Oscillations and waves

Unit 2

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2.1 Periodic motion (basic concepts) (page 53)	• Describe the periodic motion of a vibrating object in qualitative terms, and analyse it in quantitative terms (e.g. the motion of a pendulum, a vibrating spring, a tuning fork). • Define simple harmonic motion (SHM) and describe the relationship between SHM and circular motion. • Derive and use expressions for the frequency, periodic time, displacement, velocity and acceleration of objects performing SHM. • Draw and analyse x–t, v–t and a–t graphs for SHM. • Use Newton's second law and Hooke's law to derive $\omega = \sqrt{k/m}$. • Describe the effects: free oscillations, damping, forced oscillations and resonance. • Analyse the components of resonance and identify the conditions required for resonance to occur in vibrating objects and in various media, including the effects of damping on resonance. • Explain the energy changes that occur when a body performs SHM. • Draw and interpret graphs showing the variation of kinetic energy and potential energy of an object performing SHM. • Relate the energy of an oscillator to its amplitude. • Solve problems on SHM involving period of vibration and energy transfer.
2.2 Wave motion (page 80)	• Describe the characteristics of a mechanical wave and identify that the speed of the wave depends on the nature of medium. • Use the equation $v = \sqrt{T/\mu}$ to solve related problems • Describe the characteristics of a travelling wave and derive the standard equation $y = A\cos(\omega t + \phi)$ • Define the terms phase, phase speed and phase constant for a travelling wave. • Explain and graphically illustrate the principle of superposition, and identify examples of constructive and destructive interference. • Identify the properties of standing waves and for both mechanical and sound waves, explain the conditions for standing waves to occur, including definitions of the terms node and antinode. • Derive the standing wave equations. • Calculate the frequency of the harmonics along a string, an open pipe and a pipe closed at one end. • Explain the modes of vibration of strings and solve problems involving vibrating strings. • Explain the way air columns vibrate and solve problems involving vibrating air columns.

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Section	Learning competencies
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2.3 Sound, loudness and the human ear (page 97)	• Define the intensity of sound and state the relationship between intensity and distance from the source. • Describe the dependence of the speed of sound on the bulk modulus and density of the medium. Use $v = \sqrt{B/\rho}$ • Give intensity of sound in decibels, and define the terms threshold of pain and threshold of hearing. • Describe the intensity level versus frequency graph to know which the human ear is most sensitive to. • Explain the Doppler effect, and predict in qualitative terms the frequency change that will occur in a variety of conditions. • Explain some practical applications of the Doppler effect.

KEY WORDS

simple harmonic motion *the periodic oscillation of an object about an equilibrium position, such that its acceleration is always directly proportional in size but opposite in direction to its displacement*

oscillating *vibrating about a central position*

equilibrium position *the position of an oscillating object when at rest*

restoring force *the force on a displaced object that acts towards its original position*

A great many things in the world around us oscillate (vibrate) backwards and forwards, up and down, side to side, in and out, etc. Atoms within molecules vibrate and the size of these vibrations is proportional to temperature. Oscillations of charges produce electromagnetic waves: e.g. a current oscillating up and down an aerial produces radio waves. Vibrations of our vocal chords produce sound waves, as do vibrations of strings and of air inside tubes in musical instruments. Parts of machinery, e.g. in washing machines and in cars, vibrate, sometimes when we don't want them to!

When engineers build large structures like skyscrapers and bridges, they have to understand how the wind or the ways people walk across them will make them oscillate. It is impossible to stop such structures oscillating altogether, but if engineers don't design their structures to control these vibrations, they might end up shaking themselves to pieces.

Most of these oscillations are periodic. This means that they keep doing exactly the same thing in the same amount of time again and again. In some cases, usually for large objects or structures, each cycle (backwards and forwards, up and down, side to side, in and out, etc.) of the oscillation could take many seconds or even much longer. These are low-frequency oscillations. In other cases there can be hundreds, thousands or even thousands of billions of complete vibrations every second. These are high-frequency oscillations. This predictable time period can be very useful. For example, the predictable time period of pendulums, of masses on springs or of quartz crystals is used to count the passing of time in clocks and watches.

The way things oscillate can be quite complex, but many oscillations are very close to a special form of periodic motion called **simple harmonic motion**, and more complicated motion can be shown to be simply a sum of simple harmonic motions at different frequencies. This unit will analyse a few examples of oscillating objects performing simple harmonic motion in some mathematical detail.

2.1 Periodic motion (basic concepts)

By the end of this section you should be able to:

- Describe the periodic motion of a vibrating object in qualitative terms, and analyse it in quantitative terms (e.g. the motion of a pendulum, a vibrating spring, a tuning fork).
- Define simple harmonic motion (SHM) and describe the relationship between SHM and circular motion.
- Derive and use expressions for the frequency, periodic time, displacement, velocity and acceleration of objects performing SHM.
- Draw and analyse x–t, v–t and a–t graphs for SHM.
- Use Newton's second law and Hooke's law to derive $\omega = \sqrt{k/m}$.
- Describe the effects: free oscillations, damping, forced oscillations and resonance.
- Analyse the components of resonance and identify the conditions required for resonance to occur in vibrating objects and in various media, including the effects of damping on resonance.
- Explain the energy changes that occur when a body performs SHM.
- Draw and interpret graphs showing the variation of kinetic energy and potential energy of an object performing SHM.
- Relate the energy of an oscillator to its amplitude.
- Solve problems on SHM involving period of vibration and energy transfer.

Periodic oscillations

If something is **oscillating** (vibrating) this means that it is moving backwards and forwards, up and down, side to side, in and out, etc, around some central position. This central position is called the equilibrium position and it is the position of the object when it is at rest.

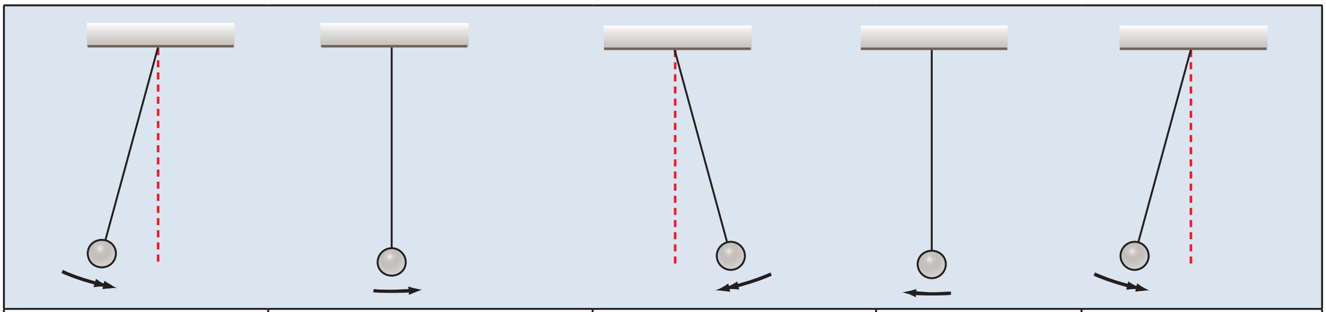
Whenever an object is displaced from its **equilibrium position** there is a force that acts towards its original position. This force is often referred to as a **restoring force**, as it tries to restore the system to its equilibrium position. This is much easier to understand if we look at some simple examples.

DID YOU KNOW?

The pendulum clock was invented in 1656 by Dutch scientist Christiaan Huygens. Huygens was inspired by investigations of pendulums by Galilei Galileo, beginning around 1602. Galileo discovered the key property that makes pendulums useful timekeepers: isochronism, which means that the period of swing of a pendulum is approximately the same for different sized swings. Up until the 1930s, the pendulum clock was the world's most accurate timekeeper, but they must be stationary to operate as any motion or accelerations will affect the motion of the pendulum, causing inaccuracies, and so they could never be used for portable devices. They are now out of date of course; we now have more accurate devices, though still using simple harmonic motion.

***Figure 2.1** A simple pendulum clock*

How does a pendulum work?



(a)	(b)	(c)	(d)	(e)
(a) If the pendulum bob is pulled to one side and released, it accelerates back towards its equilibrium position.	(b) When the pendulum bob gets back to the equilibrium position, it is moving relatively fast and, although there is no resultant force now, its inertia keeps it moving.	(c) The pendulum bob keeps moving, slowing down all the time, until it is at the same height as it started and accelerates back towards the equilibrium position.	(d) The pendulum bob passes through the equilibrium position again, going back the other way.	(e) The pendulum bob arrives back at where it started. It has completed one cycle, and will now do the same again, and again . . .

***Figure 2.2** Oscillation of a pendulum when the bob is pulled to one side and released*

KEY WORDS

resultant force *the overall force acting on an object*

acceleration *rate of change of velocity*

A simple pendulum is made by hanging a mass, known as the bob, on a string from a fixed support, as shown in Figure 2.2.

If we let the mass hang without swinging, it will hang directly below the support with all forces on it balanced. This position, where the **resultant force** acting on the bob is zero, is known as the equilibrium position.

If we give the bob a small initial displacement by pulling it to one side and then release it, there will be a resultant force, due to the weight of the bob and the tension acting in the string. This force pulls it back towards the equilibrium position. This causes **acceleration** towards the equilibrium position (opposite to the direction of displacement).

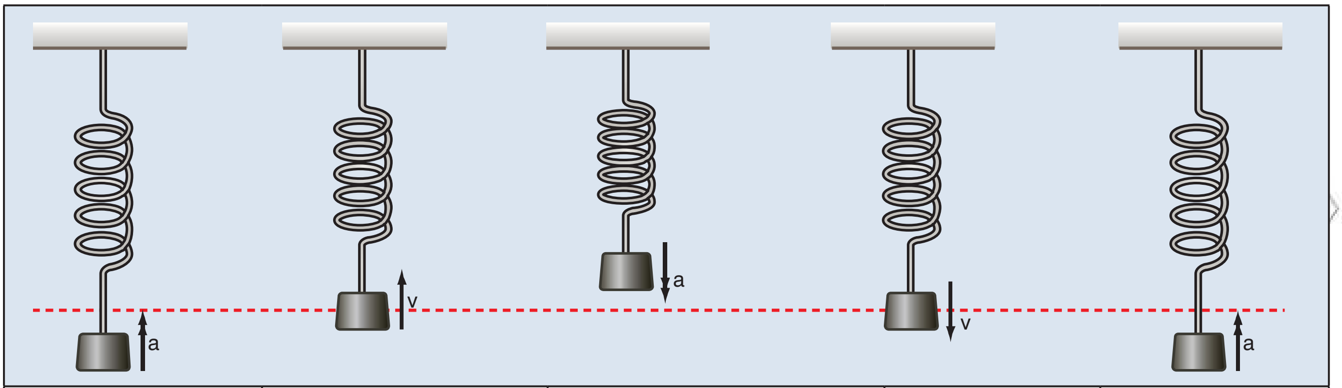
When the bob reaches the equilibrium position, the resultant force is now zero, but the bob is moving and can't stop instantly. Its inertia keeps it moving through the equilibrium position, and if there is no significant friction of air resistance, it will keep moving, slowing down all the time until it is as high as it was when it started.

It now has a displacement equal and opposite to its starting displacement. However, as displacement is a vector quantity it is now a negative value. If the initial displacement was 3 cm, the displacement after one swing (half an oscillation) will be –3 cm.

In exactly the same way, it will swing back to where it started to complete one complete cycle of the oscillation. It will now repeat this process again and again.

It is important to notice the force causing the oscillation always acts towards the equilibrium position.

How does a mass on a spring oscillate?



(a) If the mass is pulled down, the tension in the spring is greater the weight of the mass. The resultant force is upwards and the mass accelerates upwards when released.	(b) When the mass gets back to the equilibrium position, it is moving relatively fast and although the weight of the mass balances the tension in the spring and there is no resultant force now, its inertia keeps it moving.	(c) The mass keeps moving, slowing down all the time, until it is as far above the equilibrium point as it started below. The weight of the mass is greater than the tension in the spring, the resultant force is now downwards and the mass accelerates downwards.	(d) The mass passes through the equilibrium position again, going back the other way.	(e) The mass arrives back at where it started. It has completed one cycle, and will now do the same again, and again . . .

***Figure 2.3** Oscillation of a mass–spring system when the mass is displaced downwards and released*

If a mass is hung from a support by a spring and allowed to settle until it is stationary, it will hang with the spring stretched so that the restoring force (in this case the tension in the spring) is equal and opposite to the weight of the mass. This is the equilibrium position.

If we now pull the mass down, the tension in the spring will be greater than the weight of the mass. The resultant force on the mass is upwards and so, if we let go, it accelerates upwards. When the mass gets back to the equilibrium position it is moving and, although there is no resultant force here, its inertia keeps it moving.

The mass keeps moving, slowing down all the time, until it is as far above the equilibrium point as it started below. The tension in the spring is now less than the weight of the mass, the resultant force is now downwards and the mass accelerates downwards. The mass passes through the equilibrium position again, and carries on until it arrives back at where it started. It has completed one cycle, and will now do the same again, and again …

Activity 2.1: Mass–spring forces

Sketch a diagram of the forces acting on a mass–spring system:

a) when in equilibrium position

b) at the bottom of its oscillation

c) at the top of its oscillation.

This process would also happen if the spring was horizontal on a low friction surface.

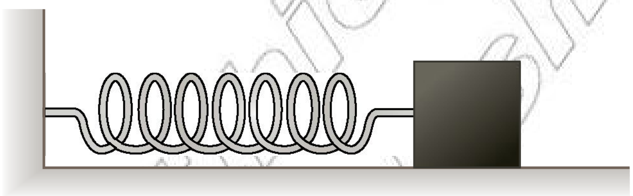


***Figure 2.4** A horizontal mass–spring system*

***Figure 2.5** Vehicle suspensions can act like mass–spring systems.*

Discussion activity

We have used the words displacement and acceleration in describing the motions of the pendulum and the mass–spring system. These are vector quantities: their directions are very important. When the bob or mass is moving away from the equilibrium position and slowing down, which direction is the acceleration in? What can you say about the direction of the acceleration (i) relative to the equilibrium position, and (ii) relative to the direction of displacement? What do you know in general about the acceleration of an object and the resultant force acting on it?

How do we define SHM?

Simple harmonic motion (SHM) is a periodic oscillation of an object about an equilibrium position such that its acceleration is always directly proportional in size but opposite in direction to its displacement. (The acceleration is always towards the equilibrium position.)

This defining relationship is shown in Figure 2.6. This graph is much simpler than many graphs that will follow later in this unit, but **it is the most important**.

It follows from Hooke's law that the restoring force has the same relationship to the displacement (as forces and acceleration are directly proportional). The greater the displacement from the equilibrium position the greater the restoring force, and this force acts in the opposite direction to the displacement.

SHM

- **The acceleration is proportional to the displacement.**
- **The acceleration is in the opposite direction to the displacement.**

Consider the mass–spring system. When the spring is most extended, it is furthest from its equilibrium position. At this point the restoring force is also at its greatest, but it acts in the opposite direction.

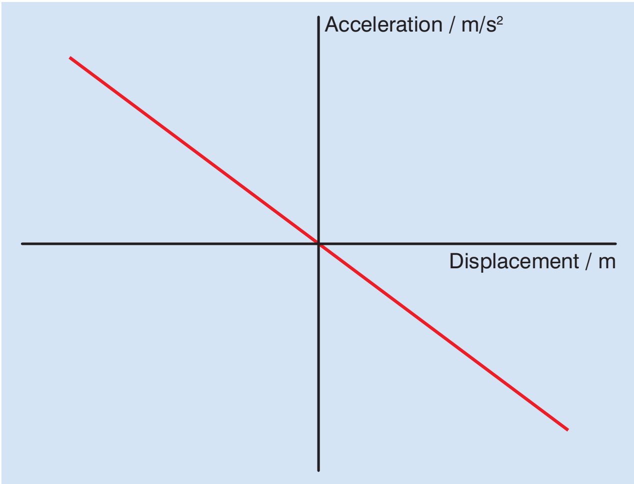


Figure 2.6 *Defining relationship for SHM: acceleration is directly proportional but opposite in sign to displacement.*

What does SHM look like?

If we plot how the **displacement** of an object performing simple harmonic motion varies with time, we find that the variation is **sinusoidal**, as shown in Figure 2.7. Note that the displacement goes positive and negative as the mass oscillates either side of the equilibrium position.

The size of the maximum displacement in either direction is called the **amplitude** A. The time to perform one complete cycle of the oscillation is called the **time period** T.

KEY WORDS

displacement *the distance moved in a specific direction*

sinusoidal *an oscillation that can be described mathematically using sine or cosine functions*

amplitude *the maximum displacement of the wave from the equilibrium position*

time period *the time taken for one complete cycle of an oscillation*

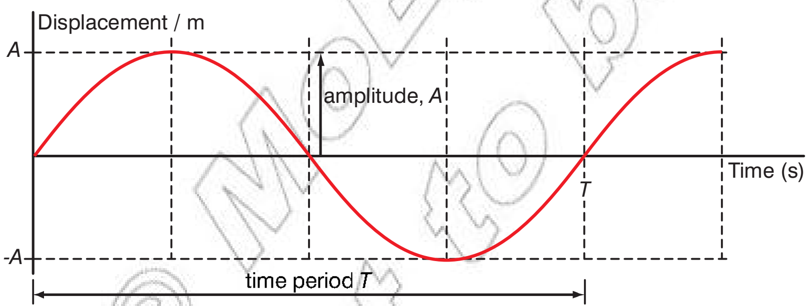


Figure 2.7 *Variation of displacement with time for simple harmonic motion*

When we say the oscillation is sinusoidal, we mean that the displacement is described mathematically using sine or cosine functions:

$$x = A \sin\left(2\pi \frac{t}{T}\right) \text{ or } x = A \cos\left(2\pi \frac{t}{T}\right),$$

where A is the amplitude of the oscillation and T the time period. Either could be used, but throughout the rest of this chapter we will use,

$$x = A \sin\left(2\pi \frac{t}{T}\right),$$

although the cosine function gives a better description if the SHM is started by displacing the oscillator and then releasing it.

If $x = A \sin\left(2\pi \frac{t}{T}\right)$, with $\left(\frac{2\pi}{T}t\right)$ expressed in radians:

when $t = 0$	$x = A \sin\left(\frac{2\pi}{T}0\right)$	$= A \sin(0)$	$= 0$
$t = \frac{T}{4}$	$x = A \sin\left(\frac{2\pi}{T}\frac{T}{4}\right)$	$= A \sin\left(\frac{\pi}{2}\right)$	$= A$
$t = \frac{T}{2}$	$x = A \sin\left(\frac{2\pi}{T}\frac{T}{2}\right)$	$= A \sin(\pi)$	$= 0$
$t = \frac{3T}{4}$	$x = A \sin\left(\frac{2\pi}{T}\frac{3T}{4}\right)$	$= A \sin\left(\frac{3\pi}{2}\right)$	$= -A$
$t = T$	$x = A \sin\left(\frac{2\pi}{T}T\right)$	$= A \sin(2\pi)$	$= 0$

Looking carefully at the information above you can see how in one oscillation the displacement starts at 0 rises to a positive amplitude, falls back to zero, falls to a negative amplitude and then rises back to zero.

