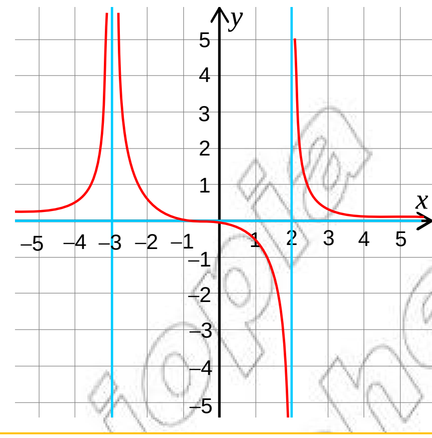
Unit 2



RATIONAL EXPRESSIONS AND RATIONAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know methods and procedures in simplifying rational expressions.*
- *understand and develop efficient methods in solving rational equations and inequalities.*
- *know basic concepts and specific facts about rational functions.*

Main Contents

INTRODUCTION

WE NOW TURN OUR ATTENTION TO FRACTIONAL FORMS. A QUOTIENT OF TWO ALGEBRAIC E DIVISION BY 0 EXCLUDED, IS CALLED A **fractional expression**. IF BOTH THE NUMERATOR AND DENOMINATOR ARE POLYNOMIALS, THE FRACTIONAL EXPRESSION IS CALLED A **rational expression**.

JUST AS RATIONAL NUMBERS ARE DEFINED IN TERMS OF QUOTIENTS OF INTEGERS, EXPRESSIONS ARE DEFINED IN TERMS OF QUOTIENTS OF POLYNOMIALS.

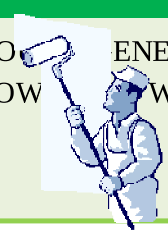
HISTORICAL NOTE

John Bernoulli (1667 – 1748)

The method of partial fractions was introduced by John Bernoulli, a Swiss Mathematician who was instrumental in the early developments of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

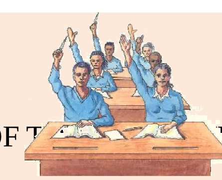
OPENING PROBLEM

ABERRA, WORKING ALONE, CAN PAINT A SMALL HOUSE IN 6 HO[illegible]ENET CAN PAINT THE SAME HOUSE IN 12 HOURS. IF THEY WORK TOGETHER, HOW [illegible] WILL IT TAKE THEM TO PAINT THE HOUSE?



2.1 SIMPLIFICATION OF RATIONAL EXPRESSIONS

ACTIVITY 2.1



DETERMINE THE DOMAIN (OR UNIVERSAL SET) OF EACH OF T[illegible]ING EXPRESSIONS.

A $x^2 + 3x - 4$ **B** LOG(x + 2) **C** $\sqrt{1-5x}$ **D** $\dfrac{3x}{x+5}$

2.1.1 Rational Expressions

Definition 2.1

A **rational expression** IS THE QUOTIENT $\frac{P(x)}{Q(x)}$ OF TWO POLYNOMIALS $P(x)$ AND $Q(x)$, WHERE $Q(x) \neq 0$. $P(x)$ IS CALLED THE **numerator** AND $Q(x)$ IS CALLED THE **denominator**.

SOME EXAMPLES OF RATIONAL EXPRESSIONS ARE THE FOLLOWING (RECALL, A NON-ZERO CO
POLYNOMIAL OF DEGREE 0):

Example 1 WHICH OF THE FOLLOWING ARE RATIONAL EXPRESSIONS?

A $\dfrac{x-2}{2x^2-3x+4}$ **B** $\dfrac{1}{x^4-1}$ **C** $\dfrac{x^3+3x-6}{4}$ **D** $\sqrt{1-5x}$

Solution ALL EXCEPT **D** ARE RATIONAL EXPRESSIONS

Example 2 EVALUATE THE RATIONAL EXPRESSION $\dfrac{2x-5}{3x+9}$ FOR THE GIVEN VALUES OF x:

A $x = 5$ **B** $x = -6$

Solution

A AT $x = 5$, $\dfrac{2x-5}{3x+9} = \dfrac{2(5)-5}{3(5)+9} = \dfrac{10-5}{15+9} = \dfrac{5}{24}$

B AT $x = -6$, $\dfrac{2x-5}{3x+9} = \dfrac{2(-6)-5}{3(-6)+9} = \dfrac{-12-5}{-18+9} = \dfrac{-17}{-9} = \dfrac{17}{9}$

Domain of a rational expression

ACTIVITY 2.2

DO THE FOLLOWING ACTIVITIES:

A FIND THE DOMAIN OF $\dfrac{x^2+2x}{5x}$.

B FACTORIZE THE NUMERATOR AND DENOMINATOR OF $\dfrac{x^2+2x}{5x}$

C SIMPLIFY $\dfrac{x^2+2x}{5x}$.

D WHEN ARE $\dfrac{x^2+2x}{5x}$ AND ITS SIMPLIFIED FORM EQUAL?

E DO STEPS **A – D** FOR THE RATIONAL EXPRESSION $\dfrac{9x^2-4}{9x^2+9x-10}$.

Note:

IN EXAMPLE 2 ABOVE, SINCE THE DENOMINATOR $3x + 9 = 0$, FOR $x = -3$, $\frac{2x-5}{3x+9}$ IS UNDEFINED WHEN $x = -3$. THEREFORE, THE DOMAIN OF $\frac{2x-5}{3x+9}$ IS $\{x : x$ IS A REAL NUMBER AND $x \neq -3\}$.

Steps to find the domain of a rational expression:

1. SET THE DENOMINATOR OF THE EXPRESSION EQUAL TO ZERO AND SOLVE.
2. THE DOMAIN IS THE SET OF ALL REAL NUMBERS EXCEPT THOSE FOUND IN STEP 1.

Example 3 FIND THE DOMAIN OF EACH OF THE FOLLOWING RATIONAL EXPRESSIONS:

A $\frac{19}{3x}$ **B** $\frac{x^2-9}{x^2-7x+10}$

Solution

A SET THE DENOMINATOR EQUAL TO ZERO AND SOLVE: $3x = 0 \Rightarrow x = 0$.

THUS, THE DOMAIN IS $\{x : x$ IS A REAL NUMBER AND $x \neq 0\}$ OR $\mathbb{R} \setminus \{0\}$.

B SET THE DENOMINATOR EQUAL TO ZERO AND SOLVE:

$$x^2 - 7x + 10 = 0 \quad \text{(FACTOR)}$$
$$(x-5)(x-2) = 0 \quad \text{(SET EACH FACTOR EQUAL TO 0 AND SOLVE)}$$
$$x - 5 = 0 \text{ OR } x - 2 = 0$$
$$x = 5 \text{ OR } x = 2$$

THUS, THE DOMAIN IS $\{x : x$ IS A REAL NUMBER AND $x \neq 2, x \neq 5\} = \mathbb{R} \setminus \{2, 5\}$

Fundamental Property of Fractions

IF a, b AND k ARE REAL NUMBERS WITH $b, k \neq 0$, THEN $\frac{ka}{kb} = \frac{a}{b}$.

Note:

USING THE ABOVE PROPERTY AND ELIMINATING ALL COMMON FACTORS FROM THE NUMERATOR AND DENOMINATOR OF A GIVEN FRACTION, IS REFERRED TO AS REDUCING (OR SIMPLIFYING) THE FRACTION TO ITS LOWEST TERM.

Definition 2.2

WE SAY THAT A RATIONAL EXPRESSION IS **reduced to lowest terms** (OR **in its lowest terms** OR **in simplest form**), IF THE NUMERATOR AND DENOMINATOR DO NOT HAVE ANY COMMON FACTOR OTHER THAN 1.

Note:

IT IS IMPORTANT TO EMPHASIZE THAT $\frac{9x^2-4}{9x^2+9x-10}=\frac{3x+2}{3x+5}$ ONLY IF $x\neq\frac{-5}{3}$ AND $x\neq\frac{2}{3}$.

THOUGH $\frac{3x+2}{3x+5}$ IS UNDEFINED AT $x=\frac{-5}{3}$ ONLY, THE ORIGINAL EXPRESSION $\frac{9x^2-4}{9x^2+9x-10}$ IS UNDEFINED AT $x=\frac{-5}{3}$ AND $x=\frac{2}{3}$. WE ARE ONLY ALLOWED TO REDUCE $\frac{3x-2}{3x-2}$ TO 1, PROVIDED THAT $3x-2\neq 0$.

To simplify a rational expression:

1. FIND THE DOMAIN.
2. FACTORIZE THE NUMERATOR AND DENOMINATOR COMPLETELY.
3. DIVIDE THE NUMERATOR AND DENOMINATOR BY ANY COMMON FACTOR (I.E. CANCEL LIKE TERMS).

Example 4 SIMPLIFY THE FOLLOWING.

A $\frac{2y^2+6y+4}{4y^2-12y-16}$. **B** $\frac{x^4+18x^2+81}{x^2+9}$. **C** $\frac{1-a}{7a^2-7}$.

Solution

A THE UNIVERSAL SET IS $\mathbb{R}\setminus\{-1, 4\}$.

THUS, $\frac{2y^2+6y+4}{4y^2-12y-16}=\frac{2(y+2)(y+1)}{4(y-4)(y+1)}=\frac{y+2}{2(y-4)}$, FOR $y\neq 4$ AND $y\neq -1$.

B $\frac{x^4+18x^2+81}{x^2+9}=\frac{(x^2+9)(x^2+9)}{x^2+9}=x^2+9$, FOR ALL $x\in\mathbb{R}$.

