural number n.

3. $2 + 4 + 6 + \ldots + 100 = 2(1 + 2 + 3 + \ldots + 50) = 2\left(\dfrac{50\times 51}{2}\right) = 2550$

4. Answers to the opening problem

   a. The next five triangular numbers are 15, 21, 28, 36 and 45.

   b. The i$^{\text{th}}$ triangular number is given by $\sum_{n=1}^{i} n = 1+2+3+\ldots+i$

   c. To show that the sum of the first n triangular numbers is given by $\sum_{i=l}^{n} Ti = \dfrac{n(n+1)(n+2)}{6}$, we use the principle of mathematical induction.

To show $T_1 + T_2 + \ldots + T_n = \dfrac{n(n+1)(n+2)}{6}$, (i) where T_n is the n^{th} triangular number. When $n = 1$, $1 = \dfrac{1(1+1)(1+2)}{6} = \dfrac{6}{6} = 1$, it is true.

Suppose it is true for $n = k$ i.e. $T_1 + T_2 + \ldots\ldots + T_k = \dfrac{k(k+1)(k+2)}{6}$

Now adding the next triangular number T_{k+1} on both sides we get:

$$T_1 + T_2 + T_3 + \ldots\ldots + T_k + T_{k+1} = \frac{k(k+1)(k+2)}{6} + T_{k+1}$$

$$= \frac{k(k+1)(k+2)}{6} + (1+2+3+\ldots.+k+1)$$

$$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$ (*here see definition of* T_k *and use Exercise* 7.4 *no*1*above)*

$$= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(k+3)}{6} = \frac{(k+1)((k+1)+1)((k+1)+2)}{6}$$

But this is the formula (i) above itself when n is replaced by $k + 1$.

Therefore, $\sum_{i=1}^{n} Ti = \dfrac{n(n+1)(n+2)}{6}$ holds for any natural number n.

d. $\sum_{i=1}^{40} Ti = \dfrac{40(41)(42)}{6} = 11,480$

5. The following table shows the situation

Number of rows	**Number of boxes**
1	6
2	8
3	10
. . . n	$2n + 4$

Thus $6 + 8 + 10 + \ldots\ldots + (2n + 4) = 4n + 110$

$\Rightarrow 3 + 4 + 5 + \ldots\ldots + (n + 2) = 2n + 55$ (*dividing by* 2)

$\Rightarrow 1 + 2 + 3 + 4 + 5 + \ldots\ldots + (n + 2) = 2n + 58$ (*adding* (1 + 2) *on both sides*)

$\Rightarrow \dfrac{(n+2)(n+3)}{2} = 2n + 58$ (*using formula in 1 above*)

$\Rightarrow n^2 + 5n + 6 = 4n + 116 \Rightarrow n(n + 1) = 10\,(11)$

$\Rightarrow n = 10$

Thus, there are 10 rows.

6. The first even natural number 2 is given as 2(1) suppose the k^{th} even natural number is given as $2k$. Then the $(k + 1)^{\text{th}}$ even natural number is $2k + 2 = 2(k + 1)$

$\therefore \forall n \in \mathbb{N}$, the n^{th} even natural number is given by $2n$.

7. The first odd natural number is 1 which is given as $2(1) - 1$. Suppose the k^{th} odd natural number is given by $2k - 1$. Then, the $(k + 1)^{\text{th}}$ odd natural number is

$(2k - 1) + 2 = 2k - 1 + 2 = 2k + 2 - 1 = 2(k + 1) - 1$

$\therefore \forall n \in \mathbb{N}$, the n^{th} odd natural number is given by $2n - 1$.

8. For $n = 1$,

$6^1 - 1 = 6 - 1 = 5$, 5×1, i.e $6^1 - 1$ is a multiple of 5.

Suppose $6^k - 1$ is a multiple of 5, i.e $6^k - 1 = 5m$, for some $m \in \mathbb{N}$

Then $6^{k+1} - 1 = 6(6^k - 1) + 5 = 6(5m) + 5 = 5(6m + 1)$

Hence, for each $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

9. For $n = 1$ $\quad 2^{1-1} = 1 \le 1!$

Suppose it holds for k. i.e,

$2^{k-1} \le k!$. Then $2^k \le 2(k!) \le (k + 1)k! = (k+1)!$ …… $(2 \le k + 1)$

Hence $2^{n-1} \le n!$, for all $n \in \mathbb{N}$. (Note that $2^k = 2^{(k+1)-1}$).

10. For $a = 1$

$1^3 = 1 = \dfrac{1^2(1+1)^2}{4}$

Suppose it is true for k. i.e $1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Then $1^3 + 2^3 + \text{- - - -} + k^3 + (k+1)^3 = \dfrac{k^2(k+1)^2}{4} + (k+1)^3$

$$= (k+1)^2\left(\frac{k^2}{4} + (k+1)\right) = (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2((k+1)+1)^2}{4}$$

Therefore, $1^3 + 2^3 + \text{- - - -} + n^3 = \sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

Finally, when you ask the students to try to do the review exercises, you may assign questions 9(g), 9(h) and question 10 (b - e) to high ability learners, if there are such students in the class.

Answers to Review Exercises on Unit 7

1. a.

p	**q**	$\neg$**p**	**p** $\Rightarrow$ **q**	$\neg$**p** $\vee$ **q**
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore\ p \Rightarrow q \equiv \neg p \vee q$

b.

p	**q**	**p** $\Rightarrow$ **q**	**q** $\Rightarrow$ **p**	**p** $\Leftrightarrow$ **q**	$(\mathbf{p} \Rightarrow \mathbf{q}) \wedge (\mathbf{q} \Rightarrow \mathbf{p})$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\therefore\ p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

c.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\therefore\ \neg(p \wedge q) \equiv \neg p \vee \neg q$

d.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Thus, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

2. a.

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$\therefore\ (p \wedge q) \Rightarrow p$ is a tautology.

b.

p	q	$p\vee q$	$p\Rightarrow(p\vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$\therefore p\Rightarrow(p\vee q)$ is a tautology.

c.

p	q	$\neg p$	$p\vee q$	$\neg p\wedge(p\vee q)$	$\neg p\wedge(p\vee q)\Rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

$\therefore\ (\neg p\wedge(p\vee q))\Rightarrow q$ is a tautology.

d.

p	q	$p\Rightarrow q$	$p\wedge(p\Rightarrow q)$	$(p\wedge(p\Rightarrow p))\Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$\therefore\ (p\wedge(p\Rightarrow q))\Rightarrow q$ is a tautology.

3. a. Let S (x): x is a student in this class and F (x): x can speak French

   Then the statement will be: $(\exists x)(S(x)\wedge F(x))$

   b. Let S (x): x is a student in this class and D (x): x knows how to drive a car

   Then the statement will be: $(\forall x)(S(x)\Rightarrow D(x))$.

   c. Let S (x): x is a student in this class and B (x): x has a bicycle.

   Then the statement will be: $(\exists x)(S(x)\wedge B(x))$.

4. a. F b. T c. T d. F
   e. T f. F
5. a. T b. F c. T d. T
   e. T f. T g. T h. T
6. a. T b. F c. T d. F
   e. T f. F g. T
7. a.

p	q	$\neg q$	$q\Rightarrow p$	$\neg q\Leftrightarrow p$	$(q\Rightarrow p)\wedge(\neg q\Leftrightarrow p)$	$[(q\Rightarrow p)\wedge(\neg q\Leftrightarrow p)]\Rightarrow p$
T	T	F	T	F	F	T
T	F	T	T	T	T	T
F	T	F	F	T	F	T
F	F	T	T	F	F	T

The argument is valid.

b. 1. $p \Rightarrow \neg q$ (Premise)

2. $p \wedge r$ (Premise)

3. p (Simplification, 2)

4. r (Simplification, 2)

5. $\neg q$ (Modes ponens, 1, 3)

6. $\neg q \Leftrightarrow r$ (Bi-implication, 5, 4)

$\therefore$ The argument is valid.

c.

p	q	r	$p \Rightarrow q$	$p \Rightarrow r$	$q \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	T

The argument is a fallacy because, in the 6th row, while the premises $p \Rightarrow q$ and $q \Rightarrow r$ are true, the conclusion $q \Rightarrow r$ is false.

d.

p	q	$\neg p$	$\neg q$	$(p \Rightarrow q) \wedge \neg p$	$[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

The argument is a fallacy

e.

p	q	r	$p \Rightarrow q$	$r \Rightarrow q$	$p \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

The argument is a fallacy because, on the second row, $p \Rightarrow r$ is False while $p \Rightarrow q$ and $r \Rightarrow q$ are both True.

f.

p	q	$\neg q$	$p \vee q$	$(p \vee q) \wedge p$	$[(p \vee q) \wedge p] \Rightarrow \neg q$
T	T	F	T	T	F
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	T

The argument is a fallacy.

8. a. Let p: you send me an e - mail

q: I will finish my homework

r: I will go to sleep early

s: I will wake up early

The argument in symbolic form becomes

$$\begin{array}{r} p \Rightarrow q \\ \neg p \Rightarrow r \\ r \Rightarrow s \\ \hline \neg q \Rightarrow s \end{array}$$

Now 1. $p \Rightarrow q$ (premise)

2. $\neg q \Rightarrow \neg p$ (contra positive, 1)

3. $\neg p \Rightarrow r$ (premise)

4. $\neg q \Rightarrow r$ (syllogism, 2, 3)

5. $r \Rightarrow s$ (premise)

6. $\neg q \Rightarrow s$ (syllogism, 4, 5)

$\therefore$ The argument is valid.

b. Let p: Alemu has an electric car

q: Alemu drives a long distance

r: Alemu's car needs to be recharged

s: Alemu will visit an electric station

Then, the argument in symbolic form becomes:

$$\begin{array}{l} (p \wedge q) \Rightarrow r \\ r \Rightarrow s \\ q \\ \neg s \\ \hline \neg p \end{array}$$

1.	$(p \wedge q) \Rightarrow r$	(premise)
2.	$r \Rightarrow s$	(premise)
3.	$(p \wedge q) \Rightarrow s$	(syllogism, 1,2)
4.	$\neg s$	(premise)
5.	$\neg(p \wedge q)$	(modes Tollens, 3,4)
6.	$\neg p \vee \neg q$	(De Margnus, 5)
7.	q	(premise)
8.	$\neg p$	(Disjunctive syllogism, 6, 7)

$\therefore$ The argument is valid.

9. a. Let x and y be odd.

$\Rightarrow x = 2\ell + 1$ and $y = 2k + 1$, for some $\ell, k \in \mathbb{Z}$

$\Rightarrow xy = (2\ell + 1)(2k + 1) = 4\ell k + 2\ell + 2k + 1$

$= 2(2\ell k + \ell + k) + 1 = 2m + 1$ where $m = 2\ell k + \ell + k$

$\Rightarrow xy$ is odd

b. Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers. Then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

Now $a, b, c, d \in \mathbb{Z}$ and $b \neq 0 \wedge d \neq 0$

$\Rightarrow ac \in \mathbb{Z}$ and $bd \in \mathbb{Z}$ and $bd \neq 0$

$\Rightarrow \frac{ac}{bd} \in \mathbb{Q} \Rightarrow \frac{a}{b} . \frac{c}{d} \in \mathbb{Q}$

c. It is not true.

Counter example:

$\sqrt{2}$ is irrational. But, $\sqrt{2} . \sqrt{2} = 2$ is not irrational.

d. Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers

$\Rightarrow a, b, c, d \in \mathbb{Z}$ and $b \neq 0 \wedge d \neq 0$

$\Rightarrow bd \neq 0$ and $bd \in \mathbb{Z}$

Now $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, by definition

As $(ad + bc) \in \mathbb{Z}$ and $bd \in \mathbb{Z}, bd \neq 0$

$\frac{ad + bc}{bd} \in \mathbb{Q} \Rightarrow \frac{a}{b} + \frac{c}{d} \in \mathbb{Q}$

e. Using proof by contrapositive,

n is odd $\Rightarrow n = 2k + 1$, for some $k \in \mathbb{Z}$

$\Rightarrow n^3 = 8k^3 + 12k^2 + 6k + 1$

$\Rightarrow n^3 + 5 = 8k^3 + 12k^2 + 6k + 6$

$= 2\,(4k^3 + 6k^2 + 3k + 3)$

$\Rightarrow n^3 + 5$ is even

$\therefore n^3 + 5$ odd $\Rightarrow n$ is even.

f. False

Counter example, $k = 7$ is prime, while $k + 2 = 9$ is not prime.

g. Using proof by contra-positive,

$p = q \Rightarrow pq = p^2$ and $\frac{p+q}{2} = p$

$\Rightarrow \sqrt{pq} = p$ and $\frac{p + q}{2} = p$

$\Rightarrow \sqrt{pq} = \frac{p + q}{2}$

$\therefore$ if $\sqrt{pq} \neq \frac{p+q}{2}$ then $p \neq q$

h. $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{(n-r)!(r)!}$

$\Rightarrow \binom{n}{r} = \binom{n}{n-r}$. The assertion is proved

10. a. For $n = 1$

$2^0 = 1 = 2^1 - 1$

Suppose it is true for k i. e $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

$$\Rightarrow \sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2\left(2^{k+1}\right) - 1 = 2^{k+2} - 1$$

$$= 2^{(k+1)+1} - 1$$

Hence, the assertion is true, $\forall n \in \mathbb{N}$.

b. $1^2 = \frac{1\ (1\ +\ 1)\ (2\ (1)\ +\ 1)}{6}$. Thus for $n = 1$ it is true

Suppose it is true for k.

i.e, $1^2 + 2^2 + \ldots. + k^2 = \frac{k(k+1)(2k+1)}{6}$

$$\Rightarrow 1^2 + 2^2 + \ldots.+ k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)\left(2k^2 + k + 6k + 6\right)}{6}$$

$$= \frac{(k+1)\left(2k^2 + 4k + 3k + 6\right)}{6}$$

$$= \frac{(k+1)\left(2k(k+2) + 3(k+2)\right)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\left((k+1)+1\right)\left(2(k+1)+1\right)}{6}$$

Hence, the assertion is true, for all $n \in \mathbb{N}$.

c. For $n = 1$

$1 \times 2 = 2 = \frac{1\ (1\ +\ 1)\ (2\ (1)\ +\ 1)}{3}$

Suppose it holds for k.

i.e, $1 \times 2 + 2 \times 3 + \ldots\ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

Then $(1 \times 2) + (2 \times 3) + \ldots.+ k(k+1) + (k+1)\left((k+1)+1\right)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)\left((k+1)+1\right)\left((k+1)+2\right)}{3}$$

Hence $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$.

d. For $n = 1$, it is true because,

$$1^2 = \frac{1\ (2\ (1)\ -\ 1)\ (2\ (1)\ +\ 1)}{3} = 1$$

Suppose it is true for k.

i.e, $1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

$$\Rightarrow 1^2 + 3^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 2k + 3k + 3)}{3}$$

$$= \frac{(2k+1)(2k(k+1) + 3(k+1))}{3}$$

$$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

Hence, the assertion is true for each $n \in \mathbb{N}$.

e. for $n = 1$

$\frac{1}{n(n+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1}$ it holds

Suppose it holds for k.

i.e $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Then for $k + 1$ it becomes.

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Thus, the assertion is true for each $n \in \mathbb{N}$.

UNIT 8 FURTHER ON STATISTICS

INTRODUCTION

The word statistics has two meanings; statistics as numerical data and statistics as methods.

a. Statistics as numerical data: are numerical statements of facts that are obtained by inquiry, measurement or observation.

Example: Statistics of students may include total number of students, their weight height, etc.

b. Statistics as methods is defined as the collection, organization, presentation, analysis and interpretations of data.

In this unit, you first need to give a brief revision of the purpose of the field statistics in different sectors of social and economical situations. After this revision, it is important to point out that the unit mainly focuses on introducing students to different methods of collecting data and sampling techniques. The unit also discusses the various forms of presenting data.

Students should also be encouraged: to present a given set of data using graphs, and to compute different measures of central tendency and measures of variations. They are also expected to relate measures of central tendency and measures of dispersion in measuring consistency and symmetry.

At the end of each main point, it is helpful to provide students with examples that are applicable to real life.

Unit Outcomes

After completing this unit, students will be able to:

- *know basic concepts about sampling techniques.*
- *construct and interpret statistical graph.*
- *know specific facts about measurement in statistical data.*

Suggested Teaching Aids for Unit 8

Since Statistics is one of the fields that are practically applied in our day-to-day experience, there might be a lot to apply depending on where the discussion is applied. Some of the teaching aids that are useful for teaching statistics and particularly this unit are: Colored chalks (white board markers), chalk board, different colored objects (marbles), playing cards, dice, coins, paper for drawing graphs, color pencils, and straight edged ruler for drawing. Different graphs, charts, different statistical data and figures of different organizations, text and other references and calculators are also useful. In addition, you can also use different software such as MS-EXCEL, SPSS, Fathom, etc.

8.1 SAMPLING TECHNIQUES

Periods allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *describe the three methods/ techniques of sampling.*
- *explain the advantages and limitations of each technique of sampling.*

Vocabulary: Population, Sample, Random sampling, Systematic sampling, Stratified sampling, Sampling technique

Introduction

In this sub-topic, the students are expected to identify the three sampling techniques, namely, simple random sampling, systematic sampling and stratified sampling, and they will also determine advantages and limitations of each technique. In order to teach these techniques, it is recommended to let the students say or write their understanding before they discuss each. Finally, it will be helpful to practice each in some practical problems.

Teaching Notes

Start the subtopic, it will be better to revise the purpose of the field statistics in different sectors of social and economic situations. You can then proceed by refreshing students' memory with definitions of statistical terms such as population and sample; and by encouraging students to do Activity 8.1whose explanation is given below.

Answers to Activity 8.1

The question of the Activity is as follows. Ministry of Agriculture wants to study the productivity of using irrigation farming. If you were asked to study this, obviously, you would start from collecting data. If so, discuss the following questions.

1. Why do you need to collect data?
2. How would you collect the data?
3. From where would you collect the data?

Collecting data is the basis for conducting statistical process. Without having an aggregate of facts or collected information, it is difficult to perform statistics. Hence, it is essential to collect data. The answer to the question: "How would you collect data?" is actually wide. But, at this stage, we can think of different mechanisms of data collection such as using questionnaire, interview, observation, discussion, performing experiment, etc. We can collect data either from a population or from a sample (which is considered to represent a population). But here you are expected to explain to the students that trying to collect data from the entire population is difficult, time consuming and cumbersome. Thus, data would be collected from what is called a sample, which is part of the population and is assumed to be representative of the population. The data in this case can be collected either primarily from the farmers themselves or from secondary sources such as the wereda bureau of agriculture and natural resources. Following this discussion, encourage the students in identifying the concepts of a population and a sample.

Right after the students have understood the concept of population, sample and the need for sampling, you can let them discuss some of the sectors in which statistics is useful. Some examples are given in the student textbook. Here you also need to let students understand the ideas of sample size, homogeneity and independence, equally likely, and representativeness of a sample to a population. The way such issues can be addressed and how sample must be selected from a population will be discussed in subsequent subtopic. Before discussing the sampling techniques, you can reinforce their understanding by forming a discussion group and letting them do Activity 8.2.

Answers to Activity 8.2

Athletics federation decides to construct athletics academies in some parts of Ethiopia. For this purpose, it needs to study the potential source of athletes so as to decide where to build the academies.

1. Is it possible to study the whole population of the country? Why?
2. How would the Federation collect a sample from the entire population?

3. What characteristics must be fulfilled by the sample?

The possible answer to this activity can be:

1. It is not practically possible to study the whole population. It is difficult because the population is too many and trying to study is time consuming and costly.
2. The Federation can collect a sample by using any of the sampling techniques such as: simple random sampling, stratified sampling, systematic sampling, stratified sampling, purposive sampling, etc.
3. The sample must represent the whole population and it must be unbiased.

After ensuring the understanding of the students on the use of samples in statistical process, you can describe the ways how a sample is taken from a population. At this stage, you need to describe the three types of sampling techniques and methods of sampling, i.e random sampling (in which every member of the population has an equal chance of being selected), systematic sampling and stratified sampling. Although there could be various characteristics that a sample must fulfill, one of the leading characteristics is an issue of bias when choosing a sample. You then discuss the advantages and limitations of each sampling technique. After explaining the different sampling techniques, you can duplicate random numbers from statistics books and give them to students to practice to use them. You can also encourage them to use randomization by using playing cards or lottery methods. While discussing these, it will be essential for students to recognize the probability sampling and non-probability sampling. While students do the examples on the student textbook, you can add the following questions for high achievers.

Do we have other sampling techniques (both probability and non-probability)? You can let them think of cluster sampling from the probability sampling. Let them also recognize purposive and accidental sampling from the non-probability sampling.

Assessment

You can ask students to list the sampling techniques, and to describe the advantages and disadvantages of each of the three sampling techniques. You can do this through class activities, group discussions, giving assignments, or giving tests.

Answers to Exercise 8.1

1. Statistics as methods is defined as the collection, organization, presentation, analysis and interpretations of data.
2. **Population** in statistics means the complete collection of items (individuals) under consideration, whereas **Sample** in statistics is the limited number of items

taken from the population on which the study/investigation is carried out. (Note: A sample can be of any size or may consist of the entire population)

3. The three sampling techniques are simple random sampling, systematic sampling and stratified sampling. Simple Random Sampling is a technique by which each and every observation has an equal chance of being selected. In Systematic Sampling, first we need to sort each observation according to some magnitude then randomly select one object and continue to select others at some specified length from prior selection. In Stratified Sampling, there is known and identified strata (or observed difference) among the observations. After identifying each strata then select a sample from each strata by using either simple random sampling or systematic sampling.

4. There are other types of sampling that can be classified as probability and non-probability sampling. Cluster sampling can be considered as probability sampling, whereas quota sampling, accidental sampling techniques, etc are some of the non-probability sampling techniques.

5. a. The sample interval is 5 (dividing 100 by 20).

   b. The members of the sample are 4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, 79, 84, 89, 94, 99.

6. They are already listed in the student text but encourage them to bring more advantages and disadvantages by referring statistics books.

8.2 REPRESENTATION OF DATA

Periods allotted: 2 periods

Competencies

At the end of this sub-unit students will be able to:

- *describe the different ways of representations of data .*
- *explain the purpose of each representation of data.*

Vocabulary: Frequency distribution, Histogram, Bar chart

Introduction

Students have discussed some of the tabular and chart representations of data in grades 9 and 11. Here, they will see some more types and ways of data representations and are expected to explain the purposes of each representation. It is recommended that students collect practical data from their surrounding and represent the data using different ways so that they can understand the concept behind each concept.

Teaching Notes

You can start this sub-unit by considering examples from the different ways of representations of data (both for discrete and continuous) that were discussed in previous grades and by showing models (from government and non-governmental organizations) of different representations such as pictographs, frequency distribution table, bar chart, histogram, etc. You can enforce the students understanding by letting them do Activity 8.3.

Answers to Activity 8.3

The answer to activity 8.3 is as follows

Weight (in kgs)	Number of students
47 – 51	5
52 – 56	6
57 – 61	4
62 – 66	5
67 – 71	2

You can guide them to see how representing raw data makes it easier for interpretation and observation. You can also discuss in good detail the advantage of representing data. The intention here is to make the students be able to describe the different ways of representations of data and to give some interpretations from graphical presentations of data. They can present various forms of their class by way of describing graphically such as weight, age, sex, etc and discuss some interpretations.

Some illustrative examples are delivered in the student textbook to highlight how useful data representations are for interpreting data. You can use examples 2, 3 and 4 of the student textbook for discussion. While all students try to understand the interpretations given beneath each chart, you can ask high achievers to give other possible interpretations from each graphical representation. Example, age does not make any difference in Gambia which means that the prevalence is not changing with respect to age. The overall prevalence is decreasing from Somalia to Kenya.

Assessment

You can ask students how the tabular method (frequency distribution) and pictorial methods of representations of data are helpful and when one method is preferable than the other in presenting the required information. You can do this by giving various representations of same data and letting students identify by themselves through class activities, group discussions, or giving assignments.

Answers to Exercise 8.2

1. a. The discrete frequency distribution is

Weight (in kg)	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
No. of students	2	1	3	4	4	2	2	3	1	3	1	1	3

b. You need to take a random number of classes (here we consider 6) and the frequency distribution with 6 classes is

Weight (in kg)	**Number of students**
48 – 49	3
50 – 51	7
52 – 53	6
54 – 55	5
56 – 57	4
58 – 60	5
Total	30

2. i. a. Between 50 and 55, there are 18 students

b. Less than 53, there are 14 students

c. More than 54, there are 12 students

d. Between 55 and 60, there are 12 students

ii. Between 50 and 51is the highest frequent group.

8.3 CONSTRUCTION AND INTERPRETATION OF GRAPHS

Periods allotted: 6 periods

Competencies

At the end of this sub-unit, students will be able to:

- *construct graphs of statistical data.*
- *identify statistical graph.*
- *explain the significance of representing a given data in different types of graphs.*
- *draw histogram for a given frequency distribution.*
- *sketch frequency polygon for a given frequency distribution.*
- *sketch frequency curve for a given frequency distribution.*

- *draw bar chart.*
- *construct line graph for data related to time.*
- *construct pie chart for a given data.*

Vocabulary: Frequency distribution, Histogram, Frequency polygon, Frequency curve, Bar chart, Line graph, Pie chart.

Introduction

Previously, students discussed the tabular and graphical representation where they saw some representative examples. However, there are more graphical representations that students need to know. In addition to their previous knowledge, they will see more on histograms and will also see more others such as frequency polygon and frequency curves. Some other charts such as Bar charts, line graphs and Pie charts will also be presented.