Activity 2.2: Displacement using cosine

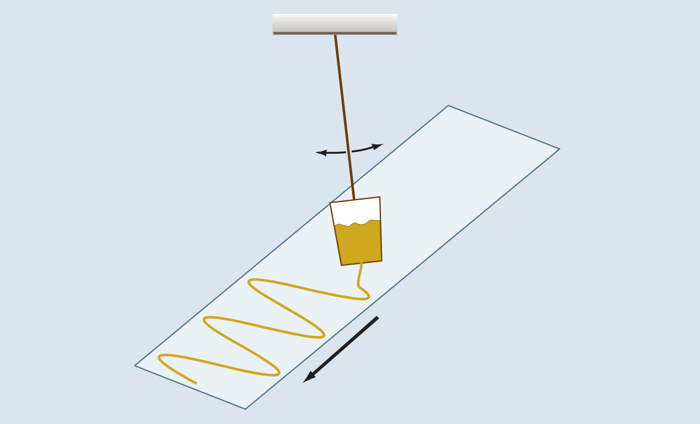
Use the same method above to show how the displacement varies if cosine were to be used instead of sine. Sketch the corresponding displacement–time graph.

Discussion activity

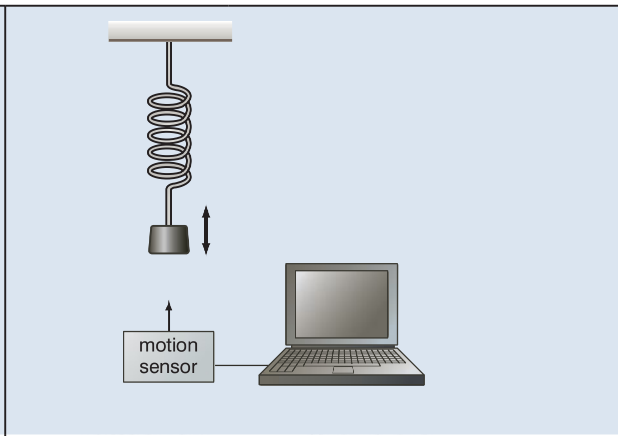
A sinusoidal motion looks fairly complicated, so why is simple harmonic motion called simple? Looking at Figure 2.7 should give you a clue.

How can we observe SHM?

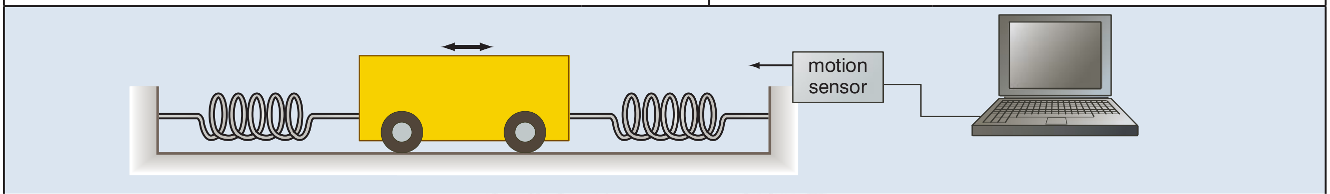
Figure 2.8 shows a number of ways of obtaining a graph of displacement against time for oscillators performing SHM.



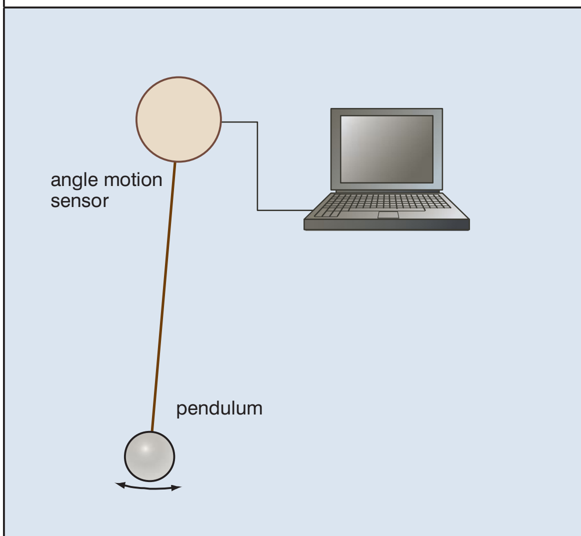
Hang a small bucket with a hole in the bottom on a rope. Fill the bucket with sand so that a stream of sand runs out onto a long sheet of paper underneath. Start the bucket swinging and pull the paper at a constant speed and the sand will draw a sinusoidal wave.



If you have a motion sensor connected to a PC, there are several ways to record variation of displacement with time. One way is to place the sensor facing upwards underneath a mass hanging on a spring.



A dynamics trolley moving backwards and forwards with a anchored spring attached to both ends can make a more consistent target for a motion sensor.



An angular motion sensor is an easy way of observing how a pendulum swings.

Figure 2.8 *Experiments to observe SHM*

Think about this...

If the frequency of a mass–spring system is 50 Hz how many times in 1 second will the mass pass through its equilibrium position?

Activity 2.3: Displacements

For the same pendulum calculate the displacement after:

a) 1.2 s

b) 3.4 s

Explain the significance of the negative value.

Think about this...

It is a good idea to avoid using the word 'fast' when describing oscillations. Even if the frequency is low, if the amplitude of the oscillation is large, the oscillator will be moving quickly when it goes through the equilibrium position, and so the word fast could still apply. This can lead to confusion.

To give a good clear scientific description, simply talk about high or low frequencies and large or small amplitudes.

KEY WORDS

frequency *the number of cycles per second*

Frequency and time period

The **frequency**, f, of an oscillation is the number of cycles it completes per second. The unit is the hertz, symbol Hz. A frequency of 50 Hz would correspond to 50 complete oscillations per second.

Frequency is related to time period by:

- $f = \dfrac{1}{T}$

and so our mathematical expression for displacement can be written as

- $x = A \sin (2\pi f t)$.

Worked example 2.1

A pendulum has a frequency of 4.0 Hz and amplitude of 5.0 cm. Determine the displacement after 4.6 seconds.

- $x = A \sin (2\pi f t)$. *State the equation of SHM*
- $x = 0.050 \sin (2\pi \times 4.0 \times 4.6)$ *Substitute the known values*
- $x = 0.029$ m or 2.9 cm. *Solve the equation and give the units*

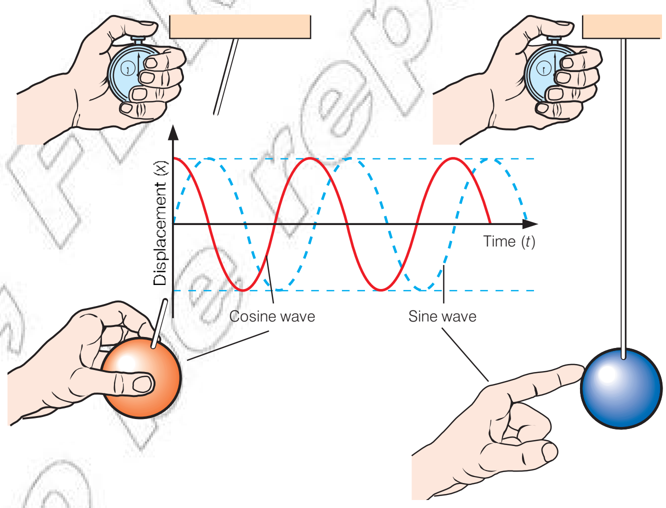


***Figure 2.9** Oscillations of an object*

Discussion activity

One feature of SHM, particularly useful in building clocks, is that, for perfect SHM, the frequency or time period does not vary with amplitude. So if an oscillator does lose energy and its amplitude fall over time, the time period will not change.

Circular motion, SHM and angular frequency

If a point P moves around in circle of radius A, as shown in Figure 2.10, starting from point C, then the height of point P, after it has turned through angle θ is given by

- $h = A\sin\theta$

We now need to introduce a new quantity: angular speed ω. Angular speed is the rate of change of angle turned θ with time, in exactly the same way that linear velocity v is the rate of change of linear displacement s with time. ω is measured in radians per second (rad/s).

- $\omega = \dfrac{\theta}{t}$

If the point P is rotating at angular speed ω radians per seconds then, after time t seconds, the total angle turned, in radians, is

- $\theta = \omega t$

and so we can rewrite the equation for the height of point P as

- $h = A\sin(\omega t)$

If P goes round in one complete cycle, the angle turned is 2π. If P is rotating with a frequency of f cycles per second, the total angle turned per second is $f \times 2\pi$ radians. Hence

- $\omega = 2\pi f$

In the equation $x = A\sin(2\pi ft)$ we can replace $2\pi f$ by ω and write:

- $x = A\sin(\omega t)$

which is the same as our expression for the height of point P rotating at angular speed ω. The height h of point P going round in a circle and the displacement x of an object performing SHM are therefore the same, as shown in Figure 2.11.

Think about this...

We must use radians when considering all the equations of SHM. Remember 2π radians is equal to 360° or one complete oscillation.

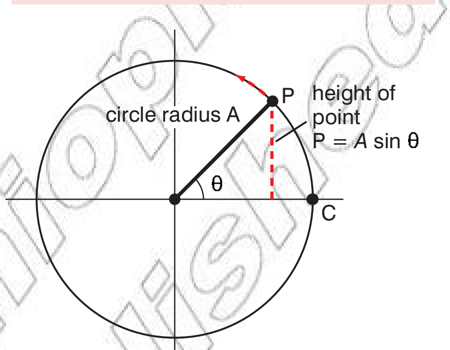


***Figure 2.10** The height h of point P when it has turned through angle θ from starting point C is given by h = Asin (ωt).*

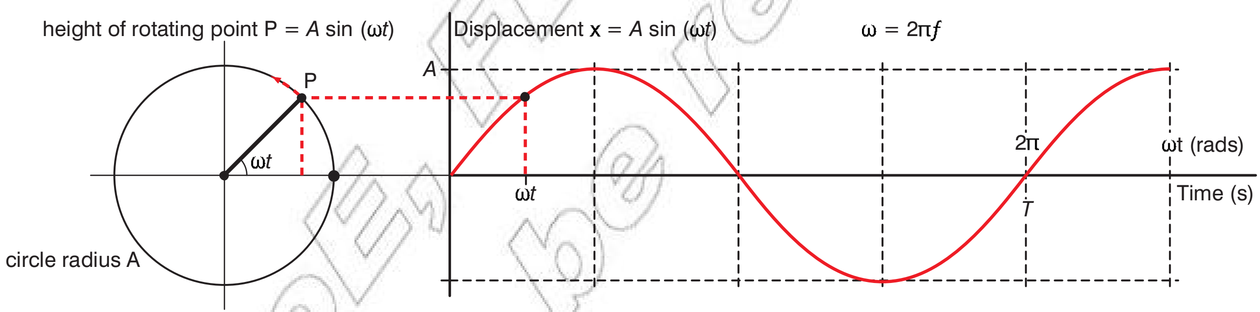


***Figure 2.11** After time t, the displacement x of an object performing SHM of amplitude A and frequency f is the same as the height h of a point P performing circular motion with radius A and angular speed ω = 2πf.*

The relationship between **angular speed** and time period is

- $\omega = \dfrac{2\pi}{T}$

Because of its relationship to frequency, ω is sometimes called **angular frequency**.

KEY WORDS

angular speed *the rate of change of angle turned with time*

angular frequency *the rate of change of angular displacement*

Discussion activity

Figure 2.12 shows a piston moving backwards and forwards inside a cylinder connected by a rod, hinged at both ends, to a rotating wheel.

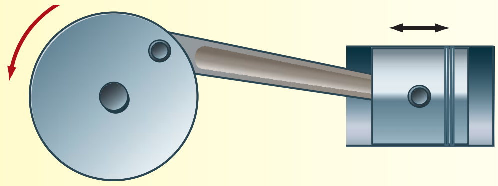


Figure 2.12 *Motion of a piston*

Under what conditions does the motion of the piston approximate to SHM?

KEY WORDS

rate of change *The speed at which a variable changes over a specific period of time*

Displacement, velocity and acceleration in SHM

The velocity of the oscillating mass is the **rate of change** of displacement. It becomes zero at the limits of the oscillation. For example, at the top of a pendulum's swing.

In general, velocity can be found as the gradient of the displacement–time graph. At the maximum displacement (the amplitude) the gradient is zero – consequently the velocity is zero. Therefore, when $x = \pm A$ the velocity is zero.

The maximum velocity occurs when the oscillating object passes through the equilibrium position. Again this may be seen on the displacement–time graph. The gradient of the line is at its greatest when it passes through the equilibrium position therefore the velocity is greatest at this point.

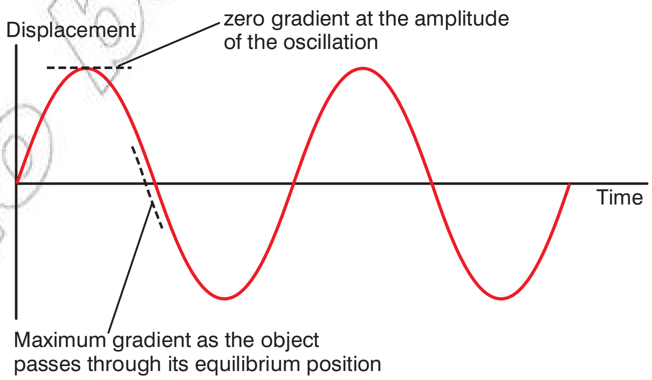


Figure 2.13 *The gradient of the displacement–time graph is equal to the velocity of the object.*

The equation of the oscillation shown in Figure 2.13 is $x = A\sin(\omega t)$. Therefore the velocity may be found by:

- $v = \omega A\cos(\omega t)$

and the maximum size of velocity (v_o)

- $v_o = \omega A$

This equation is obtained when cos(ωt) = 1. This happens whenever the mass passes through the equilibrium position.

Acceleration is the **rate of change** of velocity, or the gradient of the velocity time graph. This can be shown to give

- $\boldsymbol{a = -\omega^2 A\sin(\omega t)}$

and the maximum **size** of acceleration (a_o)

- $a_o = \omega^2 A$

This equation is obtained when sin(ωt) = 1. This happens whenever the mass reaches its maximum displacement.

Since $A\sin(\omega t) = x$, this is the same as

- $\boldsymbol{a = -\omega^2 Ax}$

This is the defining equation for SHM.

We have already stated that acceleration is directly proportional and opposite in sign to displacement. We now see that the constant of proportionality is $-\omega^2$. Remember this is also equal to $(2\pi f)^2$ or $(2\pi/T)^2$.

A graph of acceleration plotted against time will look like an upside down version of the graph of displacement, emphasising the crucial point that acceleration is always in the opposite direction to displacement.

Graphs of displacement, velocity and acceleration against time t or angle ωt are shown overleaf in Figure 2.14.

Two key points to note and check for yourself looking at these is that:

- **the velocity at any time is the gradient of the displacement-time graph at that time and the acceleration at any time is the gradient of the velocity–time graph at that time, and**
- **the acceleration is directly proportional to and opposite in sign to the displacement.**

If the oscillator starts from the limit of oscillation at $x = A$, then displacement is better described using a cosine wave and the equations for displacement, velocity and acceleration become:

SHM equation summary

Remember you can use either the sine or cosine function to describe the displacement of a system oscillating with SHM. You need to consider when the timing of the oscillation begins.

- Timing starts with system in its equilibrium position → sine
- Timing starts with system at its maximum displacement → cosine

Think about this...

The equation for velocity can be found using differential calculus.

$$v = \frac{ds}{dt} = \frac{d}{dt}(A\sin(\omega t))$$
$$= \omega A\cos(\omega t)$$

Think about this...

The equation for acceleration can be found using differential calculus.

$$a = \frac{dv}{dt} = \frac{d}{dt}\omega A\cos(\omega t)$$
$$= -\omega^2 A\sin(\omega t)$$

The equation can be written as $a = -\omega^2 x$.

This is a differential equation and we have just shown that $x = A\sin(\omega t)$ is a solution of this differential equation. It can be shown that $x = A\cos(\omega t)$ is an equally valid solution, and that a more general solution is $x = A\cos(\omega t + \theta_o)$, where θ_o is an initial "phase" angle.

The equations are summarised below:

	Using sin to describe displacement	Using cos to describe displacement
Displacement	$x = A\sin(\omega t)$	$x = A\cos(\omega t)$
Velocity	$v = \omega A\cos(\omega t)$	$v = -\omega A\sin(\omega t)$
Acceleration	$a = -\omega^2 A\sin(\omega t)$	$a = -\omega^2 A\cos(\omega t)$

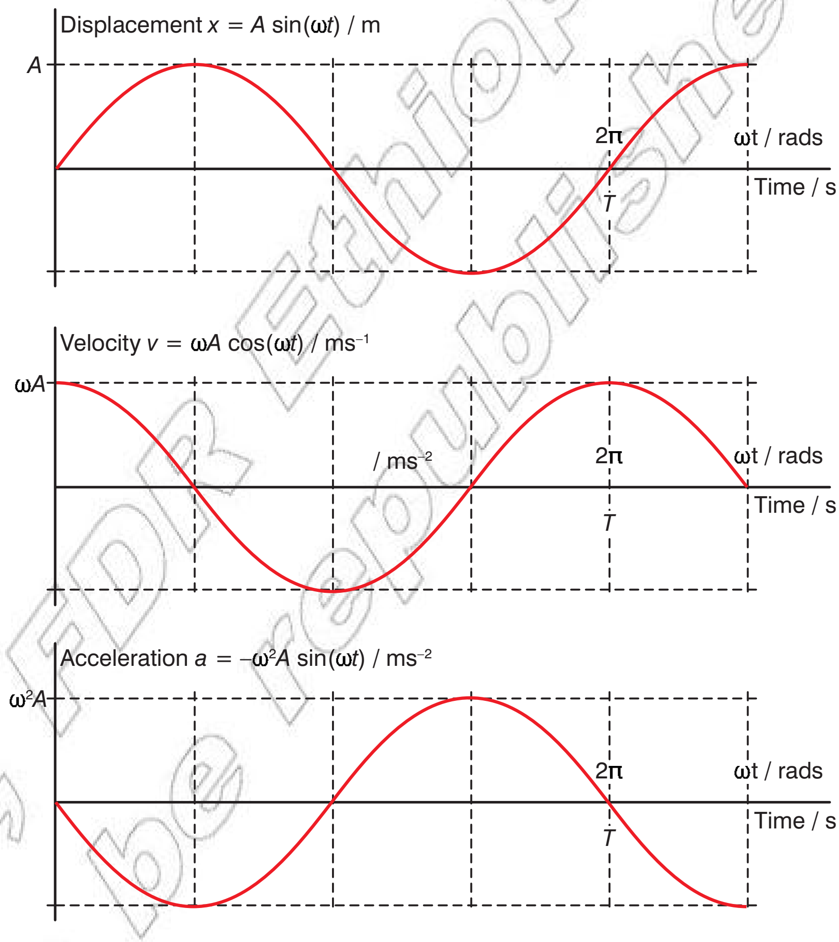


***Figure 2.14** Graphs of displacement, velocity and acceleration against time t or angle ωt for an object performing SHM*

Worked example 2.2

An object moves with simple harmonic motion of amplitude 11 cm and a time period of 2.4 s. Calculate:

a) the frequency

b) the angular frequency

c) the maximum velocity of the object

d) the maximum acceleration of the object

e) the displacement, velocity and acceleration, after 0.5 s, if the object starts from the limit of oscillation at $x = A$.

a) Frequency

- $f = \frac{1}{T}$ *State the relationship to be used*
- $f = \frac{1}{2.4} = 0.42$ Hz *Substitute in known values and solve, giving the units*

b) Angular frequency

- $\omega = 2\pi f = \frac{2\pi}{T}$ *State the relationship to be used*
- $\omega = \frac{2\pi}{2.4} = 2.6$ rad/s *Substitute in known values and solve, giving the units*

c) Maximum velocity

- $v_0 = \omega A$ *State the relationship to be used*
- $v_0 = \frac{2\pi \times 0.11}{2.4} = 0.29$ m/s *Substitute in known values and solve, giving the units*

d) Maximum acceleration

- $a = \omega^2 A$ *State the relationship to be used*
- $a = \left(\frac{2\pi}{2.4}\right)^2 \times 0.11 = 0.75 \text{ m s}^{-2}$ *Substitute in known values and solve, giving the units*

e) Using the cosine equation for displacement, after 0.5 s

- $x = A\cos(\omega t)$ *State the relationship to be used*
- $x = 0.11 \times \cos\left(\frac{2\pi}{2.4} \times 0.5\right)$
  $= 0.11 \times \cos(1.31)$
  $= 0.11 \times 0.259 = 0.028$ m

Substitute in known values and solve, giving the units

The use the appropriate equation to find the velocity, in this case:

- $v = -\omega A\sin(\omega t)$
- $v = -\left(\frac{2\pi}{2.4}\right) \times 0.11 \times \sin(1.31)$
  $= -\left(\frac{2\pi}{2.4}\right) \times 0.11 \times 0.966$
  $= -0.28 \text{ m s}^{-1}$

Substitute in known values and solve, giving the units

Finally use the defining equation for SHM to find the acceleration.