C $\frac{1-a}{7a^2-7}=\frac{-a+1}{7(a^2-1)}=\frac{-(a-1)}{7(a-1)(a+1)}=-\frac{1}{7(a+1)}$, FOR $a\in\mathbb{R}\setminus\{-1, 1\}$.

Exercise 2.1

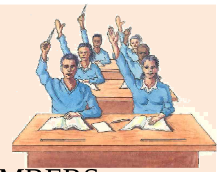
STATE THE DOMAIN AND SIMPLIFY EACH OF THE FOLLOWING:

A $\frac{4x-12}{4x}$ **B** $\frac{6x^2+23x+20}{2x^2+5x-12}$ **C** $\frac{x^3+3x^2}{x+3}$

D $\frac{x^3-27}{x^4+3x^3-27x-81}$ **E** $\frac{x^2-5x+6}{3x^3-2x^2-8x}$ **F** $\frac{x^4-8x}{3x^3-2x^2-8x}$

2.1.2 Operations with Rational Expressions

ACTIVITY 2.3



DO THE FOLLOWING IN GROUPS.

PERFORM THE FOLLOWING OPERATIONS ON RATIONAL NUMBERS.

A $\frac{5}{8}+\frac{7}{8}$ **B** $\frac{3}{4}+\frac{5}{6}$ **C** $\frac{11}{12}-\frac{4}{12}$ **D** $\frac{7}{10}-\frac{2}{5}$

Note:

RATIONAL EXPRESSIONS OBEY THE SAME RULES AS RATIONAL NUMBERS, FOR ADDITION, SU
MULTIPLICATION AND DIVISION.

Addition and subtraction of rational expressions

LET $P(x)$, $Q(x)$, AND $R(x)$ BE POLYNOMIALS SUCH THAT $Q(x) \neq 0$, THEN

$$\frac{P(x)}{Q(x)}+\frac{R(x)}{Q(x)}=\frac{P(x)+R(x)}{Q(x)} \quad \text{AND} \quad \frac{P(x)}{Q(x)}-\frac{R(x)}{Q(x)}=\frac{P(x)-R(x)}{Q(x)}.$$

Example 5 FOR EACH RATIONAL EXPRESSION, STATE THE UNIVERSAL SET AND SIMPLIFY.

A $\frac{x-5}{x+1}+\frac{x+5}{x+1}$ **B** $\frac{2x-3}{x^2+6x+9}-\frac{-x+6}{x^2+6x+9}$

C $\frac{3a+13}{a+4}-\frac{2a+7}{a+4}$ **D** $\frac{a^2-1}{a^2-7a+12}-\frac{8}{a^2-7a+12}$

Solution

A $\frac{x-5}{x+1}+\frac{x+5}{x+1}=\frac{(x-5)+(x+5)}{x+1}=\frac{2x}{x+1}$, FOR $x \neq -1$.

B $\frac{2x-3}{x^2+6x+9}-\frac{-x+6}{x^2+6x+9}=\frac{(2x-3)-(-x+6)}{x^2+6x+9}=\frac{3x-9}{x^2+6x+9}$, FOR $x \neq -3$.

C $\frac{3a+13}{a+4}-\frac{2a+7}{a+4}=\frac{(3a+13)-(2a+7)}{a+4}=\frac{a+6}{a+4}$, FOR $a \neq -4$.

D $\frac{a^2-1}{a^2-7a+12}-\frac{8}{a^2-7a+12}=\frac{(a^2-1)-8}{a^2-7a+12}=\frac{a^2-9}{a^2-7a+12}$

$$=\frac{(a-3)(a+3)}{(a-3)(a-4)}=\frac{a+3}{a-4}, \text{ FOR } a \neq 3 \text{ AND } 4.$$

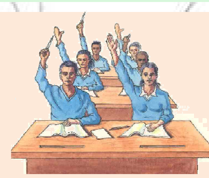
Exercise 2.2

PERFORM THE INDICATED OPERATIONS AND SIMPLIFY.

A $\dfrac{2x-3}{x^2+5x}-\dfrac{3x-5}{x^2+5x}$ **B** $\dfrac{x^2+3x}{x^2+2x-15}-\dfrac{2x+12}{x^2+2x-15}$

C $\dfrac{12}{5x}+\dfrac{x-2}{5x}$ **D** $\dfrac{6y+11}{4y^2+12y-7}-\dfrac{4y+4}{4y^2+12y-7}$

ACTIVITY 2.4



DO THE FOLLOWING ACTIVITIES:

A FACTORIZE THE DENOMINATORS AND FIND THE DOMAIN OF $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}$.

B WHAT IS THE LEAST COMMON MULTIPLE (LCM) OF THE DENOMINATORS?

C APPLY THE RULE FOR ADDITION OF RATIONAL NUMBERS TO EXPRESS $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}$ IN THE FORM $\dfrac{P(x)}{Q(x)}$

D SIMPLIFY YOUR RESULT.

Steps to add and subtract rational expressions with unlike denominators:

1. FACTORIZE THE DENOMINATORS COMPLETELY.
2. FIND THE LCM.
3. BUILD EACH RATIONAL EXPRESSION INTO AN EQUIVALENT EXPRESSION WITH THE DENOMINATOR EQUAL TO THE LCM.
4. ADD AND SUBTRACT THE NUMERATORS AND WRITE THE RESULT OVER THE COMMON DENOMINATOR.
5. SIMPLIFY THE NUMERATOR AND FACTORIZE IT TO REDUCE IT

Example 6 PERFORM THE INDICATED OPERATIONS AND SIMPLIFY.

A $3+\dfrac{2}{3y+6}+\dfrac{y-3}{y^2-4}$ **B** $\dfrac{3}{2c-1}-\dfrac{1}{c+2}-\dfrac{5}{2c^2+3c-2}$

Solution

A WE FIRST FIND THE LCM BY FACTORIZING EACH DENOMINATOR:

$$\frac{3}{1}\cdot\frac{3(y-2)(y+2)}{3(y-2)(y+2)}+\frac{2(y-2)}{3(y+2)(y-2)}+\frac{3(y-3)}{3(y-2)(y+2)}$$

$$= \frac{9(y-2)(y+2)+2(y-2)+3(y-3)}{3(y-2)(y+2)}$$

$$= \frac{9y^2-36+2y-4+3y-9}{3(y-2)(y+2)} = \frac{9y^2+5y-49}{3y^2-12}$$, FOR $y \neq$ -2 AN

B NOTICE THAT $2c^2+3c-2=(2c-1)(c+2)$. THUS, THE LCM IS $(2c-1)(c+2)$ AND

$$\frac{3}{2c-1}-\frac{1}{c+2}-\frac{5}{2c^2+3c-2} = \frac{3}{2c-1}-\frac{1}{c+2}-\frac{5}{(2c-1)(c+2)}$$

$$= \frac{3(c+2)}{(2c-1)(c+2)}-\frac{2c-1}{(2c-1)(c+2)}-\frac{5}{(2c-1)(c+2)}$$

$$= \frac{3(c+2)-(2c-1)-5}{(2c-1)(c+2)} = \frac{c+2}{(2c-1)(c+2)} = \frac{1}{2c-1}$$ FOR $c \neq -2$ AND $\frac{1}{2}$.

Exercise 2.3

PERFORM THE INDICATED OPERATIONS AND SIMPLIFY.

A $\frac{25y^2}{5y-4}+\frac{16}{4-5y}$ **B** $\frac{1}{x^2}+\frac{1}{x^2+x}$ **C** $u+1+\frac{1}{u+1}$

D $\frac{2}{a^2}-\frac{3}{ab}+\frac{4}{b^2}$ **E** $\frac{3}{x^2-x}-\frac{2}{x^2+x-2}$ **F** $\frac{x}{(x+1)^2}+\frac{2}{x+1}$

G $\frac{-50x^2-55x+8}{15x^2+x-2}-\frac{25x}{5x+2}+\frac{25x^2+15x}{3x-1}$ **H** $\frac{6}{z+4}-\frac{2}{3z+12}$

ACTIVITY 2.5

DO THE FOLLOWING IN GROUPS.

1 PERFORM THE FOLLOWING OPERATIONS ON RATIONAL NUMBERS.

A $\frac{3}{7}\times\frac{3}{5}$ **B** $\frac{5}{11}\times\frac{22}{45}$ **C** $\frac{4}{6}\div\frac{9}{12}$ **D** $\frac{7}{4}\div\frac{35}{6}$

2 WHAT ARE THE RULES USED TO SIMPLIFY THE EXPRESSIONS?

Multiplication of rational expressions

IF $P(x)$, $Q(x)$, $R(x)$ AND $S(x)$ ARE POLYNOMIALS SUCH THAT $Q(x) \neq 0$, $S(x) \neq 0$, THEN

$$\frac{P(x)}{Q(x)}\cdot\frac{R(x)}{S(x)} = \frac{P(x)R(x)}{Q(x)S(x)}$$

Steps to multiply rational expressions:

1. FACTORIZE THE NUMERATORS AND DENOMINATORS COMPLETELY
2. DIVIDE OUT ALL THE COMMON FACTORS.
3. MULTIPLY NUMERATOR WITH NUMERATOR AND DENOMINATOR WITH DENOMINATOR TO GET THE ANSWER.