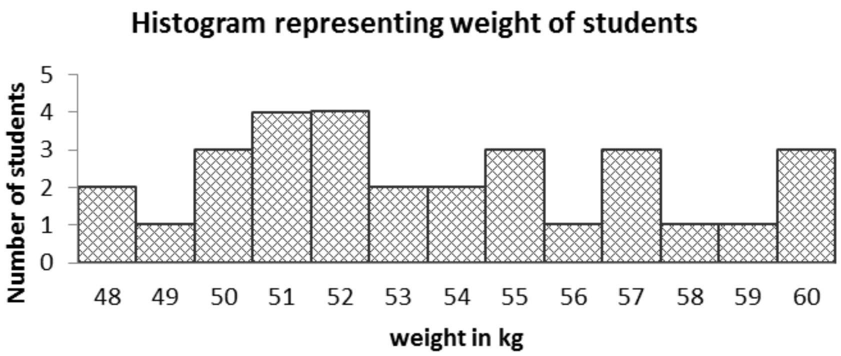
Teaching Notes

Since students have some background from their grade 9 mathematics, you can start this sub-unit by encouraging students to perform activity 8.4 and give them chance to discuss

1. methods and procedures of drawing and presenting statistical graphs in an understandable and attractive way, and
2. how to obtain the correct information i.e. how to read and interpret them.

Answers to Activity 8.4

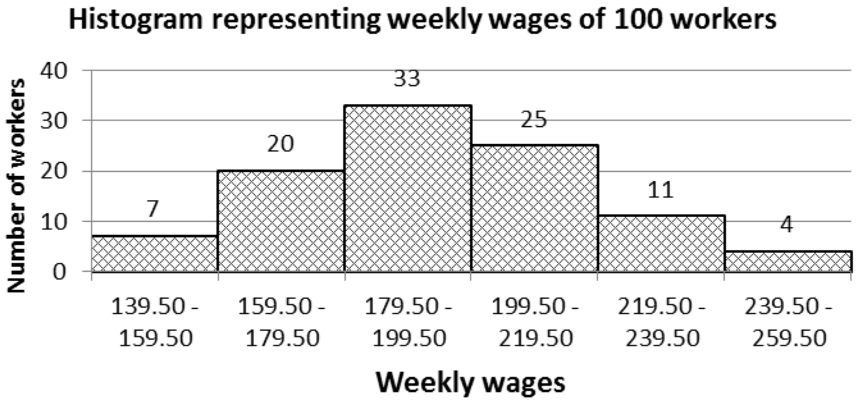
The answer for the activity is as follows.



You also discuss how frequency polygon is obtained from corresponding histogram, and also how the given curve is related to the frequency polygon. Following the discussion of histogram of ungrouped data which the students had discussed earlier, you can proceed to discussing histogram of grouped frequency distribution. Here, encourage and guide students to perform each task in Activity 8.5 and finally, consolidate their understanding on histogram of grouped frequency distribution.

Answers to Activity 8.5

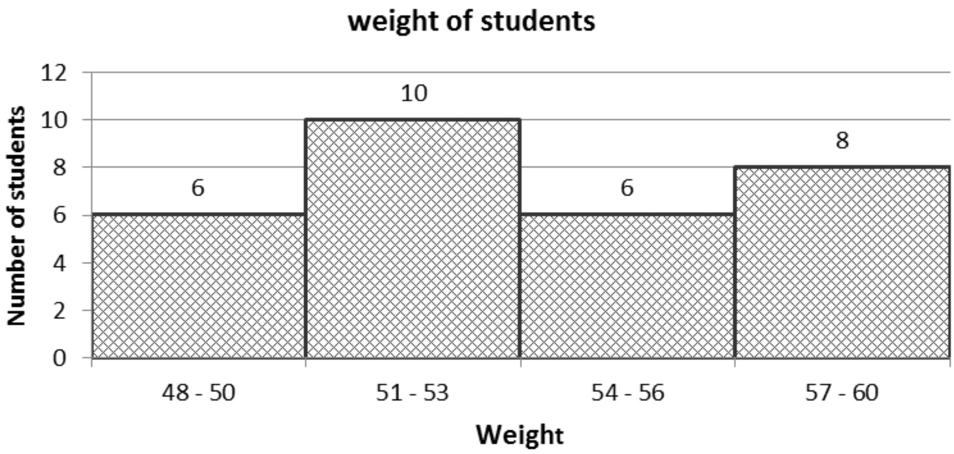
For answering activity 8.5, we will have the following histogram.



You can also ask them to group the data given in Activity 8.4 say into 4 groups. If we assume 4 classes which give the following frequency distribution

Weight of students	Number of students
48 - 50	6
51 - 53	10
54 - 56	6
57 - 60	8

Notice that the class width may not be always equal especially at the beginning or at the end. The histogram of the distribution is



Following the understanding and ability of students in drawing a histogram, you can proceed into frequency polygons which can easily be constructed by using line segments and joining the mid points of each bar in the histogram. Illustrative example is given as Example 3 on page 334 and Example 4 of page 335 of the student textbook. Let them also notice that they can draw frequency polygons by simply connecting frequencies at each class marks. An important idea is also cumulative frequency curve (**Ogive curve**) that is useful to determine values less than or more than any reference value in the range of the data. Example 5 of the students textbook illustrates this. When you make sure that students are able to draw each type of graphs, you need to take note for students to describe the nature of frequency polygon and frequency curve of a grouped frequency distribution in terms of symmetry, though it will be discussed in good detail later. You can explain or let the students discuss each idea and particularly frequency polygon and frequency curve. Immediately, following this, you can let them check symmetry, and skeweness of some data.

The next point of discussion is representation of data using diagrams (charts) that include Bar charts, line graphs and Pie charts. In order to start this discussion you can let the students do Activity 8.6 which will help students to discuss Bar charts and identify the relationship and the difference between histogram and bar chart.

At this stage, it will be sufficient if the students can identify that, in Bar charts, there is an equal gap between each Bar and in histogram there is no gap between each rectangular area. You are also expected to guide students to draw bar charts, line graphs and pie charts through the examples offered in the student textbook and by using additional real life problems. It is better if students do the drawings by themselves in the form of group work or assignment which you can summarize later for consolidating their understanding. They need also identify the three types of bar charts namely simple bar charts, component bar charts, and multiple Bar charts.

Answers to Activity 8.6

1. Bar Chart is a rectangular bar where each bar represents data or group and its height represents the corresponding frequency.
2. Similarity: both consider rectangular Bar to represent data.

   Difference: there is equal gap between each bar in the Bar chart but, in histogram there is no gap between each Bar.

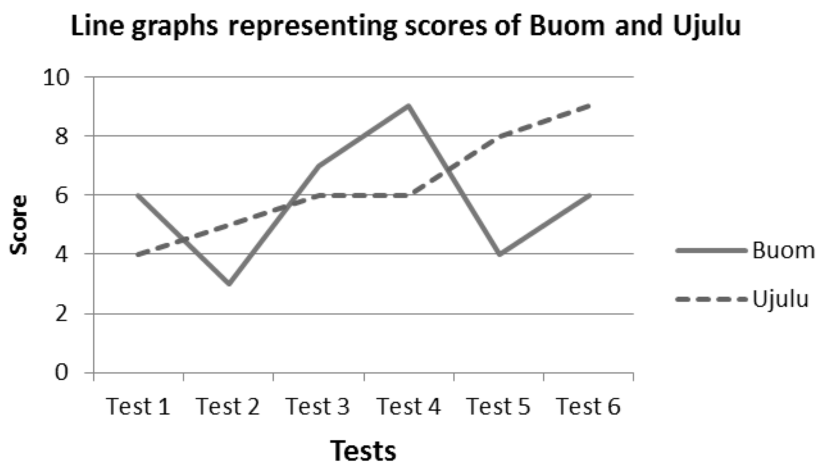
Following these discussions on charts, another graph which is very useful in data analysis, especially for comparing different data series are line graphs. First they need to discuss how they draw line graph. After they understand how to draw, you can give them an example that guides how line graphs are useful to compare data.

Example: Consider the following data representing scores of Buom and Ujulu on successive six tests counted out of 10 each.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6
Buom	6	3	7	9	4	6
Ujulu	4	5	6	6	8	9

Compare the academic achievement of these two students by using line graphs?

Solution: The following represents the line graphs of the two students.



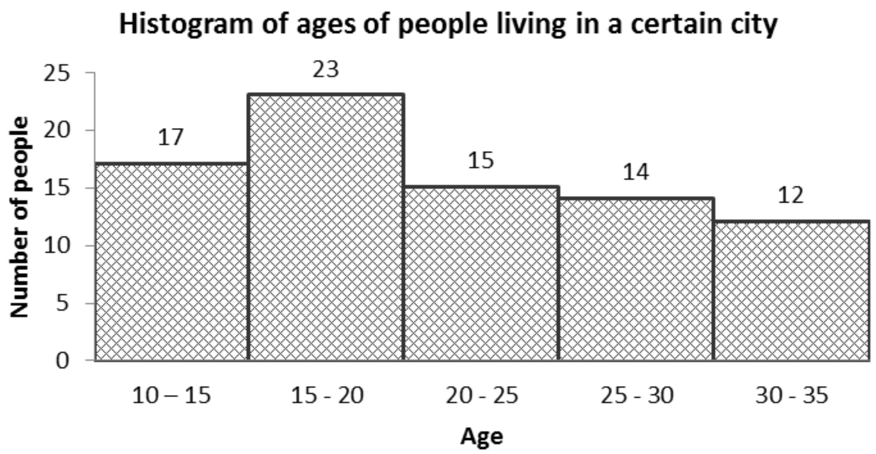
Discuss with the students that Ujulu has steadily increased her achievement while Buom has fluctuating achievement. Let them recognize how useful line graphs are in comparing data. For high achievers, you can hint them to read more of other forms of data representation such as horizontal bars, stem-and-leaf, Box-plot representations.

Assessment

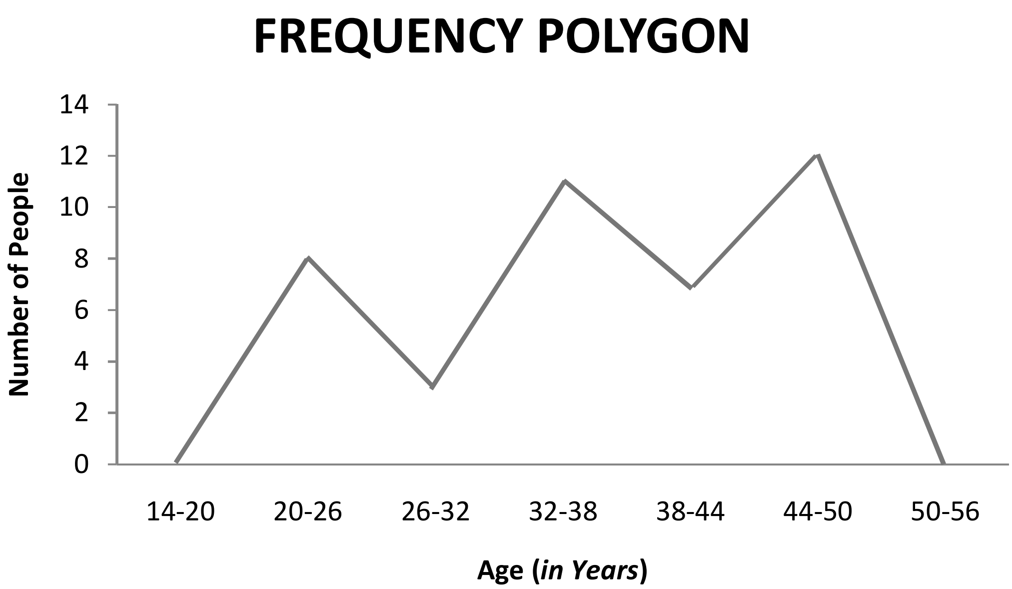
You can give exercise problems to students on sketching different types of charts either from given frequency distribution or by giving them real life problem whose data will be collected by the students themselves that will be organized into frequency distribution. You can do this through group assignment or project work.

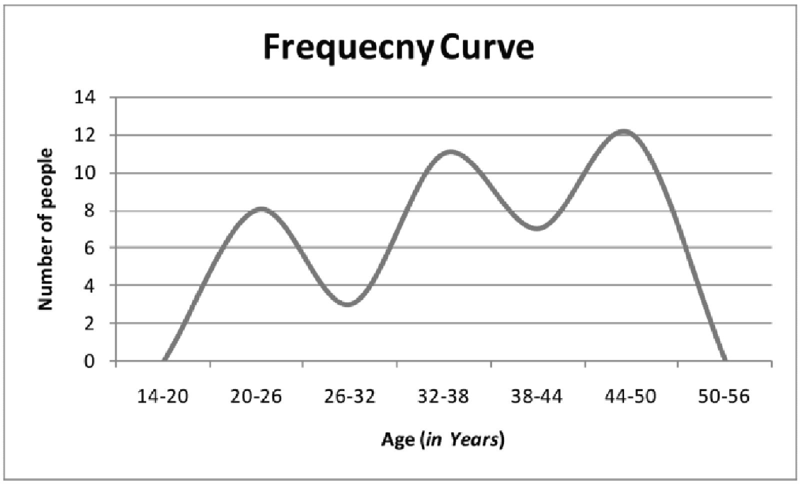
Answers to Exercise 8.3

1.

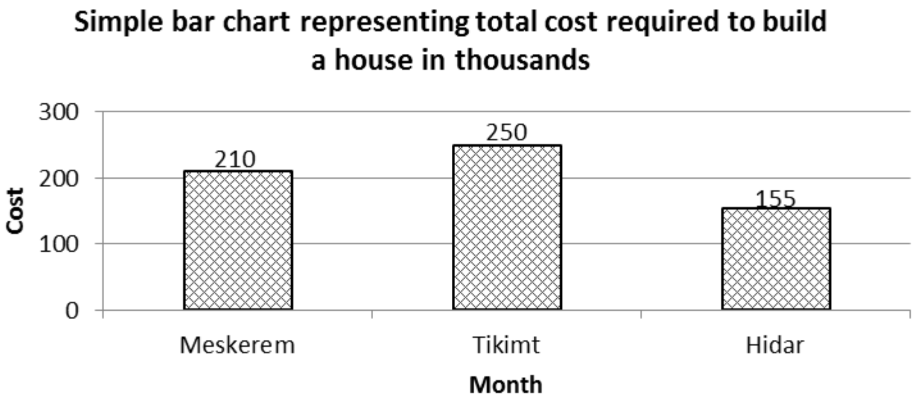


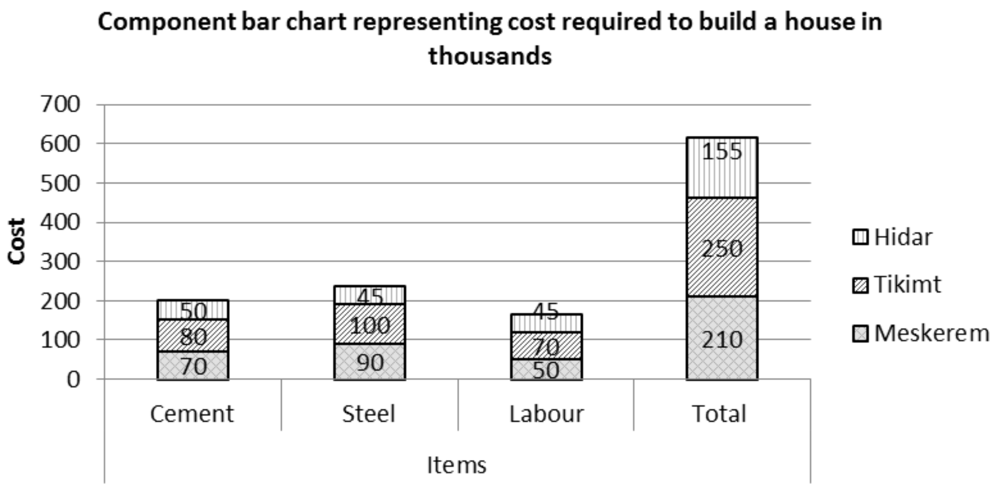
2.

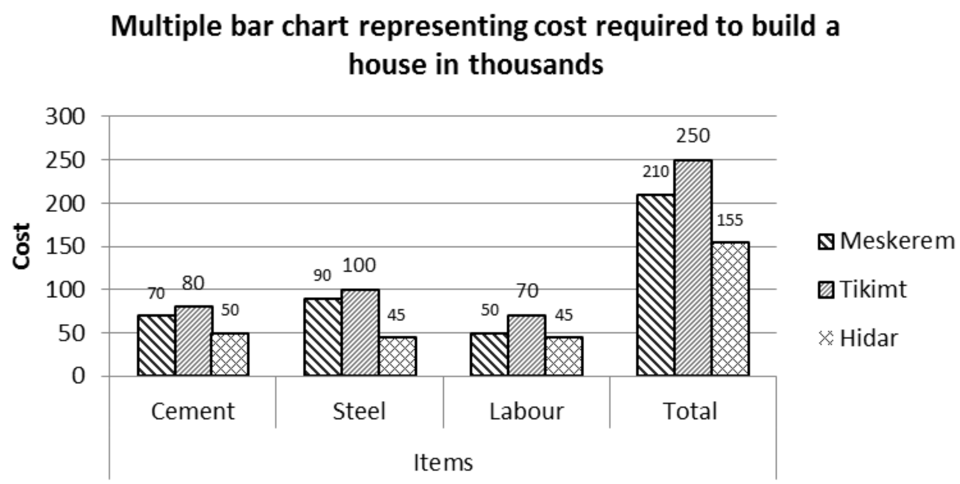



Frequecny Curve
14
12
10
8
6
4
2
0
Number of people
14-20
20-26
26-32
32-38
38-44
44-50
50-56
Age (in Years)

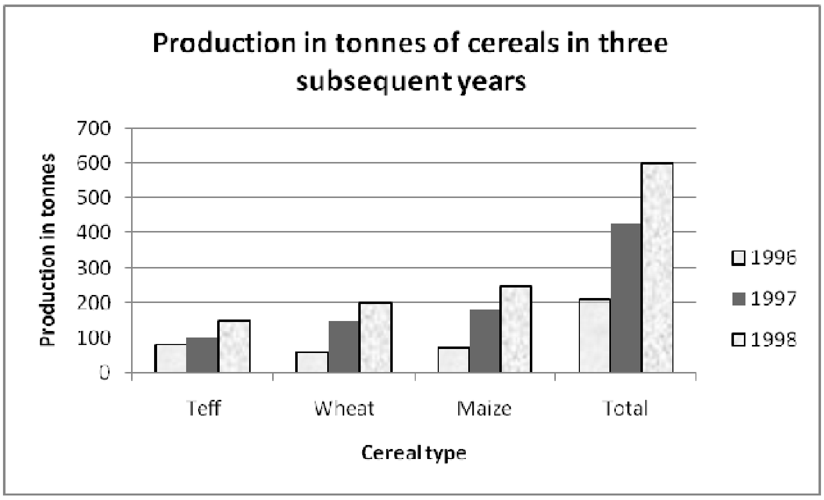
3.


Simple bar chart representing total cost required to build a house in thousands
300
200
100
0
Cost
210
250
155
Meskerem
Tikimt
Hidar
Month


Component bar chart representing cost required to build a house in thousands
700
600
500
400
300
200
100
0
Cost
50
80
70
45
100
90
45
70
50
155
250
210
Cement
Steel
Labour
Total
Items
Hidar
Tikimt
Meskerem



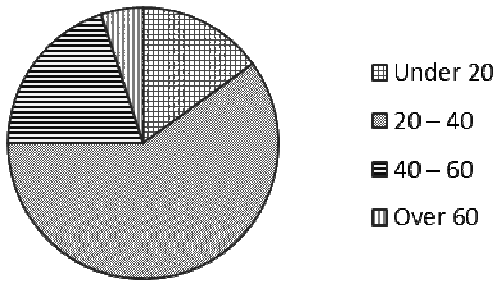
4. This is identical with question number 3. One representation can be the following.



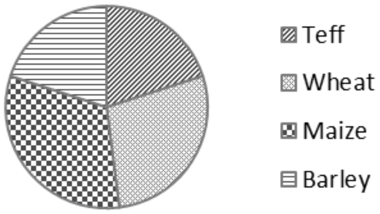
5.

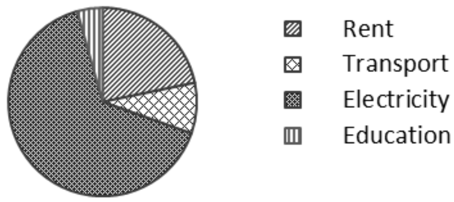
Age	Number of people	Degree
Under 20	15	54
20 – 40	60	216
40 – 60	20	72
Over 60	5	18

The pie chart of this looks like



6. a. b.





8.4 MEASURES OF CENTRAL TENDENCY AND MEASURES OF VARIABILITY

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *compute the three mean deviations of a given data.*
- *describe the relative significance of mean deviation as a measure for dispersion.*
- *calculate the inter-quartile range for a given data.*
- *describe inter-quartile range as a measure of variability in values of a given set of data.*
- *describe the usefulness of standard deviation in interpreting the variability of a given data.*

Vocabulary: Mean, Median, Mode, Range, Inter-quartile range, Mean deviation

Introduction

Students have discussed previously measures of central location and dispersion with particular focus on range, standard deviation and variance. In this sub-topic, they will discuss further on the concepts of central tendency and measures of dispersion. They will also consider dispersion measures such as quartile deviation and mean deviation, and, in general, deviations at the three measures of central tendency (mean, median and mode).

Teaching Notes

You may begin this section by encouraging students to do Activity 8.7. With the help of examples of ungrouped data and grouped frequency distribution, give a brief revision of calculating the mean, median, mode, quartiles, range, inter-quartile range and standard deviation.

Answers to Activity 8.7

a. Mean = 57.27 b Median = 55 c. Range = 30

d. $Q_1 = 50$ e. $Q_3 = 63$ f. SD = 8.559

To help students revise these ideas, it is recommended that discussion is conducted on the examples given in the student textbook from page 344 to 346. You can also use Exercise 8.4 for revision purpose and assessing student competencies.

Assessment

You can give several exercise problems on both measures of central tendency and in interpreting a given data. You also need to ask students to describe which of the measurements gives enough information about a given data. In addition, you may give real life problems whose data will be collected by the students themselves that will be organized into frequency distribution, so that they will calculate various measures of central tendency. You can do this through group assignment or project work.

Answers to Exercise 8.4

1. a. Mean = 67.1 b. Mean = 22
2. 14
3. 22
4. 18
5. a. Mean $= \dfrac{7\times3+10\times2+11\times4+15\times8+19\times6}{3+2+4+8+6} = \dfrac{319}{23} = 13.87$

   b. Mean $= \dfrac{20\times8+30\times12+40\times20+50\times10+60\times6+70\times4}{8+12+20+10+6+4}$

   $= \dfrac{2460}{60} = 41$

   c. First, find the class mark of each class. Then, multiply each midpoint by the corresponding frequency to calculate the mean as follows:

$$\text{Mean} = \frac{4.5\times5+14.5\times10+24.5\times8+34.5\times13+44.5\times4}{5+10+8+13+4} = 24.75$$

6. a. To calculate the median, first sort the data in increasing order. This will give 1, 2, 6, 7, 8, 10. From this, we see that the median is the average of the two middle values which is 6.5. There is no mode for this data.

b. Divide the total frequency by 2 to get the median value, i.e. the median is the $\left(\frac{n+1}{2}\right)^{th} = \left(\frac{22}{2}\right)^{th} = 11^{th}$ item which is 15. The mode is also 15 since it is the highest frequent value.

c. The median is $\left(\frac{39}{2}\right)^{th}$ item = 19.5$^{\text{th}}$ item. It is in the 4$^{\text{th}}$ class.

Therefore, the median class is 10 – 12.

$B_L = 9.5$, $n = 39$, $\frac{n}{2} = 19.5$, $f_c = 17$, $i = 3$, $cf_a = 11$

Therefore, median = $B_L + \left(\frac{\frac{n}{2} - cf_a}{f_c}\right)i = 9.5 + \left(\frac{19.5-11}{17}\right)3 = 11$

The modal class is 10 – 12, which is the highest frequent. Thus mode is $M_o = B_L + \left(\frac{d_1}{d_1+d_2}\right)i$ where $B_L = 9.5$, $d_1 = 9$, $d_2 = 6$ and $i = 3$.

Thus mode = $B_L + \left(\frac{d_1}{d_1+d_2}\right)i = 9.5 + \left(\frac{9}{9+6}\right)3 = 11.3$

7. a. $Q_1 = 11.25$, Q_2 = half way of the data = 15.5, and $Q_3 = 20.75$

b. $Q_k = B_L + \left(\frac{\frac{kn}{4} - cf_a}{f_c}\right)i$ from which,

$$Q_1 = B_L + \left(\frac{\frac{n}{4} - cf_a}{f_c}\right)i = 9.5 + \left(\frac{5.75-3}{4}\right)5 = 12.94,$$

$$Q_2 = 14.5 + \left(\frac{11.5-7}{6}\right)5 = 18.25$$

$$Q_3 = 19.5 + \left(\frac{17.25-13}{7}\right)5 = 22.54\,.$$

When the students revise measures of central location in good detail, you can proceed into discussing measures of dispersion. Here, two new ideas will be introduced, namely, inter-quartile range and mean deviation. Before introducing mean deviation, encourage the students do Activity 8.8 and take notice of the use of taking the absolute values of the deviations. It is also advisable for students to realize mean deviation as one of the measures of variation that takes each observation into account. It is also remarkable to discuss mean deviation from the mean, the median and the mode, and describe when these are useful in practical problems. Several examples are given in the student textbook. When all students discuss these examples, you can ask the following questions those students who finish their task earlier or are recognized clever students.