- $a = -\omega^2 A\cos(\omega t)$ *State the relationship to be used*
- $a = -\left(\frac{2\pi}{2.4}\right)^2 \times 0.11 \times \cos\left(\frac{2\pi}{2.4} \times 0.5\right) = -0.20 \text{ m s}^{-2}$

Substitute in known values and solve, giving the units

How does velocity depend on displacement?

We already know that the velocity is zero when $x = \pm A$ at the limits of the oscillation, and that it has its maximum size of $v_o = \omega A$ when the mass passes through the equilibrium position.

To get a general expression we need to use a trigonometric identity:

- $\sin^2\theta + \cos^2\theta = 1$

If $v = -\omega A\cos(\omega t)$ we have $v^2 = \omega^2 A^2 \cos^2(\omega t)$

If $x = A\sin(\omega t)$ we have $\omega^2 x^2 = \omega^2 A^2 \sin^2(\omega t)$

and so

- $v^2 + \omega^2 x^2 = \omega^2 A^2 \cos^2(\omega t) + \omega^2 A^2 \sin^2(\omega t)$
- $v^2 + \omega^2 x^2 = \omega^2 A^2[\cos^2(\omega t) + \sin^2(\omega t)]$
- $v^2 + \omega^2 x^2 = \omega^2 A^2 \times 1$
- $v^2 + \omega^2 x^2 = \omega^2 A^2$
- $v^2 = \omega^2 A^2 - \omega^2 x^2$
- $v = \pm\omega\sqrt{A^2 - x^2}$

This equation shows us that at any given displacement (x) an oscillating object may have +/– a specific velocity. This is easy to explain.

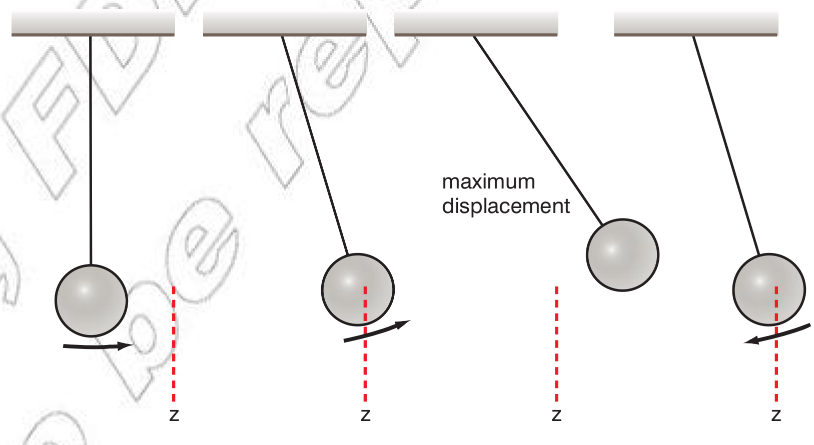


Figure 2.15 As part of any oscillation the mass will pass through the same point twice.

If you consider a simple pendulum swinging towards its maximum displacement, on its way up it passes through point Z. It then stops and swings back through Z in the opposite direction. Therefore at any given displacement a pendulum bob may have a velocity equal to $+v$ or $-v$.

If the displacement is 0 (i.e. as the mass passes through the equilibrium position) $x = 0$. Therefore $v = \pm\omega\sqrt{A^2 - x^2}$ becomes $v = \pm\omega\sqrt{A^2 - 0^2}$. This simplifies to $v_o = \pm\omega A$.

How do we calculate the time periods of real examples of SHM?

In analysing real systems that perform SHM to find their time period we always follow the same procedure. We imagine the oscillating mass being displaced from the equilibrium position by displacement x and analyse the forces acting on it as a function of its displacement.

If the system does perform SHM, this resultant force will be a restoring force proportional to x:

- $F = -kx$

where k is a constant of proportionality depending on the parameters of the system. (This equation can be used as an alternative definition of SHM.)

If we now apply (from Newton's second law), we can replace F to write:

- $ma = -kx$
- $a = -\frac{k}{m}x$

Comparing this with the defining equation for SHM

- $a = -\omega^2 x$

we see that

- $\omega^2 = \frac{k}{m}$

This gives us the angular frequency ω, and from this we can obtain frequency f or time period T by

- $f = \frac{\omega}{2\pi}$

or

- $T = \frac{2\pi}{\omega}$

What is the time period of a mass–spring system?

Here, we consider a mass m suspended by a spring of spring constant k. This analysis is complicated a little by the fact that the spring is already stretched when it is in the equilibrium position but, as we shall see, terms that this causes in the equations cancel out, and the analysis ends up looking like the general analysis above.

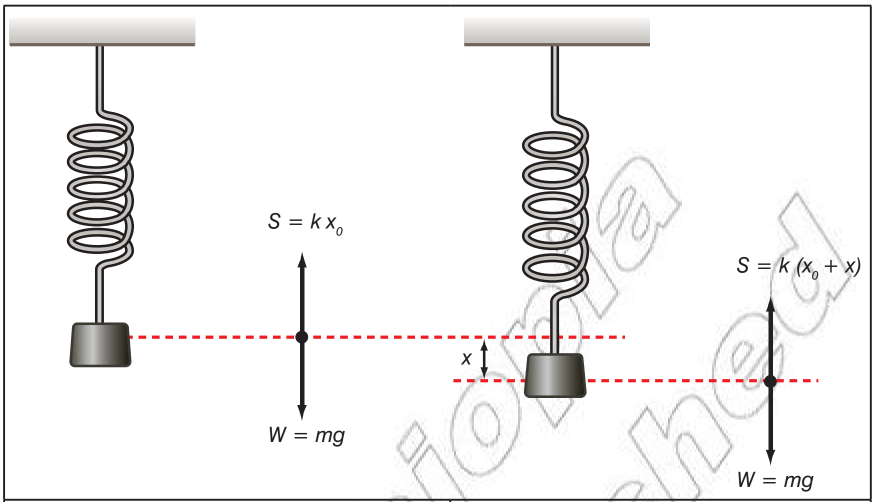
At equilibrium the tension in the spring is equal and opposite to the weight of the mass

- $S = W$
- $kx_o = mg$

and the resultant force downwards on the mass is zero.

Think about this...

It is interesting to note the time period of a mass spring system is independent of the gravitational field strength. Take a mass spring system to another planet and the time period of its oscillations will be the same.



(a) Spring and mass at equilibrium with spring stretched so that the tension *S* in the spring is equal and opposite to the weight of the mass.	(b) Mass displaced *x* downwards from equilibrium position. Tension in spring now greater than the weight of the mass.

Figure 2.16 *A mass–spring system*

When mass is displaced x downwards, the tension in the spring, and hence the resultant force downwards (in the direction of the displacement), is

- $F = W - S$
- $F = mg - k(x_o + x)$
- $F = mg - kx_o - kx$

But, since $kx_o = mg$, this is

- $F = mg - mg - kx$
- $F = - kx$

Newton's second law tells us that, and so

- $ma = - kx$

and hence

- $a = -\frac{k}{m}x$

Comparing this with the general defining equation for SHM $a = -\omega^2 x$, and recalling that $\omega = \frac{2\pi}{T}$ we have

- $\omega^2 = \frac{k}{m}$
- $\frac{2\pi}{T} = \omega = \sqrt{\frac{k}{m}}$
- $\frac{T}{2\pi} = \sqrt{\frac{m}{k}}$
- $T = 2\pi\sqrt{\frac{m}{k}}$

We have of course assumed that Hooke's law ($S = k \times$ extension) is obeyed. As long as it is, and provided that we can ignore energy losses in the spring and due to air resistance, the mass–spring system performs perfect SHM.

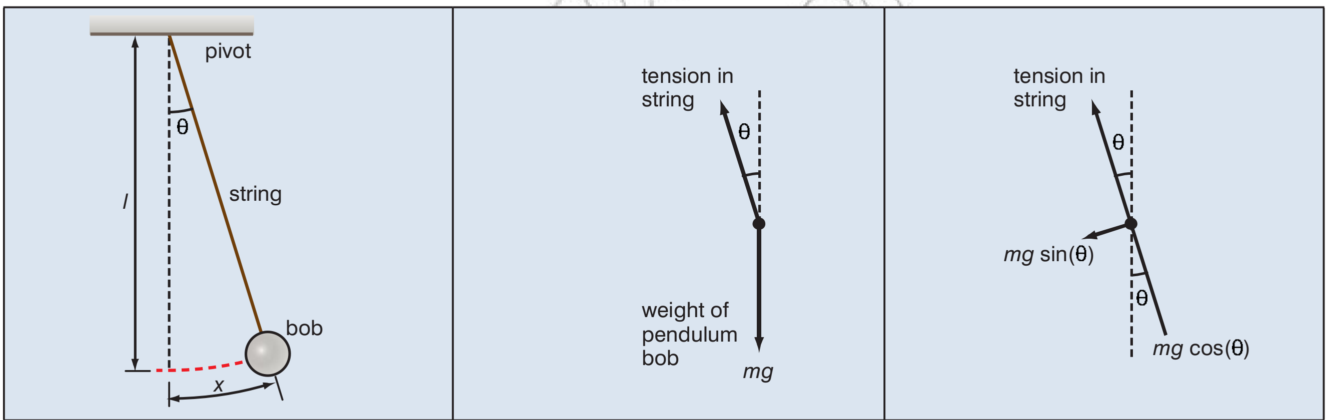
If we make the amplitude of the oscillations too large, however, and we exceed the elastic limit of the spring the above equations are no longer valid and the time period will probably start to become a little longer.

What is the time period of a simple pendulum?

A simple pendulum comprises a single mass m, which we treat as a point mass on a string, length l, (or frictionlessly pivoted rod) whose mass we ignore. This is clearly an approximation and analysis of a simple pendulum is made a little more complicated by the need to make a few more approximations.

To find the time period of a simple pendulum consider the motion of the bob in a circle radius l about the pivot. We analyse the forces acting on the pendulum bob for a displacement x along the circular path that the bob follows, which corresponds to an angular displacement θ, as shown in Figure 2.17. From the definition of angle measurement, in radians

- $\theta = \frac{x}{l}$



(a) Pendulum displaced by angle θ. Note: the pendulum bob follows a circular path	(b) Free body force diagram for pendulum bob.	(c) Free body force diagram with resolved into radial and tangential components relative to circular path of bob

Figure 2.17 *Angular displacement of a pendulum bob*

There are two forces acting on the pendulum bob: the tension in the string acting towards the pivot and the weight on the bob acting vertically downwards. Since the motion of the bob must be at right angles to the string, we know that forces in the direction of the string, radial with respect to the pivot, cannot contribute to the acceleration of the bob along its circular path, at right angles to the string, tangential with respect to the pivot.

If we resolve the weight, mg, of the bob parallel and perpendicular to the string, as shown in Figure 2.17c, the component $mg\cos\theta$

parallel to the string makes no contribution to the tangential motion of the bob. The resultant tangential restoring force is the component of the bob's weight perpendicular to the string.

- $F = -mg\sin\theta$

The negative sign tells us that the force is back towards the equilibrium position. The acceleration of the bob is

- $a = \frac{F}{m} = -g\sin\theta$

Now, we have to make another important approximation. For small angles (less than 10°), if we express θ in radians:

- $\theta \approx \sin\theta$

And so, for small angles of swing

- $a = -g\theta = -g\frac{x}{l}$

and so, comparing this to the defining equation for SHM, $a = -\omega^2 x$, we obtain

- $\omega^2 = \frac{g}{l}$
- $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$
- $T = 2\pi\sqrt{\frac{l}{g}}$

It is important to remember what approximations we have made to arrive at the above expression for the time period. We have assumed that the mass of the string can be ignored and that the mass can be treated as a point mass.

For a pendulum where the bob is small compared to the length of the string but has a much greater mass, this is a good approximation. If these assumptions are not valid, then we have a compound pendulum, that requires a different approach, but motion is still SHM. If the angular amplitude of oscillation is not small the approximation that $\theta \approx \sin\theta$ ceases to be valid and the motion, though still periodic, ceases to be SHM. As the amplitude increases, the restoring force for larger displacements will become less required for SHM and the time period will increase.

Big Ben, a famous clock in London, England, has a very large pendulum and the bob has a flat top. Very fine adjustments can be made to its period by adding coins on top of the bob. How does this work?

For a compound pendulum (such as a swinging metre rule) the time period is better expressed using the relationship:

- $T = 2\pi\sqrt{\frac{I}{mgL}}$

where

I = moment of inertia of pendulum

m = mass of pendulum

L = distance from the pivot to the centre of mass of the pendulum.

Discussion activity

Pendulum clocks tend to use quite large masses. Why?

Figure 2.18 *Big Ben in London*

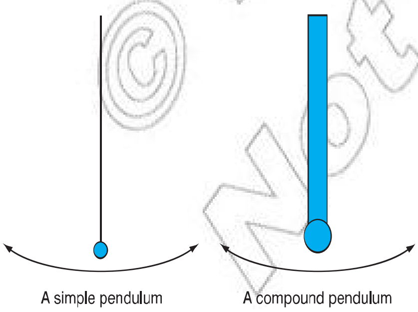
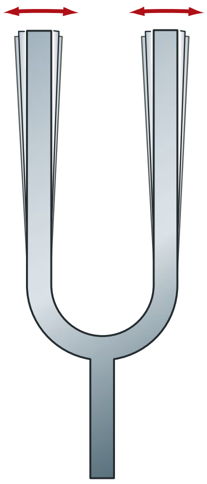


Figure 2.19 *A simple pendulum vs. a compound pendulum*



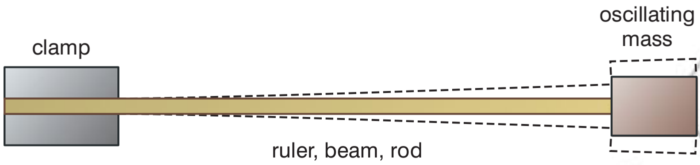


Figure 2.20 *Further examples of systems that perform SHM*

DID YOU KNOW?

When we analyse radial forces acting on the pendulum bob, the tension in the string is only equal to $mg\cos\theta$ when the bob is at the limit of its swing. When the pendulum bob is swinging at velocity v in a circle of radius l, there is a centripetal acceleration towards the pivot and hence a resultant centripetal force:

$$\text{Tension} - mg\cos\theta = \frac{mv^2}{l}$$

Forced oscillations and resonance

A **free oscillation** occurs when an oscillator is displaced from its equilibrium position and released so that it can oscillate freely, with no external forces acting on it.

The oscillator then oscillates at its natural frequency.

Forced or driven oscillations occur when a **periodic driving force** acts on an oscillator. This will make the oscillator oscillate at the frequency of the periodic driving force rather than at its natural frequency. As long as the frequency of the periodic driving force is not the same as the oscillator's natural frequency, the amplitude of the oscillations is usually relatively small.

If the frequency of the periodic driving force is the same as the oscillator's natural frequency, energy is transferred easily into the oscillation and the amplitude of the oscillation becomes large, sometimes very large.

This phenomenon is called **resonance**. Resonance occurs when:

- $f = f_o$

where

f = driving frequency

f_o = natural frequency of the system.

The natural frequency of the oscillator is often referred to as the **resonant frequency**. A plot of driven amplitude against driving frequency peaks at the resonant frequency, as shown overleaf in Figure 2.21.

KEY WORDS

free oscillation *when a body is displaced from its equilibrium position and allowed to oscillate without any external forces acting on it*

periodic driving force *a force of constant frequency acting on an oscillator*

resonance *the tendency of a system to oscillate with larger amplitudes when the frequency of the periodic driving force is the same as the natural frequency of the oscillator*

resonant frequency *the natural frequency of an oscillator*

DID YOU KNOW?

In 1940, the Tacoma Narrows Bridge in the USA collapsed within 6 months of being opened after the way the wind flowed over it caused a periodic twisting that ripped it apart. At the time, it was the third longest suspension bridge in the world. This is sometimes described as being classical example of resonance, but this isn't quite true. Simple resonance was already well understood by the bridge designers. The catastrophic vibrations that destroyed the bridge were due to a more complicated phenomenon known as aeroelastic flutter. Lessons learnt from the collapse of the Tacoma Narrows Bridge have affected the designs of suspension bridges ever since.

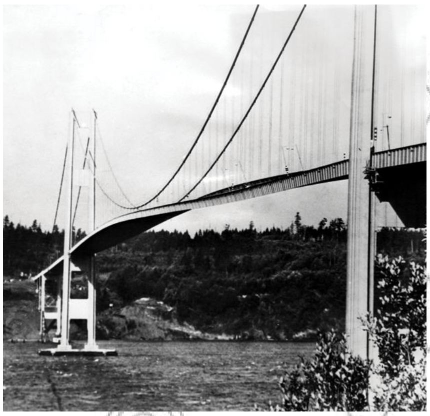


Figure 2.23 *The Tacoma Narrows Bridge just before its collapse.*

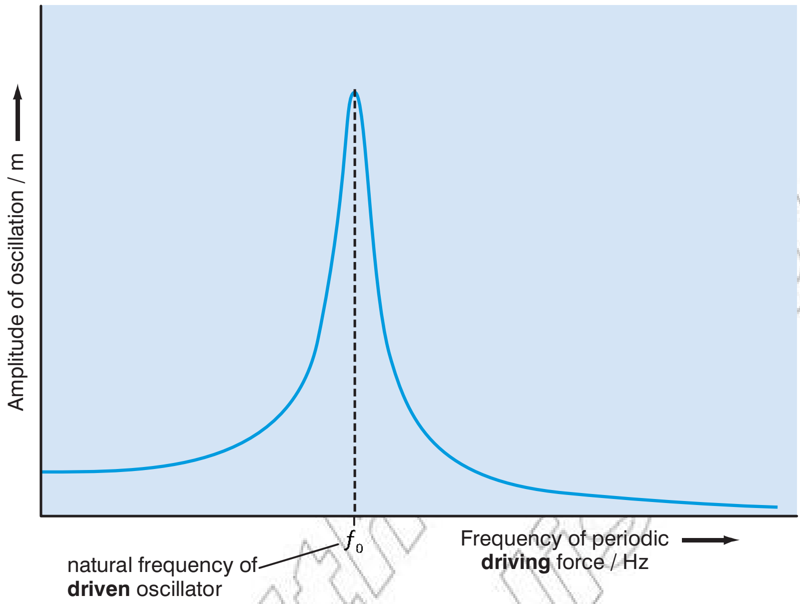


Figure 2.21 *This plot of driven amplitude against driving frequency peaks at the resonant frequency.*

This can be demonstrated using the experimental setup shown in Figure 2.22. The vibrator moves the top of the spring up and down with small amplitude, providing a periodic driving force, with the frequency being set by the signal generator. If the frequency of the signal generator is varied slowly, small oscillations of the mass are observed except at the natural frequency of the mass–spring system, when the oscillations become very large.