Example 7 EVALUATE AND SIMPLIFY:

A $\dfrac{5x+5}{x-2}\cdot\dfrac{x^2-4x+4}{x^2-1}$ **B** $\dfrac{4x+20}{x^2+10x+25}\cdot\dfrac{x+2}{4x+8}$

Solution

A BY FIRST FACTORIZING THE NUMERATORS AND DENOMINATORS, WE GET:

$$\frac{5x+5}{x-2}\cdot\frac{x^2-4x+4}{x^2-1}=\frac{5(x+1)}{x-2}\cdot\frac{(x-2)(x-2)}{(x-1)(x+1)}=\frac{5(x-2)}{(x-1)},\text{ FOR } x\neq -1,\ 1 \text{ AND } 2.$$

B FACTORIZING THE NUMERATOR AND DENOMINATOR YIELDS:

$$\frac{4x+20}{x^2+10x+25}\cdot\frac{x+2}{4x+8}=\frac{4(x+5)}{(x+5)(x+5)}\cdot\frac{x+2}{4(x+2)}=\frac{1}{x+5},\text{ FOR } x\neq -5 \text{ AND } -2.$$

Division of rational expressions

IF $P(x)$, $Q(x)$, $R(x)$ AND $S(x)$ ARE POLYNOMIALS SUCH THAT $R(x)\neq 0$, $Q(x)\neq 0$, $S(x)\neq 0$ THEN

$$\frac{P(x)}{Q(x)}\div\frac{R(x)}{S(x)}=\frac{P(x)}{Q(x)}\cdot\frac{S(x)}{R(x)}=\frac{P(x)S(x)}{Q(x)R(x)}$$

Example 8 PERFORM THE FOLLOWING OPERATIONS AND SIMPLIFY:

A $\dfrac{36x^2-48x+16}{3x^2+13x-10}\div\dfrac{4x^2-12x+9}{2x^2+7x-15}$ **B** $\dfrac{a}{a-b}\div\dfrac{b}{a-b}$

Solution

A FIRST, YOU HAVE TO INVERT THE SECOND FRACTION AND THEN FACTORIZE EACH EXPRESSION AND SIMPLIFY.

$$\frac{36x^2-48x+16}{3x^2+13x-10}\times\frac{2x^2+7x-15}{4x^2-12x+9}=\frac{4(3x-2)(3x-2)}{(3x-2)(x+5)}\times\frac{(2x-3)(x+5)}{(2x-3)(2x-3)}$$

$$=\frac{4(3x-2)}{2x-3}\ \text{ FOR } x\neq -5,\ \frac{2}{3} \text{ AND } \frac{3}{2}.$$

B FIRST, INVERT THE SECOND FRACTION AND MULTIPLY:

$$\frac{a}{a-b}\div\frac{b}{a-b}=\frac{a}{a-b}\times\frac{a-b}{b}=\frac{a}{b},\ \text{ FOR } a\neq b \text{ AND } b\neq 0.$$

Exercise 2.4

1 EVALUATE AND SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS. STATE THEIR DOMAINS.

A $\dfrac{x^2-x-12}{x^2-9}\times\dfrac{3+x}{4-x}$ **B** $\dfrac{x^3-27}{x^2-9}\times\dfrac{x+3}{x^2+3x+9}$

2 PERFORM THE INDICATED OPERATIONS AND SIMPLIFY:

A $\dfrac{x^2-7x+12}{4-x}\times\dfrac{5}{x^2-9}$ **B** $\dfrac{2x^2-3x-2}{x^2-1}\div\dfrac{2x^2+5x+2}{x^2+x-2}$

C $\dfrac{x^2-x-6}{3x^2-12}\div\dfrac{x^2-3x}{2-x}$

2.1.3 Decomposition of Rational Expressions into Partial Fractions

SO FAR, YOU HAVE BEEN COMBINING RATIONAL EXPRESSIONS USING ADDITION, MULTIPLICATION, SUBTRACTION AND DIVISION RULES. NEXT, YOU WILL CONSIDER THE REVERSE PROCESS-DECOMPOSING A RATIONAL EXPRESSION INTO SIMPLER ONES.

WE OBTAIN THE SUM OF FRACTIONS $\dfrac{2}{x-2}$ AND $\dfrac{3}{x+1}$ AS FOLLOWS:

$$\frac{2}{x-2}+\frac{3}{x+1}=\frac{5x-4}{(x-2)(x+1)}=\frac{5x-4}{x^2-x-2}$$

THE REVERSE PROCESS OF WRITING $\dfrac{5x-4}{x^2-x-2}$ AS A SUM OR DIFFERENCE OF SIMPLE FRACTIONS

(FRACTIONS WITH NUMERATORS OF LESSER DEGREE THAN THEIR DENOMINATORS) IS FREQU IMPORTANT IN CALCULUS. EACH SUCH SIMPLE FRACTION IS CALLED A **partial fraction**, AND THE PROCESS ITSELF IS CALLED **decomposition into partial fractions**.

Definition 2.3

IN A RATIONAL EXPRESSION $\dfrac{P(x)}{Q(x)}$, IF THE DEGREE OF $P(x)$ IS LESS THAN THAT OF $Q(x)$, THEN $\dfrac{P(x)}{Q(x)}$ IS CALLED A **proper rational expression**. OTHERWISE IT IS CALLED **improper**.

FROM YOUR PREVIOUS KNOWLEDGE OF ALGEBRA, YOU KNOW THAT ANY RATIONAL EXPRESS WRITTEN AS THE SUM OF A POLYNOMIAL AND A PROPER RATIONAL EXPRESSION.

TO DECOMPOSE A RATIONAL EXPRESSION $\frac{P(x)}{Q(x)}$, THE DEGREE OF $P(x)$ MUST BE LESS THAN THE DEGREE OF $Q(x)$. IN A CASE WHERE THE DEGREE OF $P(x)$ IS GREATER THAN OR EQUAL TO THE DEGREE OF $Q(x)$, YOU HAVE ONLY TO DIVIDE $P(x)$ BY $Q(x)$ TO OBTAIN $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, WHERE THE DEGREE OF $R(x)$ IS LESS THAN THAT OF $Q(x)$. THE DECOMPOSITION IS THEN DONE ON $\frac{R(x)}{Q(x)}$.

Example 9 EXPRESS $\frac{2x^3+10x^2-3x+1}{x+3}$ AS A SUM OF A POLYNOMIAL AND A PROPER RATIONAL FRACTION.

Solution USING LONG DIVISION,

$$2x^3 + 10x^2 - 3x + 1 = (x + 3)(2x^2 + 4x - 15) + 46.$$

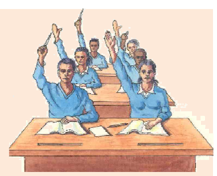
THUS, $\frac{2x^3+10x^2-3x+1}{x+3} = (2x^2+4x-15) + \frac{46}{x+3}$.

MOREOVER, YOU NEED TO RELY ON THE FOLLOWING DEFINITION TO DO THE PARTIAL DECOMPOSITION:

Definition 2.4

TWO POLYNOMIALS OF EQUAL DEGREE ARE EQUAL TO EACH OTHER, IF AND ONLY COEFFICIENTS OF TERMS OF LIKE DEGREE ARE EQUAL.

ACTIVITY 2.6



1 FACTORIZE $x^3 - 3x^2 + 2x$

2 FACTORIZE EACH OF THE FOLLOWING (IF FACTORIZABLE)

A $x^2 - 6x + 9$ B $15x^2 + 14x - 8$ C $x^2 - x + 2$

3 FOR EACH OF THE QUADRATIC POLYNOMIALS $ax^2 + bx + c$ IN QUESTION 2 ABOVE, FIND $b^2 - 4ac$. WHICH QUADRATIC POLYNOMIAL CANNOT BE FACTORIZED FURTHER? CAN W THE SIGN OF $b^2 - 4ac$ TO DECIDE WHICH QUADRATIC POLYNOMIALS CAN BE FACTORIZED?

4 FACTORIZE $x^4 + 7x^3 + 12x^2 - 7x - 13$.

✍Note:

$ax^2 + bx + c$ IS NOT REDUCIBLE IN REAL NUMBERS, IF $b^2 - 4ac < 0$.

Theorem 2.1 Linear and quadratic factor theorem

FOR A POLYNOMIAL WITH REAL COEFFICIENTS, THERE ALWAYS EXISTS A COMPLETE FACT INVOLVING ONLY LINEAR AND/OR QUADRATIC FACTORS (RAISED TO SOME POWER OF NATU $k \geq 1$), WITH REAL COEFFICIENTS, WHERE THE LINEAR AND QUADRATIC FACTORS ARE NOT RELATIVE TO REAL NUMBERS.

SO, ONCE YOU HAVE DECIDED THAT PARTIAL FRACTION DECOMPOSITION IS TO BE DONE FOR EXPRESSION, YOU FACTORIZE THE DENOMINATOR AS COMPLETELY AS POSSIBLE. THEN, FOR E IN THE DENOMINATOR, YOU CAN USE THE FOLLOWING TABLE TO DETERMINE THE TERM(S) Y IN THE PARTIAL FRACTION DECOMPOSITION. THE TABLE GIVES THE VARIOUS CASES THAT CA

	Factor in the Denominator	Corresponding term in the Partial Fraction
1	$ax + b$	$\frac{A}{ax+b}$, A CONSTANT
2	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$, $A_1, A_2, \cdots, A_k$ ARE CONSTANTS
3	$ax^2 + bx + c$ (WITH $b^2 - 4ac < 0$)	$\frac{Ax+B}{ax^2+bx+c}$, A, B ARE CONSTANTS
4	$(ax^2+bx+c)^k$ (WITH $b^2 - 4ac < 0$)	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$, $A_1, A_2, \cdots, A_k$, $B_1, B_2, \cdots, B_k$ ARE CONSTANTS.