Compare and contrast mean deviation about the mean, the median and the mode. Discuss the condition in which each of these measures are preferred.

Finally, let the students identify the advantages and limitations of each of the measure they discussed earlier.

Answers to Activity 8.8

1. Mean = 5.6
2. Deviation of each data from the mean is:

x	**2**	**6**	**4**	**9**	**5**	**7**	**3**	**6**	**8**	**6**
x- M(x)	**-3.6**	**0.4**	**-1.6**	**3.4**	**-0.6**	**1.4**	**-2.6**	**0.4**	**2.4**	**0.4**

3. 0
4. 1.68

Assessment

You can assess students' understanding on deviations about the mean, median and mode. You can do these by giving exercise problems similar to the questions in Exercise 8.5 or some other ones in a quiz/test form.

Answers to Exercise 8.5

1. a The mean deviation about the mean is 3.877
   The mean deviation about the median is 3.714
   The mean deviation about the mode is 3.714

b. The mean deviation about the mean is 2.6
The mean deviation about the median is 2.6
The mean deviation about the mode is 2.6

2. a. The mean deviation about the mean is 1.0744
The mean deviation about the median is 1
The mean deviation about the mode is 1

b. The mean deviation about the mean is 1.41
The mean deviation about the median is 1.4
The mean deviation about the mode is 1.54

c. The mean deviation about the mean is 4.83
The mean deviation about the median is 4.87
The mean deviation about the mode is 4.91

d. The mean deviation about the mean is 3.15
The mean deviation about the median is 2.94
The mean deviation about the mode is 2.74

Cognizant of the fact that students have understood mean deviation, they will discuss inter-quartile range which is important in data analysis under special condition. That is, another essential consideration of variation, especially when there is an open ended data, is the inter-quartile range. At this juncture, encourage students to do activity 8.9 and to identify the difference between range and inter-quartile range, and let your students do exercise problems on determining inter-quartile range.

Answers to Activity 8.9

City A 10, 14, 15, 16, 16, 17, 20 (Arranging)

$$Q_1 = \left(\frac{7+1}{4}\right)^{th} \text{item} = 2^{nd}\,\text{item} = 14$$

$$Q_3 = 3\left(\frac{7+1}{4}\right)^{\text{th}} \text{item} = 6^{th}\ \text{item} = 17$$

IQR = $Q_3 - Q_1 = 17 - 14 = 3$

City B 13, 14, 15, 15, 16, 16, 17

$$Q_1 = \left(\frac{7+1}{4}\right)^{th} \text{item} = 2^{\text{nd}}\ \text{item} = 14$$

$$Q_3 = 3\left(\frac{7+1}{4}\right)^{th} \text{item} = 6^{\text{th}}\ \text{item} = 16$$

IQR = = $Q_3 - Q_1 = 16 - 14 = 2$

Since IQR city B < IQR city A

City A has more variable temperature.

Finally, encouraging students to do Activity 8.10 and with a brief revision of computing standard deviation of ungrouped data and grouped frequency distribution, let students come to the conclusion that, among the measures of dispersion (variability) of data, the standard deviation is particularly useful in conjunction with the so-called normal distribution.

Answers to Activity 8.10

1. Range = 55

   Inter-quartile range = 28.75

   Mean deviation = 14.32

   Standard deviation = 17.6977 if we consider sample and 16.7896 if we consider population.

2. Help the students to understand that standard deviation is comparatively the best measure and that range is crude and weak measure of variation.

8.5 ANALYSIS OF FREQUENCY DISTRIBUTION

Periods allotted: 2 periods

Competencies

At the end of this subunit, students will be able to:

- *compare two groups of similar data.*
- *determine the consistency of two similar groups of data with equal mean but different standard deviations.*
- *describe the application of coefficient of variation in comparing two groups of similar data.*

Vocabulary: Mean, Standard deviation, Coefficient of variation

Introduction

So far students have discussed measures of central location and measures of dispersion separately. They might have some understanding on how these can describe data. However, there is another concept which is important for making judgments about data called coefficient of variation that are measures by joint consideration of measures of central location and measures of dispersion. The students will discuss particularly the coefficient of variation that relates standard deviation and the mean.

Teaching Notes

At this point concern can be given to comparison of two or more sets of data. This is because comparing sets of data issues of unit are important. Before dwelling on comparison, you can start this lesson by asking the students to do Activity 8.11 as a result of which they will be able to explain how they compare two or more groups of data. It also helps them deal with comparing data with respect to consistency.

Answers Activity 8.11

	If we consider the data as a sample		If we consider the data as a population	
	A	B	A	B
Standard Deviation	2.738	0.5	2.582	0.471
Mean	5	5	5	5
CV	0.5477	0.1	0.516	0.094

Therefore, Data B is more consistent than Data A. The reason for this is that it has less CV.

Here students need to note that mean alone may not completely describe a given data. There could be data whose mean are equal but they differ in their standard deviation. From such a data set, the one with lesser standard deviation is considered to be more consistent and the one with higher standard deviation is highly variable (less consistent). This will be true only when our data sets have the same unit. Yet, there can be data sets that have different units (example kg and cm) and make it difficult to compare each. If we have data sets that have different units, then we can consider the quotient of the standard deviation to mean which will help them compare consistency (as a relative measure) of the sets of data. Finally, students need to come to the fact that, when we calculate the relative measure of variation through coefficient of variation given as a quotient of the standard deviation and the mean, the data (series) having greater value of the quotient is less consistent and the one with lesser value of the quotient is more consistent, and using examples encourage the students to compare two distributions with equal means but different standard deviation, and other series of data with different mean and different standard deviation. You can ask your students to compare their results in mathematics test and chemistry test. For example, you may also ask them to compare their height and weight. With these examples they can try to understand how coefficient of variation (which is unit less number) is useful to compare data sets.

Assessment

To assess students learning, you can give exercise problems on comparing two similar groups of data to determine consistency in values by computation and ask students to describe their situation in their own words.

Answers to Exercise 8.6

1. a. Standard deviation of Team A = 21.18, and standard deviation of Team B = 39.06

b. From this, we can see that Team A which has less standard deviation is more consistent than Team B.

2. a. CV of A = $\frac{\sigma_A}{\overline{\overline{X}}_A} = \frac{120}{8000} = 0.015$ and CV of B = $\frac{\sigma_B}{\overline{\overline{X}}_B} = \frac{140}{8000} = 0.0175$

b. Company B has variable income since its standard deviation is larger than that of Company A. or its CV is higher than that of company A.

8.6 USE OF CUMULATIVE FREQUENCY CURVE

Periods allotted: 4 periods

Competencies

At the end of this subunit, students will be able to:

- *describe the relationship among mean, median and mode for grouped data using its frequency curve.*
- *use cumulative frequency graphs to determine the dispersion of values of data (in terms of mean, median and standard deviation).*
- *determine the variability of value of data in terms of quartiles by using cumulative frequency graph.*

Vocabulary: Symmetry, Skewness, Karl Pearson's coefficient of skewness, Bowley's coefficient of skewness.

Introduction

So far, students have discussed the three measures of central tendency and skewness of the distribution of data set, which may be symmetrical, skewed to the left and skewed to the right. They discussed skeweness by observing histograms or frequency curves as presented on pages 336 – 338 in the student text book. Here, they will discuss how they can use measures of central tendency and measures of dispersion to determine the skeweness of a distribution.

Teaching Notes

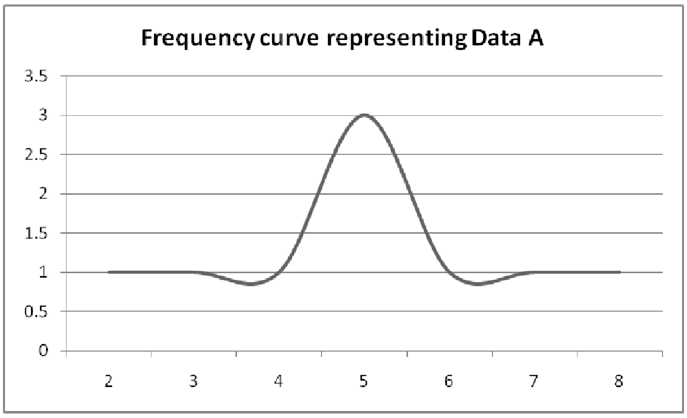
You may start the lesson by first asking the students to do Activity 8.12 and then discuss the result by comparing the mean, the mode and the median. Then, revise the three types of frequency curves (symmetrical, skewed to the left, and skewed to the right) discussed in previous section of this unit.

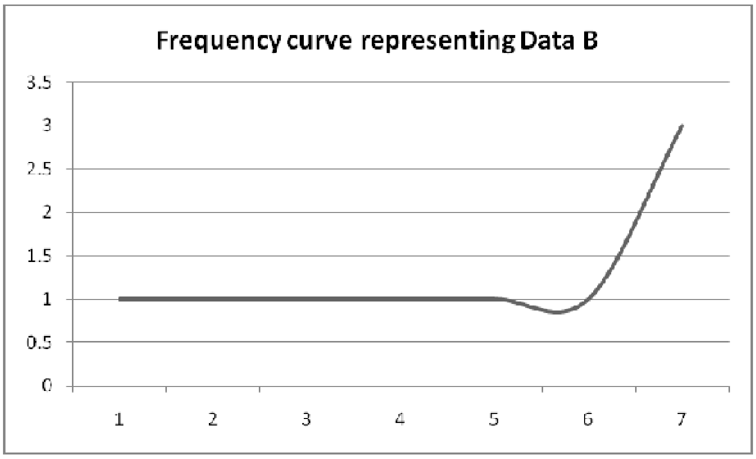
Activity 8.12

1.

	Data	
	A	B
Mean	5	5
Median	5	5
Mode	5	8

2.





The observation can be illustrated as follows.

1. The mean of Data A and Data B are the same.
2. The Median of Data A is the same to median of Data B
3. The Mode of Data B is higher than Mode of Data A.

You then discuss how different values of the mean, median, and mode of a frequency distribution for a grouped data are indicative of the form of the curve in terms of skewness and guide the students to come to the conclusion that if mean = median = mode, then the distribution is symmetrical. Otherwise, it is non symmetrical or is skewed.

Once again, if mean < median < mode, then the distribution is skewed to the left or else it is skewed to the right.

After recognizing these facts, you need to let them do examples on determining skeweness through comparing mean, median and mode. When you feel the students have understood well the ideas of skewness and relationship between the location of the mean, median and mode, you can proceed to discuss determining skewness through mathematical approach by using Karl Pearson's coefficient of skewness which involves mean, median and standard deviation.

With active participation of the students, you may also discuss the use of quartiles in determining skewness. One of the approaches to measuring skewness using quartiles is determining Bowley's coefficient of skewness. The characteristic determination of the values of these coefficients of skewness is presented in the student text with some illustrative examples.

You can ask for high achivers to determine skeweness through other measures involving quartiles. You can guide them to consult statistics books. You can also ask them the following questions: If a distribution is symmetrical then,

a. Mean = median = Q_2, and
b. $Q_1 + Q_3 = 2\text{median} = 2Q_2$

What relationship can you draw from these? Can we say $Q_1 = Q_3 = Q_2$? The answer is No. What we can say is that the distance from Q_2 to Q_1 and the distance from Q_2 to Q_3 are equal.

Assessment

Give different types of frequency curves and ask students to describe the relationships among the measures of location of the data presented by each graph. You can also give several exercise problems on determining the dispersion of values of a data by using the measures of location and the standard deviation of the data given. It is also possible to ask students about variability of values of a given data from computation.

Answers to Review Exercises on Unit 8

1. Given as an answer to Question 1 of Exercise 8.1
2. They are probability sampling techniques that are free from personal bias of the investigator.

   They give each observation an equal chance of being selected.
3. Frequency polygon is formed by line segments, whereas frequency curve is a curve that does not form polygon. But, both are represented by frequency of each data value.
4. They present data in an organized way, help to easily understand the interpretation.
5. Skewness explains the degree of symmetry about the measures of central tendency.
6. Simple bar chart presents the data in a single entity as it appears. Multiple bar chart presents data categorically, and component bar chart which is similar to multiple bar chart presents all classifications in a category in one bar.
7. **Mean deviation** is simply the mean of the deviations of each data from the average. If the average is mean, it is mean deviation. If the average is median, then the mean deviation is about the median, and if the average is mode, then the mean deviation is about the mode. In regard to their advantage, the following can be considered.

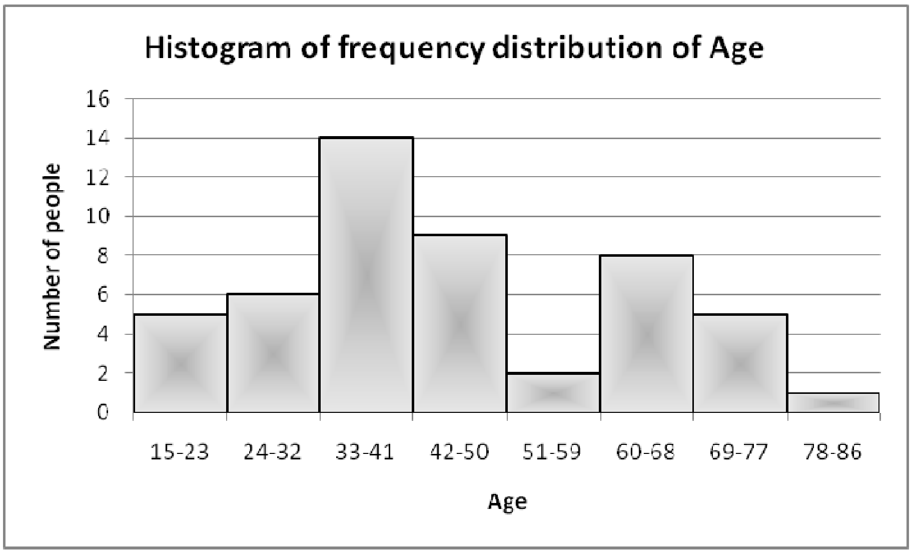
   Advantage of mean deviation about the mean: Can be used for an aggregate of facts for which mean is the best representative. About the mode is advantageous in qualitative data where highest frequent value is best representative. About median is of advantage especially when mean is subject to be affected by extreme value.
8. Range cannot be computed in an open ended data. Both are crude measures of variation that never consider all the data under consideration.
9. Because it considers all the data under consideration and it fulfills all the algebraic properties.
10. a. Discrete

    b. By assuming 8 classes for our case, class width = $\frac{65}{8} = 8.125 \approx 9$ and the frequency distribution will be as follows.

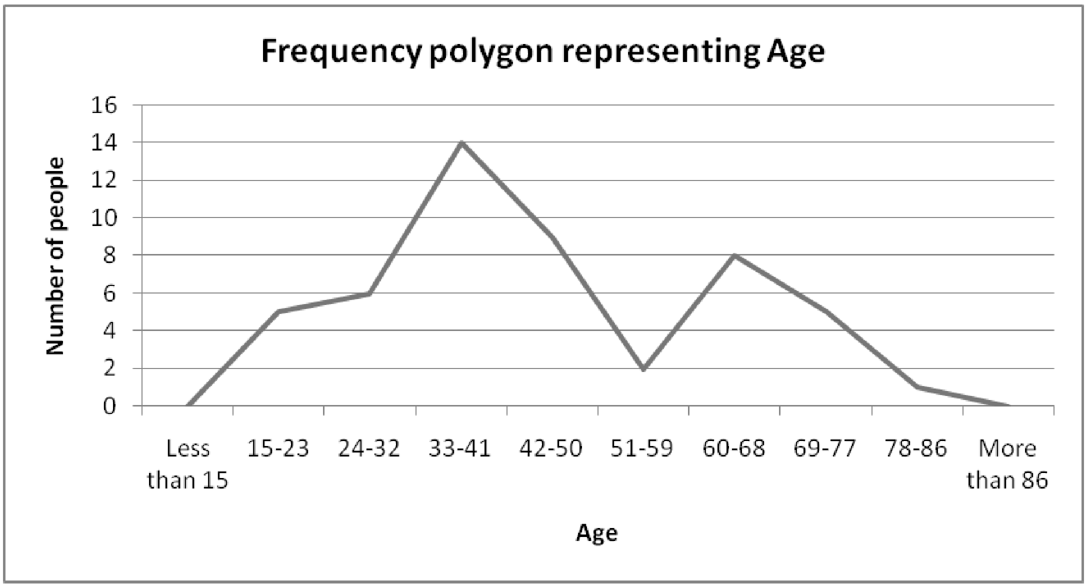
Age	15 - 23	24 - 32	33 - 41	42 - 50	51 - 59	60 - 68	69 - 77	78 - 86
Frequency	5	6	14	9	2	8	5	1

Note: It is possible to choose other number of classes as well and produce different frequency distribution.

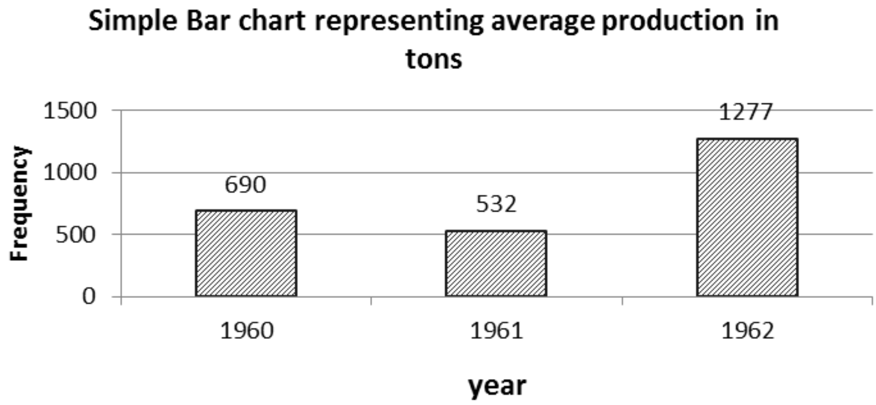
c.



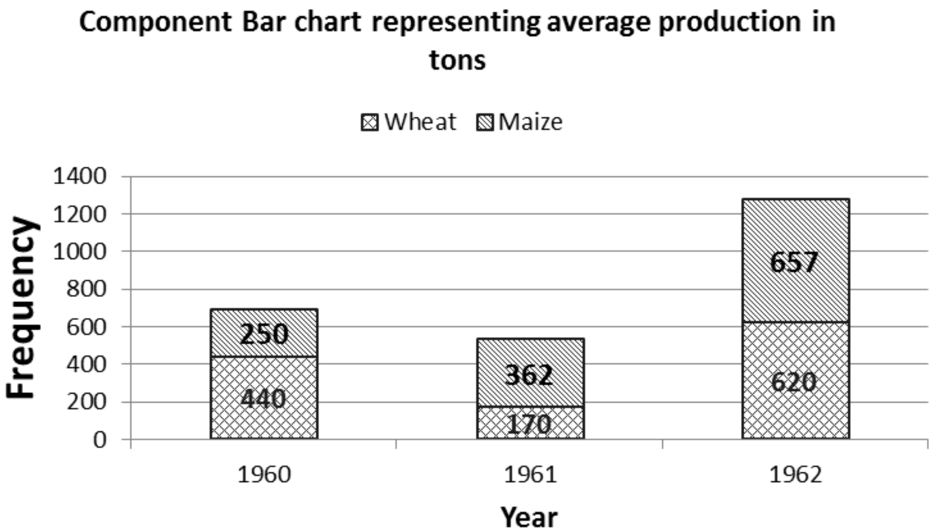
The frequency polygon that represents the above frequency distribution is



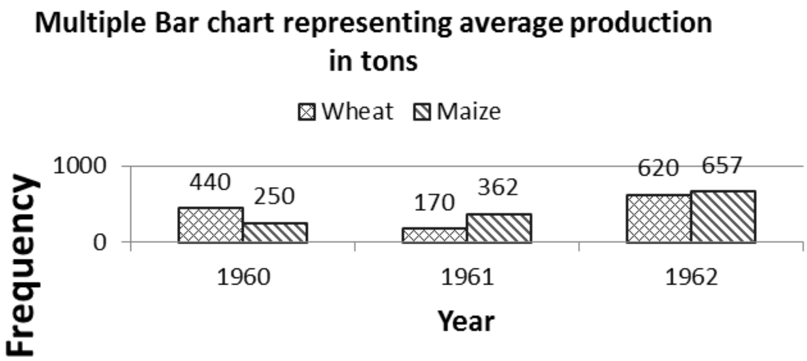
11. a.


Simple Bar chart representing average production in tons
1500
1000
500
0
Frequency
690
532
1277
1960
1961
1962
year

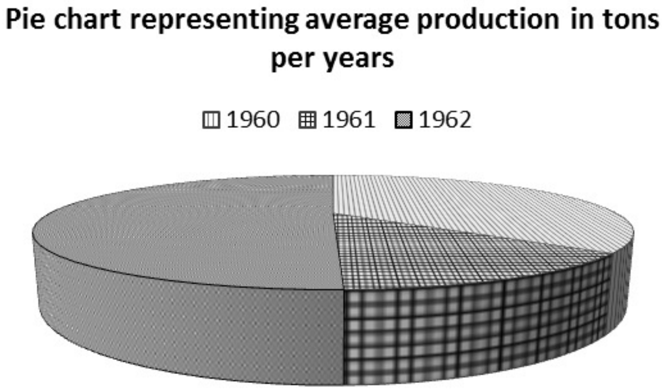
b.


Component Bar chart representing average production in tons
Wheat
Maize
1400
1200
1000
800
600
400
200
0
Frequency
250
440
362
170
657
620
1960
1961
1962
Year

c.


Multiple Bar chart representing average production in tons
Wheat
Maize
1000
0
Frequency
440
250
170
362
620
657
1960
1961
1962
Year

d.



12. a. Mean = 22.68

Median = 22.96

Mode = 23.25

Q_1 = 15.75

Q_3 = 29.31

b. Mean Deviation about the mean = 7.16

Mean Deviation about the Mode = 7.01

Mean Deviation about the Median = 7.09

Range = 39

Inter-Quartile range = 13.56

Standard deviation = 8.688

Coefficient of variation = 0.38

Pearson's coefficient of skewness = –0.0963 implying that the distribution is negatively skewed (skewed to the left).

13. The coefficient of variation for A = 0.166667 and that of B = 0.285714

Therefore, workers of company A are more consistent in their performance because they have lesser coefficient of variation.

14. i.

Mean	4.866667	MD about the mean	1.675556
Mode	6	MD about the mode	1.866667
Median	6	MD about the median	1.866667

ii. Range = 9 – 1 = 8

Inter Quartile Range = $Q_3 - Q_1 = 6 - 3 = 3$

iii. The Standard Deviation = 2.0296 if we consider the data as a sample and 1.9955 if we consider them as a population.

iv. The Coefficient of Variation = 0.41.

UNIT 9 MATHEMATICAL APPLICATIONS FOR BUSINESS AND CONSUMERS

INTRODUCTION

The main task of this unit is to enable students use their mathematical knowledge to solve their daily life calculation problems such as determining unit cost and total cost of items to be purchased by choosing the most economical purchase, applying percent decrease and percent increase in business to their advantage. The unit also aims at enabling students calculate the initial expense and ongoing expenses of owning a house using mortgage, and calculating total hourly wages, salaries and commissions.

Unit Outcomes

After completing this unit, students will be able to:

- *find unit cost, the most economical purchase and total cost.*
- *apply percent decrease to business discount.*
- *calculate the initial expense of buying a house and ongoing expenses of owning a house.*
- *calculate commissions, total hourly wages and salaries.*

Suggested Teaching Aids in Unit 9

In addition to the monthly payment table (table 9.1) attached to the students textbook, it would be necessary to have a scientific calculator during each lesson of this unit. It may be sufficient to have one calculator for a group of five or six students.

9.1 APPLICATIONS TO PURCHASING

Periods Allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *find unit cost.*
- *find the most economical purchase.*
- *find total cost.*

Vocabulary: Unit cost, Total cost, Economical purchase

Introduction

This subunit deals with unit cost and total cost which leads students to be able to determine economical purchase of items whenever there are alternatives. This is one aspect of application of their knowledge of mathematics in their daily life. It begins with the definition of unit and of total cost and considers different examples to be followed through doing appropriate exercises.

Teaching Notes

Start the lesson by asking the students to answer the questions in activity 9.1. Next, encourage the students to give their own examples about unit cost and total cost and then define unit cost and total cost with their active participation.

Discuss the examples in the student's text book and try to add more examples of your own or ask the students to give similar examples of their own based on items that they buy in our daily life. Discuss with the students that the most economical purchase is often found by comparing unit costs using the examples in the student's text or any other examples of your own.

Answers to Activity 9.1

1. a. Birr 37.5 b. Birr 12.5
2. a. Birr 0.95 b. Birr 10.45

Answers to Exercise 9.1

1. $(5 \times \text{Birr } 26.80) + (3 \times \text{Birr } 18.40) + (5 \times \text{Birr } 6.30) = \text{Birr } 488.70$
2. Buying 4kg of nails for Birr 25.80 is economical.
3. Birr 200. 45
4. a. $(6 \times \text{Birr } 7.25) + (6 \times \text{Birr } 3.625) = \text{Birr } 65.25$

   b. $(12 \times \text{Birr } 7.25) - \text{Birr } 65.26 = \text{Birr } 21.75$
5. a. $30 \text{ hrs} \times (\text{Birr } 35.40/\text{hr}) = \text{Birr } 1062.00$

   b. $\text{Birr } 991.20 \div 35.4 \text{ Birr / hr} = 28 \text{ hours}$

Assessment

- You can ask them orally about unit cost and total cost and about their calculations.
- In order to check whether the expected competencies are met or not,give them exercise problems on finding unit, total cost and determination of most economical purchase

9.2 PERCENT INCREASE AND PERCENT DECREASE

Periods Allotted: 4 periods

Competency

At the end of this sub-unit, students will be able to:

- *apply percent increase and percent decrease to business.*

Vocabulary: Percentage, Percentage increase, Percentage decrease, Profit less Percentage profit, Percentage loss, Cost price, Selling price, Markup, Merchandising business, Discount, Regular price, Sale price.