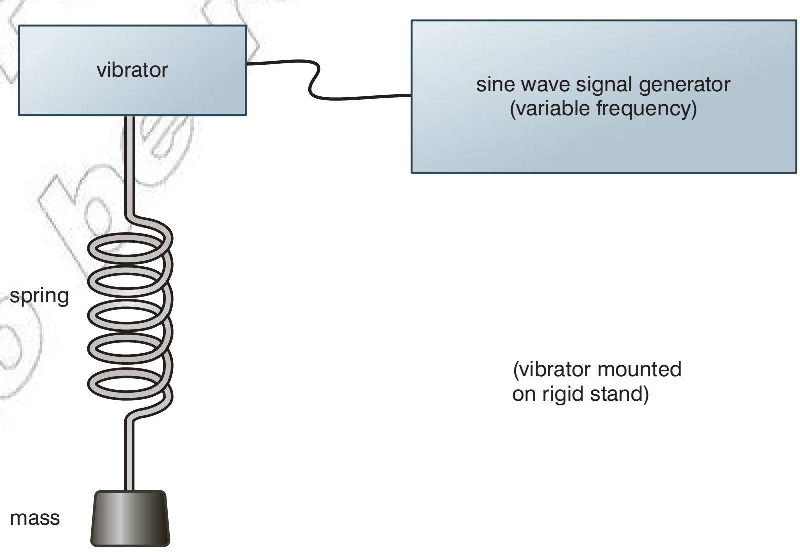


Figure 2.22 *Experiment to demonstrate resonance*

If you've pushed someone on a swing you will be familiar with providing a driving force at the natural frequency of the oscillator. If you stand behind the swing and give a small push just as it reaches the limit of the backward swing you are, by taking your timing from the swing itself, naturally pushing the swing at its natural frequency and, with just gentle pushes, you can quickly build up a large amplitude.

If you try pushing the swing at any other frequency, higher or lower, you will find it much harder work to cause even a small amplitude oscillation.

Resonance occurs in many man-made machines and structures. For example, car engines, or unbalanced wheels will create a periodic driving force affecting the whole car, the frequency of this driving force will increase with the car's speed. If this frequency becomes the same as the natural frequency of some part of the car that can oscillate, then that oscillation can become very large, and can be the cause of annoying rattles that occur at specific car speeds.

If resonance does occur, the large amplitude oscillations can cause damage. Bridges can collapse or at least oscillate violently, driven by wind or the regular pace of people walking across them. Troops of marching soldiers often stop marching and walk across bridges out of step to avoid causing this. One way to reduce the effects of resonance and its potentially damaging effects is to design machines so that, if they do oscillate, the natural frequency and the frequency of any periodic driving force are never the same.

Figure 2.24 *Glasses can be made to shatter if they vibrate at their resonant frequency*

A washing machine is a good example of this. The drum that the clothes go into is suspended on springs and can oscillate as a mass–spring system. During the wash cycle the drum revolves slowly, at a rate well below any natural oscillation frequencies, but during the spin cycle the high speed rotation, particularly if the load is unbalanced, could cause a lot of vibration. Most washing machines have a large mass, sometimes made from concrete, strapped to the drum. This extra mass lowers the natural frequency so that it is well below any driving frequency caused by the high speed spin. Sometimes the machine will vibrate violently but very briefly when it starts to spin as the rotation rate passes through the natural frequency.

Damping of oscillations

Another way to reduce oscillations is to introduce **damping forces**. Damping forces are resistive, energy dissipating, forces that oppose motion by always being in the opposite direction to the velocity.

KEY WORDS

damping forces *resistive forces that oppose the motion of an oscillator by acting in the opposite direction to its velocity*

Air resistance and friction are typical examples of damping forces and are the reason why pendulums naturally stop swinging and masses on springs stop oscillating.

The damping force is given by:

- $F_d = -bv$

where

b = the damping coefficient and is dependent on the medium providing the damping

v = the velocity of the object through the medium.

This equations shows how the resistive force is directly proportional, but opposite, to the velocity. As a result the amplitude of the oscillation will decay exponentially, as shown overleaf in Figure 2.25 (a). Note that the period of the oscillation does not

Discussion activity

Once a suspension bridge has been built it is very difficult to change its natural frequency of oscillation. Why?

change as the amplitude gets smaller. Heavier damping causes a more rapid decay of amplitude as shown in Figure 2.25(b).

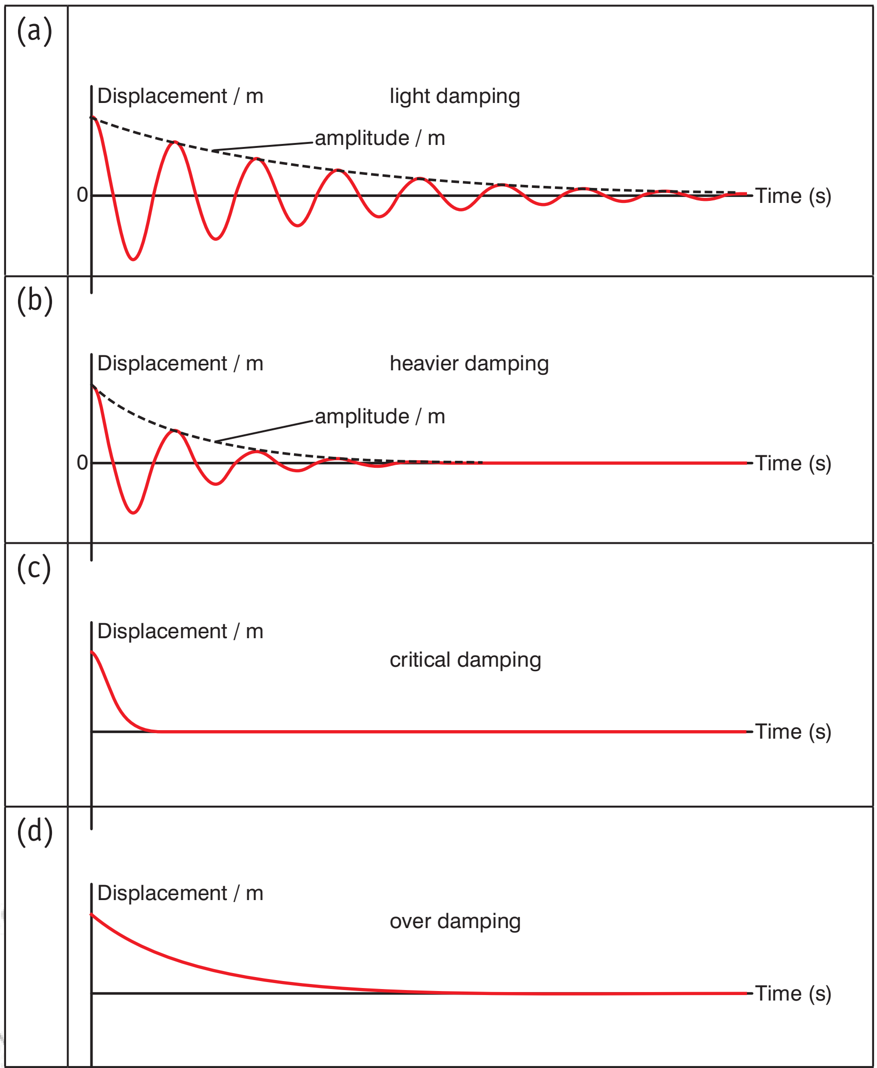


Figure 2.25 *Plots of displacement against time for an oscillator that is displaced and then released, for different amounts of damping.*

KEY WORDS

overdamping *damping that prevents oscillation entirely and only allows the oscillator to return slowly to its equilibrium position*

An example of deliberate damping can be found in a car suspension system. A piston inside cylinder, as shown in Figure 2.26, containing viscous oil can move but the faster it moves the greater the resistance to movement. If such damping is very heavy it can prevent oscillation altogether, so that if the 'oscillator' is displaced it can only return very slowly to the equilibrium position. This is known as **overdamping** and is shown in Figure 2.25(d).

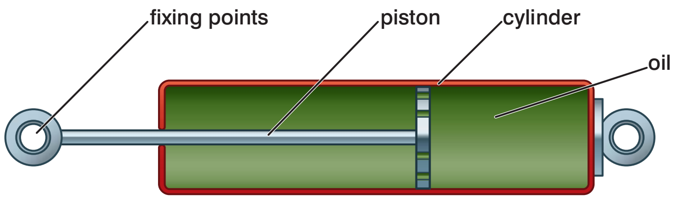


Figure 2.26 *A simple viscous damper. The piston can move inside the cylinder but the faster it moves the greater the resistance to movement.*

Activity 2.4: Damping

Identify the type of damping in the following cases and justify your answer.

a) Pendulum in air

b) Pendulum in water

c) Pendulum in thick treacle

Damping in a car suspension is not normally so heavy, as this would produce a very 'hard' and uncomfortable ride for the passengers. The damping shown in Figure 2.25(b), on the other hand, would provide a very bouncy ride; this would be called **underdamping**. The damping in a car suspension is always a compromise somewhere near to the critical damping shown in Figure 2.25(c). **Critical damping** is the amount of damping that leads to the oscillator settling back to a stationary state at the equilibrium position in the shortest possible time.

Damping reduces the effects of resonance. As the periodic driving force transfers energy into the oscillator the damping mechanism dissipates the energy. The resonance peak in the graph of driven amplitude against driving frequency becomes lower and relatively wider, as shown in Figure 2.27. It can also be seen that damping also causes a very small reduction in the natural frequency of the oscillator.

KEY WORDS

underdamping *damping that allows the oscillator to move back and forth through its equilibrium position before returning to rest*

critical damping *the amount of damping that allows the oscillator to return to its equilibrium position in the shortest possible time*

conservation of energy *the total amount of energy in an isolated system remains constant over time*

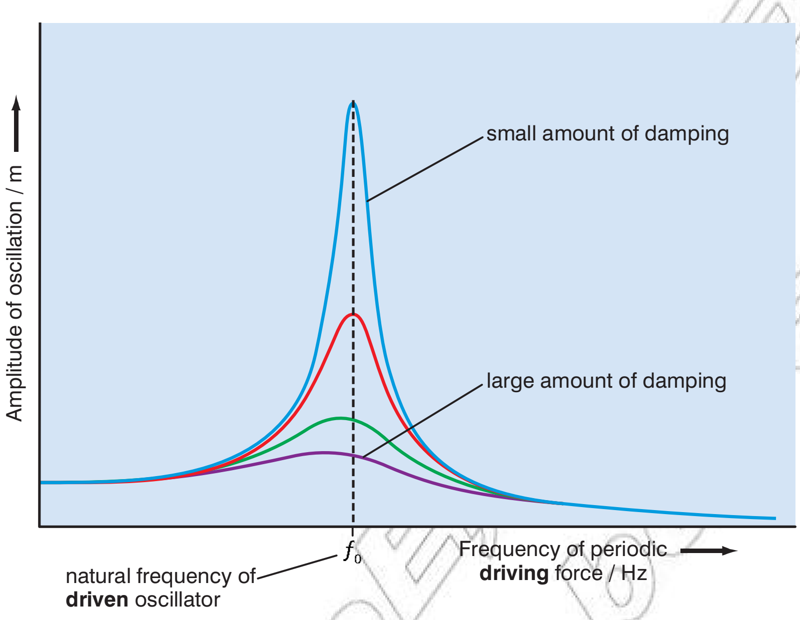


Figure 2.27 *Driven amplitude against driving frequency for forced oscillations of an oscillator with different amounts of damping.*

Energy in SHM

Any oscillator performing SHM has energy and the law of **conservation of energy** tells us that, in the absence of any external forces or damping, that energy must be constant even if it may be changing in form.

When the oscillator is passing through the equilibrium position, when , the resultant force acting on it is zero and it has no potential energy but it is moving at the maximum velocity and has kinetic energy. When the oscillator is at the limit of oscillation, when $x = A$, and the velocity is temporarily zero, the kinetic energy must be zero, but the force acting on the oscillator is at a maximum and the oscillator's energy is all stored as potential energy.

We know that the kinetic energy at any time is given by

- $E_k = \frac{1}{2}mv^2$

To obtain an expression for the potential energy (PE) of the oscillator for any displacement x we need to calculate how much work has been done against the restoring force to move it to that displacement. We can do using the relationship work equals force time distance, but we have to take into account that the restoring force is not one constant value but increases with x.

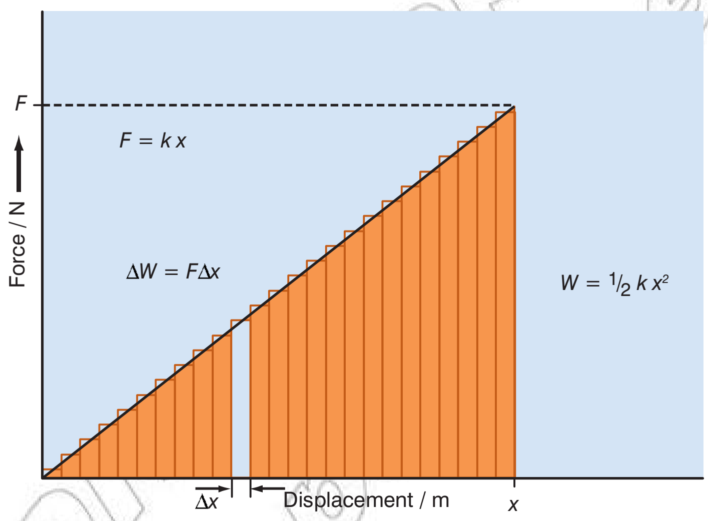


Figure 2.28 *Calculation of PE. The work done against the restoring force in moving the oscillator from the equilibrium position to displacement x is the area under the graph of force against displacement.*

Figure 2.28 shows how we can calculate the work done. For any small increase Δx in x the work done is $\Delta W = F\Delta x$, where $F = kx$ is the force given by . We can see that this contribution to the total work done is a portion of the total area under the graph of force against displacement and that the total work done is the total area under the graph, and hence

- $PE = \frac{1}{2}kx^2$

The total energy at any time is the sum

- $\text{Total energy} = PE + E_k = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

This total energy is equal to the kinetic energy when $x = 0$ and $v = v_0$, or to the potential energy when $x = A$, i.e.

- $\text{Total energy} = \frac{1}{2}mv_0^2 = \frac{1}{2}kA^2$
- $\text{Total energy} = \frac{1}{2}m\omega^2A^2$

Note that the total energy of an oscillator is proportional to the amplitude squared.

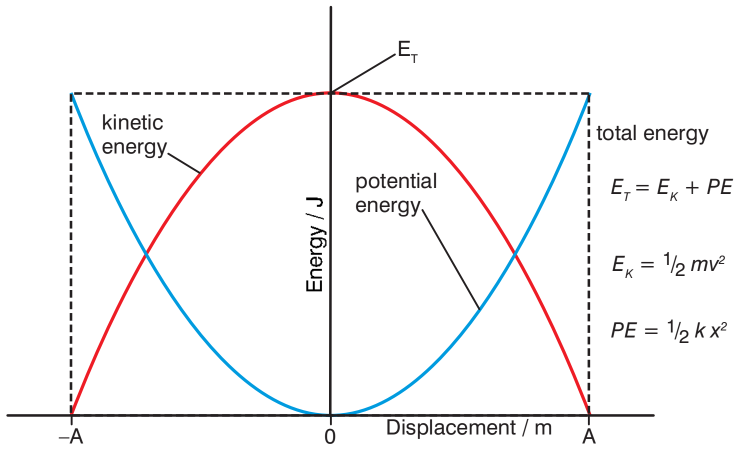


Figure 2.29 *Variation of kinetic energy and potential energy of an oscillator with displacement, showing that the sum remains constant*

This variation of E_k and PE with displacement is shown in Figure 2.29.We can show that these expressions for energy are consistent with our earlier expression for the value of velocity v for any given displacement x:

- $v = \omega\sqrt{A^2 - x^2}$ Starting with our equation for velocity
- $v^2 = \omega^2(A^2 - x^2)$ Making v^2 the subject
- $v^2 = \omega^2 A^2 - \omega^2 x^2$ Multiplying out the brackets

As this is a mass–spring system we can substitute in k/m for ω^2, giving

- $v^2 = \frac{k}{m} A^2 - \frac{k}{m} x^2$

Substituting v^2 into our E_k equations gives:

- $\frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2$

Therefore

- $E_k = total\ energy - PE$

Worked example 2.3

A block of mass 2.2 kg is attached to a spring with a spring constant of 40 N/m. It is pulled down a distance of 30 cm. Find the blocks kinetic energy as it passes through the equilibrium and determine its velocity at this point.

- $\frac{1}{2}mv^2 = \frac{1}{2}kA^2 - \frac{1}{2}kx^2$

Using the relationship above, but is this case as the block is passing through its equilibrium position x = 0 so the relationship simplifies to:

- $\frac{1}{2}mv^2 = \frac{1}{2}kA^2$

Substitute in known values and solve, giving:

- $E_k = ½kA^2 = ½ \times 40 \times 0.3^2 = 1.8$ J

The velocity can then easily be determined using the equation for kinetic energy:

- $E_k = ½mv^2$
- $v = \sqrt{2E_K/m}$ *Rearrange to make v the subject*
- $v = \sqrt{((2 \times 1.8) / 2.2)}$ *Substitute known values*
- $v = 1.3$ m/s *Solve and give the unit*

Summary

In this section you have learnt that:

- Simple harmonic motion (SHM) is a periodic oscillation of an object about an equilibrium position such that its acceleration is always directly proportional in size but opposite in direction to its displacement. The defining equation is
  $a = -\omega^2 x$
  where
  $\omega = 2\pi f = \frac{2\pi}{T}$.
- For an oscillator performing SHM:
  time period does not depend on amplitude
  $x = A\sin(\omega t)$
  $v = \omega A\cos(\omega t)$
  $a = -\omega^2 A\sin(\omega t)$
  $v = \omega\sqrt{A^2 - x^2}$
- For a mass–spring system $T = 2\pi\sqrt{\frac{m}{k}}$.
- For a pendulum $T = 2\pi\sqrt{\frac{l}{g}}$.
- An oscillator will oscillate at its natural frequency if displaced and allowed to oscillate freely without external forces and damping.
- Forced oscillations occur if an oscillator is driven by a periodic driving force.
- Resonance occurs when an oscillator is driven at its natural frequency
- Damping forces are opposite in direction to velocity and dissipate energy, causing an exponential decay in the amplitude of free oscillations.
- Damping reduces the amplitude of driven oscillations, reducing the effects of resonance
- Total energy = $PE + E_k = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$
- The total energy of an oscillator is proportional to amplitude squared.

Review questions

1. An object moving with simple harmonic motion has an amplitude of 3 cm and a frequency of 30 Hz. Calculate:
   a) the time period of the oscillation,
   b) the acceleration in the centre and at the maximum displacement of an oscillation, and
   c) the velocity in the centre and at the maximum displacement of an oscillation.
2. How long does a pendulum need to be to have a time period of 1 second? Explain all the approximations and assumptions you make to carry out this calculation.
3. A mass of 500 g is suspended on a vertical spring of spring constant $k = 10$ N/m.
   a) Calculate the frequency at which the mass will oscillate if displaced downwards a small distance and released.
   b) A periodic driving force of variable frequency f is applied to the top of the spring. Sketch and explain a graph of amplitude against frequency f for the oscillation of the mass.
   c) On the same axes, sketch a graph of amplitude against frequency f for the oscillation of the mass if a relatively large piece of cardboard is taped horizontally to the mass.
   d) On the same axes, sketch a graph of amplitude against frequency f for the oscillation of the mass if its size is increased to 1 kg.
4. A simple pendulum has a length of 1.2 m and the bob has a mass of 800 g. The pendulum swings with an amplitude of 14 cm. Calculate:
   a) the velocity of the pendulum bob at the centre of its swing
   b) the kinetic energy of the pendulum bob at the centre of its swing
   c) the kinetic energy and the potential energy of the pendulum bob when it is 8 cm from the centre of its swing.
5. Describe the key features of the different forms of damping the general effect of damping on resonance.