Example 10 DECOMPOSE EACH OF THE FOLLOWING RATIONAL EXPRESSIONS INTO PARTIAL FRACTIONS:

A $\dfrac{5x+7}{x^2+2x-3}$ **B** $\dfrac{6x^2-14x-27}{(x+2)(x-3)^2}$ **C** $\dfrac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

D $\dfrac{x^3-4x^2+9x-5}{(x^2-2x+3)^2}$ **E** $\dfrac{x^3}{(x+1)(x+2)}$

Solution

A THE DENOMINATOR $x^2 + 2x - 3 = (x - 1)(x + 3)$. THE TWO FACTORS $(x - 1)$ AND $(x + 3)$ ARE DISTINCT. THUS, WE APPLY PART 1 OF THE TABLE TO GET:

$$\frac{5x+7}{x^2+2x-3} = \frac{5x+7}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

TO FIND THE CONSTANTS A AND B, WE COMBINE THE FRACTIONS ON THE RIGHT SIDE OF THE ABOVE EQUATION TO OBTAIN

$$\frac{5x+7}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

SINCE THESE EXPRESSIONS HAVE THE SAME DENOMINATOR, THEIR NUMERATORS MUST BE EQUAL.

THUS, $5x+7 = A(x+3)+B(x-1) = (A+B)x+3A-B$. USING DEFINITION 2.4, WE HAVE

$A+B=5$ AND $3A-B=7$, WHICH GIVES $A=3$ AND $B=2$.

HENCE, $\frac{5x+7}{x^2+2x-3} = \frac{3}{x-1}+\frac{2}{x+3}$.

(THIS CAN EASILY BE CHECKED BY ADDING THE TWO FRACTIONS ON THE RIGHT.)

B USING PARTS 1 AND 2 OF THE TABLE, WE WRITE

$$\frac{6x^2-14x-27}{(x+2)(x-3)^2} = \frac{A}{x+2}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$$

$$= \frac{A(x-3)^2+B(x+2)(x-3)+C(x+2)}{(x+2)(x-3)^2}$$

THUS, $6x^2-14x-27 = A(x-3)^2+B(x+2)(x-3)+C(x+2)$ WHICH HOLDS FOR ALL x.

IN PARTICULAR, IF $x=3$, THEN $-15=5C$, WHICH GIVES $C=-3$ AND IF $x=-2$, THEN $25=25A$, WHICH GIVES $A=1$.

THERE ARE NO OTHER VALUES OF x THAT WILL CAUSE TERMS ON THE RIGHT TO BE EQUAL TO ZERO. SINCE ANY VALUE x CAN BE SUBSTITUTED TO PRODUCE AN EQUATION RELATING A, B, AND C, WE LET $x=0$ AND OBTAIN

$-27 = 9A-6B+2C$ (SUBSTITUTE $A=1$ AND $C=-3$)

$-27 = 9-6B-6 \Rightarrow B=5$

THUS, $\frac{6x^2-14x-27}{(x+2)(x-3)^2} = \frac{1}{x+2}+\frac{5}{x-3}-\frac{3}{(x-3)^2}$.

C FOR x^2-x+1, $b^2-4ac=-3<0$. THUS, IT CANNOT BE FACTORIZED FURTHER IN THE REAL NUMBERS. USING, PARTS 1 AND 3 OF THE TABLE:

$$\frac{5x^2-8x+5}{(x-2)(x^2-x+1)} = \frac{A}{x-2}+\frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x-2)}{(x-2)(x^2-x+1)}$$

THUS, FOR ALL x, $5x^2-8x+5 = A(x^2-x+1)+(Bx+C)(x-2)$.

IF $x=2$, THEN 9 = 3A WHICH GIVES A = 3.

IF $x=0$, THEN USING $A=3$, WE HAVE 5 = 3 – 2C SO THAT C = –1

IF $x = 1$, THEN USING $A = 3$ AND $C = -1$, YOU HAVE B = 2.

HENCE, $\dfrac{5x^2 - 8x + 5}{(x-2)(x^2 - x + 1)} = \dfrac{3}{x-2} + \dfrac{2x-1}{x^2 - x + 1}$.

D SINCE $x^2 - 2x + 3$ CANNOT BE FACTORIZED FURTHER IN THE REAL NUMBERS, YOU PROCEED TO USE PART 4 OF THE TABLE, AS SHOWN BELOW

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} = \frac{Ax + B}{x^2 - 2x + 3} + \frac{Cx + D}{(x^2 - 2x + 3)^2}$$

$$= \frac{(Ax + B)(x^2 - 2x + 3) + (Cx + D)}{(x^2 - 2x + 3)^2}$$

THUS, FOR ALL x, $x^3 - 4x^2 + 9x - 5 = (Ax + B)(x^2 - 2x + 3) + Cx + D$

$$= Ax^3 + (B - 2A)x^2 + (3A - 2B + C)x + (3B + D)$$

EQUATING COEFFICIENTS OF TERMS OF LIKE DEGREE, WE OBTAIN

$A = 1$; $B - 2A = -4$; $3A - 2B + C = 9$ AND $3B + D = -5$

FROM THESE EQUATIONS WE FIND THAT $A = 1$, $B = -2$, $C = 2$ AND $D = 1$. NOW YOU CAN WRITE

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} = \frac{x - 2}{x^2 - 2x + 3} + \frac{2x + 1}{(x^2 - 2x + 3)^2}.$$

E THIS IS NOT A PROPER RATIONAL EXPRESSION.

NOTE THAT $(x + 1)(x + 2) = x^2 + 3x + 2$. DIVIDE x^3 BY $x^2 + 3x + 2$. IT GIVES A QUOTIENT $x - 3$ AND REMAINDER $7x + 6$.

THEREFORE, $\dfrac{x^3}{(x+1)(x+2)} = x - 3 + \dfrac{7x+6}{(x+1)(x+2)}$.

NOW USING THE USUAL METHOD, $\dfrac{7x+6}{(x+1)(x+2)} = \dfrac{-1}{x+1} + \dfrac{8}{x+2}$.

HENCE, $\dfrac{x^3}{(x+1)(x+2)} = (x-3) - \dfrac{1}{x+1} + \dfrac{8}{x+2}$

Exercise 2.5

WRITE EACH OF THE FOLLOWING RATIONAL EXPRESSIONS IN PARTIAL FRACTIONS:

A $\dfrac{7x+6}{x^2 + x - 6}$ **B** $\dfrac{5x+7}{(x-1)(x^2 + x + 2)}$ **C** $\dfrac{3x+5}{(x-2)^2}$

D $\dfrac{(x+3)^2}{(x^2+1)(x+3)}$ **E** $\dfrac{x^2 + 4x - 3}{x - 2}$ **F** $\dfrac{7x^2 - 11x + 6}{(x-1)(2x^2 - 3x + 2)}$

2.2 RATIONAL EQUATIONS

YOU ALREADY KNOW HOW TO SOLVE LINEAR AND QUADRATIC EQUATIONS. IN THIS SUBUNI DISCUSS THE SOLUTION OF RATIONAL EQUATIONS.

ACTIVITY 2.7

STATE THE UNIVERSAL SET AND SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $\frac{2}{3}=\frac{x}{3}$ **B** $x+2-3(x-2)=0$

C $\frac{x}{3}+\frac{3x}{4}=2$ **D** $2(10x+3)=5x+6$

E THE STAFF MEMBERS OF A SCHOOL AGREED TO CONTRIBUTE 2000 BIRR TO MAKE UP A FUND TO HELP NEEDY STUDENTS IN THE SCHOOL. SINCE THEN, TWO N MEMBERS HAVE JOINED THE STAFF, AND AS A RESULT, EACH MEMBER'S SHARE BEEN REDUCED BY 5 BIRR. HOW MANY MEMBERS ARE NOW ON THE STAFF?

Definition 2.5

A **rational equation** IS AN EQUATION THAT CAN BE REDUCED TO THE FORM $\frac{P(x)}{Q(x)}=0$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS AND $Q(x) \neq 0$.

Note:

TO SOLVE A RATIONAL EQUATION, YOU CAN MULTIPLY BOTH SIDES OF THE EQUATION BY THE DENOMINATORS FOR THOSE VALUES OF THE VARIABLE FOR WHICH THE LCM IS NON-ZERO. AN THING TO KEEP IN MIND IS THAT THOSE VALUES THAT CAUSE THE DENOMINATOR TO BECOME SOLUTIONS TO THE EQUATION. A NUMBER THAT LOOKS TO BE A SOLUTION BUT CAUSES THE D OF THE ORIGINAL EQUATION TO BECOME 0 IS CALLED AN **extraneous solution**.

To solve rational equations, you follow the following steps:

1. FACTORIZE ALL THE DENOMINATORS AND DETERMINE THEIR LCM.
2. RESTRICT THE VALUES OF THE VARIABLE THAT MAKE THE LCM EQUAL TO 0.
3. MULTIPLY BOTH SIDES OF THE RATIONAL EQUATION BY THE LCM AND SIMPLIFY.
4. SOLVE THE RESULTING EQUATION.
5. CHECK THE ANSWERS AGAINST THE RESTRICTED VALUES IN STEP 2. ANY SUCH VALUE MUST BE EXCLUDED FROM THE SOLUTION.