Introduction

The main task of this subunit is to familiarize students with the common daily life business concepts such as profit, loss, percentage increase or decrease, percentage profit or loss so that they will be able to apply their mathematical knowledge to perform different calculations related to the daily life business and make their life easier. It begins with a brief revision of percentage and introduces the terms mentioned above by turns by considering different examples.

Teaching Notes

Start the subunit by reviewing percent. Percent can be introduced as a fraction whose denominator is 100. Because a given fraction has infinitely many equivalent forms, the one with denominator 100 is commonly chosen to express comparisons or ratios. For example $\frac{1}{2} = \frac{50}{100}$, which means that one out of two can be expressed as 50 out of hundred or 50%. After introducing the word percent and defining percentage, encourage students to discuss activity 9.2 first in groups and then considering different examples such as scores in a test or increase or decrease in prices of different items in daily life. Then introduce percentage increases and decreases followed by percentage profit and loss. The difference between profit and markup should be noted. When you ask the students to do the exercises 9.2 and 9.3, please note that questions numbers 12, 13 and 20 of Exercise 9.2 and question number 7 of Exercise 9.3 can be assigned for fast learners or gifted students if there are such students in the class.

Answers to Activity 9.2

1. 238
2. 5%
3. He did best in English as he scored 90%
4. 176

Answers to Exercise 9.2

1. 280
2. Birr 6,000
3. 3 games
4. 12 out of 20 or 60% passed.
5. Let the number of those who appeared be x. If 62% of those who appeared passed, it means that 38% of those who appeared failed.

   Therefore $\frac{38}{100}x = 513 \Rightarrow x = 1350$

Now let the total number of candidates be y. If 10% of the candidates were absent, it means that 90% of the candidates appeared.

Therefore $\frac{90}{100} \times y = 1350 \quad \Rightarrow y = 1500$

Hence, the total number of candidates enrolled was 1500.

6. Actual profit = Birr 1500 – Birr 1200 = Birr 300.

   Percent profit = $\frac{300}{1200} \times 100\% = 25\%$

7. Loss = Birr 4,200 – Birr 3,000 = Birr 1200

   Percent loss = $\frac{1200}{4200} \times 100\% = 28\frac{4}{7}\%$

8. Total selling price = 200 × 1.75 = Birr 350. Which is less than the cost price. Therefore there is a loss.

   Loss = 400 – 350 = Birr 50

   Percent loss = $\frac{50}{400} \times 100\% = 12\frac{1}{2}\%$

9. Percent reduction = $\frac{450 - 360}{450} \times 100\% = 20\%$

10. Percent increase = $\frac{13.75 - 5.50}{5.50} \times 100\% = 150\%$

11. The previous area = 15cm × 25cm = 375 cm.

    The new dimensions are 15cm × 1.2 = 18cm and 25cm × 1. 2 = 30cm.

    Therefore, the new area = 18cm × 30cm = 540cm^2.

    Hence the percent increase in area = $\frac{540 - 375}{375} \times 100\% = 44\%$

12. Let Birr x be the old price of a kg of teff

    $\Rightarrow$ the old price of 1kg of teff = $\frac{x}{a}$ Birr.

    The new price of 1kg of teff = $(1.1)\frac{x}{a}$ Birr

    $\Rightarrow$ Birr x buys $\frac{a}{1.1}$ kg of teff

    But $\frac{a}{1.1} = (0.909)\, a = 90.9\% \times a$

Therefore, the amount that the Birr x buys is 90.9% of the old amount. That means the consumption must be reduced by 9.1%

13. Let M and F represent the ages of the husband and his wife respectively.

    Then $M = (1.1)\,F \Rightarrow F = \frac{1}{1.1}\,M = (0.909)\,M$

    Which means that the age of his wife is 90.9% of his age. Therefore his wife is 9.1% younger than him.

14. The total amount after 3years will be $200\,(1.04)^3 = 224.97$

    Therefore, the compound interest = 224.97 – 200 = Birr 24.97

15. The income tax to be paid = $\frac{35}{100}(3,000)$ = Birr 1050.

    Therefore, his net income = salary – income tax + allowance

    = (3,000 – 1050) + 250

    = Birr 2200 assuming that the allowance is not taxed.

16. Let the price before VAT be Birr x.

    Then $x + 0.15x = 2{,}800 \Rightarrow x = \frac{2,\ 800}{1.15}$ = Birr 2434. 78

    Therefore, the price of the TV before VAT is Birr 2434.78

    Hence the VAT amount will be Birr 365.22

17. Let the original price be Birr x. Then the price after 20% reduction will be $0.8x$ Birr.

    Therefore, $0.8x = 90 \Rightarrow x = \frac{90}{0.8}$ = Birr 112.50

    Hence the original price was Birr 112.50

18. Selling price = Cost price + profit

    = Birr 250 + $\frac{25}{100}(250)$

    = Birr 312.50

19. Profit = $\frac{18}{100}(80{,}000)$ = Birr 14, 400.

Cost price = selling price – profit = 80,000 – 14,400
= Birr 65,600

20. Let the present population be P.

The annual increment will be 0.05P

Then population increment in t years will be $0.05\text{P} \times t$

The population will double when 0.05 pt = P.

$$\Rightarrow t = \frac{\text{p}}{0.05\text{p}} = \frac{1}{0.05} = 20 \text{ years}$$

Therefore, the population will double in 20 years.

21. If the allowance is not taxed, her monthly salary will be:

income – allowances = 2400 – 150 = Birr 2250.

Therefore the rate of taxation = $\frac{337.5}{2250} = 0.15 = 15\%$

Answers to Exercise 9.3

1. a. Discount = Birr 22, sale price = Birr 198
   b. Discount = Birr 90, sale price = Birr 360
   c. Discount is $14\frac{2}{7}\%$, Discount = Birr 600.
   d. Discount is 25%, sell price = Birr 450
   e. Price = 10,666.67 Birr, Discount Birr 2666.67
2. a. Markup = Birr 20, Rate of markup = 9%
   b. Selling price = Birr 200, Rate of markup = 25%
   c. Cost price = Birr 700, Rate of markup = $42\frac{6}{7}\%$
   d. Selling price = 330 Birr, Mark up = Birr 30.
   e. Cost price = Birr 576, markup = Birr 144
   f. Selling price = Birr 766.67, cost price = Birr 666.67
3. Selling price = cost price + markup = Birr 950 + 0.3 (Birr 950)
   = Birr 1235

4. Net (selling) price = Birr 8000 – 0.15 (Birr 8,000) = Birr 6800
5. Amount of discount = Birr 35 – 0.12 (Birr 35) = Birr 30.80
6. Marked price = $\frac{\text{birr}\,12,\ 000}{0.9}$ = Birr 13, 333.33
7. C is the best deal because:

   The total discount in case A is 53.25%

   The total discount in case B is 52% while

   The total discount in case C is 55%

Assessment

- Give the students various exercise problems on calculations of cost, selling price, markup and markup rate.
  For example, a shop owner purchases a TV for birr 3200 and sells it for birr 3700. What is the cost price, the selling price, the markup and what markup rate does the owner use?
- Give them various exercise problems on calculation of regular price, sale price, discount and discount rate. For example, a newly constructed alternative road reduced the normal 11 hours travel time between two cities by $2\frac{1}{2}$ hours. What percent decrease in the travel time does this represent?
- You can also design assessment questions that are analogous to the problems given in the exercises in the student text book.

9.3 REAL ESTATE EXPENSES

Periods Allotted: 4 periods

Competencies

At the end of this sub-unit, students will be able to:

- *calculate initial expenses of buying a home.*
- *calculate ongoing expenses of owning a home.*

Vocabulary: Mortgage, Installment plan, Down payment, Installment charge, Loan origination fee, Loan appraisal fee, Principal, Interest, Amortization

Introduction

Owning a house is every one's dream. On the other hand, a house is one of the most expensive items that someone purchases. Therefore, many people are able to afford paying for a house by taking a long term loan from a bank. This subunit deals with the details of owning a house through mortgage and installment plan, aiming at enabling the students to identify the different associated terminologies and to calculate the initial and the ongoing expenses of owning a house, using their mathematical knowledge.

Teaching Notes

The lesson on real estate expenses shall start with a careful introduction of the common terminologies associated. Encourage students to discuss these terms by considering different examples. Then, ask them to do Activity 9.3 which again helps to motivate the students to proceed. A clear understanding of these terms will help the students to have a better understanding of the lesson.

Answers to Activity 9.3

Down payment = 20% (795, 615) = Birr 159, 123

Total amount to be financed = Birr 795, 615 – Birr 159,123

= Birr 636, 492

Loan origination fee (3points) = 3% (Birr 636,492) = Birr 19,095

Total closing costs = Birr 32,795

Total amount of mortgage loan = Birr 636, 492 + Birr 32,795

= Birr 669, 287

Next, introduce the ongoing expense of owning a house as the continuing monthly expense involved in owning a house after the down payment and the loan origination payment. Emphasize that the ongoing monthly payment depends on three main factors. They are, **the amount of mortgage loan**, **the interest rate and the loan period**. And then, introduce amortization as a process in which a loan will be paid up completely in a given length of time of equal payments that includes the compound interest, noting that the monthly mortgage payment includes the payment of both the principal and the interest on the mortgage where the interest is charged on the unpaid balance of the loan.

Then, introduce the amortization formula which is given by;

$$P.P = P\left(\frac{i}{1-\left(1+i\right)^{-n}}\right)$$

where $P.P$ = principal payment (usually per month)

P = principal (original loan)

I = interest rate per payment interval

n = number of payments made

Note that: i. If the loan period is 10 years and the payment is made per month, then the number of payments $n = 10 \times 12 = 120$

ii. If the annual interest rate is 6%, then the interest rate per month will be $\frac{6\%}{12} = 0.005$

Finally when you ask the students to do the exercises either as a class work in groups or as a home work, it is better to give question number 5 of Exercise 9.5 for gifted students if there are such students in the class.

Answers to Exercise 9.4

1. $P.P = P\left(\frac{i}{1-(1+i)^{-n}}\right)$ where $i = 0.02$, $p = 95{,}000$, $n = 48$

   Therefore, the monthly payment ($p.p$) = $95{,}000\left(\frac{0\cdot 02}{1-(1\cdot 02)^{-48}}\right)$ = Birr 3097.17

2. $P = 300{,}000$, $i = \frac{3\%}{12} = 0.0025$ and $n = 5 \times 12 = 60$

   Therefore, the monthly payment ($P.P$) = $300{,}000\left(\frac{0\cdot 0025}{1-(1\cdot 0025)^{-60}}\right)$

   = Birr 5390.61

Show to the students how the calculation is simplified by using the attached table. Use examples to illustrate this.

Answers to Exercise 9.5

1. a. We get 0.0066530 corresponding to 30 years and 7%.

   Therefore, $P.P = 80{,}000 \times 0.0066530$ = Birr 532.24

   b. We read 0.0205165 corresponding to 5 years and 8.5 %

   Therefore, $P.P$ (monthly payment) = $150{,}000 \times 0.0205165$

   = Birr 3077.48

2. a. Monthly payment = $20{,}000 \times 0.0084386$ = Birr 168.77

   b. Monthly payment = $160{,}000 \times 0.0073899$ = Birr 1182.38

   c. Monthly payment = $450{,}000 \times 0.0143471$ = Birr 6456.20

   d. Monthly payment = $1{,}000{,}000 \times 0.0080462$ = Birr 8046.20

3. Monthly payment = $450{,}000 \times 0.0107461$ = Birr 4835.75

4. Down payment = 15% (140,000) = Birr 21,000

   Mortgage = 140,000 – 21,000 = Birr 119,000

   Monthly payment = $119{,}000 \times 0.0093213$ = Birr 1109.23

5. $P.P = 2500 \times r = 82.44 \Rightarrow r = 0.032976$

   Because 36 months = 3 years, we look for 0.032976 on the table in the row containing year 3.

   Thus, we find 0.032976 in the column of 11.5%.

   Therefore, the rate she received was 11.5%

Assessment

- Ask the students oral questions concerning purchase price, down payment, mortgage, loan origination fee and ongoing expenses.
- Give them exercise problems on down payment, mortgage and loan origination fee and check the answers.
- Give them exercise problems on calculation of ongoing expenses of owning a house. For example, Bikila purchased a house for birr 350,000 and made a down payment of birr 52,500. The bank charges an annual interest rate of 9% on Bikila's 30 years mortgage. Find the monthly mortgage payment.

9. 4 WAGES

Periods Allotted: 4 periods

Competency

At the end of this sub-unit, students will be able to:

- *calculate commissions, total hourly wages, and salaries*

Vocabulary: Commission, Wage, Salary, Regular pay, Overtime pay, Gross pay.

Introduction

People work to earn money for their living and for many other reasons. Payments for work are also made in different ways. This subunit introduces the most common ways of getting payments for different types of work so that students will be able to calculate different payments using their knowledge of mathematics. As a result they will be able to decide on their choice whenever different payments are available.

Teaching Notes

You can start the lesson by explaining the three ways of receiving payments for doing work. Namely, commissions, hourly payments and salary. Then encourage students to discuss these three payments by raising their own examples. You may then ask the students how they understand these payments.

Then extend the discussion by raising the different ways of receiving salaries such as weekly, every other week or monthly and also raise an example of a commission by considering a broker who receives a commission on the selling price of a house or a car based on percent. After discussing different examples including those in the student's text with active participation of the students, you may ask them to do the Exercises 9.6 and 9.7 one after the other, as convenient, to be done in class in groups or as a homework. The answers given by the students shall be checked.

Answers to Exercise 9.6

1. Commission = 6% (140,000) = Birr 8400
2. Commission = 15% (2,000) + 5% (4,600 – 2,000)

   = 300 + 130 = Birr 430
3. Commission = 10% (150,000) + 15% (50,000) + 20% (340,000 – 200,000)

   = 15,000 + 7,500 + 28,000 = Birr 50,500
4. Week's salary = basic weekly salary + commission

   = 400 + 5 % (50,000) + 10 % (80,000 – 50,000)

   = 400 + 2,500 + 3,000 = Birr 5900
5. Commission earned by Abdulaziz = 3% (521, 780.00)

   = Birr 15,653.40

Commission earned by Yohannes

$= 3\% \,(600{,}000) + 5\% \,(814{,}110.90 - 600{,}000.00)$

$= 18{,}000 + 10{,}705.55$

$=$ Birr 28,705.55

Commission earned by Sheriff

$= 3\% \,(600{,}000) + 5\% \,(400{,}000) + 7.5\% \,(500{,}000)$

$= 18{,}000 + 20{,}000 + 37{,}500$

$=$ Birr 75,500.00

Commission earned by Elias $= 3\% \,(600{,}000) + 5\% \,(386{,}352.2)$

$= 18{,}000 + 19{,}317.61$

$=$ Birr 37,317.61

Answers to Exercise 9.7

1. $30\frac{1}{4} \times 20.75 =$ Birr 627. 70

2. $604.50 \div 32\frac{1}{2} = 18.6$

   Therefore the rate is Birr 18.60 per hour.

3. Naomi's gross pay = Birr 1020.00

   Genet worked for 5 hours on Thursday.

   Ayantu's hourly rate was Birr 25.00

4. Gross pay $= (40 \times 20) + \frac{1}{2}(30) + 4(40) =$ Birr 975.00

5. Annual salary $= 450 \times \frac{52}{2} =$ Birr 11,700.00

Answers to Review Exercises on Unit 9

1.	a.	Birr 12.00	b.	24 pencils		
2.		Birr 83.10				
3.	a.	Birr 25.00	b.	Birr 30.00	c.	20%

4. 7.5%

5. He earns Birr 38.75

6. Wassihun's salary = Birr 1920.00

7. 35.71%

8. The wholesaler pays Birr 2400.00

   An electrician pays Birr 2800.00

   A consumer pays Birr 3400.00

9. His payment per month will be: Birr 1330.6

10.

Employee	Total hrs	Regular hrs	Overtime hrs	Regular wage	over wage	Gross pay
Abdissa	48	40	8	320	96	**416**
Takeste	36	36	-	288	-	**288**
Guji	37	37	-	296	-	**296**
Gizachew	**50**	**40**	**10**	**320**	**120**	**440**

11. His bill = $(200 \times 0.40) + 8.00 + 15\% \, (88.00)$ = Birr 101.20

12. a. 600g for Birr 25.00 is a better buy

    b. 200g pasta plus 20% extra for Birr 18.70 is a better buy.

REFERENCE MATERIALS:

These days search for a reference is at forefront with authentic supply of electronic references. However, with the assumption that there will be limitations in some parts to over utilize ICT, some hard copy reference materials are listed here that can help develop better learning and teaching of mathematics and these units. These books are selected assuming that they are available in many schools or at their surrounding libraries. For those who have access to the internet, e-resources are offered as a supplement to those hard copies, if not essentially preferred. You can also access additional reference materials that are available in your school library. These are simply guides to help you use them as references. However, they are not the only to be prescribed. You can also use the web sites given here for reference and demonstration.

A. Dawson and R. Parsons (1990). Understanding pure and applied mathematics II for advanced level.

A.W.Bowman et al (1987). Introduction to Statistics: A Computer Illustrated Text. IOP publishing Limited, London, UK.

Demissu Gemeda and Yismaw Alemu (1996). Basic Mathematics for Business and Social Sciences, Addis Ababa University Press, Addis Ababa.

E. Johnston & J. Mathews (1996). Calculus for Engineering and the Sciences. Preliminary version volume 1. Harper Collins College publishers.

George Woodbury (2002). An Introduction to Statistics. Cengage Learning, BUXBURY, Thomson Learning, Canada.

Howard Anton (1987). Calculus. John Wiley & Sons, Inc. USA.

Kinfegebriel Dessalegn and Zenebe Deneke (1999). The New Guide Preparatory Program Aster Nega Publishing Enterprise, Addis Ababa.

_________ (2001). Alpha problem solver series mathematics: For Preparatory Program Aster Nega Publishing Enterprise, Addis Ababa.

K.M.Ramachandran (2009). Mathematical Statistics with Applications, Academic Press, USA

L Bostock, et al (1996). GCSE Higher Mathematics. Stanley Thornes (publishers) LTD, Great Britain.

____________ (1995). Pure Mathematics I. Stanley Thornes (publishers) LTD, Great Britain

____________ (1995). Pure Mathematics II. Stanley Thornes (publishers) LTD, Great Britain

R.E. Larson and R.P. Hostetler (1991). Precalculus, 4th Ed.

R. J. Larsen (2009). An Introduction to Mathematical Statistics and Its Application, 4th Ed. Prinentice Hall, USA

Robert Ellis and Denny Gulick (1986).Calculus With Analytic Geometry. Third Edition. Harcourt Brace Jovanovich, Inc. USA.

W.A. Wallis et al (2009). Statistics: A New Approach. The Free Press, New York, USA.

http://www.coolmath.com

http://www.mhhe.com

http://www.hot.sra.edu/~matsc

http://www.aaamath.com

http://www.homepage.mac.com

http://www.wordmath.com

http://www.geometersketchpad.com

http://www.purplemath.com/modules/linprog.htm

Table of Monthly Payment

	Annual Interest Rate															
Yrs	6.0%	6.5%	7.0%	7.5%	8.0%	8.5%	9.0%	9.5%	10.0%	10.5%	11.0%	11.5%	12.0%	12.5%	13.0%	13.5%
1	0.0860664	0.0862964	0.0865268	0.0867574	0.0869884	0.0872198	0.0874515	0.0876835	0.0879159	0.0881486	0.0883817	0.0886151	0.0888488	0.0890829	0.0893173	0.0895520
2	0.0443206	0.0445463	0.0447726	0.0449959	0.0452273	0.0454557	0.0456847	0.0459145	0.0461449	0.0463760	0.0466078	0.0468403	0.0470735	0.0473073	0.0475418	0.0477770
3	0.0304219	0.0306490	0.0308771	0.0311062	0.0313364	0.0315675	0.0317997	0.0320330	0.0322672	0.0325024	0.0327387	0.0329760	0.0332143	0.0334536	0.0336940	0.0339353
4	0.0234850	0.0237150	0.0239462	0.0241789	0.0244129	0.0246483	0.0248850	0.0251231	0.0253626	0.0256034	0.0258455	0.0260890	0.0263338	0.0265800	0.0268275	0.0270763
5	0.0193328	0.0195662	0.0198001	0.0200380	0.0202769	0.0205165	0.0207584	0.0210019	0.0212470	0.0214939	0.0217424	0.0219926	0.0222445	0.0224979	0.0227531	0.0230099
6	0.0165729	0.0168099	0.0170490	0.0172901	0.0175332	0.0177784	0.0180255	0.0182747	0.0185258	0.0187790	0.0190341	0.0192912	0.0195502	0.0198112	0.0200741	0.0203390
7	0.0146086	0.0148494	0.0150927	0.0153383	0.0155862	0.0158365	0.0160891	0.0163440	0.0166012	0.0168607	0.0171224	0.0173865	0.0176527	0.0179212	0.0181920	0.0184649
8	0.0131414	0.0133862	0.0136337	0.0138839	0.0141367	0.0143921	0.0146502	0.0149109	0.0151742	0.0154400	0.0157084	0.0159794	0.0162528	0.0165288	0.0168073	0.0170882
9	0.0120058	0.0122545	0.0125063	0.0127610	0.0130187	0.0132794	0.0135429	0.0138094	0.0140787	0.0143509	0.0146259	0.0149037	0.0151842	0.0154676	0.0157536	0.0160423
10	0.0111021	0.0113548	0.0116109	0.0118702	0.0121328	0.0123986	0.0126676	0.0129398	0.0132151	0.0134935	0.0137750	0.0140595	0.0143471	0.0146376	0.0149311	0.0152274
11	0.0103670	0.0106238	0.0108841	0.0111480	0.0114155	0.0116864	0.0119608	0.0122387	0.0125199	0.0128045	0.0130924	0.0133835	0.0136779	0.0139754	0.0142761	0.0145799
12	0.0097585	0.0100192	0.0102838	0.0105523	0.0108245	0.0111006	0.0113803	0.0116637	0.0119508	0.0122414	0.0125356	0.0128332	0.0131342	0.0134386	0.0137463	0.0140572
13	0.0092472	0.0095119	0.0097807	0.0100537	0.0103307	0.0106118	0.0108968	0.0111857	0.0114785	0.0117750	0.0120753	0.0123792	0.0126867	0.0129977	0.0133121	0.0136299
14	0.0088124	0.0090810	0.0093540	0.0096314	0.0099132	0.0101992	0.0104894	0.0107837	0.0110820	0.0113843	0.0116905	0.0120006	0.0123143	0.0126317	0.0129526	0.0132771
15	0.0084386	0.0087111	0.0089883	0.0092701	0.0095565	0.0098474	0.0101427	0.0104423	0.0107461	0.0110540	0.0113660	0.0116819	0.0120017	0.0123252	0.0126524	0.0129832
16	0.0081144	0.0083908	0.0086721	0.0089583	0.0092493	0.0095449	0.0098452	0.0101499	0.0104590	0.0107724	0.0110900	0.0114117	0.0117373	0.0120667	0.0123999	0.0127367
17	0.0078310	0.0081112	0.0083966	0.0086871	0.0089826	0.0092829	0.0095880	0.0098978	0.0102121	0.0105308	0.0108538	0.0111810	0.0115122	0.0118473	0.0121862	0.0125287
18	0.0075816	0.0078656	0.0081550	0.0084497	0.0087496	0.0090546	0.0093645	0.0096791	0.0099984	0.0103223	0.0106505	0.0109830	0.0113195	0.0116600	0.0120043	0.0123523
19	0.0073608	0.0076486	0.0079419	0.0082408	0.0085450	0.0088545	0.0091690	0.0094884	0.0098126	0.0101414	0.0104746	0.0108122	0.0111539	0.0114995	0.0118490	0.0122021
20	0.0071643	0.0074557	0.0077530	0.0080559	0.0083644	0.0086782	0.0089973	0.0093213	0.0096502	0.0099838	0.0103219	0.0106643	0.0110109	0.0113614	0.0117158	0.0120738
21	0.0069886	0.0072836	0.0075847	0.0078917	0.0082043	0.0085224	0.0088458	0.0091743	0.0095078	0.0098460	0.0101887	0.0105358	0.0108870	0.0112422	0.0116011	0.0119637
22	0.0068307	0.0071294	0.0074342	0.0077451	0.0080618	0.0083841	0.0087117	0.0090446	0.0093825	0.0097251	0.0100722	0.0104237	0.0107794	0.0111390	0.0115023	0.0118691
23	0.0066885	0.0069907	0.0072992	0.0076139	0.0079345	0.0082609	0.0085927	0.0089297	0.0092718	0.0096187	0.0099701	0.0103258	0.0106857	0.0110494	0.0114168	0.0117876
24	0.0065598	0.0068654	0.0071776	0.0074961	0.0078205	0.0081508	0.0084866	0.0088278	0.0091739	0.0095248	0.0098803	0.0102400	0.0106038	0.0109715	0.0113427	0.0117173
25	0.0064430	0.0067521	0.0070678	0.0073899	0.0077182	0.0080523	0.0083920	0.0087370	0.0090870	0.0094418	0.0098011	0.0101647	0.0105322	0.0109035	0.0112784	0.0116565
26	0.0063368	0.0066492	0.0069684	0.0072941	0.0076260	0.0079638	0.0083072	0.0086560	0.0090098	0.0093683	0.0097313	0.0100984	0.0104695	0.0108443	0.0112224	0.0116038
27	0.0062399	0.0065556	0.0068772	0.0072073	0.0075428	0.0078842	0.0082313	0.0085836	0.0089410	0.0093030	0.0096695	0.0100401	0.0104145	0.0107925	0.0111738	0.0115581
28	0.0061512	0.0064702	0.0067961	0.0071287	0.0074676	0.0078125	0.0081630	0.0085188	0.0088796	0.0092450	0.0096148	0.0099886	0.0103661	0.0107471	0.0111313	0.0115185
29	0.0060701	0.0063921	0.0062130	0.0070572	0.0073995	0.0077477	0.0081016	0.0084607	0.0088248	0.0091934	0.0095663	0.0099431	0.0103236	0.0107074	0.0110943	0.0114841
30	0.0059955	0.0063207	0.0066530	0.0069922	0.0073377	0.0076891	0.0080462	0.0084085	0.0087757	0.0091474	0.0095232	0.0099029	0.0102861	0.0106726	0.0110620	0.0114541