2.2 Wave motion

By the end of this section you should be able to:

- Describe the characteristics of a mechanical wave and identify that the speed of the wave depends on the nature of medium.
- Use the equation $v = \sqrt{T/\mu}$ to solve related problems.
- Describe the characteristics of a travelling wave and derive the standard equation $y = A\cos(\omega t + \phi)$
- Define the terms phase, phase speed and phase constant for a travelling wave.
- Explain and graphically illustrate the principle of superposition, and identify examples of constructive and destructive interference.
- Identify the properties of standing waves and for both mechanical and sound waves, explain the conditions for standing waves to occur. Including definitions of the terms node and antinode.
- Derive the standing wave equations.
- Calculate the frequency of the harmonics along a strong, a open pipe and a pipe closed at one end.
- Explain the modes of vibration of strings and solve problems involving vibrating strings.
- Explain the way air columns vibrate and solve problems involving vibrating air columns.
- Analyse, in quantitative terms, the conditions needed for resonance in air columns, and explain how resonance is used in a variety of situations.
- Identify musical instruments using air columns, and explain how different notes are produced.

Figure 2.30 *Waves on water*

KEY WORDS

direction of propagation *the direction in which energy is transferred along a travelling wave*

mechanical wave *a wave that involves the oscillations of particles of a physical medium*

What is a travelling wave?

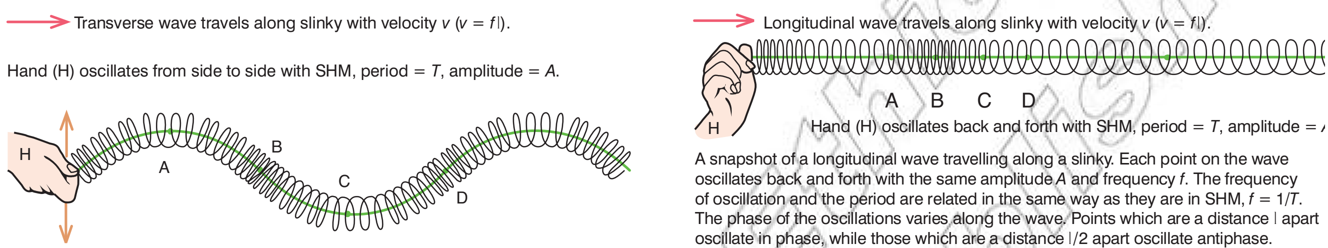
Electromagnetic and sound waves are particularly important to us, but waves on water are a little easier to observe. If we drop a pebble into a pond we see small waves or ripples radiating outwards.

If dip a stick into the middle of the pond and move it up and down with SHM the motion becomes continuous. These waves are spreading out in two dimensions and sound and electromagnetic waves spread out in three dimensions, but for the moment, we will simply consider waves travelling in one dimension. We will later consider stationary, or standing, waves, but first we need to understand a little about travelling waves.

A travelling wave transfers energy, and sometimes information, from one place to another, in what is called the **direction of**

propagation. An oscillation at the source of energy causes an oscillation to travel through space. For electromagnetic waves this oscillation is of electric and magnetic fields and does not need a medium. In a **mechanical wave** that involves the oscillations of particles of a physical medium, as the particles pass on energy, they undergo temporary displacements but no permanent change in the position. For example, when ripples travel across a pond the water molecules oscillate vertically but do not move in the direction of the wave.

Transverse and longitudinal waves?



A snapshot of a transverse wave travelling along a slinky. Each point on the wave oscillates from side to side with the same amplitude A and frequency f. The frequency of oscillation and the period are related in the same way as they are in SHM, $f = 1/T$. The phase of the oscillations varies along the wave. Points which are a distance l apart oscillate in phase, while those which are a distance l/2 apart oscillate antiphase.

A snapshot of a longitudinal wave travelling along a slinky. Each point on the wave oscillates back and forth with the same amplitude A and frequency f. The frequency of oscillation and the period are related in the same way as they are in SHM, $f = 1/T$. The phase of the oscillations varies along the wave. Points which are a distance l apart oscillate in phase, while those which are a distance l/2 apart oscillate antiphase. Point B on the wave is at a point of **compression** – the points to the left of B are displaced to the right of their equilibrium position, while those to the right of B are displaced to the left of their equilibrium position. The reverse is true of point D, which is at a point of rarefaction – the points to the left of D are displaced to the left of their equilibrium position, while those to the right are displaced to the right of their equilibrium position.

Figure 2.31 *Waves along a slinky*

We can demonstrate a wave travelling along a stretched slinky, as shown in Figure 2.31. We can create two distinctly different types of travelling wave.

A **transverse wave** is one in which the oscillation, the temporary displacement of mass or field strength, is at right angles to the direction of propagation. Electromagnetic waves and waves travelling along a string or rope are examples of transverse waves. Waves on water can appear to be transverse if the amplitude is small, but in reality they are more complicated and involve the water moving in circles.

KEY WORDS

transverse wave *wave where the oscillations are perpendicular to the direction of wave motion*

longitudinal wave *wave where the oscillations are parallel to the direction of wave motion*

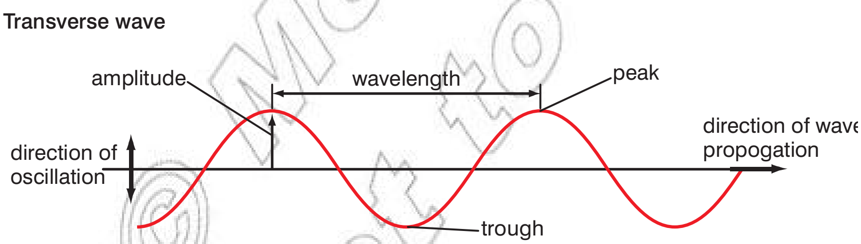


Figure 2.32 *A transverse wave*

A **longitudinal wave** is one in which the oscillation, the temporary displacement of mass, is backwards and forwards along the path of wave propagation/net energy transfer. Sound is a longitudinal wave.

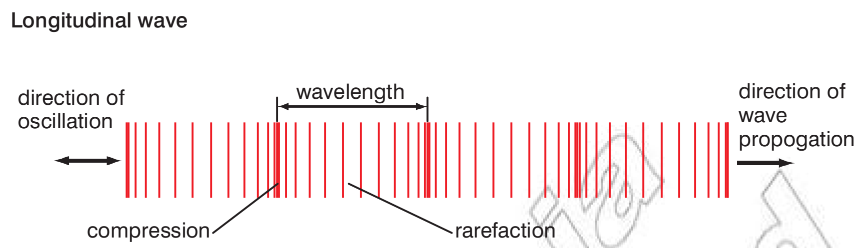


Figure 2.33 *A longitudinal wave*

A wave is carried by a chain of oscillators, each passing on its energy to the next oscillator. Hence, just as the energy of a single oscillator is proportional to the square of the amplitude of the oscillation, the power or rate of energy transfer of a wave is proportional to amplitude squared.

Wave speed

The frequency of a wave can be defined in two equivalent ways. It is the frequency of the individual oscillators that pass the energy along, the number of times particles go up and down or backwards and forwards per second.

It is also the number of complete waves, the number of wavelengths that pass any given point per second. If the wavelength is λ, and f wavelength pass a point per second, then the speed of the wave must be given by the wave equation:

- $v = f\lambda$

Worked example 2.4

If a sound wave with a frequency of 500 Hz passes through a liquid at a speed of 1500 m/s, then its wavelength must be

- $\lambda = \frac{v}{f}$
- $\lambda = \frac{1500}{300} = 3$ m

Wave speed through different media

The speed of any travelling wave depends on the media it is travelling through (more on the speed of sound in chapter 2.3).

For a mechanical wave travelling along a string the speed of the wave depends on the **tension** of the string and the **mass per unit length** (sometimes called **linear density**).

- $v = \sqrt{\frac{T}{\mu}}$

where

μ = mass per unit length given by $\mu = \frac{m}{l}$ in kg/m

T = tension in the string in N.

The formula given above shows us that the 'tighter' the string the faster the waves will travel down its length. Additionally the 'lighter' the string, (the smaller its mass/length ratio), the faster the waves will travel down its length.

The **phase speed** of a wave is the rate at which the phase of the wave travels through space. Any given phase of the wave (for example, the crest or the trough) will appear to travel at the phase velocity. The phase velocity is given in terms of the wavelength λ (lambda) and period T as

- $v_{phase} = \lambda / T$

KEY WORDS

tension *a measure of the force tending to stretch a string*

mass per unit length *a measure of the distribution of the mass of a string along its length*

linear density *a measure of mass per unit length*

phase speed *the rate at which the phase of the wave travels through spa*

How do we describe a travelling wave mathematically?

An oscillation at the source causes a travelling wave, which causes oscillations along its path. A mathematical description of the wave must give an expression for the temporary displacement Y at any distance x along the path of the wave at any time t.

If a wave is sinusoidal, then a snapshot of the wave, i.e. a side view at an instant in time, looks as shown below in Figure 2.34(a). If the wave now travels on from the position shown it causes particles to oscillate with SHM at points A and B but, because the wave has to travel a quarter of a wavelength further to reach point B, the oscillation at point B always lags behind that at point A by quarter of a cycle, $\pi/2$ radians, as shown in Figures 2.34 (b) and (c).

If points A and B were half a wavelength apart than the oscillation at B would lag half a cycle, π radians, behind that at A. If A and B were a whole wavelength apart that the oscillation at B would be a whole cycle behind that at A and therefore be effectively back in step with it.

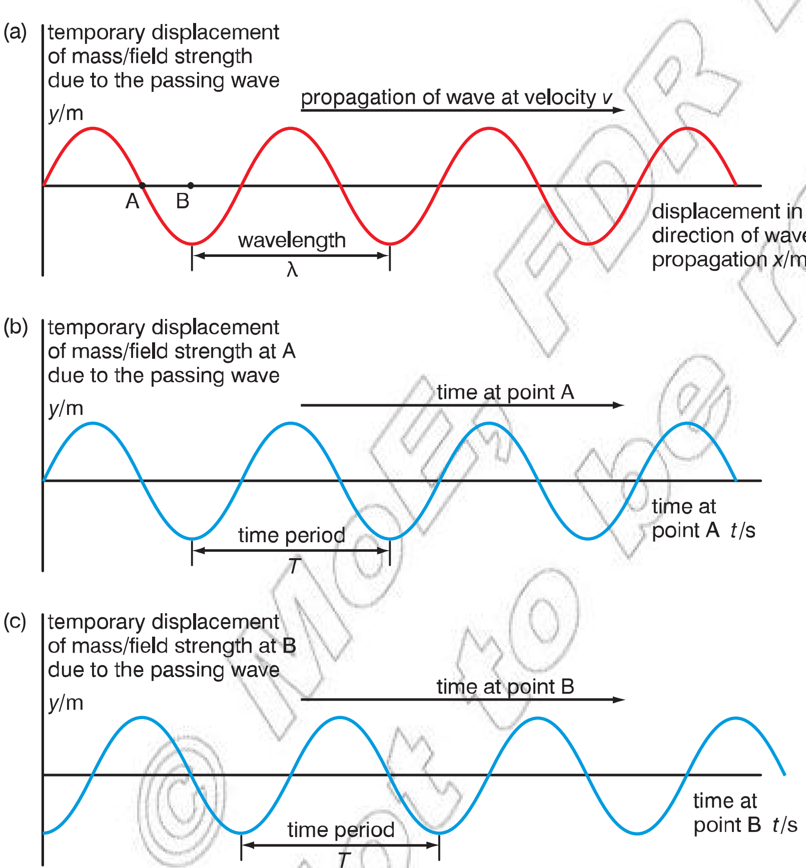


Figure 2.34 A side view of a transverse wave at a single instant in time. As the wave is sinusoidal, the wave causes particles to oscillate with SHM at points A and B. Over time there is a phase shift, this does not change the shape of the move but moves it back and forth along the x-axis.

Figure 2.34 shows a series of snapshots of the travelling wave at successive instants in time, quarter of a cycle apart. At time $t = 0$, the wave is a sine wave described by the equation

- $Y = A\sin\left(2\pi \frac{x}{\lambda}\right)$

When $t = \frac{T}{4}$, the wave has moved a quarter of a wavelength to the right and is described by the equation

- $Y = A\sin\left(2\pi\frac{x}{\lambda} - \frac{\pi}{2}\right)$

We can confirm that this is the correct expression by checking the values that this gives.

When $x = 0$ $\sin\left(2\pi\frac{x}{\lambda} - \frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

When $x = \frac{\lambda}{4}$, $\sin\left(2\pi\frac{x}{\lambda} - \frac{\pi}{2}\right) = \sin\left(2\pi\frac{1}{4} - \frac{\pi}{2}\right) = \sin(0) = 0$

and so on.

When $t = \frac{T}{2}$, the wave has moved a half of a wavelength to the right and is described by the equation

- $Y = A\sin\left(2\pi\frac{x}{\lambda} - \pi\right)$

In general, after time t:

- $Y = A\sin\left(2\pi\frac{x}{\lambda} - 2\pi\frac{t}{T}\right)$

or

- $Y = A\sin\left(2\pi\frac{x}{\lambda} - 2\pi ft\right)$

This is a very useful description. By substituting in a value of x for the position of a point along the wave's path, we can obtain an expression for the oscillation at that point, or by substituting in a value for t at a particular instant we can obtain an expression describing the shape of the wave at that instant.

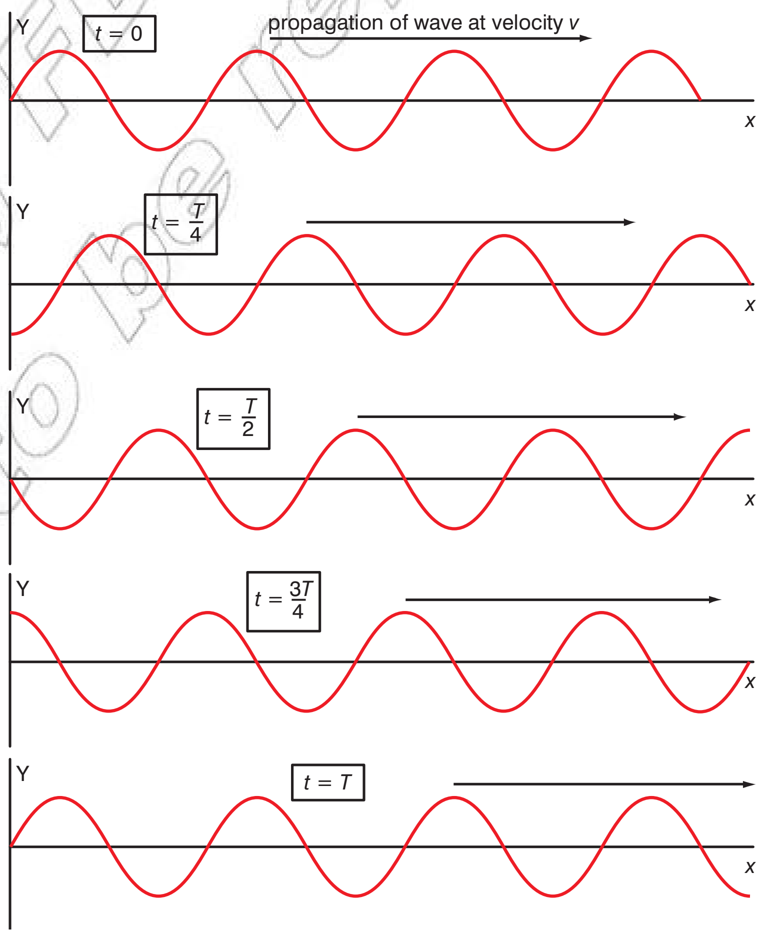


Figure 2.35 *A travelling wave shown at a series of successive times*

Just like SHM it does not really matter whether a sine or cosine function is used to describe the travelling wave. A cosine version of the travelling wave equation may be seen below:

- $Y = A\cos(\omega t + \phi)$

In this equation ϕ is the phase constant of the wave, this effectively moves the waves back and forth along the *x*-axis.

Principle of superposition

If two or more waves pass through a single point then the resultant instantaneous displacement at that point is the sum of the displacements that would be created separately by each wave, taking signs into account. The waves pass on through the point and each other and continue on unaffected. This works for two waves passing through each other at any angle, as shown in Figure 2.36, and for waves passing through each other in opposite directions along a string, as shown in Figure 2.37.

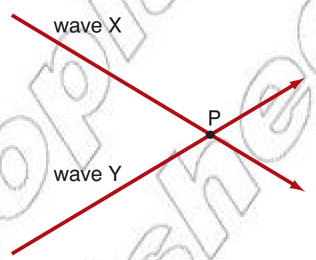
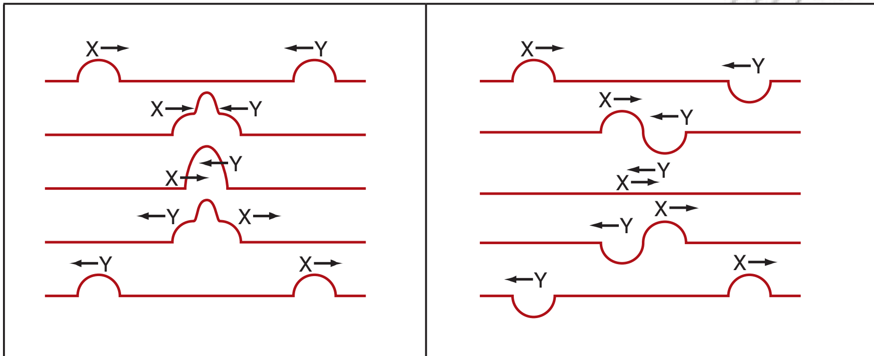


Figure 2.36 *Two waves, X and Y, pass through each other unaffected, but the instantaneous displacement at point P is the sum of the of the displacements that would be created separately by each wave.*



(a) Pulses of the same sign crossing.	(b) Pulses of opposite sign crossing

Figure 2.37 *Pulses passing through each other going in opposite directions along a string and demonstrating superposition as they cross*

Constructive and destructive interference

Interference is another word for superposition. When two waves arrive at the same point they interfere with each other: the instantaneous displacement at that point is the sum of the displacements that would be created separately by each wave. If we consider just two waves, the result is an oscillation whose amplitude depends on the relative **phase** of the two waves.

The **phase difference** between two oscillations is an angular measurement of the difference in their timing, best understood by thinking about the circular motion link to SHM shown in Figure 2.11. Since a whole cycle involves an angular change of 2π radians, a half cycle difference is a phase difference of π, and a quarter cycle difference is a phase difference of $\pi/2$.

If the two waves are causing independent oscillations that go up and down, and pass through the equilibrium point at the same times, as shown overleaf in Figure 2.38, they are said to be in phase with each other.

KEY WORDS

phase *a measurement of the position of a point on a wave after a particular time. Two sine waves are said to be in phase when corresponding points of each reach maximum or minimum displacements at the same time.*

phase difference *the angular difference in timing between 2 waves*

KEY WORDS

antiphase *where two sine waves are performing the opposite motion to each other. The phase difference between them is 180 degrees.*

constructive interference *the production of large oscillations by the superposition of two waves that are in phase with each other*

destructive interference *the cancelling out of oscillations caused by the superposition of two waves that are in antiphase*

If they are always performing completely the opposite motion to each other, as shown in Figure 2.38, they are said to be completely out of phase, 180° or π radians out of phase with each other, or in **antiphase**.