Example 1 SOLVE EACH OF THE FOLLOWING EQUATIONS:

A $\dfrac{2}{x+1}=\dfrac{3}{x-2}$ **B** $\dfrac{x}{x+4}-\dfrac{4}{x-4}=\dfrac{x^2+16}{x^2-16}$

C $\dfrac{10}{x(x-2)}+\dfrac{4}{x}=\dfrac{5}{x-2}$ **D** $\dfrac{3a-5}{a^2+4a+3}+\dfrac{2a+2}{a+3}=\dfrac{a-1}{a+1}$

Solution

A YOUR RESTRICTIONS ARE $x\neq -1$ AND $x\neq 2$. NOW, MULTIPLY BOTH SIDES OF THE EQUATION BY THEIR LCM $(x+1)(x-2)$:

$$\frac{2}{x+1}=\frac{3}{x-2}\Rightarrow\left(\frac{2}{x+1}\right)\left(\frac{(x+1)(x-2)}{1}\right)=\left(\frac{3}{x-2}\right)\left(\frac{(x+1)(x-2)}{1}\right)$$

$$2(x-2)=3(x+1)\quad\Rightarrow\; x=-7$$

THIS DOES NOT CONTRADICT OUR RESTRICTIONS THAT $x\neq -1$ AND $x\neq 2$.

THUS, OUR SOLUTION SET IS $\{-7\}$.

B YOUR RESTRICTIONS ARE $x\neq -4$ AND $x\neq 4$. NOW MULTIPLY BOTH SIDES BY THE LCM $(x-4)(x+4)$, WHICH WILL GET RID OF THE DENOMINATORS:

$$x(x-4)-4(x+4)=x^2+16\Rightarrow\; x^2-8x-16=x^2+16\Rightarrow -8x=32\Rightarrow\; x=-4$$

THIS IS AGAINST OUR RESTRICTION $x\neq -4$, AND MUST BE EXCLUDED FROM OUR SOLUTION. SINCE THERE ARE NO OTHER VALUES IN OUR SOLUTION, THE SOLUTION IS $\varnothing$.

C THE LCM HERE WILL BE $x(x-2)$, AND x CANNOT BE 0 OR 2. MULTIPLYING BOTH SIDES OF THE EQUATION BY THIS DENOMINATOR:

$$\left(\frac{10}{x(x-2)}\right)\left(\frac{x(x-2)}{1}\right)+\left(\frac{4}{x}\right)\left(\frac{x(x-2)}{1}\right)=\left(\frac{5}{x-2}\right)\left(\frac{x(x-2)}{1}\right)$$

$$\Rightarrow 10+4(x-2)=5x\Rightarrow 10+4x-8=5x\Rightarrow\; 4x+2=5x$$

$$\Rightarrow x=2.$$

BUT $x=2$ IS NOT ALLOWED. THUS, THE SOLUTION SET IS $\varnothing$.

D SINCE $a^2+4a+3=(a+3)(a+1)$, THE LCM IS $(a+3)(a+1)$, WHERE a CANNOT BE -3 OR -1. NOW YOU CAN MULTIPLY BOTH SIDES BY THE LCM

$$\left(\frac{3a-5}{(a+3)(a+1)}\right)\left(\frac{(a+3)(a+1)}{1}\right)+\left(\frac{2a+2}{a+3}\right)\left(\frac{(a+3)(a+1)}{1}\right)=\left(\frac{a-1}{a+1}\right)\left(\frac{(a+3)(a+1)}{1}\right)$$

$$\Rightarrow 3a-5+(2a+2)(a+1)=(a-1)(a+3)$$

WHEN SIMPLIFIED THIS GIVES $a^2+5a=0$ OR $a(a+5)=0$.

THIS GIVES US $a = 0$ OR $a = -5$.

THESE DO NOT CONTRADICT OUR RESTRICTIONS $a \neq -3$ AND $a \neq -1$.

THUS, OUR SOLUTION SET IS $\{-5, 0\}$.

Example 2 ONE INTEGER IS FOUR LESS THAN FIVE TIMES ANOTHER. THE SUM OF THEIR RECIPROCALS IS $\frac{2}{3}$. WHAT ARE THE INTEGERS?

Solution WHEN WE ENCOUNTER SUCH WORD PROBLEMS OR PROBLEMS OF REAL LIFE, WE FIRST ASSIGN VARIABLES TO THE UNKNOWNS. NOW, LET THE UNKNOWN INTEGE x AND y. THEN ONE IS FOUR LESS THAN FIVE TIMES ANOTHER CAN BE WRITTEN $x = 5y - 4$, FOR x, y INTEGERS.

THE SUM OF THEIR RECIPROCALS IS $\frac{1}{x} + \frac{1}{y} = \frac{2}{3}$.

SUBSTITUTING FOR x, WE GET: $\frac{1}{5y-4} + \frac{1}{y} = \frac{2}{3}$.

THIS RATIONAL EQUATION REDUCES TO THE QUADRATIC EQUATION

$5y^2 - 13y + 6 = 0$, WITH SOLUTIONS $y = 2$ AND $y = \frac{3}{5}$.

BUT, SINCE $\frac{3}{5}$ IS NOT AN INTEGER, THE ONLY SOLUTION FOR y IS 2. THEREFORE WE GET $x = 6$.

THUS THE REQUIRED INTEGERS ARE 2 AND 6.

Exercise 2.6

1 STATE THE UNIVERSAL SET AND SOLVE EACH OF THE FOLLOWING RATIONAL EQUATIONS:

A $\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}$

B $\frac{x-6}{x} = \frac{x+4}{x} + 1$

C $\frac{4}{a} = \frac{1}{a^2+4a} - \frac{a+3}{a^2+4a}$

D $\frac{2}{x-4} - \frac{3}{x+1} = \frac{6}{x-1}$

E $\frac{3x-2}{5} = \frac{4x}{7}$

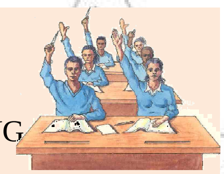
F $\frac{x+4}{x-5} - \frac{1}{x+5} = \frac{10}{x^2-25}$

2 TWO PLANES LEAVE AN AIRPORT FLYING AT THE SAME RATE. IF THE FIRST PLANE FLIES 1.5 HOURS LONGER THAN THE SECOND PLANE AND TRAVELS 2700 MILES WHILE THE SECOND PLAN ONLY 2025 MILES, FOR HOW LONG WAS EACH PLANE FLYING?

3 A TREE CASTS A SHADOW OF 34 FEET AT THE TIME WHEN A 3 FOOT TALL CHILD CASTS A SHADOW OF 1.7 FEET. WHAT IS THE HEIGHT OF THE TREE?

2.3 RATIONAL FUNCTIONS AND THEIR GRAPHS

ACTIVITY 2.8



IDENTIFY THE TYPES (NAMES) OF EACH OF THE FOLLOWING STATE THEIR DOMAINS.

A $f(x) = 3x + 5$ **B** $g(x) = 4 - x + 3x^2$

C $f(x) = log(x + 1)$ **D** $g(x) = 2^{3x+2}$

E $f(x) = 5\cos x$ **F** $g(x) = \sqrt{9 - x^2}$

2.3.1 Rational Functions

Definition 2.6

A **rational function** IS A FUNCTION OF THE FORM $f(x) = \dfrac{p(x)}{q(x)}$, WHERE $p(x)$ AND $q(x)$ ARE POLYNOMIALS AND $q(x) \neq 0$.

Example 1 WHICH OF THE FOLLOWING ARE RATIONAL FUNCTIONS?

A $f(x) = \dfrac{x-2}{2x^3 + x^2 - x}$ **B** $g(x) = \dfrac{x^2 + 3x + 2}{1}$ **C** $h(x) = \sqrt{9 - x^2}$

Solution f AND g ARE RATIONAL FUNCTIONS, WHILE h IS NOT.

$g(x) = \dfrac{x^2 + 3x + 2}{1}$ IS THE SAME AS $y = x^2 + 3x + 2$, SO, ANY POLYNOMIAL FUNCTION IS A RATIONAL FUNCTION.

Note:

A A RATIONAL FUNCTION IS SAID TO BE IN LOWEST TERMS, IF $p(x)$ AND $q(x)$ HAVE NO COMMON FACTOR OTHER THAN 1.

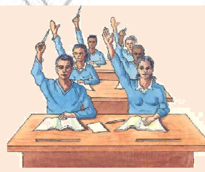
B THE DOMAIN OF A RATIONAL FUNCTION f IS THE SET OF ALL REAL NUMBERS EXCEPT THE VALUES OF x THAT MAKE THE DENOMINATOR $q(x)$ ZERO.

Example 2 GIVE THE DOMAIN OF THE FUNCTION $f(x)=\dfrac{x+1}{2x^2+5x-3}$.

Solution THE DENOMINATOR $2x^2+5x-3=0 \Rightarrow (2x-1)(x+3)=0 \Rightarrow x=\frac{1}{2}$ OR $x=-3$.

THUS, OUR DOMAIN IS THE SET OF ALL REAL NUMBERS EXCEPT $\frac{1}{2}$ AND -3, BECAUSE $\frac{1}{2}$ AND -3 BOTH MAKE THE DENOMINATOR EQUAL TO 0.