Random Number Table

13962	70992	65172	28053	02190	83634	66012	70305	66761	88344
43905	46941	72300	11641	43548	30455	07686	31840	03261	89139
00504	48658	38051	59408	16508	82979	92002	63606	41078	86326
61274	57238	47267	35303	29066	02140	60867	39847	50968	96719
43753	21159	16239	50595	62509	61207	86816	29902	23395	72640
83503	51662	21636	68192	84294	38754	84755	34053	94582	29215
36807	71420	35804	44862	23577	79551	42003	58684	09271	68396
19110	55680	18792	41487	16614	83053	00812	16749	45347	88199
82615	86984	93290	87971	60022	35415	20852	02909	99476	45568
05621	26584	36493	63013	68181	57702	49510	75304	38724	15712
06936	37293	55875	71213	83025	46063	74665	12178	10741	58362
84981	60458	16194	92403	80951	80068	47076	23310	74899	87929
66354	88441	96191	04794	14714	64749	43097	83976	83281	72038
49602	94109	36460	62353	00721	66980	82554	90270	12312	56299
78430	72391	96973	70437	97803	78683	04670	70667	58912	21883
33331	51803	15934	75807	46561	80188	78984	29317	27971	16440
62843	84445	56652	91797	45284	25842	96246	73504	21631	81223
19528	15445	77764	33446	41204	70067	33354	70680	66664	75486
16737	01887	50934	43306	75190	86997	56561	79018	34273	25196
99389	06685	45945	62000	76228	60645	87750	46329	46544	95665
36160	38196	77705	28891	12106	56281	86222	66116	39626	06080
05505	45420	44016	79662	92069	27628	50002	32540	19848	27319
85962	19758	92795	00458	71289	05884	37963	23322	73243	98185
28763	04900	54460	22083	89279	43492	00066	40857	86568	49336
42222	40446	82240	79159	44168	38213	46839	26598	29983	67645
43626	40039	51492	36488	70280	24218	14596	04744	89336	35630
97761	43444	95895	24102	07006	71923	04800	32062	41425	66862
49275	44270	52512	03951	21651	53867	73531	70073	45542	22831
15797	75134	39856	73527	78417	36208	59510	76913	22499	68467
04497	24853	43879	07613	26400	17180	18880	66083	02196	10638
95468	87411	30647	88711	01765	57688	60665	57636	36070	37285
01420	74218	71047	14401	74537	14820	45248	78007	65911	38583
74633	40171	97092	79137	30698	97915	36305	42613	87251	75608
46662	99688	59576	04887	02310	35508	69481	30300	94047	57096
10853	10393	03013	90372	89639	65800	88532	71789	59964	50681
68583	01032	67938	29733	71176	35699	10551	15091	52947	20134
75818	78982	24258	93051	02081	83890	66944	99856	87950	13952
16395	16837	00538	57133	89398	78205	72122	99655	25294	20941
53892	15105	40963	69267	85534	00533	27130	90420	72584	84576
66009	26869	91829	65078	89616	49016	14200	97469	88307	92282
45292	93427	92326	70206	15847	14302	60043	30530	57149	08642
34033	45008	41621	79437	98745	84455	66769	94729	17975	50963
13364	09937	00535	88122	47278	90758	23542	35273	67912	97670
03343	62593	93332	09921	25306	57483	98115	33460	55304	43572
46145	24476	62507	19530	41257	97919	02290	40357	38408	50031
37703	51658	17420	30593	39637	64220	45486	03698	80220	12139
12622	98083	17689	59677	56603	93316	79858	52548	67367	72416
56043	00251	70085	28067	78135	53000	18138	40564	77086	49557
43401	35924	28308	55140	07515	53854	23023	70268	80435	24269
18053	53460	32125	81357	26935	67234	78460	47833	20496	35645

MINIMUM LEARNING COMPETENCIES (MLCs)

No	Content	MINIMUM LEARNING COMPETENCIES (MLCs)
1	**ALGEBRA** **Mathematical Applications in Business**	◊ ***For social science stream only*** • find unit cost • find the most economical purchase • find total cost • apply percent increase and percent decrease to business • apply percent increase and percent decrease to business • calculate initial expenses of buying a home • calculate ongoing expenses of owning a home • calculate commissions, total hourly wages, and salaries
2	**RELATION AND FUNCTION** **Sequences and Series**	• revise the notion of sets and functions • explain the concepts sequence, term of a sequence, rule (formula of a sequence) • compute any term of a sequence using rule(formula) • draw graphs of finite sequences • determine the sequence, use recurrence relations such as, $u_{n+1} = 2\mathbf{u_n} + 1$,given $\mathbf{u_1}$ • generate the Fibonacci sequence and investigate its uses, appearance in real life • define arithmetic progressions and geometric progressions • determine the terms of arithmetic and geometric sequences • use the sigma notation for sums • compute partial sums of arithmetic and geometric progressions • apply partial sum formula to solve problems of science and technology • define a series • decide whether a given geometric series is divergent or convergent • show how infinite series can be divergent or convergent • show how recurring decimals converge • discuss the applications of arithmetic and geometric progressions (sequences) and series in science and technology and daily life
3	**LOGIC**	◊***For Natural science stream only*** • recall what they have studied about statements and logical connectives in the previous grade • revise open statement • understand the concept of quantifiers • determine truth values of statements with quantifiers. • define argument and validity • check the validity of a given argument • use rules of inference to demonstrate the validity of a given argument • distinguish between the nature of different types of mathematical proofs • apply the right type of proof to solve the required problem • apply the principle of mathematical induction for proving • identify a problem and determine whether it could be proved using principle of mathematical induction or not

4	**STATISTICS AND PROBABILITY**	◊***For social science stream only*** • describe the three methods/techniques of sampling • explain the advantages and limitation of each techniques of sampling • describe the different ways of representations of data • explain the purpose of each representation of data • Construct graphs of statistical data • identify statistical graph • explain the significance of representing a given data in different types of graphs • draw histogram for a given frequency distribution • Sketch frequency polygon for a given frequency distribution • sketch frequency curve for a given frequency distribution • draw bar chart • construct line graph for data related to time • construct pie chart for a given data • compute the three mean divations of a given data • describe the relative significance of Mean divation as a measure of dispersion • calculate the inter-quartile range for a given data • describe inter-quartile range as a measure of variability in values of a given set of data • describe the usefulness of standard deviation in interpreting the variability of a given data • compare two groups of similar data • determine the consistency of two similar group of data with equal mean but different standard deviations • describe the application of coefficient of variation in comparing two groups of similar data • describe the relationship among mean, median and mode for grouped data by using its frequency curve. • use cumulative frequency graphs to determine the dispersion of values of data (in terms of its Mean, Median and Standard deviation) • determine the variability of value of data in terms of quartiles by using cumulative frequency graph • describe the relationship among mean, median and mode for grouped data by using its frequency curve
5	**CALCULUS** **Limits of sequence of numbers**	• define upper and lower bound of number sequences • find out the least upper (greatest lower) bound of sequences • define limit of a number sequence • consolidate their knowledge on the concept of sequences stressing on the concept of null sequence • apply theorems on the convergence of bounded sequences • prove theorem about the limit of the sum of two convergent sequences • apply theorems on the limit of the difference, product, quotient of two convergent sequences • define limit of a function

		• determine the limit of a given function at a point • find out the limit of the sum, difference, product and quotient of two functions • define continuity of a function in interval • describe the properties of continuous functions • use properties of continuous functions to determine the continuity of various functions • consolidate what they have studied on limits • solve problems on limit and continuity to stabilize what have learnt in the unit
	Introduction to Differential Calculus	• find the rate of change of one quantity with respect to another • sketch different straight line and curved graphs and find out slopes at different points of each graph • define differentiability of a function at a point x_0 • explain the geometrical and mechanical meaning of derivative • set up the equation of tangent line at the point of tangency, using the concept of derivative • find the derivative of elementary functions over an interval • find the derivatives of power, simple trigonometric, exponential and logarithmic functions • apply the sum and difference formulae of differentiation of functions • apply the product and quotient formulae of differentiation of functions • apply the chain rule and differentiate composition of functions • find the 2^{nd} and the n^{th} derivative of a function • consolidate and stabilize what has been studied in the unit
	Application of Differential Calculus	• consolidate the concept zero(s) of a function • find critical numbers and maximum and minimum values of a function on a closed interval • explain the geometric interpretations of Rolle's theorem and mean value theorem • find numbers that satisfy the conclusions of mean value theorem and Rolle's theorem • solve problems on application of differential calculus • interpret and apply differential calculus on problems • involving the rate of change • consolidate what has been learnt in this unit
	Introduction to Integral Calculus	• differentiate between the concepts differentiation and integration • use the properties of indefinite integrates in solving problems of integration • integrate simple trigonometric functions • use different techniques of integration for computation of integrals • compute area under a curve • use the concept of definite integral to calculate the area under a curve • state fundamental theorem of calculus

		• apply fundamental theorem of calculus to solve integration problems • state the properties of definite integrals • apply the properties of definite integrals for computations of integration • apply the knowledge on integral calculus to solve problems
6	**GEOMETRY** **Coordinate Geometry and Vectors**	◊***For Natural science stream only*** • construct the coordinate axes in space • identify planes determined by the axes in space • identify the octants determined by the planes and axes • read the coordinates of a point in space • describe how to locate a point in space • plot a point whose coordinates are given • give the equations for the planes determined by the axes • show graphically how to find the distance between two points in space • compute distance between two given points in space • determine coordinates of the mid-point of a segment in space • describe the equation of a sphere • derive equation of a sphere • solve problems related with sphere • add, subtract vectors and multiply by a scalar in space • use the unit vectors **i**, **j** and **k** while representing a vector • describe the properties of addition to solve exercise problems • show the closure property on their own • find the length of a vector in space • find the scalar product of two vectors in space • evaluate and show the angle between two vectors in space

Federal Democratic Republic of Ethiopia

Ministry of Education

Mathematics Syllabus

Grade 12

2009

General Introduction

Mathematics learning at the second cycle of secondary education (Grades 11 and 12) should contribute to the students' growth into good, balanced and educated individuals and members of society. At this cycle, they should acquire the necessary mathematical knowledge and develop skills and competencies needed in their further studies, working life, hobbies, and all-round personal development. Moreover the study of mathematics at this level shall significantly contribute to the students' lifelong learning and self-development throughout their lives. These aims can be realized by closely linking mathematics learning with daily life, relating theory with practice; paying attention to the practical application of mathematical concepts, theorems, methods and procedures by drawing examples from the fields of agriculture, industry and sciences like physics, chemistry and engineering.

Mathematics study in grades 11 and 12 should be understood as the unity of imparting knowledge, developing abilities and skills and forming convictions, attitudes and habits. Therefore, the didactic-methodical conception has to contribute to all these sides of the educational process and to consider the specifics of students' age, the function of the secondary school level in the present and prospective developmental state of the country, the pre-requisites of the respective secondary school and the guiding principles of the subject mathematics.

- In determining the general methodical approach of topics or special teaching methods for single periods, due consideration has always to be given to the **orientation of the main objectives**:
  - acquisition of solid knowledge on mathematical concepts, theorems, rules and methods assigned in the syllabus.
  - acquisition of reliable capability in working with this knowledge more independently in the field of problem solving.

The main activity for achieving these objectives includes engaging student in mental activity during classroom learning.

- Teaching has to consider students' interest that is related to their range of experience, actual events of the country and local reality so as to help them answer questions originating from daily life.
- Problem solving is a suitable means for engaging students in mental activity. This has to be understood as a complex process, including the activities of the teachers and students. A teacher can be engaged in selecting or arranging the problems, planning their use in the classroom and organizing the process of solving problems while the students are engaged in solving the problems and in checking the results gained.
- While planning and shaping classroom learning the teacher has to observe that the application of newly acquired knowledge and capability is a necessary part in the process of complete recognition and solid acquisition of the new subject matter. Thus, application has to be carried out during presentation and stabilization.
- Introduction to a new topic and presentation of new subject matter have to be carried out using knowledge and experiences of students by encouraging students to actively

participate in the teaching learning process by familiarizing students with the new subject matter and help them to understand and appreciate its use.

- Stabilization has to be understood as the fundamental process of mathematics learning. It has to be regarded as a principle of shaping in all stages of teaching and as precondition for mental activity of students and the enhancement of capabilities applying knowledge more independently.
- Within the total process of stabilization, exercising (in relation with revision, deepening and systematization) hold a central position. In mathematics, exercising is to be understood in a wider sense. In the first place, it is aimed at the formation of skills; but it is also oriented towards fixing knowledge (including subject matter dealt with previously) and habituating to certain modes of working and practicing behavior. Furthermore last exercising has to facilitate the development of definite strategies of problems solving, being relatively independent of subject matter. Shaping of exercises has to be concentrated on assigning sufficient time, analyzing the real performances of students overcoming weak points of knowledge and capability, using all fundamental forms of exercises (daily activity, miscellaneous and complex exercises) systematically.
- Under the aspect of Students' preparation for Tertiary Education, it is necessary to prepare them step by step for mastering these demands. So, in shaping the teaching-learning process priority has to be given to methods which promote students' activities of cognition and reduce their mechanical rote learning. Students have to be asked, for instance, to report ways of solving a problem they have used with explanation and reason. Students have to be acquainted with forms of cooperative work between peer groups, with the application of the deductive approach, with preparation of project papers, with seminary instruction and discussion forums about special themes.
- The teacher has to observe the following peculiarities of grades 11 and 12 (as compared with mathematics learning of grades 9 and 10).
    - deeper penetration into modern and general mathematical theories,
    - higher level of abstraction and generalization,
    - higher demands with regard to logical strictness in the treatment of subject matter and the exactness of mathematical language (including terminology and symbolism)
    - closer relations to neighboring disciplines and ranges of application, especially to Physics, Technology and Agriculture.
    - the time allotment for grades 11 and 12 is made for 33 weeks (165 periods). The remaining weeks have to be used for revision, systematization and evaluation.
    - units 1, 2, 3, 4, 5, 6 and 7 of Grades 11 and units 1, 2, 3, 4 and 5 of Grades 12 are common to both natural science and social science stream students, while units 8 and 9 of Grades 11 and units 6 and 7 of Grades 12 are to be offered only to natural science stream students and units 10and 11of Grades 11 and units 8 and 9 of Grades 12 are only for social science stream students.

Cycle Objectives

Objectives of Mathematics Learning in the Second Cycle of Secondary Education (Grade12)

At the end of the second cycle of secondary education, students should be able to:

- apply the mathematical knowledge and capabilities gained to solve problems independently.
- develop mental abilities and high skills and competencies in calculations, especially, in the field of logical thinking, reasoning, proving, defining and use of mathematical language, terminologies and symbols correctly.
- develop an appreciation for the importance of mathematics as a field of study by learning its historical development, scope and its relationship with other disciplines.
- develop scientific outlook and personality characteristics such as working activities with algorithms, exactness, neatness, honesty and carefulness according to self-prepared plans for solving problems in line with the needs of the society.

Allotment of Periods
For Units and Sub-units of Mathematics Grade 12

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
Unit 1: Sequence and Series	1.1 Sequences (sequences of numbers)	3	18
	1.2 Arithmetic sequence and Geometric sequence.	3	
	1.3 The sigma notation (Σ) and partial sums.	6	
	1.4 The notion of "Infinite series"	4	
	1.5 Application of Sequence and series	2	
Unit 2: Introduction to Limits and Continuity	2.1 Limits of sequence of numbers	12	28
	2.2 Limits of functions	6	
	2.3 Continuity of a function	5	
	2.4 Exercises on the application of limits	3	
	2.5 Miscellaneous exercises	2	
Unit 3: Introduction to Differential Calculus	3.1 Introduction to Derivatives • understanding rates of change • Graphical definition of derivative • Formal definition (Differentiability at a point) • Differentiability over an interval	10	27
	3.2 Derivatives of different functions. • Differentiation of power, simple trigonometric, exponential and logarithmic functions.	3	
	3.3 Derivatives of combinations and compositions of functions	12	
	3.4 Miscellaneous exercise	2	
Unit 4: Applications of Derivatives	4.1 Extreme values of functions	13	25
	4.2 Minimization and maximization problems	6	
	4.3 Rate of change	6	
Unit 5: Introduction to Integral Calculus	5.1 Integration as inverse process of differentiation • Integral of: - Constant - Power - Trigonometric - Exponential and logarithmic functions	7	30
	5.2 Techniques of integration • Elementary substitution	9	

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
	• Partial fractions • Integration by parts 5.3 Definite integrals, area and fundamental theorem of calculus 5.4 Applications of integral calculus	 8 6	
Unit 6: Three Dimensional Geometry and Vectors in Space	**(Topics for Natural Science Stream only)** 6.1 Coordinate axes and coordinate planes in space 6.2 Coordinates of a point in space 6.3 Distance between two points in space 6.4 Mid - point of a segment in space 6.5 Equation of sphere 6.6 Vectors in space	 2 2 2 1 2 8	17
Unit 7: Mathematical Proofs	**(Topics for Natural Science Stream only)** 7.1 Revision on logic 7.2 Different types of proofs 7.3 Principles and applications of mathematical induction 7.4 General exercises	5 4 4 2	15
Unit 8: Further on Statistics	**(Topics for Social Science Stream only)** 8.1 Sampling techniques 8.2 Representation of data 8.3 Construction of graphs and interpretation 8.4 Measures of central tendency and variation of a set of data, including grouped data. (Mean, Median, Mode, Range, Inter quartile rang and Standard deviation from the data itself or from given totals) 8.5 Analysis of frequency distributions with equal means but different variances (coefficients of variation). 8.6 Use of cumulative frequency graph to estimate median value, quartile and inter quartile range of a set of data.	 3 2 6 5 2 4	22
Unit 9: Mathematical Applications for Business and Consumers	**(Topics for Social Science Stream only)** 9.1 Applications to purchasing 9.2 Percent increase and percent decrease 9.3 Real estate expenses 9.4 Wages	 3 4 4 4	15

Introduction

Mathematics study at Grade 12 level is mainly aimed at exposing students to higher mathematical knowledge and competencies necessary to enable them peruse with their higher education. The first part, which is common to both natural science and social science streams students is introduction to calculus where the basic concepts of differential and integral calculus are introduced with intuitive explanations and examples followed by formal definitions. Very important theorems which are essential to the development of the subject are stated carefully with illustrative examples without their proofs. It is believed that this is sufficient to enable the student to grasp the contents and importance of these theorems and apply them intelligently. The second part is stream specific, where each of the two streams will have two special units (three dimensional geometry, vectors in space and mathematical proof for natural science stream students, whereas further on statistics and application for business and consumers for social science stream students). No one can master any branch of mathematics without much practice in problem solving, and hence it is essential that students are encouraged and assisted to attempt all of the given exercise problems.

Objectives

At Grade 12 level, students should be able to:

- apply the knowledge and capability gained to solve problems independently.
- use high skills in calculations.
- work with algorithms, according to plans for problem - solving and use methods of self - checking.
- develop mental abilities, especially in the field of logical reasoning, proving, defining and using mathematical language correctly.
- work activities with exactness and neatness with respect to the above general outcomes, the following grade specific outcomes are expected at the end of learning Grade 12 mathematics.

Students should be able to:

- be familiar with number sequences, arithmetic and geometric sequences and partial sums of number sequences.
- develop competences and skills in computing any term of a number sequence and also find out possible rules from given terms.
- apply the knowledge of sequences and series to solve practical and real life problems.
- perform examinations for convergence of number sequences and determine respective limit with the help of the studied laws for limits.
- determine simple cases of limits of a function at a finite point.
- determine the differentiability of a function at a point.
- find the derivatives of some selected functions over given intervals.

- find the second and n^{th} derivatives of power, polynomial and rational functions.
- make use of differential calculus to find out local/absolute maximum and minimum of a function.
- apply differential calculus in solving maximization and minimization problems.
- use their knowledge on differential calculus to solve problems involving rate of change.
- integrate different polynomial, simple trigonometric, exponential and logarithmic functions.
- apply the knowledge of integral calculus to solve real life mathematical problems.
- apply facts and principles about coordinates in space to solve related problems.
- evaluate and show the angle between two vectors in space.
- develop the knowledge of logic and logical connectives.
- apply the principle of mathematical induction for a problem that needs to be proved inductively.
- construct and interpret statistical graphs.
- compute the three mean deviations of a given data.
- describe the relative significance of mean deviation as a measure of dispersion.
- determine the consistency of two similar groups of data with equal mean but different standard deviations.
- describe the relationship among mean, median and mode for grouped data.
- find unit cost, the most economical purchase and the total cost.
- apply percent decrease to business.
- calculate the initial expenses of buying a home and ongoing expenses of owing a home.

Unit 1: Sequences and Series *(18 periods)*

Unit outcomes: Students will be able to:

- revise the notion of sets and functions.
- grasp the concept of sequence and series.
- compute any term of sequences from given rule.
- find out possible rules (formulae) from given terms.
- identify the types of sequences and series.
- compute partial and infinite sums of sequences.
- apply the knowledge of sequence and series to solve practical and real life problems.

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***1. Sequences and Series*** **1.1. Sequences** *(3 periods)*		
• revise the notion of sets and functions.	• Revision on Sets and Functions	• Start the lesson by revising the concepts of sets and functions (relations, symbols, graphs) using different examples in a form of discussion with active participation of students.	• Ask questions to revise 'sets' and 'functions'
• explain the concepts sequence, term of a sequence, rule (formula of a sequence)	• Number sequence	• Define "number sequence" as a special function whose domain is the set of natural numbers.	• Give exercise problems on sequences as class and home works and check solutions.
• compute any term of a sequence using rule (formula). • draw graphs of finite sequences.		• Introduce "term of a sequence, n^{th} term of a sequence" (rule (formula) of a sequence), finite and infinite sequences, and graphs of finite number sequences with the active participation of students giving enough activities.	• corrections are given based on the feedback from students.
• determine the sequence, use recurrence relations such as $u_{n+1} = 2\mathbf{u_n} + 1$ given $\mathbf{u_1}$ • generate the Fibonacci sequence and investigate its uses (application) in real life.	• Recurrence relations (used in numerical methods) • Fibonacci sequence (1200AD and first man to create western number system)	• Let students exercise using different examples such as: $u_{n+1} = 2\mathbf{u_n} + \mathbf{1}$, $u_1 = 3$, $n \geq 1$ $u_2 = 2u_1 + 1 = 7$, $u_3 = 15$, hence 3, 7, 15... $\mathbf{u_{n+1}} = \mathbf{u_n} + \mathbf{u_{n-1}}$, $\mathbf{u_1} = 1$, $\mathbf{u_2 = 2}$ thus $u_3 = 3$, $u_4 = 3 + 2 = 5$ sequence is 1,2,3,5,8,13, ...	• Let students write many sequences and series and accordingly use the formulae to get the n^{th} term of the sequence.

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
• define arithmetic progressions and geometric progressions. • Determine the terms of arithmetic and geometric sequences	**1.2.Arithmetic Sequence and Geometric Sequence** *(3 periods)*	• Define "Arithmetic progressions $\{A_k\}$ and geometric progressions $\{G_k\}$" • Derive and introduce the k^{th} term $A_k = A_1+(k - 1)d$ and $G_k= G_1r^{k-1}$ of an arithmetic progression and a geometric progression respectively. • Discuss monotonically increasing and monotonically decreasing sequences with active participation of students. • Let students practice on exercise problems of arithmetic progressions and geometric progressions.	• Give different exercise problems on arithmetic progressions and geometric progressions as class and home works and check their solutions. Corrections are given depending on the feedback from students.
• use the sigma notation for sums.	**1.3 The Sigma Notation and Partial Sums** *(6 periods)* • The sigma notation (Σ)	• Introduce the sigma notation which stands for " the sum of" defining $\sum\limits_{i=1}^{n} x_i$ as $x_1 + x_2 + x_3 + \ldots + x_n$ • Discuss operations on sums, multiplication of a sum by constant. i.e. 1. $\sum\limits_{i=1}^{n}(x_i + y_i) = \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} y_i$ 2. $\sum\limits_{i=1}^{n} kx_i = k\sum\limits_{i=1}^{n} x_i$ 3. $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{k} x_i + \sum\limits_{i=k+1}^{n} x_i (1\leq k \leq n)$	• Give different exercise problems on the use of the summation (sigma) notation as class and home works. **Example** 1.Express each of the following sums in Σ notation. a) 1+4+9+16+25 b) $(-2)^{13} + (-2)^{14} + (-2)^{15} + \ldots + (-2)^{21}$ 2. Find each of the following sums. a) $\sum\limits_{i=2}^{5} \frac{1}{2^n}$ b) $\sum\limits_{n=1}^{4} 3n+5$

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
• find the n^{th} partial sum of a sequence. • use the symbol for the sum of sequences.	• Partial sum of sequences.	• Introduce the n^{th} partial sum of a sequence and manner of writing the n^{th} partial sum S_n of the sequence $\{A_k\}$, $k\in N$ as $S_n=\sum_{k=1}^{n} A_k$ • Assist students in practicing on calculations of partial sums.	• Giving different exercise problems on calculations of partial sums. **Example** 1. What is the 7^{th} partial sum of the sequence $-3, -5, -8, \ldots$ 2. Find S_{10} for $S_n = \sum_{i=1}^{n} 2n-1$
• compute partial sums of arithmetic and geometric progressions • apply partial sum formula to solve problems of science and technology	• Computations of partial sums.	• Introduce the formulae for the sum of arithmetic and geometric progressions. • Discuss the applications of arithmetic and geometric progressions in science and technology and daily life. • Solve equations occurring in this connection with the help of log table. Problems on population, investment, development, taxation, etc. should be included here.	• Give various exercise problems as class and home works and check their solutions, giving corrections depending on the feedback from students. **Example** If an investment starts with Birr 2,000,000 and additional amount of Birr 25,600 is added to it at the beginning of each subsequent year, what will be the total amount invested at the end of the 6^{th} year?