If two oscillations are in phase with each other we get **constructive interference** giving a large amplitude oscillation.

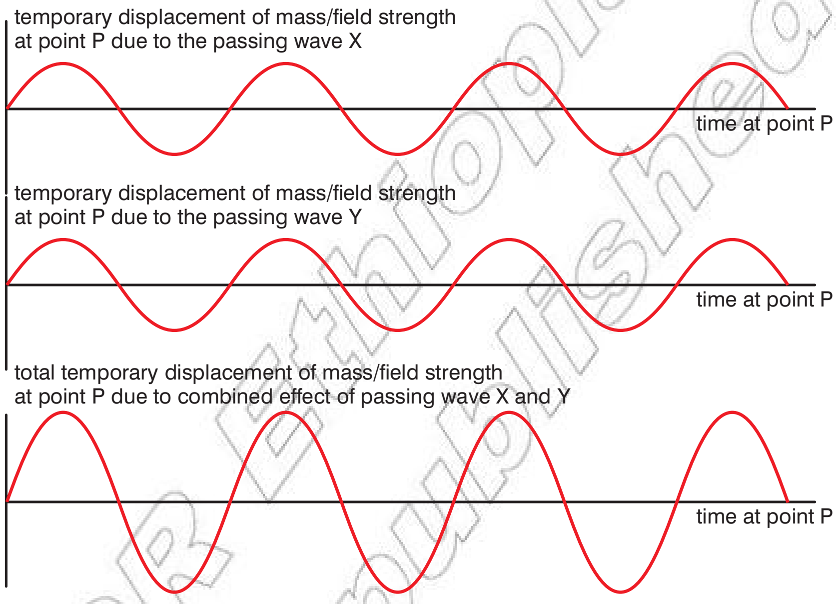


Figure 2.38 *Constructive interference: two oscillations in phase with each other combine to produce a larger oscillation at the same frequency.*

If the two oscillations are in antiphase we get **destructive interference** or cancellation leading to a small or zero resultant oscillation. Note that we only get complete cancellation if the two oscillations are in perfect antiphase and have the same amplitude.

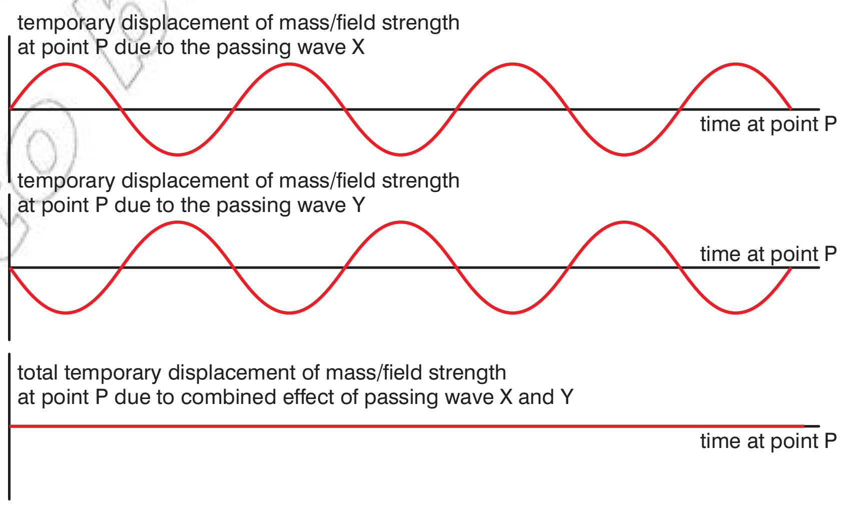


Figure 2.39 *Destructive interference: two oscillations in antiphase cancel each other to produce a small or zero resultant oscillation.*

Think about this...

Waves at sea are caused by the action of wind on the surface of the water. In big oceans on a calm day you see big waves that have travelled thousands of kilometres from a storm centre thousands of kilometres away. Sometimes these waves are nice and regular. Sometimes you can experience a very irregular pattern of waves getting stronger and weaker and very difficult to predict. Why is this?

Reflections of waves

When a travelling wave reaches a sudden change in medium it will be at least partially reflected. In some circumstances this reflection can be total. When a wave travelling along a string reaches the end, it will be completely reflected if the end of the string is either firmly clamped so that it cannot move at all or if the end is completely free to move. If the end of the string is connected to a second string of different mass per unit length, some energy will be reflected and some will be transmitted on along the next string.

If the end of the string is fixed so that it cannot oscillate at all, then the sum of the incident and reflected wave must be always zero and so the reflected wave must be in antiphase. The reflection includes a phase shift of π radians. If the end of the string is completely free to move, we still get 100% reflection, but with no phase shift. The same rules apply to sound waves in narrow tubes, as in musical instruments. Where the end of the tube is closed there is no net oscillation and the sound wave is reflected with a phase shift of π radians; where the end of the tube is open there is a large net oscillation and the sound wave is reflected with no phase shift.

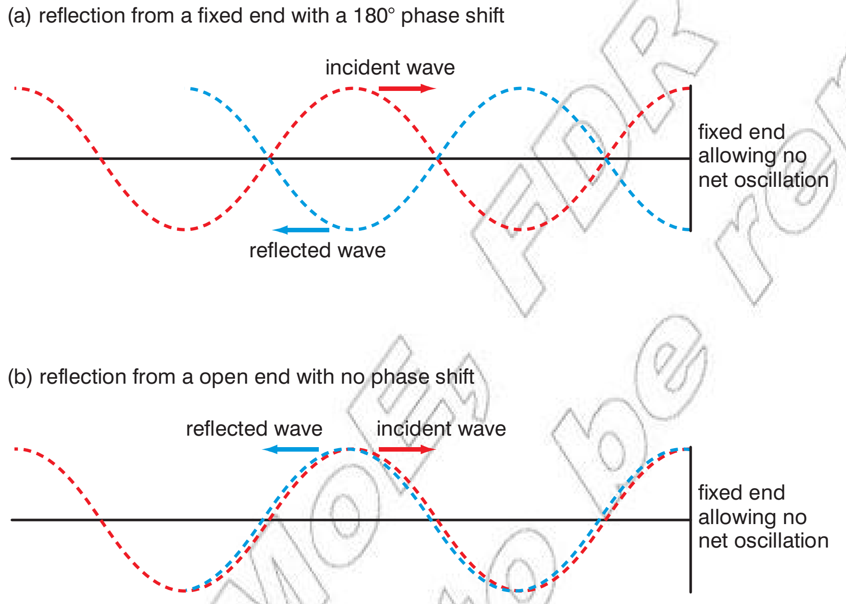


Figure 2.40 *Reflections at the end of a string, or air column*

Standing waves on strings

Musical instruments use standing, or stationary, waves, either on strings or in air columns, to generate sound at different frequencies. If a string is fixed at both ends and it is plucked, or has a bow drawn across it, then waves will travel away along the string to be reflected at the ends. This produces waves travelling in opposite directions along the string and they will interfere with each other. The waves that travel away from the initial point of plucking will be at a wide range of different frequencies, and for most of these frequencies the reflections will never interfere constructively with each other and they will disappear. At some specific wavelengths however, depending on the length of the string, the waves travelling in opposite directions will interfere to produce a standing wave, as shown in Figure 2.41. If we can observe the oscillation of the string, which can be done using high speed photography or using a stroboscope, we do not see travelling wave but a wave shape that stays in one place, a stationary, or standing, wave.

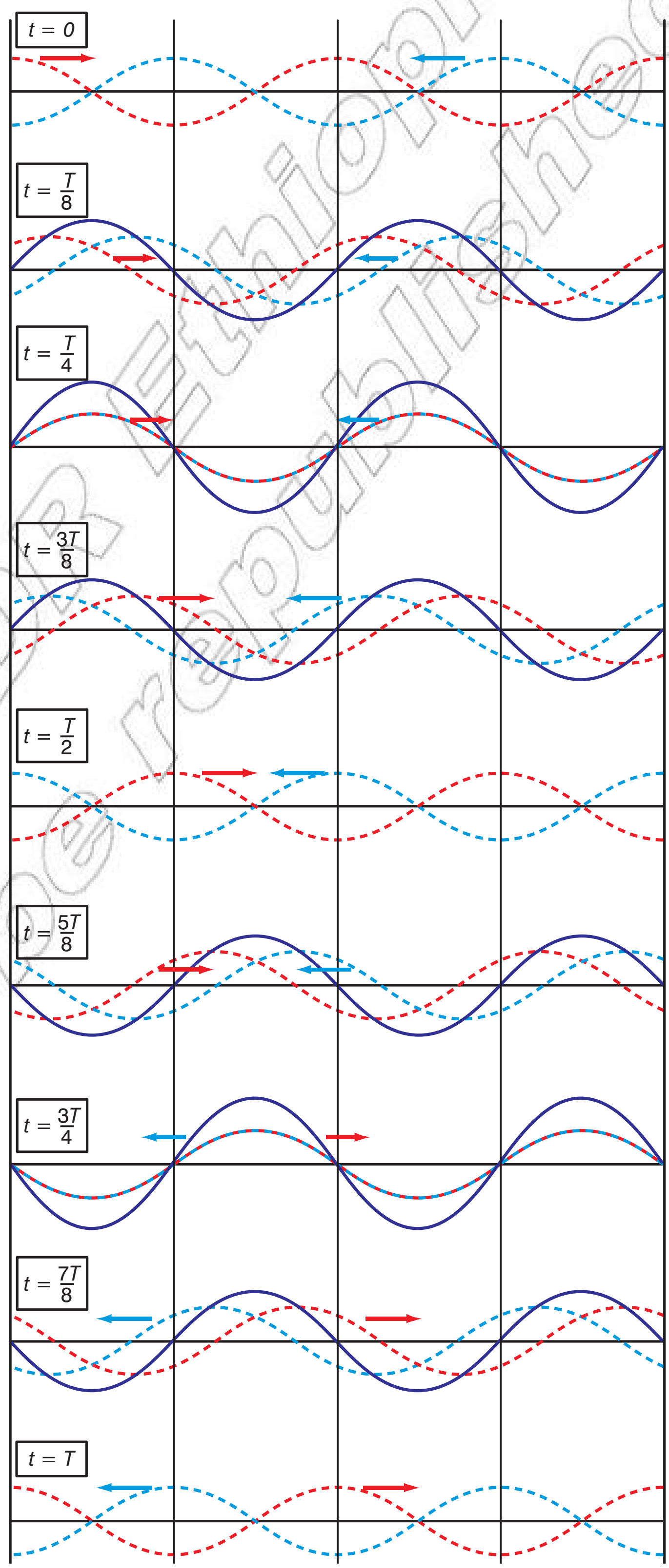


Figure 2.41 *Formation of a stationary wave by superposition of two waves of equal amplitudes and wavelength travelling in opposite directions .The red and blue waves are the two travelling waves, which we do not see in reality. The black line is the standing wave that we do see.*

At points where the wave travelling to the right and the wave travelling to the left are always in antiphase, as at the fixed ends, superposition produces no net oscillation. These points are called **nodes**. There are positions however where the two waves are always in phase with each other and here superposition produces large oscillations.

The points where the two waves are always perfectly in phase and the net oscillations largest are called **antinodes**, as shown in Figure 2.42. All points of the string between any two adjacent nodes, half a wavelength, oscillate in phase with each other.

KEY WORDS

nodes *the points where two superimposed waves are in antiphase and there is no net oscillation*

antinodes *the points where two superimposed waves are in phase and the net oscillations are largest*

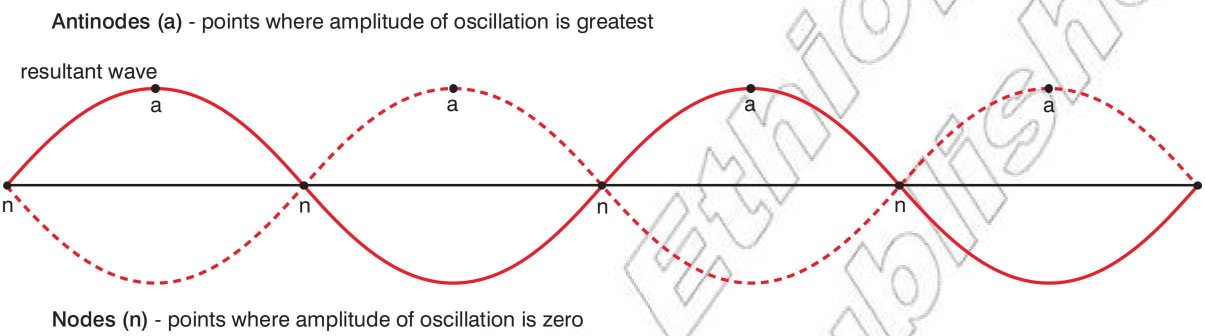
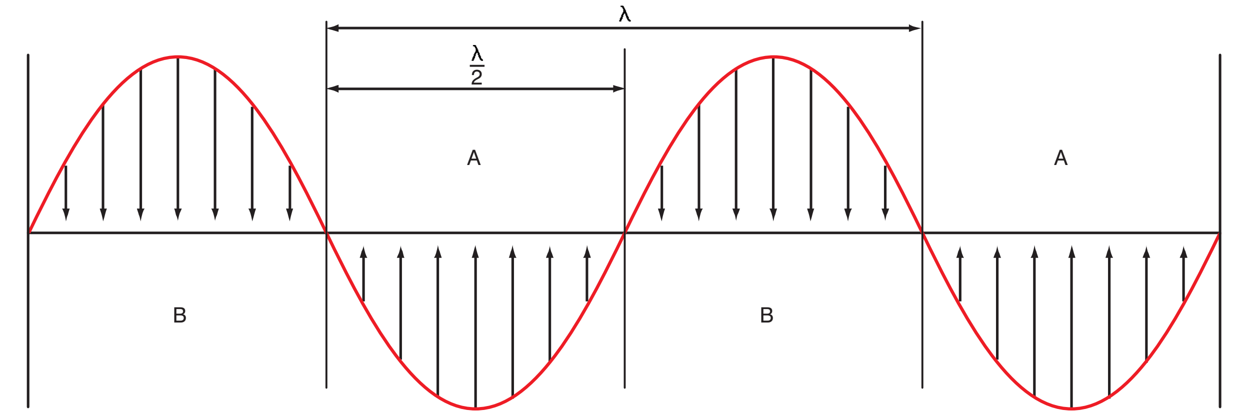


Figure 2.42 *Nodes and antinodes on a stationary wave. The distance between two successive nodes and antinodes is half a wavelength.*

Adjacent half wave length sections are in antiphase with each other, as shown in Figure 2.43.



All particles in the string in sections A oscillate in phase with each other – those in the centre with the biggest amplitude	All particles in the string in sections B oscillate in phase with each other – those in the centre with the biggest amplitude
Particles in section A oscillate in anti-phase with particles in section B, i.e. their oscillations have a phase difference of π or 180°.	

Figure 2.43 *Oscillations in different sections of a stationary wave*

The location of nodes and antinodes are very important.

- Nodes occur when the distance along the string is equal to $n\lambda / 2$
- Antinodes occur when the distance along the string is equal to $(n+½)\ \lambda / 2$.

Where n = an integer number 0, 1, 2, 3, etc.

The mathematics of standing waves

Two travelling waves moving in opposite directions can be represented by the equations below:

- $y_1 = y_0 \sin(kx - \omega t)$

and

- $y_2 = y_0 \sin(kx + \omega t)$

where

y_0 is the amplitude of the wave,

ω is the angular frequency measured in radians per second (we could use $2\pi f$ instead),

k is equal to $2\pi / \lambda$ (as seen in the travelling wave equations discussed earlier)

x and t are variables for position and time, respectively.

So the resultant wave y equation will be the sum of y_1 and y_2:

- $y = y_0 \sin(kx - \omega t) + y_2 = y_0 \sin(kx + \omega t)$

We can use a trigonometric identity to simplify this to:

- $y = 2y_0 \cos(\omega t) \sin(kx)$

This equation shows not only that the wave oscillates in time, but has also these oscillations vary in the x direction. That is as you move further along the wave (in the x direction) the oscillations vary. Several maxima occur at $x = \lambda/4, 3\lambda/4, 5\lambda/4$, these are the antinodes. Where as at $x = 0, \lambda/2, \lambda, 3\lambda/2$, the function is zero and so the amplitude is always zero – these are the nodes.

Wavelength and the length of string

A standing wave happens if the distance for a wave to travel in a complete circuit, from one point to one end, back to the other end and finally back to where it started, is a whole number of wavelengths. We get stationary waves if the length of the string is a whole number of half wavelengths, i.e. we get stationary waves if

- $n = \dfrac{2L}{\lambda}$

where L is the length of the string and n is an integer, or when

- $\lambda = \dfrac{2L}{n}$

Using velocity $v = f\lambda$ we can show that we get standing waves on the string when

- $\frac{v}{f} = \frac{2L}{n}$

or

- $f = n\frac{v}{2L}$

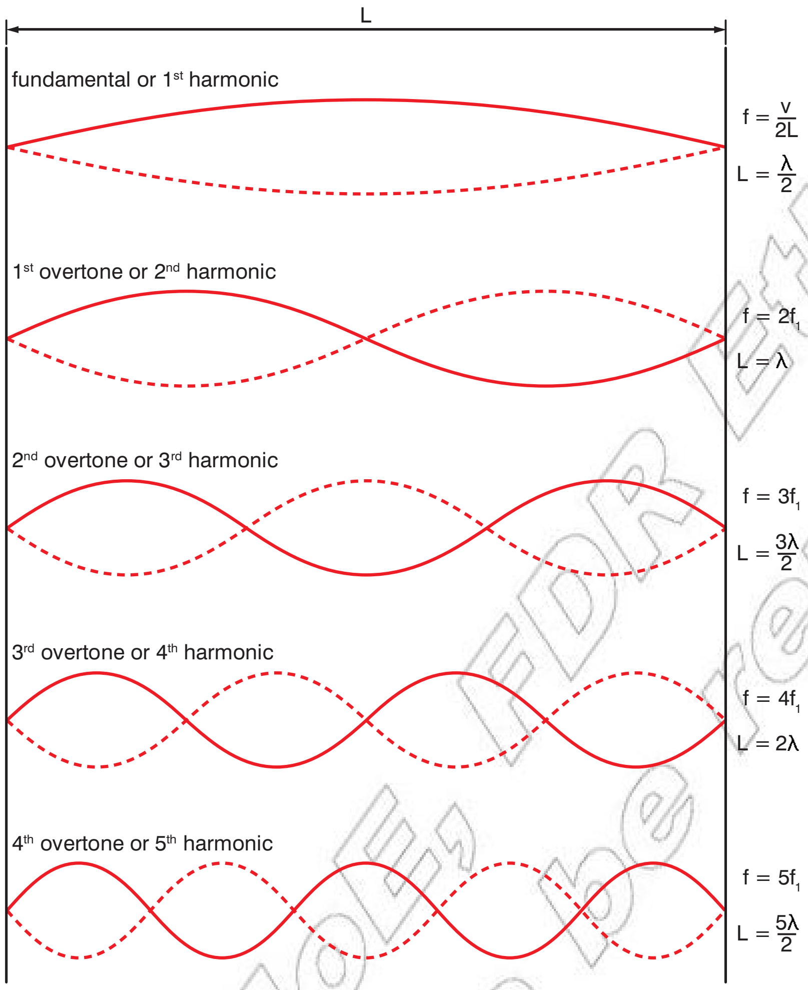


***Figure 2.44** Modes of vibration of a string fixed at both ends*

Figure 2.44 shows the standing waves that can be formed on a string fixed at both ends for values of n from 1 to 5. The lowest frequency of oscillation, when $\lambda = 2L$ and there is just one antinode is known as the fundamental frequency, f_1. All the other possible oscillation frequencies are integer multiples of this fundamental frequency and are known as **harmonics**. For example, the oscillation at $2f_1$ is known as the second harmonic, that at $3f_1$ as the third harmonic, and so on. The first harmonic is the fundamental frequency. The harmonic number is the same as the number of antinodes. A string does not necessarily oscillate at only one of these frequencies; it can oscillate at several different harmonic frequencies at the same time. The resulting shape of the oscillating string can look quite complex.