ACTIVITY 2.9



GIVEN THE RATIONAL FUNCTIONS $f(x)=\dfrac{1}{x}$ AND $g(x)=\dfrac{x}{x-2}$, FIND THE FOLLOWING FUNCTIONAL VALUES AND PLOT THE CORRESPONDING POINTS ON THE COORDI

1. A $f(2)$ B $f(-3)$ C $f(0.4)$ D $f(-1.5)$
2. A $g(0)$ B $g(3)$ C $g(-2)$ D $g(2.5)$

Group Work 2.1

DO THE FOLLOWING IN GROUPS.

CONSIDER THE FUNCTION $f(x)=\dfrac{1}{x}$.

1. WHAT IS ITS DOMAIN?
2. A FILL IN THE FOLLOWING TABLE FOR VALUES OF x TO THE LEFT OF 0:

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$						$\to -\infty$

   B FILL IN THE FOLLOWING TABLE FOR VALUES OF x TO THE RIGHT OF 0:

x	1	0.5	0.1	0.01	0.001	$\to 0$
$f(x)$						$\to \infty$

3. COMPLETE THE FOLLOWING SENTENCES:

   AS x APPROACHES 0 FROM THE LEFT, $f(x)$ ____ WITHOUT BOUND.

   AS x APPROACHES 0 FROM THE RIGHT, $f(x)$ ____ WITHOUT BOUND.

   AS x INCREASES OR DECREASES WITHOUT BOUND, $f(x)=\dfrac{1}{x}$ APPROACHES THE VALUE OF ____

4 HERE IS THE GRAPH OF $f(x)=\frac{1}{x}$.

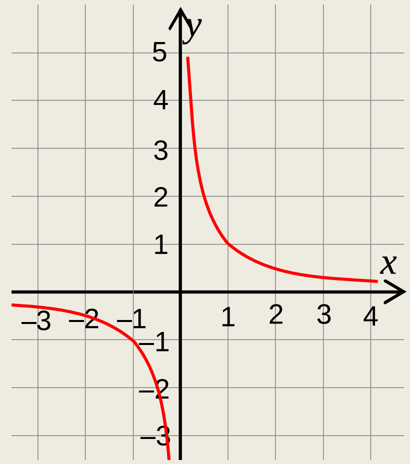


Figure 2.1

DO YOUR OBSERVATIONS CORRESPOND WITH THE GRAPH?

Note:

THESE TWO BEHAVIOURS OF f NEAR $x = 0$ ARE DENOTED AS FOLLOWS.

A $f(x) \to -\infty$ AS $x \to 0^-$

B $f(x) \to \infty$ AS $x \to 0^+$

IN THIS CASE, THE LINE $x = 0$ (THE y-AXIS) IS CALLED A **vertical asymptote** OF THE GRAPH OF f.

IN ADDITION, WE HAVE:

C $f(x) \to 0$ AS $x \to -\infty$

D $f(x) \to 0$ AS $x \to \infty$

HERE, THE LINE $y = 0$ (THE x-AXIS) IS CALLED A **horizontal asymptote** OF THE GRAPH OF f.

Definition 2.7

1 THE LINE $x = a$ IS CALLED A **vertical asymptote** OF THE GRAPH OF f IF $f(x) \to \infty$ OR $f(x) \to -\infty$ AS $x \to a$, EITHER FROM THE LEFT OR FROM THE RIGHT.

2 THE LINE $y = b$ IS CALLED A **horizontal asymptote** OF THE GRAPH OF f IF $f(x) \to b$ AS $x \to \infty$ OR AS $x \to -\infty$.

Rules for asymptotes and holes

ONCE THE DOMAIN IS ESTABLISHED AND THE RESTRICTIONS ARE IDENTIFIED, HERE ARE THE PERTINENT FACTS.

Note:

LET $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{ax^n+...+a_0}{bx^m+...+b_0}$, BE A RATIONAL FUNCTION, WHERE n IS THE LARGEST EXPONENT IN THE NUMERATOR AND m IS THE LARGEST EXPONENT IN THE DENOMINATOR.

1 THE GRAPH WILL HAVE A VERTICAL ASYMPTOTE AT $x = a$ IF $q(a) = 0$ AND $p(a) \neq 0$. IN CASE $p(a) = q(a) = 0$, THE FUNCTION HAS EITHER A HOLE AT $x = a$ OR REQUIRES FURTHER SIMPLIFICATION TO DECIDE.

2 IF $n < m$, THEN THE x-AXIS IS THE HORIZONTAL ASYMPTOTE.

3 IF $n = m$, THEN THE LINE $y = \dfrac{a}{b}$ IS A HORIZONTAL ASYMPTOTE.

4 IF $n = m + 1$, THE GRAPH HAS AN OBLIQUE ASYMPTOTE AND WE CAN FIND IT BY LONG DIVISION.

5 IF $n > m + 1$, THE GRAPH HAS NEITHER AN OBLIQUE NOR A HORIZONTAL ASYMPTOTE.

Example 3 GIVE THE VERTICAL AND HORIZONTAL ASYMPTOTES, IF THEY EXIST:

A $f(x)=\dfrac{1}{x+2}$ **B** $f(x)=\dfrac{x-2}{x^2-4}$

C $f(x)=\dfrac{x^2-1}{x^2+3x+2}$ **D** $f(x)=\dfrac{(x-1)(x+1)}{(x+1)^2(x+2)}$

Solution

A $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{1}{x+2}$. THE DOMAIN OF f IS $\{x : x \neq -2\}$.

SINCE $p(-2) \neq 0$ AND $q(-2) = 0$, $x = -2$ IS A VERTICAL ASYMPTOTE.

BESIDES DEGREE OF $p(x)$ < DEGREE OF $q(x)$. THUS, $y = 0$ IS A HORIZONTAL ASYMPTOTE..

B CONSIDER THE RATIONAL FUNCTION $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{x-2}{x^2-4}$. THE DOMAIN OF f IS ALL REAL NUMBERS EXCEPT $x = -2$ AND $x = 2$.

$p(-2) \neq 0$ AND $q(-2) = 0$. THUS, $x = -2$ IS A VERTICAL ASYMPTOTE.

$p(2) = q(2) = 0$. THUS, f HAS A HOLE AT $\left(2, \dfrac{1}{4}\right)$

CANCELLING OUT THE COMMON FACTOR $x-2$, WE OBTAIN $g(x)=\frac{1}{x+2}$, AND $g(2)=\frac{1}{4}$.

THERE $n = 1$ AND $m = 2$. THEREFORE, THE x-AXIS IS A HORIZONTAL ASYMPTOTE.

C FACTORIZING NUMERATOR AND DENOMINATOR GIVES:

$$f(x)=\frac{x^2-1}{x^2+3x+2}=\frac{p(x)}{q(x)}=\frac{(x-1)(x+1)}{(x+1)(x+2)}$$

AT $x = -1$, $p(-1) = q(-1) = 0$. REDUCING TO LOWEST TERMS WE HAVE:

$$g(x)=\frac{x-1}{x+2}$$, WHICH GIVES $g(-1) = -2 \neq 0$. THUS, f HAS A HOLE AT $(-1, -2)$.

AT $x = -2$, $p(-2) = 3 \neq 0$ AND $q(-2) = 0$. THUS $x = -2$ GIVES A VERTICAL ASYMPTOTE.

SINCE THE DEGREE OF THE NUMERATOR IS EQUAL TO THE DEGREE OF THE DENOMINAT
IS A HORIZONTAL ASYMPTOTE.

D $f(x)=\frac{p(x)}{q(x)}=\frac{(x-1)(x+1)}{(x+1)^2(x+2)}$, $p(-2) \neq 0$ AND $q(-2) = 0$. THUS $x = -2$ IS A VERTICAL ASYMPTOTE. AGAIN $p(-1) = q(-1) = 0$. HOWEVER, AFTER SIMPLIFICATION BY FACTORIZING, WE FIND THAT $x = -1$ IS A VERTICAL ASYMPTOTE.

SINCE $n = 2 < 3 = m$, THE x-AXIS IS A HORIZONTAL ASYMPTOTE.

Example 4 FIND THE OBLIQUE ASYMPTOTE OF THE FUNCTION $f(x)=\frac{x^2+1}{x-1}$

Solution SINCE THE DEGREE OF THE NUMERATOR IS ONE MORE THAN THE DEGREE OF THE DENOMINATOR, THE GRAPH OF f HAS AN OBLIQUE ASYMPTOTE. APPLYING LONG DIVISION YIELDS:

$$f(x)=\frac{x^2+1}{x-1}=(x+1)+\frac{2}{x-1}.$$

THUS, THE EQUATION OF THE OBLIQUE ASYMPTOTE IS THE QUOTIENT PART OF THE
WHICH WOULD BE $y = x+1$.

ACTIVITY 2.10

FOR EACH OF THE FOLLOWING RATIONAL FUNCTIONS, FIND THE DOMAIN AND IDENTIFY THE TYPE OF ASYMPTOTES.

A $f(x)=\frac{3}{x+4}$ **B** $f(x)=\frac{2x+1}{x}$ **C** $f(x)=\frac{x-3}{x^2-9}$

D $f(x)=\frac{x^2-x}{x+1}$ **E** $f(x)=\frac{4x}{1-3x}$ **F** $f(x)=\frac{x^2-x-2}{x-1}$

The zeros of a rational function

Definition 2.8

LET $f(x)=\dfrac{p(x)}{q(x)}$ BE A RATIONAL FUNCTION. AN ELEMENT a IN THE DOMAIN OF f IS CALLED A ZERO OF f IF AND ONLY IF $f(a) = 0$.