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
• define a series • decide whether a given geometric series is divergent or convergent.	**1.4 The notion "Infinite series"** *(4 periods)*	• Introduce infinite series using suitable examples. • Discuss the divergence or convergence of a given geometric series • problems on savings, interest, investment, taxation, etc. Should be included.	• Give different exercise problems as class and home works.
• show how infinite series can be divergent or convergent	• Divergent or convergent infinite series.	• Show how infinite series can be divergent or convergent **Example** 1,2,4,8,16,32,… 1+2 + 4 + 8 +16 +32+… are **divergent** and the sum tends to infinity. However, if $-1 < \mathbf{r} < 1$ then the series **converges** and the $\mathbf{n^{th}}$ term $\rightarrow \mathbf{0}$ In the following case $\mathbf{r} = \frac{1}{2}$ and $\mathbf{G_1} = 1$ $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ Then $S_n = \dfrac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)} = 2$ When n = 20, $S_{20} = \left(\frac{1}{2}\right)^{20} = \frac{1}{1048576} \rightarrow 0$ or $\left(\frac{1}{2}\right)^n \rightarrow 0$ as $n \rightarrow \infty$, then the series is convergent to 2. **NB** The terms also converge to **zero** in the sequence: $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots, \frac{1}{1024}, \ldots$ $\frac{1}{8192}, \ldots \frac{1}{262144}$, that is $\left(\frac{1}{2}\right)^n \rightarrow 0$ as $n \rightarrow \infty$ where $-1 < \mathbf{r} < 1$	**Example** A man saves Birr 100 each year, and invests it at the end of the year at 4 percent compound interest. How much will the combined savings and interest amount to at the end of 12 years?

Competencies	*Content*	*Teaching / Learning activities and Resources*	*Assessment*
• show how recurring decimals converge.	• Recurring decimals converge	**Example** 0.3333333…= 33/100+33/10000+ 33/1000000+... $\mathbf{u_1}$ = 33/100 **r** = 1/100 then $S_\infty = u_1 (1 – r^\infty)/ (1 –r)$ =33/100(1–0)/(99/100) =33/99=1/3 Note $1/_3{}^n \to \mathbf{0}$. as $n \to \infty$ since **r** = 1/100 then **-** 1 < r < 1 Check 1/3 = 0.333333333… **NB** $S_\infty = \dfrac{G_1}{1-r}$	
• discuss the applications of arithmetic and geometric progressions (sequences) and series in science and technology and daily life.	• **1.5 Applications of arithmetic and geometric progressions and series in science and technology and daily life**. *(2 periods)*	• Solve equations occurring in this connection with the help of log table. • Solve problems on populations, investment, development, taxation, usage of water resources, development, production, banking and insurance, etc. • Show the convergence and divergence in the binomial theorem for different values of a, n and bx in $(a \pm bx)^n$	

Unit 2: Introduction to Limits and Continuity *(28 periods)*

***Unit outcomes*:** Students will be able to:

- understand the concept "limit" intuitively.
- find out limit of a number sequence.
- determine the limit of a given function.
- determine continuity of a function over a given interval.
- apply the concept of limits to solve real life mathematical problems.
- develop a suitable ground for dealing with differential and integral calculus.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***2. Introduction to Limits and Continuity*** **2.1. Limits of sequence of numbers** *(12 periods)*		
• define upper and lower bound of number sequences. • find out the least upper (greatest lower) bound of sequences.	• Upper and lower bound of number sequences.	• Give different revision exercise problems on finding minimum and maximum elements of given sets. • Define "upper and lower bound" of number sequence using appropriate examples. • Define "least upper" and "greatest lower" bound of number sequences. • Introduce the concepts increasing and decreasing of sequences. • Illustrate how to check the boundedness of sequences. • Illustrate how to find out the least upper and the greatest lower bound of sequences using examples. • Assist students in exercising on problems of finding the least upper bound and greatest lower bound of sequences..	• Ask questions on definition of upper and lower bounds of number sequences. • Give different exercise problems on the determination of the least upper bound and the greatest lower bound of sequences and check solutions.
• define limit of a number sequence.	• Limits of Sequences intuitively.	• Discuss the concept "limit of a sequence" by using simple and appropriate examples. • Define limit of a number sequence introduce $\lim_{n\to\infty} a_n$ **e.**g. What happens to a number sequence $\left\{8-\frac{1}{n}\right\}$ as $n\to\infty$? • Discuss convergent and divergent sequences. Define	• Give different exercise problems on limit of a sequence. **Example** 1. Find the limit as n tends to infinity, $\underset{n\to\infty}{\text{limit}}\left\{\frac{6n-5}{2n+4}\right\}$

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		limit of a number sequence introducing $\lim\limits_{n\to\infty} a_n$ e.g. What happens to a number sequence $\left\{8-\frac{1}{n}\right\}$ as $n \to \infty$? • Discuss convergent and divergent sequences.	
• consolidate their knowledge on the concept of sequences stressing on the concept of null sequence.	• Null sequence	• Stabilize the concept "limit" by checking whether a given number represents the limit of a given sequence or not. • Introduce the concept of "null sequence" with the help of examples.	• Give various exercise problems on limits of sequences. **Example** 1. Give the limit of each of the following sequences. a) 3.2, 3.22, 3.222,… b) $\left\{\frac{5-3n}{n}\right\}$ 2. Determine whether each of the following sequences is a null sequence a) $\left\{\left(\frac{1}{5}\right)^n\right\}$ b) $\left\{\left(1-\frac{1}{n}\right)\right\}$
• apply theorems on the convergence of bounded sequences.	• Convergence of monotonic sequences	• Discuss the convergence of monotonic sequences and theorems on the convergence of bounded and monotonically increasing (decreasing) sequences.	• Give different exercise problems on the convergence of sequences as class and home works.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
			Example 1. Which of the following are monotonic sequences? a) $\left\{2-\frac{1}{n}\right\}$ b) $\left\{\frac{n}{n+1}\right\}$ c) $\left\{(-2)^n\right\}$ d) $\{2n\}$ 2. Which of the above sequences converge?
• prove theorem about the limit of the sum of two convergent sequences. • apply theorems on the limit of the difference, product, quotient of two convergent sequences.	• Convergence properties of sequences.	• Revise the sum, the difference, product, quotient of two sequences. • Prove theorem about the limit of the sum of two convergent sequences. • Introduce theorems on the limit of the difference, product, quotient of two convergent sequences (without proof).	• Let students reprove the theorem $\lim\limits_{n\to\infty}(a_n+b_n)$ $=\lim\limits_{n\to\infty}a_n+\lim\limits_{n\to\infty}b_n$ • Give various exercise problems requiring the application of the theorems on finding limits of differences, products, quotients as class and home works and check solutions.
		• Illustrate the application of the theorems in checking the convergence of sequences and determining the respective limits of the given convergent sequences. • Assist students in exercising the application of theorems in determining and finding out the limits of given sequences.	**Example** Find the limit of each of the following. a) $\lim\limits_{n\to\infty}\left(5-\frac{3}{n}\right)$ b) $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)$ c) $\lim\limits_{n\to\infty}\frac{8n+9}{n(n+3)}$

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
	2.2. Limits of Functions *(6 periods)*		
• define limit of a function. • determine the limit of a given function at a point.	• Limit of a function at a point	• Discuss the behavior of certain functions in an interval about a point using graph or table of values. • Illustrate how to use the definition of limit for determining the limit of a given function using different examples based on the intuitive definition of a limit.. • Assist students in exercising determination of limits of various functions.	• Ask students to repeat the informal definition of limit of a function. • Different exercise problems such as: Find $\lim_{x\to 1}\frac{x^2+2x-3}{x-1}$ are given and solutions are checked.
• find out the limit of the sum, difference, product and quotient of two functions.	• The limit of the sum, difference, product and quotient of two functions.	• Discuss basic limit theorems such as limit of the sum, difference, product and quotient of two functions. (including the case, for any real number c the $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ • Let and assist students exercise on different problems seeking the application of the theorems.	• Let students repeat the theorems in their own words. • Give various exercise problems seeking the application of the theorems for their solutions **Example** 1. Find $\lim_{x\to 16}\frac{1}{5}\sqrt{x}$ 2. Evaluate $\lim_{x\to 0}\frac{2x\cos x}{x+\cos x}$ 3. Evaluate $\lim_{x\to 0}(\cos x-\sin x)$

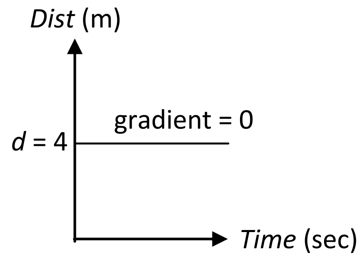
Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
	2.3 Continuity of a function *(5 periods)*		
• Define continuity of a function in an interval.		• Discuss one sided limit and non existence of limits • Show continuity and discontinuity graphically. • Discuss one sided limit and non existence of limits • Define "continuity of a function f at a point x_o", "continuity of a function f over an interval I" and "continuity of a function f at each point of the domain" • Discuss one side continuity	• Let students re-define continuity of a function - at a point x_o - over an interval I - at each point of the domain.
			• Give different exercise problems on continuity of a function. **Example** 1. Show that the function $f(x)=\frac{\sqrt{x^2+x+1}}{x-2}$ is continues at $x = 3$ without graphing. 2. Let $f(x)=\frac{x^2+3}{1+x^2}$. Determine the numbers at which f is continuous.
• describe the properties of continuous functions.	• Properties of continuous functions.	• Introduce essential properties of continuous functions. • Introduce the "Intermediate value theorem" (without proof)	• Let students re-state the essential properties of continuous functions. • Let students re-state the intermediate value theorem by citing their own examples.

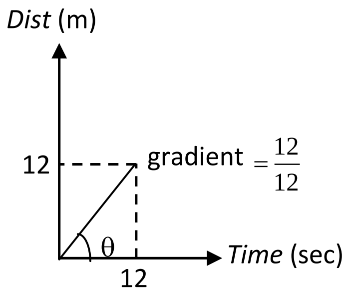
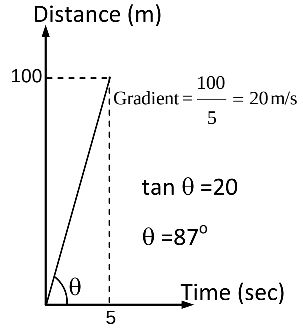
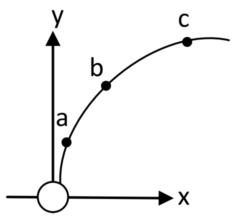
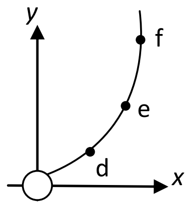
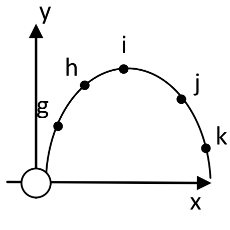
Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• use properties of continuous functions to determine the continuity of various functions.	• The intermediate value theorem	• Introduce the concept "maximum and minimum" as well as "theorem about the maximum and minimum of continuous functions in a closed interval [a, b]" (without proof) • Give and discuss different examples on the theorems mentioned above, showing how to find approximate zeros while using the intermediate value theorem using the bisection method.	• Give various exercise problems seeking the application of the intermediate value theorem, theorem about the maximum and minimum of continuous functions in a closed interval [a,b].
	2.4. Exercises on Application of Limits *(3 periods)*		
• consolidate what they have studied on limits.		• Let and assisting student solve problems related with the convergence and divergence of number sequences and sequences of partial sums. • Let and assist students solve simple problems related to limit of functions and properties of continuous functions with special attention to $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ using graphs	• Give various exercise problems related with convergence and divergence of number sequences of partial sums, and problems related to limits of functions and properties of continuous functions. • Give exercise problems containing the two important limits.
• solve problems on limit and continuity to stabilize what have learnt in the unit.	**2.5. Miscellaneous exercise** *(2 periods)*	• Assign Miscellaneous exercise problems of the unit to be done in groups, in pairs or individually and latter discuss the solutions.	• Give various exercise problems to be solved.

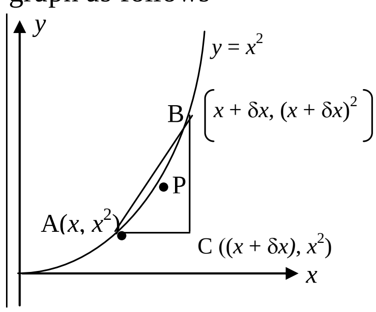
Unit 3: Introduction to Differential Calculus *(27 periods)*

Unit outcomes: Students will be able to:

- describe the geometrical and mechanical meaning of derivative.
- determine the differentiability of a function at a point.
- find the derivatives of some selected functions over intervals.
- apply the sum, difference, product and quotient formulae of differentiation of functions.
- find the derivatives of power functions, polynomial functions, rational functions, simple trigonometric functions, exponential and logarithmic functions.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***3. Introduction to Differential Calculus***		
• find the rate of change of one quantity with respect to another.	**3.1 Introduction to Derivatives** *(10 periods)* • Understanding rate of change	• Introduce differentiation as finding the rate of change of one quantity with respect to another by taking appropriate examples and considering instantaneous rates of change as opposed to average rates of change in functional values.	• Ask students to give examples of the rate of change in one quantity relative to the change in another quantity.
• sketch different straight line and curved graphs and find out slopes at different points of each graph.	• Geometrical Interpretation of derivative.	• Sketch different straight line and curved graphs and state and explain what slopes at different points of each graph are, defining slope (gradient) $= \dfrac{\text{Difference in } y \text{ values}}{\text{Difference in } x \text{ values}} = \dfrac{\Delta y}{\Delta x}$ **N.B.** At the beginning $\dfrac{\Delta y}{\Delta x}$ can be explained for straight lines as follows and later more rigorously as the limit of $\dfrac{\Delta y}{\Delta x}$ as $\Delta x \to 0$  No change in distance	

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
		 = 1m/s = speed tan θ = 1 θ = 45°     The gradients are approximately at a, b, c, d, e, f, g, h, i, j, k	

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
		$\left.\frac{dy}{dx}\right\|_{x=a} \simeq 2, \left.\frac{dy}{dx}\right\|_{x=b} \simeq \frac{1}{2}, \left.\frac{dy}{dx}\right\|_{x=c} = 0,$ $\left.\frac{dy}{dx}\right\|_{x=d} \simeq \frac{1}{2}, \left.\frac{dy}{dx}\right\|_{x=e} \simeq 1,$ $\left.\frac{dy}{dx}\right\|_{x=f} \simeq 100 \to \infty,$ as f becomes vertical $\left.\frac{dy}{dx}\right\|_{x=g} = 1, \left.\frac{dy}{dx}\right\|_{x=h} = \frac{1}{2},$ $\left.\frac{dy}{dx}\right\|_{x=i} \simeq 0, \left.\frac{dy}{dx}\right\|_{x=j} \simeq -1,$ as $k \to$ vertical, $\frac{dy}{dx} \to \infty$ • Sketch different straight lines and curved graphs distance-time graphs and show various different gradients, as above, and explain that the values of the gradients are speed (not velocity) because $\text{speed} = \frac{\text{Change in distance}}{\text{Change in time}}$ • Similarly Sketch speed - time graphs and show acceleration as the slopes. • Finally take examples such as $y = x^2$ and show how to find the slope of a point at P by using graph as follows 	

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		$BC = y_2 - y_1 = (x+\delta x)^2 - x^2$ $= 2x\delta x + (\delta x)^2$ AC = δx $\therefore \frac{dy}{dx} = \frac{BC}{AC} = \frac{2x\delta x + (\delta x)^2}{\delta x} = 2x + \delta x$ but we want AB to get smaller and smaller to the limit where $\delta x \to 0$ $\therefore \frac{dy}{dx} \to 2x \ as \ \delta x \to 0$ $\therefore$ the differential of x^2 = $2x$ $\frac{d}{dx}(x^2) = 2x$ $\therefore$ gradient at P = $2x$ for $y = x^2$	
• define differentiability of a function at a point x_o . • explain the geometrical and mechanical meaning of derivative. • set up the equation of tangent line at the point of tangency, using the concept of derivative.	• Differentiation of a function at a point.	• Discuss "limit of the quotient - difference" • Define the "differentiability of a function f at a point x_o" and first derivative $f'(x_o)$ of a function f at a point x_o. • Explain the geometrical and mechanical meaning of derivative. • Discuss using examples and exercise problems how to compute derivatives of given functions applying the concept of limit. • Introduce an algorithm for computing such derivatives. • Illustrate how to set up the equation of a tangent line at the point of contact of the line and the curve (at the point of tangency).	• Let students re-state (re-define) "differentiability of function f at a point x_0". • Ask students to explain the geometrical and mechanical meaning of derivative. • Give different exercise problems on computation of derivatives of given functions applying the concept of limit, on setting up of the equation of a tangent line to a curve at a given point.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
			Example Find the equation of the line ℓ tangent to the graph of the given function at the indicated point. 1) $f(x)= 4-3x$, $(0,4)$ 2) $f(x) = x^2+2x$, $(-3,3)$ 3) $f(x)=\sqrt{x}$; $(1,1)$
• find the derivative of elementary functions over an interval.	• Differentiation of a function over an interval.	• Introduce "differentiability of a function f over an interval I and first derivative of a function over an interval I" • Discuss with students the determination of the first derivative of some selected elementary functions. (polynomial, rational and constant functions) using appropriate examples. • Discuss one side differentiability. • Discuss the relationship between continuity of a function and differentiability, performing tests of continuity, discontinuity, and differentiability of a function. • Assist students in exercising with finding derivatives of different functions.	• Give various exercise problems on finding derivatives of polynomial, rational, constant) functions. **Example** Find the derivative of the given function at the given numbers. 1) $f(x) = x^2$ $a = \frac{3}{2}, 0$ 2) $f(t) = \cos t$ $a = 0, -\frac{\pi}{3}$ 3) Show that f is differentiable over the given interval. a) $f(x) = x^2 + x$; $(-\infty,\infty)$ b) $f(x) = 2x^3 - \sqrt{3}$; $(0, \infty)$

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• find the derivatives of power, simple trigonometric, exponential and logarithmic functions.	**3.2 Derivatives of different functions** *(3 periods)*	• Revise the concept of power, polynomial, rational, trigonometric, exponential and logarithmic functions. • Discuss the differentiation of power, simple trigonometric, exponential and logarithmic functions.	• Ask questions on the revision of polynomial and rational functions. • Give various exercise problems on the differentiation of polynomial, rational and simple trigonometrical functions. **Example** Find the derivative of each of the following functions. 1) $f(x) = 6x^5$ 2) $f(x) = \cos x$ 3) $f(x) = \ln x$ 4) $f(x) = x - \frac{3}{5}$
• apply the sum and difference formulae of differentiation of functions.	**3.3 Derivatives of combinations and compositions of functions** *(12 periods)* • Theorems on the differentiation of the sum and difference of functions.	• Revise the sum and difference of two functions in a form of discussions. • Discuss the theorems on derivatives of sums and differences of two functions being differentiable at a point x_o and over an interval I. • Discuss the application of the sum and difference formulae for two or more than two functions, by using appropriate examples.	• Ask oral questions on sums and differences of two functions. • Give different exercise problems on the application of the theorems on the differentiation of sums and differences of functions.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• apply the product and quotient formulae of differentiation of functions.	• Theorems on the differentiation of the product and quotient of functions	• Revise the product and quotient of two functions. • Discuss the theorems on the derivatives of the product and the quotient of two functions differentiable at a point x_o and over an interval I. • Discuss the application of the theorems on the derivatives of the product of two or three functions, the product of a constant and a function and the quotient of two functions.	**Example** 1) Find the derivative of each of the following functions a) $f(x)=6x^3+4x^2+5x$ b) $f(t) = t^4 - t^3$ 2) If $f(x)=x+\cos x$, then find $f'\left(\frac{\pi}{4}\right)$ • Ask oral questions on product and quotient of two functions. • Ask to restate theorems on derivatives of the product and quotient of two functions. • Give different exercise problems seeking the application of the theorems on derivatives of product and quotient functions. **Example** Find $f'(x)$ 1) $f(x)=\sin x \cos x$ 2) $f(x) = \dfrac{2x+3}{4x-1}$

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
• apply the chain rule	• The chain rule	• Discuss the chain rule and demonstrate its application using appropriate examples.	• Ask students to restate the chain rule. • Give different exercise problems on the application of the chain rule. **Example** Find the derivative of each of the following functions. 1) $f(x)=(2x-3x^3)^{-5}$ 2) $f(x) = \cos x^4$
• differentiate composition of functions.	• Differentiation of Compositions of functions.	• Revise the composition of functions. • Illustrate the differentiation of composition of functions by making use of the chain rule with different examples. • Assist students in exercising with differentiation of composite functions.	• Ask oral questions on the revision of composite functions. • Give various exercise problems on the differentiation of composite functions. **Example** Write the function $y = \sin 6x$ as a composite of two functions and find $\frac{dy}{dx}$

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
			• Give various exercise problems on finding the 2nd and the n^{th} derivative of a function as class and home works.
• find the 2nd and the n^{th} derivative of a function.	• The n^{th} derivative $f^{n}(x)$ of a function.	• Introducing the second derivative $f''(x)$ and the n^{th} derivative $f^{n}(x)$ of a function at a point x_o or in an interval I. • Discuss the derivatives of second and higher order polynomial, rational and power functions with any rational exponent.	**Example** 1) Let $f(x)=x^6-6x^4+3x-2$. Find all higher derivatives of f. 2) Let $f(x) = 4x^{½}$ Find a formula for $f''(x)$.
• consolidate and stabilize what has been studied in the unit.	**3.4 Miscellaneous Exercise** *(2 periods)*	• Give Miscellaneous Exercise problems on the unit to be done in groups, in pairs or individually.	• Various exercise problems that cover the whole topics of the unit shall be given and solutions are checked.