Worked example 2.5

What frequencies can a string vibrate at if it is 30 cm long and the velocity of travelling waves along the string is 120 m/s?

The fundamental frequency is given by:

- $f_1 = n\frac{v}{2L}$
- $f_1 = \frac{120}{2 \times 0.3}$
- $f_1 = 200$ Hz

Therefore the string can vibrate at 200 Hz, 400 Hz, 600 Hz, 800 Hz, 1000 Hz, 1200 Hz, etc.

KEY WORDS

harmonics *standing waves for which frequencies are integer multiples of the fundamental frequency*

Think about this...

The number of antinodes may be used to quickly determine the harmonic. The second harmonic has two antinodes, the third harmonic three, etc.

Figure 2.45 *Guitar strings have different thicknesses (and so mass per unit length is different) plus their tension may be altered to produce different notes.*

DID YOU KNOW?

Strings or parts of strings on a string instrument may resonate at their fundamental or harmonic frequencies when other strings are sounded. For example, an A string at 440 Hz will cause an E string at 330 Hz to resonate, because they share an overtone of 1320 Hz (3^{rd} harmonic of A and 4th harmonic of E).

The fundamental frequency of a string is clearly determined by its length, but it also depends on the velocity at which travelling waves travel along the string. As we have already shown this velocity is given by

- $v = \sqrt{\frac{T}{\mu}}$

where T is the tension in the string and μ is the mass per unit length. Hence, the fundamental frequency of a string is given by

- $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$

and so we can make a string produce a higher note if we make it shorter, increase the tension or replace it with a lighter one. These are the parameters that affect the fundamental frequency that the string in a musical instrument produces, but the tone, what makes one instrument sound so different from another arises from the harmonics that are produced at the same time as the fundamental. If a string is tuned to the musical note A (above middle C), then this means that it's fundamental frequency is 440 Hz. But it will also be producing sound waves at the harmonic frequencies 880 Hz, 1320 Hz, 1760 Hz, etc. It is the relative amplitude of these harmonics that determines the tone of the note, and this depends on the detailed design of the musical instrument and how the string is plucked or bowed.

These frequencies of oscillation can be thought of as resonant frequencies. If a sound wave hits the string at one of these frequencies, perhaps coming from another string, it will start to vibrate at that frequency. This is called a sympathetic vibration. When we cause a string to vibrate at a particular resonant frequency, we say that we have excited the corresponding resonant mode of vibration. A key difference from oscillators like pendulums and mass-spring systems is that while those oscillators have just one resonant frequency, a string has multiple resonant frequencies: its fundamental and harmonic frequencies.

Standing waves in organ pipes

Wind instruments generate standing waves in the column of air inside them. A full analysis of the vibration of the air inside a wind instrument is much more complicated than for a string. The vibrating body of air in wind instruments can be varied and complex in shape, but we can arrive at some very good approximations if we stick to straight narrow pipes.

A simple pipe will need to be open at one end, but could be open or closed at the other. Where the end of the tube is closed there is no net oscillation and the sound wave is reflected with a phase shift of π radians. There will always be a node at a closed end. Where the end of the tube is open there is a large net oscillation and the sound wave is reflected with no phase shift. There will always be an antinode at the open end. Pipes with one end closed behave differently from those with both ends open.

If one end of the pipe is closed, and the other open, we will get a node at one end and an antinode at the other. This creates two differences from standing waves on a string. The fundamental occurs when the length of the pipe is just a quarter of a wavelength, and only odd harmonics occur, as shown in Figure 2.46. These diagrams could be slightly misleading as they appear to show the waves as transverse. The vibration of air in the pipe is actually quite complex and certainly not simply transverse: diagrams should be taken as showing amplitude but not direction of oscillation.

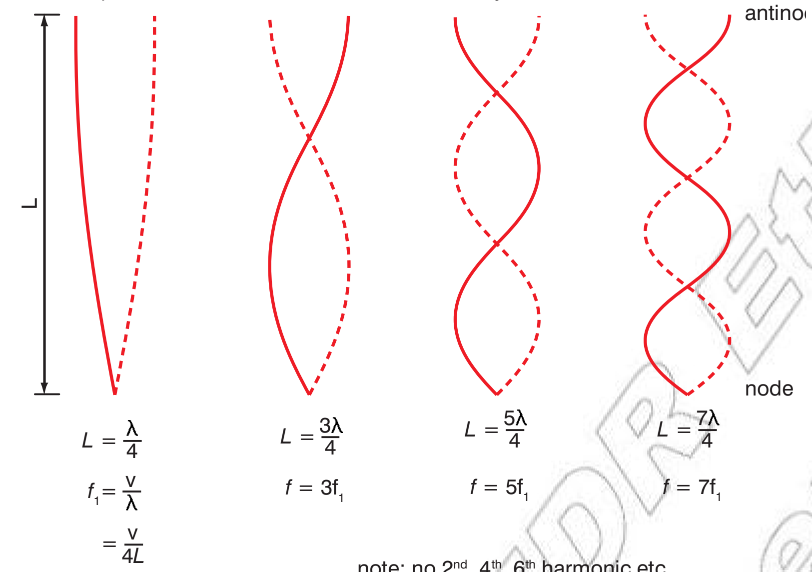


Figure 2.46 *Resonant modes of vibration for air in a narrow column with one end closed and one end open. The diagrams show amplitude of oscillation; they should not be taken as implying that the waves are simple transverse waves.*

For the *n*th harmonic: $f_n = nv/4L$ where $n = 1, 3, 5$, etc.

If both ends are open, then we get antinodes at both ends. Apart from having antinodes at the ends rather than nodes, this produces the same rules for frequencies and harmonics as a string.

The fundamental occurs for $L = \frac{\lambda}{2}$, and all harmonics, odd and even, can be produced.

Standing waves can also be created using sound in open air. If two loudspeakers are set up, facing each other and some distance apart, playing the same single tone then we have two travelling waves of the same wavelength travelling in opposite directions. This creates a standing wave, but if there is any perfect node it can only be at the centre point between the two speakers, if they are identical and producing the same amplitude.

DID YOU KNOW?

A drum skin has different resonant modes of vibration, just as a string does but in two dimensions rather than one. Where you hit the drum affects which modes are excited and hence the sound the drum produces.

Worked example 2.6

A narrow pipe is 20 cm long and is open at the top and closed at the bottom. Given that the speed of sound is 340 m/s, what frequency sounds might it be possible to produce by blowing across the top?

The wavelength at the fundamental frequency is four times the length of the pipe and therefore

- $\lambda = 0.8$ m

This corresponds to frequency of

- $f = \frac{v}{\lambda} = \frac{340}{0.8} = 425$ Hz

and therefore it might be possible to produce sounds at odd multiples of 425 Hz, i.e. at 425 Hz, 1275 Hz, 2125 Hz, etc.

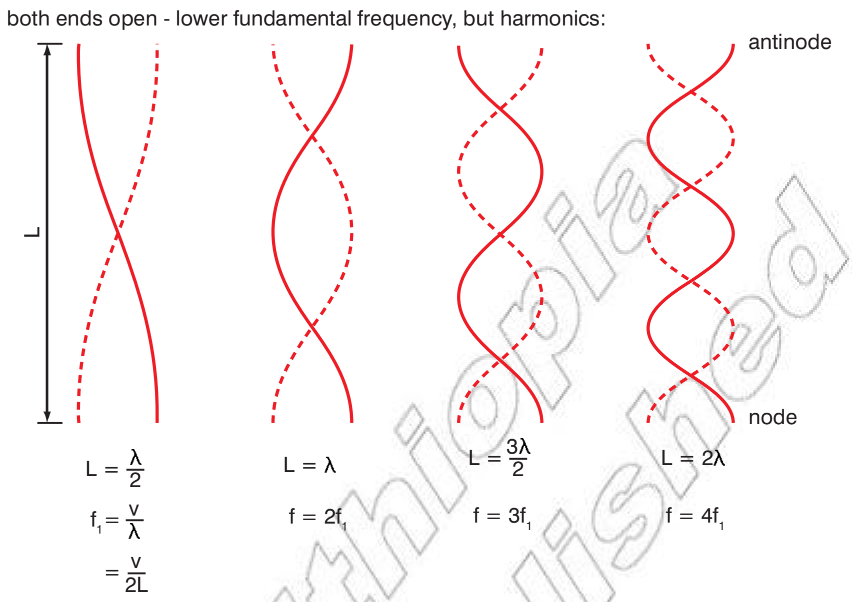


Figure 2.47 *Resonant modes of vibration for air in a narrow column with both ends open. The diagrams show amplitude of oscillation; they should not be taken as implying that the waves are simple transverse wave.*

DID YOU KNOW?

Reflections of microwaves off building, mountainsides, etc., set up standing wave patterns with nodes or places where the signal strength is very weak. This is one of the biggest problems to be overcome in making mobile phones work.

For the *n*th harmonic: $f_n = nv/2L$ where $n = 1, 2, 3$, etc.

At any other point the sound from one speaker will be louder than from the other and, although a local minimum amplitude will be produced, the complete cancellation to produce a perfect node cannot happen. Standing waves can also be created by directing sound from a single speaker towards a wall, along a path perpendicular to the wall. The wall reflects the sound wave and this produces a standing wave but, as the sound spreads out as it travels, the amplitudes of incident and reflected waves are only similar close to the wall.

Beats

DID YOU KNOW?

Musicians use the phenomenon of beats to help them tune instruments. If two instruments playing the same note are slightly out of tune, they can hear the beats. If they adjust one of the instruments closer in frequency to the other the beats become slower and slower and when the beat disappears altogether the musicians know the notes really are the same.

If two notes, differing in frequency by a few hertz are played at similar amplitudes, the phenomenon of beats can be heard. The resulting sound will be at the average of the two frequencies, but it will get louder and quieter at a frequency equal to the difference between the two frequencies. This can be predicted mathematically. If we add two sine waves at frequencies $f + \Delta f$ and $f - \Delta f$, we obtain:

- $\sin(2\pi(f - \Delta f)t) + \sin(2\pi(f + \Delta f)t) = 2\cos(2\pi\Delta f\, t)\sin(2\pi f t)$

which describes a sine wave, $\sin(2\pi f t)$, whose slowly varying amplitude is given by $2\cos(2\pi\Delta f\, t)$, where Δf is half of the frequency difference between the two sine waves. The result is shown in Figure 2.48. The time between nulls, instants of no sound, is half of the period of $\cos(2\pi\Delta f\, t)$, and is hence the time period corresponding to the frequency difference. In other words, if we play two notes at frequencies f_1 and f_2 the resulting sound has a beat frequency of

- $f_B = |f_2 - f_1| = \dfrac{1}{T_B}$

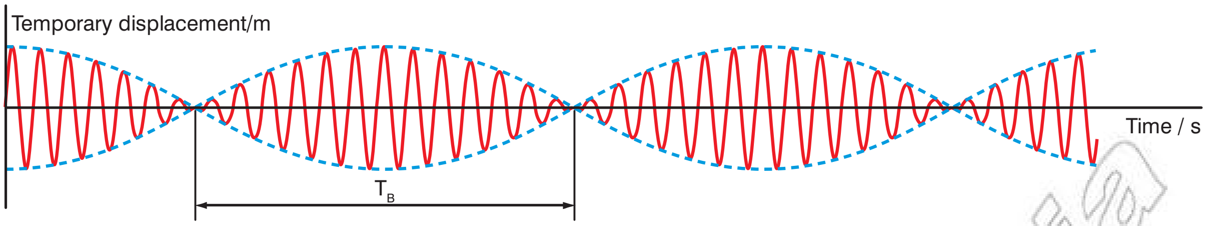


Figure 2.48 *Beats produced by the addition of two sine waves of the same amplitude but slightly different frequencies. The resultant sound is at the average of the two frequencies (red sine wave) but with an amplitude that increases and decreases at a "beat frequency" that is the difference between the two frequencies.*

Summary

In this section you have learnt that:

- A travelling wave transfers energy (and information) with no permanent movement of mass.
- A transverse wave is one in which the oscillation is perpendicular to the direction of propagation.
- A longitudinal wave is one in which the oscillation is along (parallel to) the direction of propagation.
- Wave speed $v = f\lambda$.
- The travelling wave function is $Y = A\sin\left(2\pi\frac{x}{\lambda} - 2\pi ft\right)$.
- The principle of superposition states that if two or more waves pass through a single point then the resultant instantaneous displacement at that point is the sum of the displacements that would be created separately by each wave.
- The phase difference between two oscillations is the angular difference in their timing, e.g. a half cycle difference is a phase difference of π radians.
- Constructive interference is the production of large oscillations by superposition of oscillations due to waves in phase with each other.
- Destructive interference occurs when oscillations due to waves are in antiphase with each other, and completely or partially cancel each other out.
- Nodes are points in a standing wave where no oscillation occurs.
- Antinodes are points in standing waves where the amplitude of oscillation is at a maximum.
- The distance between nodes in a standing wave is $\frac{\lambda}{2}$.

- At a fixed end of a string or closed end of an air column, travelling waves are reflected with a phase shift of π radians, giving rise to a node in the standing wave produced.
- At a free end of a string or open end of an air column, travelling waves are reflected with zero phase shift, giving rise to an antinode in the standing wave produced.
- The fundamental mode of vibration of a string or air column is the standing wave at the lowest possible frequency.
- Harmonics are integer multiples of the fundamental frequency
- A string with fixed ends supports standing waves such that $n\frac{\lambda}{2} = L$, where L is the length of the string and n is an integer.
- For a pipe with one closed end and one open end the fundamental mode of vibration occurs for $n\frac{\lambda}{4} = L$, and only odd harmonics can occur.

Review questions

1. If the speed of sound in air is 340 m/s, what is the wavelength of a sound wave at 512 Hz?
2. What is the frequency of red light with a wavelength (in free space) of 630 nm?
3. Two sinusoidal waves both have a frequency of 200 Hz. The amplitude of one is 1 cm, the amplitude of the other is 2 cm. Sketch graphs of displacement against time for the oscillations produced by each wave separately and their resultant at a point P where they cross, if
   a) they arrive at P in phase, and
   b) they arrive at P in antiphase.
4. A travelling wave on a string, of amplitude 2 mm, frequency 500 Hz and speed 300 m/s, can be described by the function

   $$Y = A\sin\left(2\pi\frac{x}{\lambda} - 2\pi ft\right)$$

   a) Sketch graphs of displacement Y against distance x for this wave, for the first 1.2 m:
      i) for time $t = 0$, and
      ii) for time $t = 0.5$ ms
   b) Sketch graphs of displacement Y against time t for the oscillation produced by this wave for the first 4 ms
      i) at the source where $x = 0$, and
      ii) at a distance $x = 30$ cm from the source.

5. A violin string is 32 cm long and has a mass per unit length of 2 g/m. What tension is required for the string to produce an A, i.e. for its fundamental frequency to be 440 Hz?
6. Imagine that you swinging backwards and forwards on a child's swing and you are listening to music coming from a loudspeaker in front of you. Explain why the music might not sound right.
7. A pipe, 68 cm long, is open at one end and closed at the other. When air is blown across the open end sound is produced at 110 Hz.
   a) What is the velocity of sound along the pipe?
   b) Blowing harder will produce a higher note. What is the next frequency that the pipe can produce?
8. In an experiment to measure the speed of sound in air, a speaker directs sound towards a wall, along a path perpendicular to the wall. The wall reflects the sound wave and this produces a standing wave. A microphone and electronic measuring device is used to measure the amplitude of the sound at different distances from the wall. Minimum values of amplitude are detected at 28 cm when the frequency used is 600 Hz.
   a) What is the measured speed of sound?
   b) Explain why the minimum values of sound are not zero and why they are more difficult to detect further from the wall.

2.3 Sound, loudness and the human ear

By the end of this section you should be able to:

- Define the intensity of sound and state the relationship between intensity and distance from the source.
- Describe the dependence of the speed of sound on the bulk modulus and density of the medium. Use $v = \sqrt{B/\rho}$
- Give intensity of sound in decibels, and define the terms threshold of pain and threshold of hearing.
- Describe the intensity level versus frequency graph to know which the human ear is most sensitive to.
- Analyse resonance conditions in air columns in quantitative terms.
- Explain the Doppler effect, and predict in qualitative terms the frequency change that will occur in a variety of conditions.
- Describe some practical applications of the Doppler Effect.

KEY WORDS

loudness *the audible strength of a sound, which depends on the amplitude of the sound wave*

intensity *the energy received by each square metre of a surface per second*

power per unit area *the power received by a square metre of a surface*

Sound loudness and intensity

The **loudness** of sound is difficult to measure scientifically. How loud a sound appears depends very on the listener, it is quite subjective. In general, the louder the sound, the greater its **intensity**.

Intensity can be defined precisely. Intensity at a point is then energy flowing through a unit area (1 m^2) per unit time (1 s). We know that the energy in an oscillation is equal to the kinetic energy at the equilibrium position

$= \frac{1}{2}mv_0^{\,2} = \frac{1}{2}m\omega^2 A^2.$

The mass of air with a cross section of 1 m^2 disturbed by a plane sound wave each second is the density of air ρ times the volume disturbed, and the volume disturbed is 1 m^2 time the distance travelled by the wave each second (its velocity). Hence, the intensity of a sound wave is given by

$I = \frac{1}{2}\rho v \omega^2 A^2,$

where ρ is the density of the air, v is the velocity of sound, $\omega = 2\pi f$ where f is the frequency and A is the amplitude of oscillation. Hence we see that intensity, measured in W/m^2, is proportional to amplitude squared and to frequency squared.

Sound intensity is defined as the sound **power per unit area**. The usual context is the measurement of sound intensity in the air at a listener's location. The basic units are watts per m^2 or W/m^2.

Alternatively the intensity of sound from source can be calculated assuming the sound spreads out equally in all directions. You may recall the intensity from a point source is given by:

- $I = \frac{P}{A}$

where

P = power of the source in W

A = area through which the sound is transmitted.

If we assume the sound travels out equally in all directions then the area covered is equal to the surface area of a sphere. So the equation becomes:

- $I = \frac{P}{A} = \frac{P}{4\pi r^2}$

where r is the distance from the source in m.

You may recall from Grade 9 this is an inverse square relationship, is you double the distance the sound intensity falls by 4 (2^2).