Example 5 FIND THE ZEROS OF THE FOLLOWING RATIONAL FUNCTIONS:

A $f(x)=\dfrac{x^2+3x+2}{x^2-2x-3}$ **B** $f(x)=\dfrac{x^2-6x+9}{x^2-9}$

Solution

A WE FIRST FACTORIZE BOTH NUMERATOR AND DENOMINATOR.

$f(x)=\dfrac{(x+1)(x+2)}{(x+1)(x-3)}$ THE DOMAIN IS $\mathbb{R}\setminus\{-1, 3\}$. NOW FOR ANY x IN THE DOMAIN, $f(x) = 0$ MEANS THE NUMERATOR $x^2 + 3x + 2 = 0$. I.E. $x = -1$ OR $x = -2$. BUT, SINCE $x = -1$ IS NOT IN THE DOMAIN OF f, THE ONLY ZERO OF f IS $x = -2$.

B FACTORIZE BOTH NUMERATOR AND DENOMINATOR: $f(x)=\dfrac{(x-3)^2}{(x+3)(x-3)}$

THE DOMAIN IS $\mathbb{R}\setminus\{-3, 3\}$. THE NUMERATOR IS ZERO AT $x = 3$. BUT SINCE 3 IS NOT IN THE DOMAIN, f HAS NO ZERO.

2.3.2 Graphs of Rational Functions

IN THIS SUBSECTION, YOU WILL USE THE ZEROS AND ASYMPTOTES OF RATIONAL FUNCTIONS TO DRAW THEIR GRAPHS.

Steps to sketch the graph of a rational function:

1. REDUCE THE RATIONAL FUNCTION TO LOWEST TERMS AND CHECK FOR ANY OPEN HOLES IN THE GRAPH.
2. FIND x-INTERCEPT(S) BY SETTING THE NUMERATOR EQUAL TO ZERO.
3. FIND THE y-INTERCEPT (IF THERE IS ONE) BY SETTING $x = 0$ IN THE FUNCTION.
4. FIND ALL ITS ASYMPTOTES (IF ANY).
5. DETERMINE THE PARITY (I.E. WHETHER IT IS EVEN OR ODD OR NEITHER).

6 USE THE x-INTERCEPTS AND VERTICAL ASYMPTOTE(S) TO DIVIDE THE x-AXIS INTO INTERVALS. CHOOSE A TEST POINT IN EACH INTERVAL TO DETERMINE IF THE FUNCTION IS POSITIVE OR NEGATIVE THERE. THIS WILL TELL YOU WHETHER THE GRAPH APPROACHES THE ASYMPTOTE IN AN UPWARD OR DOWNWARD DIRECTION.

7 SKETCH THE GRAPH! EXCEPT FOR BREAKS AT THE VERTICAL ASYMPTOTES OR CUSPS, THE GRAPH SHOULD BE A NICE SMOOTH CURVE WITH NO SHARP CORNERS.

TO DRAW THE GRAPH OF $f(x)=\dfrac{p(x)}{q(x)}$,

We need to find	Criteria
DOMAIN	$\mathbb{R}\setminus\{x: q(x)=0\}$
x - INTERCEPT	ZERO OF f
y - INTERCEPT	$x=0$ AND $0 \in$ DOMAIN OF f
VERTICAL ASYMPTOTE	$P(x)\neq 0$ AND $q(x)=0$
HORIZONTAL ASYMPTOTE	DEGREE OF $p(x) \leq$ DEGREE OF $q(x)$
OBLIQUE ASYMPTOTE	DEGREE OF $p(x)$ = DEGREE OF $q(x)+1$
PARITY	f IS ODD OR EVEN OR NEITHER

Group Work 2.2

DO THE FOLLOWING IN GROUPS. FOR EACH OF THE FUNCTIONS, FIND THE DOMAIN, x-INTERCEPT, y-INTERCEPT, THE ASYMPTOTES AND THE PARITY (IF THEY EXIST). LIST THEM IN TABLES.

A $f(x)=\dfrac{x+1}{(x-2)(x+3)^2}$ **B** $f(x)=\dfrac{x^2+5x+6}{x+1}$ **C** $f(x)=\dfrac{x-2}{x^2-4}$

Example 6 SKETCH THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x)=-\dfrac{1}{x^2}$. **B** $f(x)=\dfrac{3x^2}{(x-2)(x+1)}$

C $f(x)=\dfrac{x+1}{(x-2)(x+3)^2}$ **D** $f(x)=\dfrac{x^2+5x+6}{x+1}$.

E $f(x)=\dfrac{x-2}{x^2-4}$.

Solution

A THE FUNCTION $f(x) = -\frac{1}{x^2}$ CANNOT BE REDUCED ANY FURTHER. THIS MEANS THAT THERE WILL BE NO OPEN HOLES ON THE GRAPH OF THIS FUNCTION.

x - INTERCEPT	NONE
y - INTERCEPT	NONE
VERTICAL ASYMPTO	$x = 0$
HORIZONTAL ASYM	$y = 0$
OBLIQUE ASYMPTO	NONE
PARITY	f IS EVEN

NEXT, WE FIND AND PLOT SEVERAL OTHER POINTS ON THE GRAPH.

x	-2	-1	1	2
$y = -\frac{1}{x^2}$	$-\frac{1}{4}$	-1	-1	$-\frac{1}{4}$

THIS TABLE IS CALLED **a table of values**.

FINALLY, WE DRAW CURVES THROUGH THE POINTS, APPROACHING THE ASYMPTOTES.

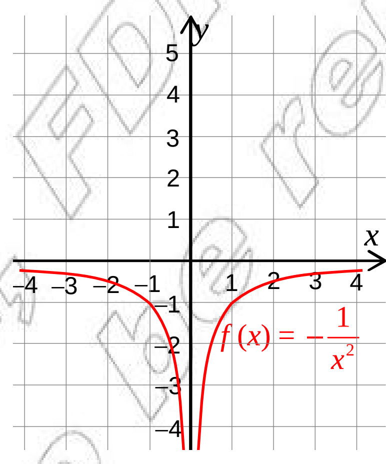


Figure 2.2

B THE FUNCTION $f(x) = \frac{3x^2}{(x-2)(x+1)}$ CANNOT BE REDUCED ANY FURTHER. THIS MEANS THAT THERE WILL BE NO OPEN HOLES ON THE GRAPH OF THIS FUNCTION.

x - INTERCEPT	$x = 0$ OR $(0, 0)$
y - INTERCEPT	$y = 0$ OR $(0, 0)$
VERTICAL ASYMPTOTE	$x = -1$ AND $x = 2$
HORIZONTAL ASYMPTOTE	$y = 3$ NOTE THAT THE GRAPH CROSSES THE HORIZONTAL ASYMPTOTE AT $x = -2$.
OBLIQUE ASYMPTOTE	NONE
PARITY	f IS NEITHER EVEN NOR ODD. YOU CAN CHECK THIS BY TAKING A TEST POINT. FOR INSTANCE, $f(-4) \neq f(4)$ AND $f(-4) \neq -f(4)$.

NEXT, WE FIND AND PLOT SEVERAL OTHER POINTS ON THE GRAPH.

x	−3	1	4	5
y	2.7	–1.5	4.8	4.17

FINALLY, WE DRAW CURVES THROUGH THE POINTS, APPROACHING THE ASYMPTOTES.

THUS, THE GRAPH OF f IS:

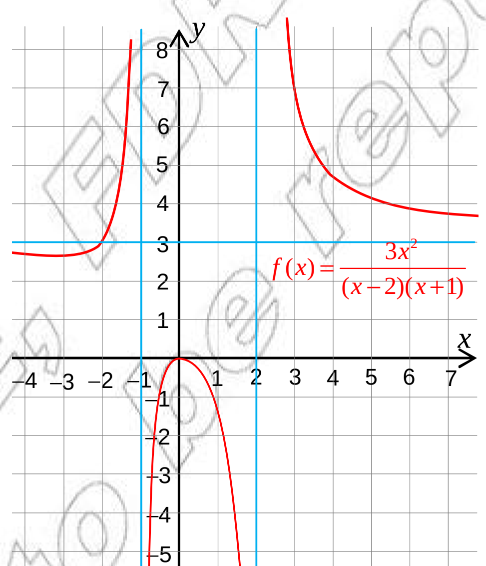


Figure 2.3

YOU HAVE ALREADY FOUND THE NECESSARY DATA TO SKETCH THE GRAPHS OF THE FUNCTIONS IN C, D, AND E IN GROUP WORK 2.4. WE ONLY NEED TO GIVE THE SKETCHES OF THE GRAPHS.

✍Note:

IF $f(x)=\dfrac{p(x)}{q(x)}$ IS IN LOWEST TERMS AND $(x-a)^n$ IS A FACTOR OF $q(x)$, THEN

✓ THE GRAPH OF f GOES IN OPPOSITE DIRECTIONS ABOUT THE VERTICAL ASYMPTOTE $x=a$ WHEN n IS ODD.

✓ THE GRAPH OF f GOES IN THE SAME DIRECTION ABOUT THE VERTICAL ASYMPTOTE $x=a$ WHEN n IS EVEN.