Unit 4: Application of Differential Calculus *(25 periods)*

Unit outcomes: Students will be able to:

- find local maximum or local minimum of a function in a given interval.
- find absolute maximum or absolute minimum of a function.
- apply the mean value theorem.
- solve simple problems in which the studied theorems, formulae, and procedures of differential calculus are applied.
- solve application problems.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***4. Applications of Differential Calculus***		
• consolidate the concept zero(s) of a function.	**4.1. Extreme values of a function** *(13 periods)* • Revision on the zeros of functions: square root functions, polynomial functions, rational functions.	• Motivate students and assist them compute zeros of:- ∗ linear and quadratic ∗ polynomial ∗ rational ∗ square root functions (where the square root functions contain radicals which are solvable by single squaring)	• Ask discussion questions on zero (s) of a function. • Give exercise problem on determination of zero(s) of linear, quadratic, polynomial, rational, and square root functions and check solutions. **Example** What are the zeros of the polynomial function $f(x)= x^3+2x^2-x-2$?
• find critical numbers and maximum and minimum values of a function on a closed interval.	**4.1.1 Critical number, and critical values**	• Define the maximum and minimum of a function on a closed interval I. • Discuss the theorem about a necessary condition $f'(x_o) = 0$ or if $f'(x_o)$ does not exist to determine the maximum and minimum values of a function on a closed interval I = $[a, b]$.	• Let students determine the existence and non-existences of critical number on an interval.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		• Let students do various exercise problems on determination of critical numbers, and maximum and minimum values of a function on a closed interval. **E.g.** $f(x)=\begin{cases} x-2x; & for\ x\geq 1 \\ 1-2x; & for\ x<1 \end{cases}$ on $[-3,3]$	• Different exercise problems on determining the number that satisfies definition of critical numbers.
• explain the geometric interpretations of Rolle's theorem and mean value theorem • find numbers that satisfy the conclusions of mean value theorem and Rolle's theorem.	• Rolle's theorem, and the mean value theorem.	• Discuss Rolle's theorem and the Mean Value theorem of differential calculus, and their geometric interpretations. • Let students do various exercise problems on conditions satisfying the Rolle's theorem and Mean Value theorems of a function on a closed interval, and problems about looking for numbers that satisfy the conclusions of Rolles's theorem and Mean Value theorem on a closed interval. • Discuss and prove theorem about the sufficient conditions $f'(x_0) \geq 0$, for all x and I and $f'(x_0) \leq 0$, for all x and I, respectively for increasing monotonity and decreasing monotority of a differentiable function on an interval I. • Let students do various problems on determining intervals where the function is increasing and where it is decreasing by applying the first derivative test given by the above theorem.	• Let students state Rolle's theorem and Mean Value theorem in their own words. **Example** (1) Verify Rolle's theorem given $f(x) = -x^2 - x - 2$ and I = [-1, -2] (2) Let $f(x) = x^3 - 3x - 2$, find a number c in (0, 3) that satisfies the conclusion of Mean Value theorem (3) Determine intervals where the function $f(x)=\frac{-4x}{x^2+2}$ is strictly increasing and strictly decreasing.
	• Local extreme values of a function on its entire domain (1^{st} derivative test.)	• Discuss theorem about the sufficient condition f' changes its algebraic sign at a critical number x_o from + ve to - ve and from -ve to +ve, respectively, for the existence of local maximum value at x_o and local minimum value at x_o. • Let students do various exercise problems on determination of	• Different exercise problems on extreme values of a function on its entire domain will be given. **e.g.** Find any maximum or

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		local extreme values, absolute extreme values of a function in its domain (if any), and turning points of its graph.	minimum value of $f(x)=\frac{3x}{x^2+9}$
	• Concavity and points of inflection (2nd derivative test)	• Define the concave upwardness, concave downwardness and inflection points of the graph of a function on an interval.	• Different exercise, problems on sketching curves by determining inflection points and concavity will be given.
		• Discuss theorem about $f''(x_0) \geq 0$, for all $x_0 \in$ I for the graph of f concave upward on I, $f''(x_0) \leq 0$ for all $x_0 \in$I, for the graph concave down ward on I and f'' changing its algebraic sign at x_0 such that $f''(x_0) = 0$ or $f''(x_0)$, does not exist for the point $(x_0, f(x_0))$ to be an inflection point. • Let students do various exercise problems on determining inflection points, intervals where the graph is concave upwards and where it is concave downwards, local and global extreme values and sketching the graph.	**Example** (1) Investigate the stationary points and intervals where the graph is concave upwards and downwards for the function. $f(x)=\frac{-4x}{x^2+2}$ (2) Sketch the curves of the functions a) $f(x)=\frac{2x}{x^2+3}$ b) $f(x) = 4x - 3x^3$
• Solve problems on application of differential calculus	**4.2 Minimization and maximization problems** *(6 periods)*	• Assist and facilitate to students in solving extreme value problems from the field of Mathematics, Natural Science, Economy and daily life. • Let students do various exercise problems on the application of extreme values. **e.g.** A tool shed with a square base and a flat roof is to have a volume of 800 cubic feet. If the floor costs Birr 6 per square foot, the roof Birr 2 per square foot, and the sides Birr 5 per square foot. Determine the dimensions of the most economical shed.	• Various exercise problems as class and home works are given, solutions are checked and corrected. **Example** (1) A right triangle has its sides 6,8 and 10 units long. What are the dimensions of a rectangle of maximum area that can be

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
			inscribed with one side of the rectangle lying along the longest side of the triangle. (2) A right circular cone is circumscribed about a sphere of radius 8 √2 meters. What must be the dimensions of the cone. (a) if its volume is minimum? (b) if its total surface area is minimum? (3) An airline company offers a round-trip group flight from New York to London. If *x* people sign up for the flight, the cost of each ticket is to be (100 - 2*x*) dollars. Find the maximum revenue the airline company can receive from the sale of tickets for the flight.
• Interpret and apply differential calculus on problems involving rate of change.	**4.3 Rate of change** *(6 periods)*	• Discuss the notations $\frac{dy}{dx} = \frac{d}{dx} f(x)$ for the derivative of a function $f(x) = y$ • Let students do exercise problems on rate of change. **e.g.** Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere then express	• Give various exercise problems as class and home works and check solutions. **Example** A board 5 feet long leans against a

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		the rate of change of volume (V) with respect to time t. 2) Let $x^2y + xy = 6$, then a) Find the rate of change of x with respect to y. b) Find the rate of change of y with respect to x. • Let students do exercise problems on the application of rate of change. **e.g.** Suppose that the radius of a spherical balloon is shrinking at $\frac{1}{2}$ centimeter per minute. How fast is the volume decreasing when the radius is 4 centimeters.	vertical wall. At the instant the bottom end is 4 feet from the wall, the other end is moving down the wall at the rate of 2 ft/sec, at the moment (a) how fast is the bottom end sliding? (b) how fast is the area of the region between the board, ground and wall changing?
• consolidate what has been learnt in this unit.		• Give Miscellaneous Exercises to be done in groups, in pairs and individually and later a whole class discussion on the solutions. **Example** A closed rectangular container with a square base is to have a volume of 2400 cubic centimeters. It costs three times as much per square centimeter for the top and bottom as it does for the sides. Find the height and base area of the most economical container.	• Exercise problems that cover the topics of this unit shall be given and checked. **e.g.** (1) Of all triangles that pass through the point Q(1, 1) and have two sides lying on the coordinate axes, one has the smallest area. Find the lengths of its sides. (2) Find the dimensions of a cylinder of surface area 54 π square units, if the volume is to be maximum?

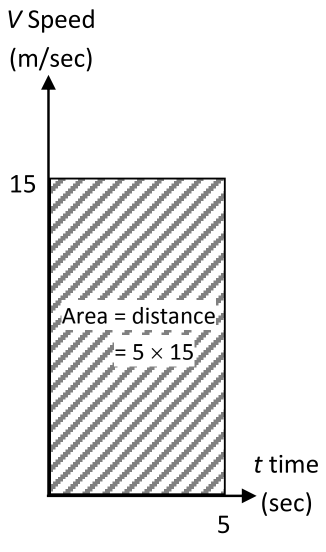
Unit 5: Introduction to Integral Calculus *(30 Periods)*

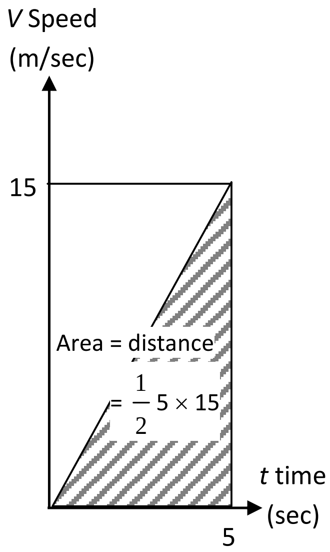
***Unit outcomes*:** Students will be able to:

- understand the concept of definite integral.
- integrate different polynomial functions, simple trigonometric functions, exponential and logarithmic functions.
- use the various techniques of integration to evaluate a given integral.
- use the fundamental theorem of calculus for computing definite integrals.
- apply the knowledge of integral calculus to solve real life mathematical problems.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***5. Introduction to integral calculus***		
• differentiate between the concepts differentiation and integration	**5.1 Integration as reverse process of differentiation** *(7 periods)* • The concept of indefinite integral. • Integration of - constant - power - exponential and logarithmic functions - simple trigonometric functions	• Define integration as the reverse operation of differentiation by using appropriate examples. Thus if $\frac{d}{dx} f(x) = F(x)$, then $F(x)$ is called the derivative of $f(x)$ and $f(x)$ is called antiderivative or an indefinite integral of $F(x)$, and in symbols, we write $\int F(x)dx = f(x)$ and that $F(x)$ is called the **integrand**. • Introduce some important standard formulae of integration. (i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq 1$ in particular $\int dx = x + c$ (ii) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c,\ n \neq -1$ (iii) $\int \frac{1}{x} dx = \log\lvert x\rvert + c$ (iv) $\int \frac{1}{x+1} dx = \log\lvert x+1\rvert + c$ (v) $\int \frac{dx}{ax+b} = \frac{1}{a}\log\lvert ax+b\rvert + c$ (vi) $\int e^x dx = e^x + c$ (vii) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$ (viii) $\int a^x dx = \frac{a^x}{\log_e x} + c$	• Ask oral questions on the definition of the definite integral. • Give various exercise problems on the application of the standard formulae of integration. **E.g.** 1) Find the antiderivative of a) $f(x) = x^6$ b) $f(x) = 5$ 2) Evaluate e ach of the following a) $\int x^5 dx$ b) $\int \frac{dx}{x^4}$ c) $\int \sqrt{x}\ dx$.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• use the properties of indefinite integrates in solving problems of integration.		• Discuss the properties of indefinite integrals a) $\frac{d}{x}\int f(x)\,dx = f(x)$ $\int f'(x)\,dx = f(x) + c$ b) Two indefinite integrals with the same derivative represent the same family of curves and so they are equal. c) $\int k f(x)\,dx = k\int f(x)\,dx$. d) $\int[f_1(x) \pm f_2(x) \pm f_3(x) \pm f_4(x) \pm ...]\,dx$ $= \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx$ $\pm \int f_4(x)\,dx \pm ...$	• Give exercise problems on the application of the properties of the indefinite integral. **E.g.** Evaluate 1) $\frac{5}{7x^{\frac{3}{4}}}dx$ 2) $\int \frac{x^3+5x^2-4}{x^2}dx$ 3) $\int \frac{(x+1)^2}{\sqrt{x}}dx$
• integrate simple trigonometric functions	• Integration of simple trigonometric functions	• Introduce and discuss the standard formulas involving integration of trigonometric functions i) $\int \sin x\,dx = -\cos x + c$ ii) $\int \cos x\,dx = \sin x + c$ iii) $\int \sec^2 x\,dx = \tan x + c$ iv) $\int \sec x \tan x\,dx = \sec x + c$ v) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$ etc.	• Give various exercise problems on the application of the standard formulae involving integration of trigonometric functions. **E.g.** Evaluate:- a) $\int(2x-3\cos x+e^x)dx$ b) $\int \sec x(\sec x+\tan x)dx$ c) $\int \frac{\sec^2\ dx}{\cos ec^2\ x}$
• use different techniques of integration for computation of integrals	**5.2 Techniques of integration** *(9 periods)*	• Discuss that so far we have only considered the problems on integration of functions in standard forms, and that integration of certain functions cannot be obtained directly if they are not in standard form. Hence we need some techniques to transform the given function to the standard form and that in this section we shall be using: • Integration by substitution. • Integration by partial fractions. • Integration by parts and elaborate each of these methods by using appropriate and sufficient number of examples.	• Ask oral questions so as students discuss the need for other techniques of integration to compute some indefinite integrals. • Give sufficient number of exercise problems to apply each technique of integration. **E.g.** (by substitution) 1) $\int \cos 4x\ dx$ 2) $\int x \sin x^2\ dx$ 3) $\int \frac{dx}{x+2\sqrt{x}}$

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
		E.g. (by substitution) Evaluate $\int (x-2)\sqrt{(x^2-4x+7)}\,dx$ **Solution:** Let $x^2 - 4x + 7 = z$. Then $(2x - 4)dx = dz$ or $2(x - 2)dx = dz$ $\Rightarrow (x - 2)dx = \frac{1}{2}\, dz$ $\therefore \int (x-2)\sqrt{(x^2-4x+7)}\,dx = \int \frac{1}{2} z^{\frac{1}{2}} dz$ $= \frac{1}{2} \times \frac{z^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3}\left(x^2-4x+7\right)^{\frac{3}{2}} + c$ • Facilitate and assist students to exercise on integration using the studied techniques of integration.	4) Evaluate each of the following integrals by making the indicated substitution. a) $\int 3x^2+(x^3+5)^9\, dx$ by letting $u = x^3 + 5$ b) $\int 3 \sin (-2x)dx$, let $u = -2x$. **E.g.** (Integration by parts) Evaluate:- 1) $\int x \cos x\, dx$. 2) $\int x \log x\, dx$ 3) $\int \frac{x}{\sqrt{x^2+a^2}}\, dx$ **E.g.** (by partial fractions) Evaluate 1) $\int \frac{dx}{x^2-9}$ 2) $\int \frac{1}{a^2-x^2}\, dx$ 3) $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\, dx$
• Compute area under a curve.	**5.3 Definite integrals, area and fundamental theorem of calculus** *(8 periods)* • Areas of regions	• Using simple examples, illustrate how to compute areas of regions with curved boundaries. **E.g.** 	• Give exercise problems on computations of areas under given curves. • Let students exercise on problems of approximation of area under a curve, say $y = x + 1$ from $x = 0$ to $x = 10$ by subdividing the given interval into equal lengths.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
			
• use the concept of definite integral to calculate the area under a curve. • state fundamental theorem of calculus • apply fundamental theorem of calculus to solve integration problems.	• The concept of definite integral • Fundamental theorem of calculus	• Introduce the concept of definite interval as a limit of a sum using appropriate examples. • Discuss the relationship between integration and area bounded by a curve, the x-axis and the limits between $x = a$ and $x = b$. • Discuss how to evaluate definite integrals with the help of appropriate examples. 1. $\int_1^4 x\,dx$ 2. $\int_0^4 x^2\,dx$ • Introduce the concept of the fundamental theorem of calculus with the help of appropriate examples.	• Give different exercise problems. • Ask students to re-state the fundamental theorem of calculus in their own words • Give exercise problems on the application of the fundamental theorem of calculus. **E.g.** Evaluate each of the following definite integrals. 1. $\int_1^4 x\,dx$ 2. $\int_0^4 x^2\,dx$ 3. $\int_{-4}^{-1} \frac{1}{x}\,dx$

• state the properties of definite integrals. • apply the properties of definite integrals for computations of integration.	• Properties of definite integrals.	• Discuss the properties of definite integrals by using appropriate examples. i) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ ii) $\int_a^b \left(f(x)+g(x)\right)dx =$ $\int_a^b f(x)dx + \int_a^b g(x)dx$ iii) $\int_a^b \left(f(x)-g(x)\right)dx =$ $\int_a^b f(x)dx - \int_a^b g(x)dx$ iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ v) $\int_a^a f(x)dx = 0$	• Ask students to re-state the properties of definite integrals. • Give different exercise problems on the application of properties of definite integrals. **E.g.** Evaluate 1) $\int_{-1}^{5} 6x^2 dx$ 2) $\int_{-3}^{3} (4x+3)dx$ 3) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$
• apply the knowledge on integral calculus to solve problems.	**5.4 Application of integral calculus** *(6 periods)*	• Illustrate the application of integral calculus in solving problems on:- • area • volume • displacement • work, etc. by using appropriate examples. **E.g.** Calculate the area under the graph of the function $f(x) = 7x+5$ between $x=\frac{1}{2}$ and $x=3$	• Give various application problems. • Give miscellaneous exercise problems that cover the whole unit.

Unit 6: Three Dimensional Geometry and Vectors in Space *(17 Periods)*

Unit Outcomes**:** Students will be able to:

- know methods and procedures in setting up coordinate system in space.
- know basic facts about coordinates and their use in determining geometric concepts in space.
- apply facts and principles about coordinates in space to solve related problems.
- know specific facts about vectors in space.

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***6. Three dimensional geometry and vectors in space***		
• construct the coordinate axes in space • identify planes determined by the axes in space. • identify the octants determined by the planes and axes.	**6.1 Coordinate axes and coordinate planes in space** *(2 periods)*	• Start the lesson by Revising the procedures in setting up the coordinate system on the plane (a two dimensional system) that the students had learnt in earlier grades. • Proceed the lesson, with active participation of the students, by considering three mutually perpendicular lines and name the point of perpendicularity by O which is called "the origin", and then introduce the three lines as the x-axis, the y-axis and the z-axis, in doing so before illustrating the situation on the black board it is better to use a model of the system. • Explain how to coordinate the axes so that the origin is assigned to 0 on each axis. • Assist students to identify the planes, i.e. the xy-plane, the xz-plane and the yz-plane which are determined by the three axes. • Guide students to identify the eight octants each formed by parts of each plane or whose bounding edges are either the positive or the negative or pair wise both the negative and positive parts of the axes.	• Form several groups of students and let them produce model of the coordinate system in space. • Ask oral questions about what the planes look like **E.g.** In how many parts does each plane divide the space. Show them. • Let the students tell any object (or part of an object) that resembles the model. • Ask students to name the bounding edge of each octant. • Give assignment in group to construct a model of three dimensional coordinate system.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		• E.g. Octant 1 is the octant whose bounding edges are the positive x-axis, the positive y-axis and the positive z-axis, then octant 2, 3 and 4 lie above the xy-plane in counter - clockwise order about the z-axis. Octants 5, 6, 7 and 8 lie below the xy-plane with octant 5 lying under octant 1.	
• read the coordinates of a point in space. • describe given low to locate a point in space. • plot a point whose coordinates are given. • give the equations for the planes determined by the axes.	**6.2 Coordinates of a point in space** *(2 periods)*	• Begin the lesson with a brief revision of assigning points to coordinates in the coordinate plane which the students are already familiar with in earlier grades. • Proceed in similar way to assign a point P in space with coordinates (ordered triple) taken from each axis, e.g. if point P has coordinates (a, b, c) where a, b, c, $\in$R read from the x-axis, the y-axis and the z-axis respectively then a is the x-coordinate, b is the y-coordinate and c is the z-coordinate of point P (it is better to begin with points taken from octant 1). • Guide students to plot a point whose coordinates are given. • Encourage and guide students to come to the conclusion that there is a one-to-one correspondence between set of points in space and set of ordered triple of real numbers. • Assist students in observing the correspondence between the sets of points in space and ordered triple of real numbers. • Assist them to describe geometric figures, concepts, relations and others in space by means of equations. **e.g.** The xy-plane = $\{(x, y, z): z = 0\}$ • Assist students in determining equations of the other two planes. i.e. xz-plane and yz-plane.	• Give exercise problems on assigning points to numbers in space and vise-versa. • Ask students to describe the nature of the coordinates peculiar to each plane. • Ask students to locate a point in space using the coordinates given. • Ask students to write the equation of the xz-plane and the yz-plane and check their progress • Let them practice on their own

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• show graphically how to find the distance between two points in space. • compute distance between two given points in space.	**6.3 Distance between two points in space** *(2 periods)*	• First revise the distance between two given points (i.e. whose coordinates are known) on the plane. Then with active participation of students, and considering two points whose coordinates are given, discuss with students on the steps to find the distance between these two given points, in doing so you may use a model of parallepiped to simplify and visualize. • Encourage students to come to the distance formula that is used to compute distance between any two given points in space and give sufficient number of exercise problems.	• Give exercise problems on calculating distance between two given points and check their work.
• determine coordinates of the mid-point of a segment in space.	**6.4 Mid-point of a segment in space***(1 period)*	• Start by revising about the coordinates of mid-point of a segment in a plane. • Assist students in finding coordinates of mid-point of a segment based on the coordinates of its end points.	• Give exercise problems on determining mid-points of a segment in space.
• describe the equation of a sphere • derive equation of a sphere • solve problems related with sphere	**6.5 Equation of a sphere** *(2 period)*	• After revising important points about sphere that the students had learnt in earlier grades, consider a sphere whose centre and radius are given in space and with the help of both the model and pictorial representation of the sphere (centre and radius shown in space), discuss with students about the derivation of the equation of the sphere. Emphasize on the fact that every point on the sphere satisfies the equation and points whose coordinates satisfy the equation lie on the sphere.	• Give exercise problems on the equation of a sphere and check their work. • Give them assignment to make a sphere and or to give practical examples of sphere.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		• Guide students in the derivation of centre-radius form of equation of any sphere in space and discuss with students about the condition that is determined by the radius; i.e. if r is the radius and centre O discuss the situation of a point P $(a_1, b_1, c_1)$with respect to the radius, that means when OP < r , OP = r and OP > r.	•
• Describe vectors in space. • use the unit vectors **i, j** and **k** while representing a vector in space.. • add, subtract vectors and multiply by a scalar in space	**6.6 Vectors In space** *(8 periods)* • Revision on vectors in plane • The notion of vectors in space.	• Start the lesson by first revising physical quantities as vectors and scalars then stress the lesson on vectors, i.e. (representation, operation, dot product and angle between two vectors), with a brief example. **e.g.** Let $\vec{a} = 2i + 3j$ and $\vec{b} = i - 2j$. Then, find $\vec{a} + \vec{b}$, $2\vec{a} - \vec{b}$, $\vec{a} . \vec{b}$ and so on. • Assist students to give some actual examples of vectors as quantities.	• Give a lot of exercise problems to perform the operations (+ & -) • Give them opportunity to represent a vector in space using a graph. • More exercises on addition and subtraction • Check their work
• describe the properties of addition to solve exercise problems. • show the closure property on their own	• Addition and subtraction of vectors in space • Properties of Addition of Vectors	• Using the above lesson as a basic clue extend the coordinate system into three dimensional and represent a vector in space in which its tail is at the origin. • Let $\vec{V} = (a, b, c)$ or $\vec{V} = ai + bj + ck$ be a vector in space whose tail is O (0, 0, 0,) and whose head is the point A (a, b, c) hence $\vec{V} = \overrightarrow{OA}$ in space. • Discuss and give more example on how to add and substract vectors using their component in space. • Discuss the properties of addition i.e. (commutative., associative and scalar multiplication). • Encourage the students to prove the closurity of the properties by giving class activity exercises.	• Give the students exercise problems on properties and let them show each properly. • Prepare different oral questions on the properties.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• find the length of a vector in space.	• Magnitude of a vector	• Start by revising the formula of distance between any two points in space. • Show students using diagrams how the length of the vector from it tail to its head is found. • Derive the formula for the magnitude of a vector that originates from the origin. e.g. Let V = ai + bj + ck then $\lvert \mathbf{v} \rvert = \sqrt{a^2 + b^2 + c^2}$	• Let the students do several exercises. • Ask oral questions to compare vectors in magnitude and direction.
• find the scalar product of two vectors in space.	• Scalar (Dot) product.	• Again remind the students that in a rectangular coordinate system (xy - plane). The dot produce of two vectors is a real number obtained by: **E.g.** Let $\vec{a} = a_1 i + a_2 j$ and $\vec{b} = b_1 i + b_2 j$ then $\vec{a} \cdot \vec{b} = \lvert a \rvert \, \lvert b \rvert \cos\theta$. or $\vec{a} \cdot \vec{b} = (a_1, b_1)\, i.i + (a_2, b_2) j.j$ $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$ • Similarly the dot product of two vectors in space is also a real number. • Show the students how the dot product can be obtained using various examples.	• Prepare questions and give chance for few students to do the dot product as a class activity. • Let the students show how to obtain dot product. • Give more exercise problems and check their work.
• evaluate and show the angle between two vectors in space.	• Angle between two vectors in space.	• Assist students in finding the scalar product and angle between vectors in space. Using the formula:- $\vec{a} \cdot \vec{b} = \lvert a \rvert . \lvert b \rvert \cos\theta$. Where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$ respectively. • Discuss and explain the angle formed between two vectors with additional examples.	• Give different types of problems. • Ask some of the students to come to the board and solve some selected problems, and give explanations.

Unit 7: Mathematical Proofs *(15 Periods)*

Unit Outcomes: Students will be able to:

- develop the knowledge of logic and logical statements.
- understand the use of quantifiers and use them properly.
- determine the validity of arguments.
- apply the principle of mathematical induction for a problem that needs to be proved inductively.
- realize the rule of inference.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
Student will be able to:	***7. Mathematical proofs***		
• recall what they have studied about statements and logical connectives in the previous grade. • revise open statement • understand the concept of quantifiers • determine truth values of statements with quantifiers.	**7.1 Revision on Logic** *(5 periods)* • Revision of statements and logical connectives • Open statements and quantifiers.	• Revise statements and logical connectives in a form of discussion and give examples like: **e.g.** p = T, q = F then p ∪ q = T and p ∩ q = F • Revise and discuss open statements. • Introduce existential and universal quantifiers. • Illustrate the use of quantifiers in changing open statements to statements. e.g. Let $p = x > 2$ and $q = x$ is odd, then find truth value for $\forall x\,(p \Rightarrow q)$, $\exists\, x\,(p\,(x) \wedge q\,(x\,))$	• Ask oral questions on statements and the logical connectives. • Give various exercise problems on the determination of truth values of statements with quantifiers. **Example** If $p(x) = x$ is prime, what is the truth value of the statement a) $(\forall x)\; p(x)$ b) $(\exists\, x)\; p(x)$?
• define argument and validity • check the validity of a given argument • use rules of inference to demonstrate the validity of a given argument.	• Arguments and validity • Rules of inference.	• Define argument and validity using appropriate examples. • Demonstrate how to check the validity of a given argument using examples. • Discuss the rules of inference and illustrate how they are used to demonstrate the validity of a given argument.	• Give various exercise problems on arguments, validity. Solutions are checked and appropriate corrections are given based on the feedback from students.
		• Assist students in exercise to determine the validity of arguments. Tell the students how to relate this lesson with real life thought.	• Give them additional exercise and solve them all.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• distinguish between the nature of different types of mathematical proofs. • apply the right type of proof to solve the required problem	**7.2. Different types of proofs** *(4 periods)*	• Discuss different types of proofs with examples, such as - Direct and indirect proof, - Proof by exhaustion method, - Disproof by counter-example, - Proof by mathematical induction. • Give various examples of the above types of proofs **E.g.** prove by counter example that $2n$ is a composite number for $n \in \mathbb{R}$. • Assist students to employ the different types of proofs to prove or disprove mathematical statements through various exercise problems.	• Give different exercise problems that necessitate students to employ the different types of proofs studied. • Give more exercise problems to prove or disprove and check their progress.
• apply the principle of mathematical induction for proving. • identify a problem and determine whether it could be proved using principle of mathematical induction or not.	**7.3 Principle and application of mathematical induction** *(4 periods)*	• Discuss and apply the first principle of mathematical induction to prove formulae or expressions. such as: - partial sum of sequences - expressions like: **e.g.** Prove that $2n^2 + 1$ is odd for $n \in \mathbb{N}$. • Discuss and explain some problems that lead to wrong conclusion while being proved by principle of mathematical induction. **e.g.** Prove or disprove that n^2-$11n$ +121 is prime number if $n \in \mathbb{N}$.	• Give different exercises and problems to be proved or disproved and check their progress. • After summary of the unit give those additional exercises problems and tests or group works.
	7.4 General Exercise problems *(2 periods)*	• Assign different types of exercise problems in different ways to be done either individually or in group.	• Give the students opportunity to do exercise problems by themselves. • Check their progress by giving test or Group work.