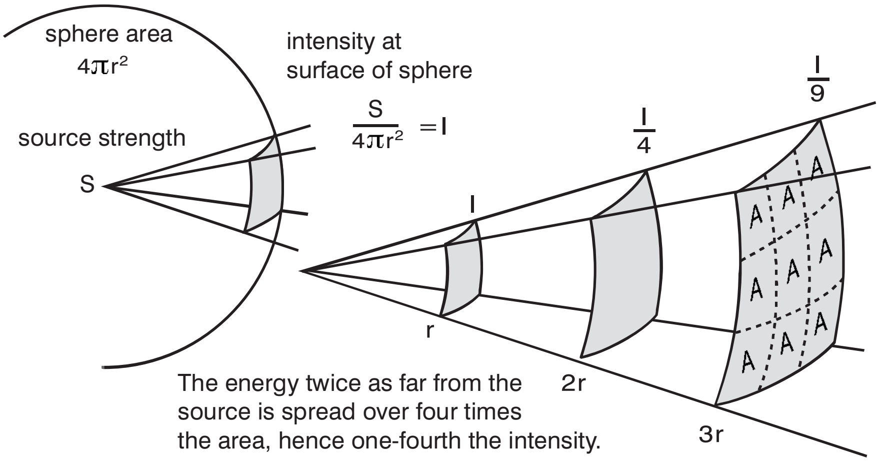


Figure 2.49 *The intensity from the source varies as an inverse square relationship.*

Worked example 2.7

A large explosion is detected 200 m away. The intensity at this distance is measured to be 400 MW/m². A few seconds later the explosion is recorded 3.2 km away. Find the power of the original explosion and the intensity recorded at the larger distance.

- $I = \frac{P}{A} = \frac{P}{4\pi r^2}$ — *State the relationship to be used*
- $P = IA = I \times 4\pi r^2$ — *Rearrange to make P the subject*
- $P = 400 \times 10^6 \times 4 \times \pi \times 200^2$ — *Substitute known values*
- $P = 2.0 \times 10^{14}$ W — *Solve for P and give the units*

We can now use the intensity equation to determine the intensity at 3.2 km.

- $I = \frac{P}{A} = \frac{P}{4\pi r^2}$ — *State the relationship to be used*
- $I = 2.0 \times 10^{14} / (4 \times \pi \times 3200^2)$ — *Substitute known values*
- $I = 1.6 \times 10^6$ W/m² — *Solve for I and give the units*

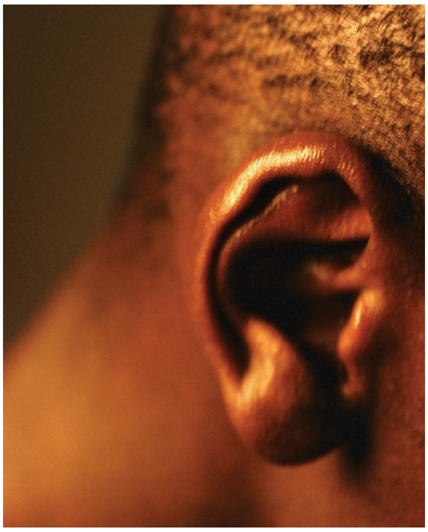


Figure 2.50 *The human ear can detect tiny changes in pressure.*

Hearing and the decibel

Many sound intensity measurements are made relative to the **threshold of hearing** intensity I_o.

This is defined as:

- $I_o = 1 \times 10^{-12}$ W/m²

One common way to measure the loudness of sound is to use the **decibel** scale. The decibel (dB) is a logarithmic unit of measurement. Decibels measure the ratio of a given intensity I to the threshold of hearing intensity I_o. This means so that I_o has the value 0 decibels (0 dB).

The intensity of a sound in dB is given by:

- $I(dB) = 10\log_{10}\left(\frac{I}{I_o}\right)$

The human ear is incredibly sensitive to sound. I_o represents a pressure change of less than one billionth of standard atmospheric pressure. In reality the actual threshold of the average human is closer to 2.5×10^{-12} W/m². This corresponds to 4 dB.

The threshold of hearing varies with frequency. The ear is most sensitive to sounds between 2000 and 5000 Hz. This can be seen on a **hearing curve**.

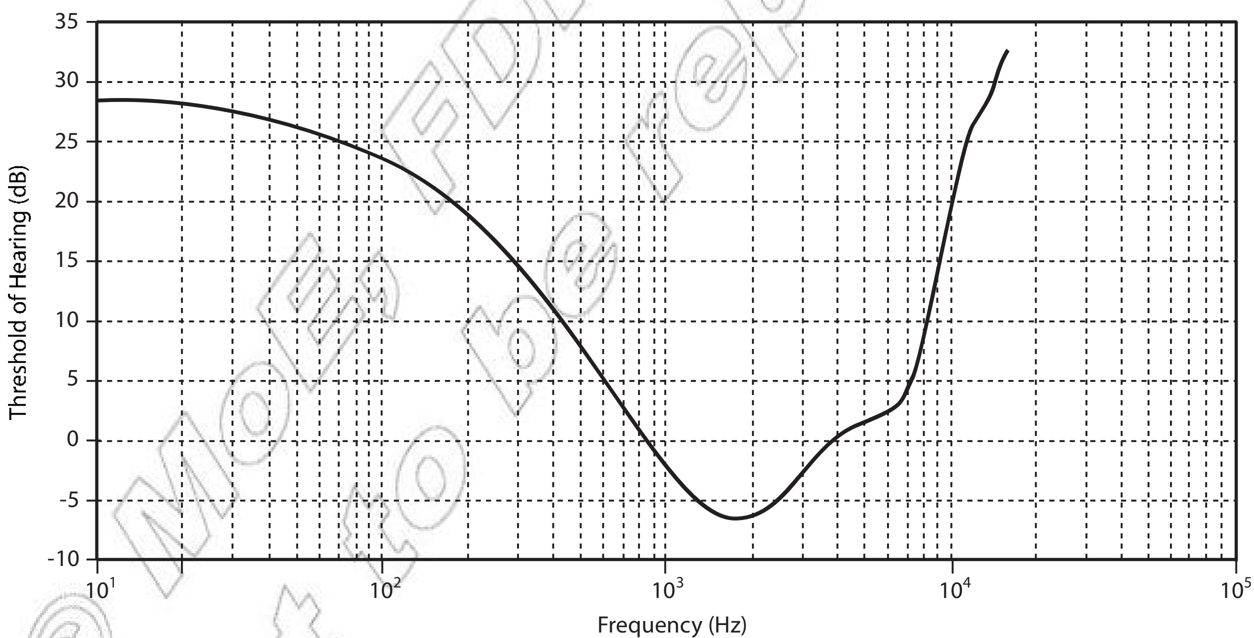


Figure 2.51 *A hearing curve shows how the threshold of hearing varies with frequency.*

This curve illustrates why certain sounds at the same intensity appear to have different volumes. The human ear is simply better a detecting some frequencies of sound than others. The exact shape of the curve depends on a number of factors including age, exposure to loud sounds and the physical characteristics of the ear.

The upper limit of human hearing is also rather subjective. A common upper limit is the **threshold of pain**. This is the point at

KEY WORDS

threshold of hearing *a standard threshold against which measurements of sound intensity are made. It is equal to 0 dB intensity.*

decibel *logarithmic unit of measurement of the loudness of sound* which pain begins to be felt by the listener. This value varies from individual to individual and for a given individual over time. In general, younger people are more tolerant of loud sounds. A common value for this limit is 130 dB.

One way to express the range of human hearing is to use the standard threshold of hearing up to the threshold of pain. This represents a huge range, from I_o to $1 \times 10^{13}\ I_o$!

The speed of sound

As discussed in the previous chapter the speed of sound depends on the medium the sound is travelling through. If the sound is travelling through a solid the speed is given by:

- $v = \sqrt{\dfrac{Y}{\rho}}$

where

Y = Young's modulus of the solid (effectively a measure of the stiffness of the solid) in Pa

ρ = density of the solid in kg/m^3

If the sound is travelling through a fluid (liquid or gas) there is a similar equation:

- $v = \sqrt{\dfrac{B}{\rho}}$

where

B = bulk modulus of the fluid (effectively a measure of the compressibility of the fluid) in Pa

ρ = density of the fluid in kg/m^3

Both equations show the speed of sound increases with the 'stiffness' of the material and decreases with the density. The table below gives typical values for the Young's/bulk modulus and densities of materials. The actual values vary depending of the exact composition of the substance.

Material	Y (GPa)	B (Pa)	ρ (kg/m^3)
Air	–	1.0×10^5	1.0
Water	–	2.2×10^9	1000
Glass	40	–	4000
Steel	160	–	7500
Diamond	442	–	3500

KEY WORDS

hearing curve *a graph which shows how threshold of hearing varies with frequency*

threshold of pain *the point at which pain, caused by sound, begins to be felt by the listener*

DID YOU KNOW?

You may recall that the speed of sound through air is given by the equation

$$v = 331\sqrt{1 + \frac{T_k}{273°\ \text{C}}}\ \text{m/s}$$

Think about this...

Just like a transverse travelling wave a longitudinal travelling wave (like sound) can be represented using a sine or cosine function.
In this case.

$$S(x, t) = A\cos\left(\frac{2\pi x}{\lambda} - 2\pi f t\right)$$

In this equation S is used instead of y to denote horizontal displacement of a particle away from its equilibrium position.

Figure 2.52 *You hear the Doppler effect whenever a siren travels passed.*

DID YOU KNOW?

The Doppler effect (or Doppler shift), is named after Austrian physicist Christian Doppler who first explained it in 1842. The Doppler effect works with light too. Analysing the light coming from distant galaxies has shown us that they are moving away from us and allowed us to calculate how fast.

Discussion activity

If you stood in the middle of a road with a blindfold on and could hear the siren of an ambulance coming towards you, you should be much more worried if the pitch did not change at all than if the pitch started to drop a little as the sound get louder. Why?

The Doppler effect

When a vehicle with siren, such as an ambulance or police car, goes past us we notice a pronounced change in pitch. As the siren approaches the pitch is higher; as it moves away from us the pitch is lower. This is known as the **Doppler effect**, or Doppler shift.

If a sound source transmits sound in all directions, like a siren, and it is stationary, then the wavelength of the transmitted sound, and hence its frequency, is same in all directions, and this is the frequency of the sound source. However, if the source is approaching a listener the sound waves are compressed. This is because, after emitting one wave front, the source moves towards the listener and emits the next wave front from a position closer to the listener. The wavelength of the sound arriving at the listener is made shorter and, since the speed at which the sound travels, the frequency, calculated by $f = v/\lambda$ is higher. The faster the source is moving the bigger the change, or shift, in frequency. If the source is moving away from the listener, the opposite happens: the wavelength is made longer and the frequency made lower. This is shown in Figure 2.53. Exactly the same effect is observed if the source remains stationary and the listener moves towards or away from the source; it is the relative velocity that matters. The increase in wavelength, $\Delta\lambda$, is given by

- $\dfrac{\Delta\lambda}{\lambda} = \dfrac{\text{relative velocity of listener and source away from each other}}{\text{velocity of sound}}$

Worked example 2.8

You measure the frequency you hear from an ambulance siren at 466 Hz. You know that the siren actually transmits sound at a frequency of 440 Hz, and the speed of sound in air is 340 m/s. What is the velocity of the ambulance relative to you?

For f = 440 Hz the wavelength is

- $\lambda = \dfrac{v}{f} = \dfrac{340}{440} = 0.7727 \text{ m}$

For f = 466 Hz the wavelength is

- $\lambda = \dfrac{v}{f} = \dfrac{340}{466} = 0.7296 \text{ m}$

Therefore $\Delta\lambda = 0.7296 - 0.7727 = -0.0431$ m, and the velocity of the ambulance is

- $\dfrac{\Delta\lambda}{\lambda} \times \text{speed of sound} = \dfrac{-0.0431}{0.7727} \times 340 = -19.0 \text{ m/s}$

i.e. the ambulance is approaching at 19.0 m/s (= 68 km/h^{1}).

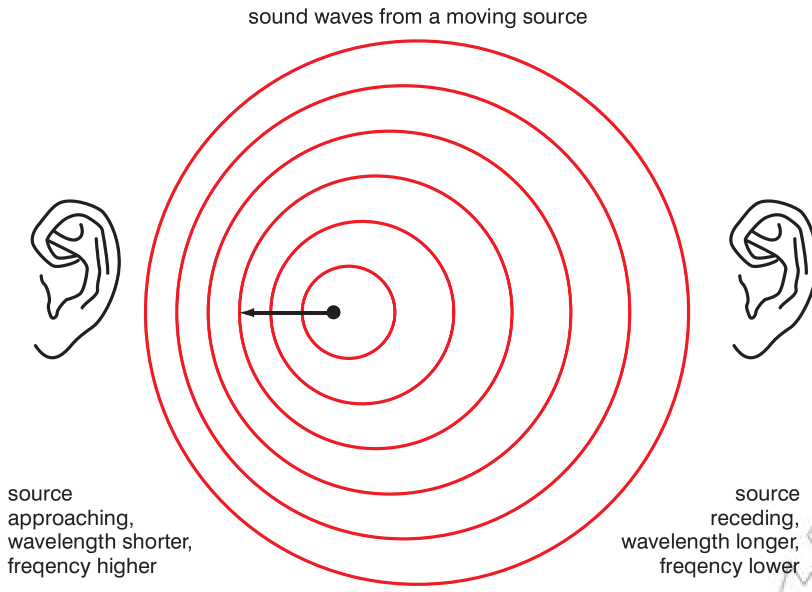


***Figure 2.53** The Doppler effect. Sound waves from an approaching source are compressed and therefore shorter, giving a higher frequency sound; waves from a receding source are stretched and therefore longer, giving a lower frequency sound.*

KEY WORDS

Doppler effect *a change in the observed frequency of a wave occurring when the source and observer are in motion relative to each other, with the frequency increasing when the source and observer approach each other and decreasing when they move apart*

There are three situations to consider:

Sound source is stationary relative to the listener

In this case the frequency received by the listener is the same as that produced by the source.

- $f_L = f_s$

Sound source is moving toward the listener (or vice-versa)

In this case the frequency received by the listener greater than the frequency produced by the source. The relationship is given by

- $f_L = f_s \dfrac{1}{1 - \frac{v_s}{v}}$

where

v_s = speed of source

v = speed of sound in air

Sound source is moving away from the listener (or vice-versa)

In this case the frequency received by the listener lower than the frequency produced by the source. The relationship is given by

- $f_L = f_s \dfrac{1}{1 + \frac{v_s}{v}}$

Applications of Doppler effect

The Doppler effect has a number of applications including:

Astronomy

Observations of the spectral lines in the visible spectrum of light from distant galaxies show a red-shift. This has been used to demonstrate the universe is expanding and is a key piece of evidence in support of the big bang theory. The Doppler effect is used to measure the speed at which stars and galaxies are approaching or receding from us.

Medical imaging and blood flow measurement

An echocardiogram is used to determine the direction and velocity of blood flow using the Doppler effect (in this case ultrasound is used).

Other flow measurements

Instruments like the laser Doppler velocimeter are used to measure velocities in a fluid flow. In this case a laser light is fired at a moving fluid. A Doppler shift is observed from reflections off of particles moving with the fluid.

Figure 2.54 *Most types of radar use the Doppler effect*

Radar

The Doppler effect is used in some types of radar. It is used to measure the velocity of a range of objects. A radar beam is fired at a moving target and reflects from the surface back to the detector. Any change in wavelength is then recorded and the object's velocity can be accurately determined. Doppler radar is used in a range of applications, including the speed of motorist, tennis serves, even the speed of a football struck towards a goal.

Summary

In this section you have learnt that:

- Intensity of sound, measured in W/m^2, is proportional to amplitude squared and to frequency squared.
- Sound intensity may be found by:

  $$I = \frac{1}{2}\rho v \omega^2 A^2 \text{ or } I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

- The threshold of hearing intensity $I_o = 1 \times 10^{-12}\ W/m^2$.
- The intensity of a sound in dB is given by:

  $$I(dB) = 10\log_{10}\left(\frac{I}{I_o}\right).$$

- One way to express the range of human hearing is to use the standard threshold of hearing up to the threshold of pain.
- The speed of sound through a solid is given by $v = \sqrt{\frac{Y}{\rho}}$.

- The speed of sound through a fluid is given by $v = \sqrt{\frac{B}{\rho}}$.
- The Doppler effect causes an increase in the frequency of the received sound wave if the sound source and listener are moving towards each other, a decrease in frequency if they are moving away from each other.
- When the source of sound is moving towards the listener
$$f_L = f_s \frac{1}{1 - \frac{v_s}{v}}$$
- When the source of sound is moving away from the listener
$$f_L = f_s \frac{1}{1 + \frac{v_s}{v}}$$

Review questions

1. Calculate the intensity in dB of a sound with an intensity of 6.2×10^{-6} W/m^2.
2. Determine the intensity of the threshold of pain for an average person.
3. Calculate the speed of sound through:
   a) water
   b) steel
   c) diamond.
4. Imagine that you swinging backwards and forwards on a child's swing and you are listening to music coming from a loudspeaker in front of you. Explain why the music might not sound right.
5. The horn of a stationary car emits sound at a frequency of 440 Hz. What frequency of note will you hear if you drive towards this car at 20 m/s? (The speed of sound in air = 340 m/s.)
6. Two cars drive along the same road towards each other, one at 15 m/s and the other at 12 m/s. Each car horn sounds at 256 Hz. Calculate the frequency that the driver of each car hears coming from the other car.
7. Describe three uses of the Doppler effect.

End of unit questions

1. A simple pendulum is made from a bob of mass 0.040 kg suspended on a light string of length 1.4 m. Keeping the string taut, the pendulum is pulled to one side until it has gained a height of 0.10 m. Calculate
   a) the total energy of the oscillation
   b) the amplitude of the resulting oscillations
   c) the period of the resulting oscillations

d) the maximum velocity of the bob

e) the maximum kinetic energy of the bob.

2. A piston in a car engine has a mass of 0.75 kg and moves with motion which is approximately simple harmonic. If the amplitude of this oscillation is 10 cm and the maximum safe operating speed of the engine is 6000 revolutions per minute, calculate:

   a) maximum acceleration of the piston

   b) maximum speed of the piston

   c) the maximum force acting on the piston.

3. An experiment is carried out to measure the spring constant of a spring. A mass of 500 g is suspended on the spring. It is pulled down a small distance and the time for 20 oscillations is measured to be 34 s.

   a) Explain why the mass performs simple harmonic motion.

   b) What is the spring constant?

   c) What is the equilibrium extension of the spring?

   d) If the mass and spring were to be moved to the surface of the Moon (where the gravitational field strength in 1.6 N/kg) what would the effect be on the time period of oscillation and on the equilibrium extension of the spring?

4. A car of mass 820 kg has an under damped suspension system. When it is driven by a driver of mass 80 kg over a long series of speed bumps 10 m apart at a speed of 3 m/s the car bounces up and down with surprisingly large amplitude.

   a) Explain why this effect occurs.

   b) Calculate the net spring constant of the car suspension system.

5. If you are given a metal rod and a hammer, how must you hit the rod to produce:

   a) a transverse wave, and

   b) a longitudinal wave?

6. A string of a musical instrument has a fundamental frequency of 196 Hz. What are the frequencies of the 2^{nd}, 3^{rd} and 4^{th} harmonics of this string.

7. A string is 1.6 m long, and waves travel along it at 2400 m/s.

   a) Sketch a labelled diagram to describe the stationary wave pattern for 4^{th} harmonic mode of vibration.

   b) Calculate the frequency of this vibration.

   c) On the same set of axes, sketch graphs of displacement against time for the oscillations 0.2 m and 0.5 m from one end of the string.

8. When tuning a piano, a musician plays a note that should be at 110 Hz while at the same time tapping a 110 Hz tuning fork and holding it next to the strings. He hears beats at 4 Hz.
   a) State and explain what frequencies the piano could be producing.
   b) Draw a sketch graph to show the resultant sound as a variation in displacement against time. Label the time axis with values.
   c) Explain how the musician now finishes tuning his piano
9. A short string will usually oscillate with smaller amplitude than a longer string. Explain the consequence for the relative loudness of different frequencies played by a string instrument if this was not the case.
10. A whistle producing a sound at 1 KHz is whirled in a horizontal circle at a speed of 18 m/s. What are the highest and lowest frequencies heard by a listener standing a few metres away, if the speed of sound in air is 340 m/s?