C

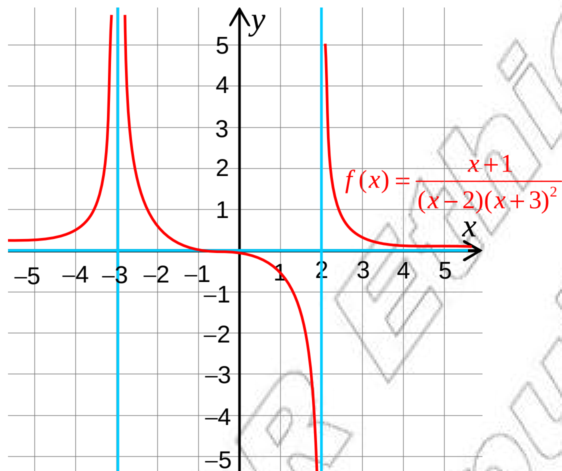


Figure 2.4

D

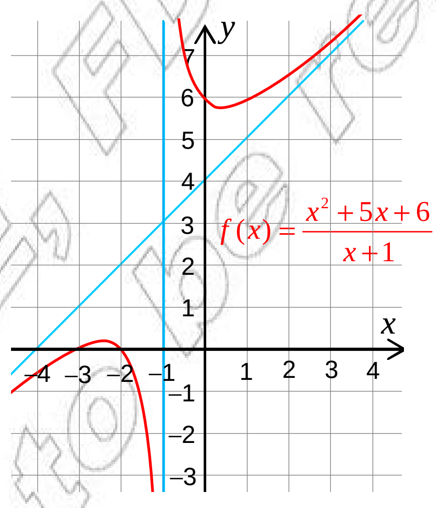


Figure 2.5

OBSERVE THAT THE GRAPH OF f APPROACHES THE LINE $y=x+4$ AS x APPROACHES ∞ OR AS x APPROACHES $-\infty$.

E

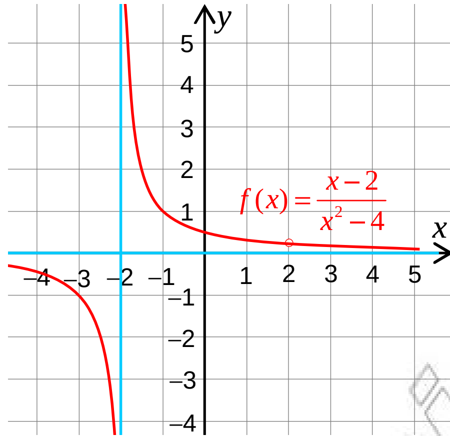
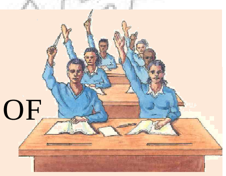


Figure 2.6

ACTIVITY 2.11



USING THE GRAPHS DRAWN IN EXAMPLE 6 ABOVE, SOLVE EACH OF THE FOLLOWING INEQUALITIES.

A **I** $-\dfrac{1}{x^2}<0$ **II** $-\dfrac{1}{x^2}>0$

B **I** $\dfrac{3x^2}{(x-2)(x+1)}<0$ **II** $\dfrac{3x^2}{(x-2)(x+1)}>0$

C **I** $\dfrac{x+1}{(x-2)(x+3)^2}<0$ **II** $\dfrac{x+1}{(x-2)(x+3)^2}>0$

D **I** $\dfrac{x^2+5x+6}{x+1}<0$ **II** $\dfrac{x^2+5x+6}{x+1}>0$

E **I** $\dfrac{x-2}{x^2-4}<0$ **II** $\dfrac{x-2}{x^2-4}>0$

Exercise 2.7

1 SKETCH THE GRAPH OF EACH OF THE FOLLOWING RATIONAL FUNCTIONS:

A $f(x)=\dfrac{x^2-x-12}{x^2-2x-8}$ **B** $f(x)=\dfrac{3x^2-5x-2}{x^2-1}$ **C** $f(x)=\dfrac{x^3+1}{x^2-1}$

D $f(x)=\dfrac{x^2+3x-4}{x-5}$ **E** $f(x)=\dfrac{2x^2-3x+2}{x^2+1}$ **F** $f(x)=\dfrac{x+2}{x^2-9}$

G $f(x)=\dfrac{-x}{x^2+x-2}$ **H** $f(x)=\dfrac{(x-1)(x+3)}{(x-2)}$ **I** $f(x)=\dfrac{x^2-9}{x-3}$

2 FOR THE RATIONAL FUNCTIONS IN QUESTION 1B, D AND H SOLVE THE INEQUALITIES $f(x)>0$ AND $f(x)<0$ FROM THEIR GRAPHS.

Key Terms

- domain
- graphs of rational functions
- horizontal asymptote
- least common multiple
- oblique asymptote
- operations on rational expressions
- partial fractions
- rational equations
- rational expression
- rational functions
- vertical asymptote
- zeros of a rational function

Summary

1 A **rational expression** IS THE QUOTIENT OF TWO POLYNOMIALS.

2 LET $P(x)$, $Q(x)$, AND $R(x)$ BE POLYNOMIALS SUCH THAT $Q(x) \neq 0$. THEN

$$\frac{P(x)}{Q(x)} + \frac{R(x)}{Q(x)} = \frac{P(x)+R(x)}{Q(x)} \quad \text{AND} \quad \frac{P(x)}{Q(x)} - \frac{R(x)}{Q(x)} = \frac{P(x)-R(x)}{Q(x)}.$$

3 IF $P(x)$, $Q(x)$, $R(x)$ AND $S(x)$ ARE POLYNOMIALS SUCH THAT $Q(x) \neq 0$, $S(x) \neq 0$, THEN

$$\frac{P(x)}{Q(x)} \cdot \frac{R(x)}{S(x)} = \frac{P(x)R(x)}{Q(x)S(x)} \quad \text{AND} \quad \frac{P(x)}{Q(x)} \div \frac{R(x)}{S(x)} = \frac{P(x)S(x)}{Q(x)R(x)} \quad \text{FOR } R(x) \neq 0$$

4 A **rational equation** IS AN EQUATION WHERE ONE OR MORE OF THE TERMS ARE FRACTIONAL ONES.

5 A **rational function** IS A FUNCTION OF THE FORM $f(x) = \dfrac{p(x)}{q(x)}$, WHERE $p(x)$ AND $q(x)$ ARE POLYNOMIALS AND $q(x) \neq 0$

6 THE LINE $x = a$ IS CALLED A **vertical asymptote** OF THE GRAPH OF f, IF $f(x) \to \pm\infty$ AS $x \to a$ FROM THE LEFT OR THE RIGHT.

7 THE LINE $y = b$ IS CALLED A **horizontal asymptote** OF THE GRAPH OF f, IF $f(x) \to b$ AS $x \to \pm\infty$.

8 AN ASYMPTOTE OF THE FORM $y = mx + b$, $m \neq 0$, IS CALLED AN **oblique asymptote**.

9 A **zero** OF $f(x) = \dfrac{p(x)}{q(x)}$ IS A VALUE a FOR WHICH $p(a) = 0$ BUT $q(a) \neq 0$.

Review Exercises on Unit 2

1 SIMPLIFY EACH OF THE FOLLOWING RATIONAL EXPRESSION.

A $\dfrac{2x-4}{x^2+x-6}$ **B** $\dfrac{x^2-x-6}{x^2+3x+2}$

C $\dfrac{x^2-5x}{x^2-25}$ **D** $\dfrac{x^3+8x^2+24x+45}{x^4+3x^3-27x-81}$

2 PERFORM THE INDICATED OPERATIONS AND SIMPLIFY.

A $\dfrac{x+5}{2}+\dfrac{x-5}{2}$ **B** $\dfrac{2x^2}{x+9}-\dfrac{162}{x+9}$

C $\dfrac{\dfrac{2}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{1}{x^2-1}}$ **D** $\dfrac{x^2-1}{x^2+3x-4}\bullet\dfrac{x^2+x-12}{x^2+4x+3}$

E $\dfrac{x^2-25}{(x-5)^2}\div\dfrac{x^2+16}{(x+4)^2}$ **F** $\dfrac{x}{x^3-1}\div\left[2-\dfrac{1}{1+\dfrac{1}{x-2}}\right]$

3 DECOMPOSE THE FOLLOWING RATIONAL EXPRESSIONS INTO PARTIAL FRACTIONS.

A $\dfrac{3}{x^2-3x}$ **B** $\dfrac{x+1}{x^2+4x+3}$ **C** $\dfrac{2x-3}{(x-1)^2}$

D $\dfrac{x+1}{x^3+x}$ **E** $\dfrac{x-1}{x^3+x^2}$ **F** $\dfrac{5x+1}{x^2(x^2+4)}$

4 STATE THE DOMAIN AND SOLVE EACH OF THE FOLLOWING EQUATIONS.

A $\dfrac{4}{x^2}=\dfrac{5}{x}-\dfrac{1}{x^2}$ **B** $\dfrac{x-6}{x}=\dfrac{x+4}{x}+1$

C $\dfrac{3}{y+3}+\dfrac{3y}{y+3}=1$ **D** $\dfrac{1}{y^2-3y}+\dfrac{1}{y-3}=\dfrac{3}{y^2-3y}$

5 STATE THE DOMAIN AND SKETCH THE GRAPH OF EACH OF THE RATIONAL FUNCTIONS. FIND INTERCEPTS AND ASYMPTOTES, IF THERE ARE ANY.

A $f(x)=\dfrac{x-3}{x+2}$ **B** $g(x)=\dfrac{3}{(x-5)^2}$ **C** $f(x)=\dfrac{x^2}{x^2+1}$

D $g(x)=\dfrac{5x}{x^2-4}$ **E** $f(x)=x+\dfrac{1}{x^2}$ **F** $g(x)=\dfrac{2x^3}{x^2+1}$