Unit 8: Further on Statistics *(22 periods)*

Unit outcomes: Students will be able to:

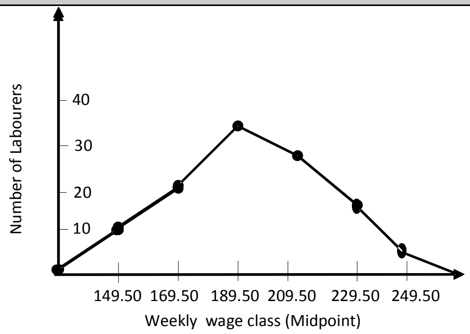
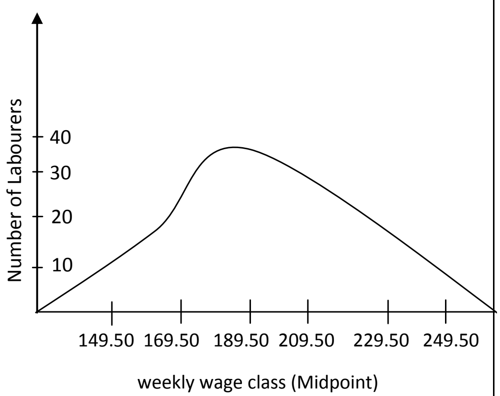
- know basic concepts about sampling techniques.
- construct and interpret statistical graphs.
- know specific facts about measurement in statistical data..

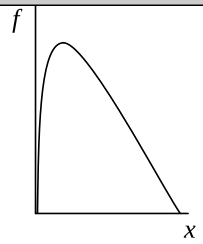
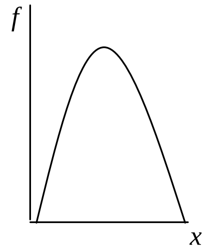
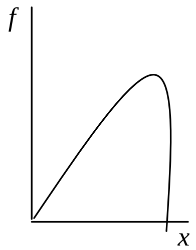
Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
Students will be able to:	***8. Further on Statistics***		
• describe the three methods/ techniques of sampling. • explain the advantages and limitation of each techniques of sampling.	**8.1 Sampling Techniques** *(3 periods)*	• After a brief revision of the purpose of the field of statistics in different sectors of social and economical situations, discuss how to collect data/ information, about the situation on which we want to study and remind students what "population" in statistics means. • With active participation of students, discuss the idea of "sample" i.e. the limited number of items taken from the population on which the study/ investigation is carried out of course, it can be any size or may consist of the entire population and in your discussion emphasize on the fact that, the sample should be representative of the whole population, so "bias" in the choice of sample members must be avoided. • Describe the three types of sampling techniques or methods of sampling viz **Random sampling** (in which every member of the population has an equal chance of being selected). **Purposive or systematic sampling** and **stratified sampling** (A combination of the previous two techniques or which is often used when the population is split into distinguishable strata or layers.) and based on and with the help of some examples explain the advantages and limitations of each techniques and also show how to avoid "bias" when choosing sample members while using the above mentioned techniques.	• Ask students to describe the advantages and limitations of each of the three sampling techniques.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• describe the different ways of representation of data. • explain the purpose of each representation of data.	**8.2 Representation of Data** *(2 periods)*	• By considering examples from the different ways of representations of data (for both discrete and continuous) that were discussed in previous grades and by showing models (from governmental or non-governmental organizations) of different representations of data viz, pictograph, frequency distribution table, bar graph, histograms, frequency polygon, cumulative frequency curve and discuss the importance and strong side of each representation in relation to (a) computational analysis and decision making purpose (b) providing information including for public awareness system purposes.	• Ask students how the tabular method (frequency distribution) and pictorial methods of representation of data are helpful and when one method is preferable than the other in presenting the required information.
• Construct graphs of statistical data • identify statistical graph. • explain the significance of representing a given data in different types of graphs.	**8.3 Construction of graphs and interpretation** (6 *periods)*	• Discuss with students about the (1) methods and procedures of drawing and presenting statistical graphs in an understandable and attractive way. (2) how to obtain the correct information i.e., how to read and interpret them. • With active participation of students discuss how a given data organized and presented in a frequency distribution (table) is represented graphically so that each graph is related to the others, though it has its own peculiar property and advantage. For instance given frequency distribution table of data, then how its frequency polygon is obtained from the corresponding histogram, and also how the ogive curve is related to the frequency polygon.	
• draw histogram for a given frequency distribution.	• Graphical representations of grouped data • Histograms	• You can start the lesson by short revision about histogram for frequency distribution of ungrouped data that the students had learnt in earlier grades. • With active participation of students, discuss how to draw a histogram for a	

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning activities and Resources</th><th>Assessment</th></tr>
<tr><td></td><td></td><td>
given frequency distribution of grouped data. For instance a histogram for the frequency distribution of weekly wages of the labourers (considered below (Table 1)) can be shown as follows.
Table 1
<table>
<tr><th>Weekly wage (class limits)</th><th>Class Boundaries (Birr)</th><th>Class midpoint</th><th>Number of Labourers</th></tr>
<tr><td>Birr 140-159</td><td>139.59 -159.50</td><td>Birr 149.50</td><td>7</td></tr>
<tr><td>160 - 179</td><td>159.50 -179.50</td><td>169.50</td><td>20</td></tr>
<tr><td>180 - 199</td><td>179.50 -199.50</td><td>189.50</td><td>33</td></tr>
<tr><td>200 - 219</td><td>199.50 -219.50</td><td>209.50</td><td>25</td></tr>
<tr><td>220 - 239</td><td>219.50-239.50</td><td>229.50</td><td>11</td></tr>
<tr><td>240 - 259</td><td>239.50-259.50</td><td>249.50</td><td>4</td></tr>
<tr><td colspan="3">Total</td><td>100</td></tr>
</table>


Note: As indicated on the histogram typically the class boundaries are entered along the horizontal axis (in contrast to ungrouped data), while the number of observations are listed along the vertical axis.
</td><td>• Give exercise problems on construction of Histograms and on how to read (get) required information from the graph.</td></tr>
<tr><td>• Sketch frequency polygon for a given frequency distribution.</td><td>• Frequency polygons</td><td>• By using appropriate examples explain the steps and principles in drawing frequency polygon, which is the line graph, of a given frequency distribution, in doing so give emphasis on the values listed along the horizontal axis and the vertical axis. As an example here is the frequency polygon for the frequency distribution given on Table 1 (The weekly wages of labourers)</td><td>• Give exercise problems on sketching frequency curve
• Ask students to describe how to obtain frequency curve of a data from a histogram of the same data</td></tr>
</table>

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
		 *Fig 2*	• .
• sketch frequency curve for a given frequency distribution	• Frequency Curves	• You may introduce the idea of the graph "Frequency Curve" of a given frequency distribution as nothing but a smoothed curve of the frequency polygon of the given frequency distribution. With active participation of students and with the help of different examples discuss how to sketch frequency curve for a given frequency distribution. For instance the frequency curve for the frequency distribution (whose frequency polygon is given above in Fig. 1) is as shown in Fig. 3 below.  *Fig. 3* • You may also introduce the three familiar types of frequency curves which are given below.	• Give exercise problems on sketching the frequency curve of a given frequency distribution. • Ask students to describe the relationship between a histogram, frequency polygon and frequency curve of a given frequency distribution.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
		 **(a) Positively skewed**  **(b) Symmetrical**  **(c) Negatively skewed** *Fig. 4*	• Let the students describe what information can they get about a data (in a frequency distribution) from symmetric and skewed frequency curves.
• draw bar chart • construct line graph for data related to time.	• **Bar chart** • **Line graph**	• With the help of examples, discuss how to draw "**bar chart**" or "*component bar* chart" which depicts amounts or frequencies for different categories of data by a series of bars (perhaps color-coded). Let first students explain the difference between "bar chart" and "histogram" then guide them to come to the fact that "histogram" always relates to data in a frequency distribution, whereas a "bar chart" depicts amount for any types of categories. Following this, introduce "**line graph**" that we use whenever the categories used represent a time segment, and it portrays changes with amounts in respect to time by a series of line segments.	• Give exercise problems on sketching Bar graphs, line graphs and pie chart as well as their advantage and limitations in providing information.

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• construct pie chart for a given data.	• **Pie chart**	• Discuss the other graphical representation of data known as "**pie chart**" which is particularly appropriate for portraying the divisions of a total amount. In this case with the help of examples introduce the most commonly used pie-chart known as "percentage pie chart" and explain the methods and procedures that the students should follow in drawing this graph and give them exercises/ examples/ to practice.	
• compute the three mean divations of a given data. • describe the relative significance of mean divation as a measure of dispersion.	**8.4 Measures of Central Tendency and Measures of Variability** *(5 periods)* • Mean Deviation	• With the help of examples of ungrouped data and grouped frequency distribution (discrete and continuous series) give a brief revision of calculating the Mean, Median, Mode, Quartile, Range and Standard deviation. • Introduce the concept of "mean deviation" i.e., the measure of dispersion which is based on all items of the distribution. • As deviation may be taken from Mean, median or mode, with active participation of the students discuss the significance of the deviation from each measures of locations in interpreting the data presented. • With the help of examples of both ungrouped and grouped data discuss with students the methods and procedures to compute each of the following a) Mean deviation about the mean b) Mean deviation about the median c) Mean deviation about the mode • In the computation of each type of the deviations, emphasize on the fact that always the absolute value of the deviation (positive value) should be taken during the calculation. With active participation of the students discuss the advantage of each deviation in interpreting the given data out of which the best result are obtained when deviation are taken from median, except when the degree of variability in the series of data is very high.	• Give several exercise problems on both measures of central tendency and measures of dispersion in interpreting a given data.
• calculate the inter-quartile	• Range and inter-quartile range	• After giving a brief revision on calculating quartiles (from measures of location) and	• Ask students to describe

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
range for a given data. • describe inter-quartile range as a measure of variability in values of a given set of data.		Range (from measures of variability or dispersion) for ungrouped data and grouped frequency distributions, define the new type of measure of variability or dispersion namely "inter-quartile range". • With the help of examples, begin with computation of inter-quartile range of ungrouped data and guide students to come to the conclusion that if Q_1 is the first (lower) quartile and Q_2 is the second quartile or median and Q_3 is the third (upper) quartile then the inter-quartile range = Q_3 -Q_1 **Example** For a given ungrouped data say: 5, 8, 8, 11, 11, 11, 14, 16 we have 5, 8, $\mathbf{Q_1}$,8, 11, $\mathbf{Q_2}$,11, 11,$\mathbf{Q_3}$ 14, 16 Q_3 = 13.25 and Q_1 = 8.0 $\therefore$ inter-quartile range = 13.25 - 8.0 = 5.25 • By considering examples of grouped data (both discrete and continuous series) discuss the procedure by which inter-quartile range is computed, and then by giving exercise problems encourage and assist students to find the inter-quartile range of a given data.	which of these measurements gives enough information about variations in values of a given data.
• describe the usefulness of standard deviation in interpreting the variability of a given data.	• Standard Deviation	• With a brief revision of computing standard deviation of ungrouped data and grouped frequency distribution, let student come to the conclusion that among the measures of dispersion (variability) of data the standard deviation is particularly useful in conjunction with the so-called normal distribution.	
• compare two groups of similar data.	**8.5 Analysis of Frequency Distribution** *(2 periods)*	• Let the students explain what they think about how to compare two groups of similar data with respect to consistency, for instance what do they observe from the following data. **E.g.** The mean and standard deviation of the gross incomes of companies in two sectors, A and B are given below.	

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>• determine the consistency of two similar groups of data with equal mean but different standard deviation.</td>
<td></td>
<td>
<table>
<tr><th>School</th><th>Mean income (in Birr)</th><th>Standard Deviation</th></tr>
<tr><td>A</td><td>8000</td><td>120</td></tr>
<tr><td>B</td><td>8000</td><td>140</td></tr>
</table>
• Guide students that having "equal means" but different "standard deviation" (as shown in the table) cannot tell directly the variability in comparing the consistency of the two data, thus introduce the new method of comparison namely "coefficient of variation" (C.V) and discuss with the help of examples how to compute it and lead students to the formula that:
C.V. of the first distribution $= \frac{\sigma_1}{\overline{x}} \times 100$
C.V of the second distribution $= \frac{\sigma_1}{\overline{x}} \times 100$
$\therefore \frac{\text{C.V of first distribution}}{\text{C.V of second distribution}} = \frac{\sigma_1}{\sigma_2}$
which means the comparison depends on the values of σ_1 and σ_2 (Note the means are equal)
</td>
<td>• Give exercise problems on comparing two similar groups of data to determine consistency in values by computation and ask students to describe their situation in the own words.</td>
</tr>
<tr>
<td>• describe the application of coefficient of variation inn comparing two groups of similar data.</td>
<td></td>
<td>• Let the students come to the fact that, the data (series) having greater value of standard deviation (or variance) is less consistent and the one with lesser value of standard deviation (or variance) is more consistent, and using examples encourage the students to compare two distributions with equal means but different standard deviations.</td>
<td></td>
</tr>
<tr>
<td>• describe the relationship among mean, median and mode for grouped data by using its frequency curve.</td>
<td>8.6 Use of Commutative Frequency Graph
(4 periods)
• Relationship among the Mean, Median and Mode</td>
<td>• You may begin the lesson by revising the three types of frequency curves discussed in this unit
• With active participation of the students discuss how different values of the mean, median and mode of a frequency distribution for a grouped data are indicative of the form of the curve in terms of skewness and guide the students to come to the conclusion that:</td>
<td>• Give different types of frequency curves and ask students to describe the relationship among the measures of location of the</td>
</tr>
</table>

Competencies	***Contents***	***Teaching / Learning activities and Resources***	***Assessment***
		a) For a unimodal distribution which is symmetrical the mean, median and mode all coincide in value (see Fig.5b below) (b) For a positively skewed distribution, the Mean is largest in value, and the Median is larger than the Mode but smaller than Mean (see Fig. 5a). c) For a negatively skewed distribution, the Mean is smallest in value and the Median is smaller than the Mode but larger than the Mean (see Fig.5c)	data presented by each graph.

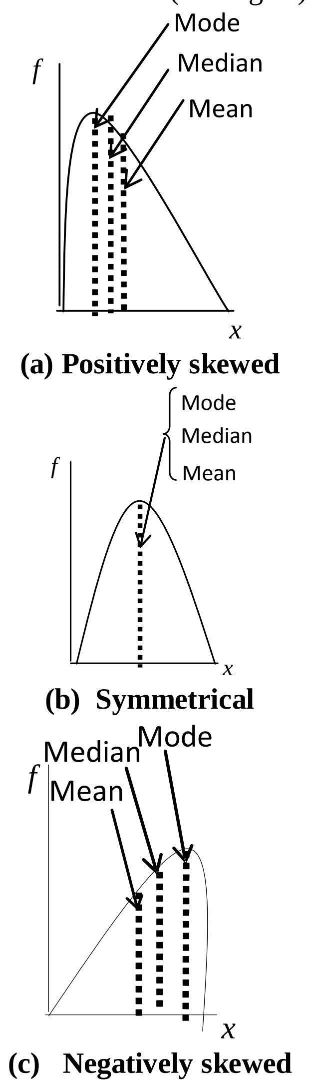


Fig. 4

Competencies	*Contents*	*Teaching / Learning activities and Resources*	*Assessment*
• use cumulative frequency graphs to determine the dispersion of values of data (in terms of its Mean, Median and Standard deviation) • determine the variability of value of data in terms of quartiles by using cumulative frequency graph.	• Mean, Median and standard deviation	• With the helps skewness of cumulative frequency graph discuss the dispersion of values of data in terms of Mean, Median and Standard deviation. • Introduce the other measure of the dispersion from symmetrical distribution which is known as "Pearson's coefficient of skewness" that is obtained by expressing the difference between the mean and the median relative to the standard deviation which means $\begin{matrix}\text{Coefficient}\\ \text{of skewness}\end{matrix} = \dfrac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$ • With active participation of students and with the help of cumulative frequency graph discuss and guide students to the fact that. a) coefficient of skewness = 0, means the distribution is symmetrical (as Mean = median) b) If it is positively skewed, the mean is always greater than the median. c) if it is negatively skewed, then the mean is always, lesser then the Median. • By using the notion of quartile and once more with the help of examples of cumulative frequency graphs guide students to come to the conclusion that, if Q_1 is the lower quartile and Q_3 is the upper quartile, than the data values are: a) positively skewed, when: Median - $Q_1 <$ Q_3 - Median b) negatively skewed, when: Median - $Q_1 >$ Q_3 - Median **Note that:** The median is the second quartile, i.e., Median = Q_2. • Discuss with students how to describe the relationship (if there is any) between the inter - quartile range (Q_3 - Q_1), and the skewness of the cumulative frequency graph.	• Give several exercise problems on determining the dispersion of values of a data by using the measures of locations and the standard deviation of the data given. • Ask students to explain (with their own words) about the variability of values of a given data from their computation (i.e. from the coefficient of skewness)

Unit 9: Mathematical Applications for Business and Consumers *(15 periods)*

Unit outcomes: Students will be able to:

- find unit cost, the most economical purchase and the total cost
- apply percent decrease to business - discount
- calculate the initial expense of buying a home and ongoing expenses of owning a home
- calculate commissions, total hourly wages and salaries.

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
Students should be able to:	***9. Applications for business and consumers***		
• find unit cost	**9.1 Applications to purchasing** *(3 periods)* • Unit cost	• Introduce unit cost by using appropriate examples and let students practice on exercises of finding unit cost. **e.g.** 10 liters of kerosene cost birr 47.50, find the unit cost	• Oral questions on the definition and calculation of unit cost are asked. • Exercise problems on finding unit cost are given, and the solutions are checked.
• find the most economical purchase		• Discuss with students that the most economical buy is often found by comparing unit costs, by taking appropriate examples such as: **e.g.** One store sells 6 cans of cola for birr 20.40, and another store sells 24 cans of the same brand for birr 79.20, Find the better buy.	• Give students exercise problems on determination of most economical purchase like: - Find the more economical purchase, 5 kilograms of nails for birr 32.50 or 4 kilograms of nails for birr 25.80.
• find total cost		• Discuss with students the problem of finding total cost. Let students arrive at the conclusion that Total cost = Unit cost × Number of units after using some appropriate examples.	• Oral questions on the formulation of the formula: Total = Unit×Number

Competencies	*Content*	*Teaching / Learning activities and Resources*	*Assessment*
			• Give various exercise problems on calculation of total cost.
• apply percent increase and percent decrease to business	**9.2 Percent increase and percent decrease** *(4 periods)*	• Revise the concept of percent increase, by using appropriate examples using statements such as "Car prices will show a 3.5% increase over last year's prices", and "Employees were given on 11% pay increase" and discuss the meanings with students. • Encourage students to give their own examples of percent increase and percent decrease and illustrate the meanings. • Revise the formula percent $\times$ base = amount and let students exercise solving problems of percent increase and percent decrease.	
• apply percent increase and percent decrease to business.		• Define the terms "cost", "selling price" and "markup" using appropriate examples and show the relation; (price) - cost = markup price and Markup $\times$ cost = markup. **E.g.** A plant nursery bought a citrus tree for birr 45 and used a markup rate of 46% What is the selling price? and also describe the terms "regular price", "sale price" and "discount" with the relation Regular price - sale price = discount, and that Discount rate$\times$regular price = discount. **E.g.** An appliance store has a washing machine that regularly sells for birr 3500 on sale for birr 2975. What is the discount rate?	• Give various exercise problems on calculation of cost, selling price, markup and markup rate. **E.g.** A bicycle store owner purchases a bicycle for birr 1050 and sells it for birr 1470. What markup rate does the owner use? • Give various exercise problems on calculation of regular price, sale price, discount and discount rate.

Competencies	*Content*	*Teaching / Learning activities and Resources*	*Assessment*
			E.g. 1) A new bridge reduced the normal 45 -minute travel time between two cites by 18 minutes. What percent decrease does this represent? 2) A bookstore is giving a discount of birr 8 on calculators that normally sell for birr 240. What is the discount rate?
• calculate initial expenses of buying a home	**9.3 Real estate expenses** *(4 periods)* • Initial expenses of buying home	• Discuss that one of the largest investments most people ever make is the purchase of a home, and that the major initial expense in the purchase is the down payment, and the amount of the down payment is normally a percent of the purchase price. • Define "the mortgage as the amount that is borrowed to buy real estate" and that the mortgage amount is the difference between the purchase price and the down payment, that is Purchase price – down payment = mortgage.	• Ask oral questions concerning "purchase price", "down payment" and "mortgage". **E.g.** A home is purchased for birr 85,000 and a down payment of birr 12750 is made. Find the mortgage
		• Discuss that another large initial expense in buying a home is the loan origination fee, which is a fee the bank charges for processing the mortgage papers. The loan origination fee is usually a percent of the	• Give exercise problems and check solutions. •

Competencies	*Content*	*Teaching / Learning activities and Resources*	*Assessment*
		mortgage and is expressed in points, which is the term banks use to mean percent. For example, "5 points" means "5 percent". points × mortgage = loan origination fee • Let students practice using exercise problems. • **E.g.** A house is purchased for birr 87,000 and a down payment, which is 20% of the purchase price, is made. Find the mortgage.	**E.g**. A home is purchased with a mortgage of birr 65,000. The buyer pays a loan origination fee of 3½ points. How much is the loan origination fee?
• calculate ongoing expenses of owning a home	• Ongoing expenses of owing home	• Discuss that besides the initial expenses of buying a home, there are continuing monthly expenses involved in owing a home. The monthly mortgage payment, utilities, insurance and taxes are some of these ongoing expenses, and that of these expenses, the largest one is normally the monthly mortgage payment. • Explain that for a fixed rate mortgage, the monthly mortgage payment remains the same throughout the life of the loan, and that the calculation of the monthly mortgage payment is based on the amount of the loan, the interest rate on the loan, and the number of years required to pay back the loan. • Explain that calculating the monthly mortgage payment is fairly difficult, so tables are usually used to simplify the calculations. **E.g.** Monthly payment table.	• Ask oral questions on ongoing expenses of owing a home.

Year	4%	5%	6%
1	0.0851499	0.0856075	0.08860664
2	0.0434249	0.0438714	0.044320
3	0.0295240	0.0299709	0.0304219
4	0.0225791	0.0230293	0.0234850
5	0.0184165	0.0188712	0.0193328
20	0.0060598	0.0065996	0.0071643
25	0.0052784	0.0058459	0.0064430
30	0.0047742	0.0053682	0.0059955

<table>
<tr><th>Competencies</th><th>Content</th><th>Teaching / Learning activities and Resources</th><th>Assessment</th></tr>
<tr><td></td><td></td><td>
<table>
<tr><th>Year</th><th>7%</th><th>8%</th><th>9%</th></tr>
<tr><td>1</td><td>0.0865267</td><td>0.0869884</td><td>0.0874515</td></tr>
<tr><td>2</td><td>0.0447726</td><td>0.0452273</td><td>0.0456847</td></tr>
<tr><td>3</td><td>0.0308771</td><td>0.0313364</td><td>0.0317997</td></tr>
<tr><td>4</td><td>0.0239462</td><td>0.0244129</td><td>0.0248850</td></tr>
<tr><td>5</td><td>0.01980012</td><td>0.0202764</td><td>0.0207584</td></tr>
<tr><td>20</td><td>0.0077530</td><td>0.0083644</td><td>0.0089973</td></tr>
<tr><td>25</td><td>0.0070678</td><td>0.0077182</td><td>0.0083920</td></tr>
<tr><td>30</td><td>0.0066530</td><td>0.0073376</td><td>0.0080462</td></tr>
</table>
E.g.
Find the monthly mortgage payment on a 30 year Birr 60,000 mortgage at an interest rate of 9%. Use the above monthly payment table.
Solution:
Birr 60,000 × 0.0080462 = Birr 482.77
Note that the monthly mortgage payment includes the payment of both principal and the interest on the mortgage.
The interest charged during any one month is charged on the unpaid balance of the loan.</td><td>• Give exercise problems on calculations of ongoing expenses of owing a house.

E.g. Bekele purchased a house for birr 120,000 and made a down payment of Birr 25,000. The savings and loan association charges an annual interest rate of 8% on Bekele's 25 year mortgage. Find the monthly mortgage payment.</td></tr>
<tr><td>• calculate commissions, total hourly wages, and salaries</td><td>9.4 Wages
(4 periods)</td><td>• Explain that commissions, hourly wages and salary are three ways to receive payment for doing work, by using appropriate examples.
E.g. As a real estate broker, Melaku receives a commission of 4.5% of the selling price of a house. Find the commission he earned for selling a home for birr 75,000</td><td>• Ask oral questions on the meanings of commissions, wages and salaries</td></tr>
</table>

Competencies	***Content***	***Teaching / Learning activities and Resources***	***Assessment***
		• Explain that an employee who receives an hourly wage is paid a certain amount for each hour worked. **E.g.** A plumber receives an hourly wage of Birr 13.25. Find the plumber's total wages for working 40 hours. • Discuss with students that an employee who is paid a salary receive payment based on weekly, biweekly (every other week), monthly or annual time schedule, and unlike the employee who receives on hourly wage, the salaried worker does not receive additional pay for working more than the regularly scheduled work day.	• Give various exercise problems on calculation of wages, commissions and salaries. • Give various exercise problems and check solutions. **E.g.** A pharmacists' hourly wage is birr 28. On Saturdays the Pharmacist earns time and half (1½ times the regular hourly wage) How much does the pharmacist earn for working 6 hours on Saturdays?

MATHEMATICS

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